**Definition 1.** *Let $a, b, c$ be non-negative integers. We use $\mathcal{G}(a, b, c)$ to denote an undirected graph with three groups of vertices, namely $A$, $B$, and $C$ with $|A| = a, |B| = b, |C| = c$ such that $A \cup B$ forms a clique and $B \cup C$ forms a clique.*

Note that $\mathcal{G}(a, b, 0)$ is a graph formed by the union of two cliques $K_a$ and $K_b$ of size $a$ and $b$, respectively.

**Theorem 1.** *Any graph with $n \leq a + b$ vertices and $m \leq \lfloor (a + b)/2 \rfloor - 1$ edges $(0 < a \leq b)$ is always a subgraph of $\mathcal{G}(a, b, 0)$ (or equivalently, a subgraph of $\mathcal{G}(b, a, 0)$).*

*Proof.* We shall prove this theorem by induction.

**(Basis Case:)** If $a = b = 1$, then $\lfloor (a+b)/2 \rfloor - 1 = 0$. Any graph with $n \leq 2$ vertices and $m \leq 0$ edges (i.e., no edges) is always a subgraph of $G(1, 1, 0)$.

**(Inductive Case:)** Suppose that the theorem holds for all $a + b \leq k$. Our target is to show that the theorem also holds for the case $a + b = k + 1$ with $a \leq b$. Consider a graph $G$ with $n \leq k + 1$ vertices and with $m \leq \lfloor (k + 1)/2 \rfloor - 1$ edges.

1. If $G$ is connected, then $n \leq m + 1 \leq (k + 1)/2 \leq b$, which is a subgraph of $K_b$, and thus a subgraph of $\mathcal{G}(a, b, 0)$.

2. Otherwise, $G$ is not connected. If $G$ has no edges, then $G$ is obviously a subgraph of $\mathcal{G}(a, b, 0)$ since $G$ has at most $a + b$ vertices. Else, let $C$ be the connected component of $G$ with the largest number $n'$ of vertices (so that $n' \geq 2$. Then, the number of edges in $C$ is at least $n' - 1$. To complete the proof, it is sufficient to show that $G - C$ is a subgraph of $\mathcal{G}(a, b - n', 0)$, as we can map $C$ as a subgraph in $K_b$.

   The number of vertices in $G - C$ is $k + 1 - n' = a + (b - n')$, and the number of edges of $G - C$ is at most

$$
\begin{aligned}
m - (n' - 1) \leq \lfloor (k + 1)/2 \rfloor - n' &= \lfloor (k + 1)/2 - n' \rfloor \\
&= \lfloor (k + 1 - n')/2 - n'/2 \rfloor \\
&\leq \lfloor (k + 1 - n')/2 - 1 \rfloor \\
&= \lfloor (k + 1 - n')/2 \rfloor - 1 = \lfloor (a + (b + n'))/2 \rfloor - 1.
\end{aligned}
$$

   By induction hypothesis, $G - C$ is a subgraph of $\mathcal{G}(a, b - n', 0)$, and consequently $G$ is a subgraph of $\mathcal{G}(a, b, 0)$.

In all cases, $G$ is a subgraph of $\mathcal{G}(a, b, 0)$. This completes the proof of the induction case, so that by the principle of mathematical induction, the theorem follows. $\square$