

# Optimization Hw2

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## Problem 1

Let  $\lambda_1$  and  $\lambda_2$  be the dual variables of the first two constraints, respectively, let  $\theta_1$  be the dual variable for the equality constraint and let  $\lambda_3, \lambda_4, \lambda_5$  be the dual variables for the constraints on the sign of the  $x$ 's. The the Lagrangian is:

$$\begin{aligned} L(x, \lambda, \theta, \delta) = & -a_1x_1 - a_2x_2 - a_3x_3 \\ & +\lambda_1(b_1x_1 + b_2x_2 + b_3x_3 - e_1) \\ & +\lambda_2(-c_1x_1 - c_2x_2 + e_2) \\ & +\theta_1(d_3x_3 - e_3) \\ & +\lambda_3(-x_1) + \lambda_4(-x_2) + \lambda_5(x_3) \end{aligned}$$

Which when rearranged gives:

$$\begin{aligned} L(x, \lambda, \theta, \delta) = & (-a_1 + b_1\lambda_1 - c_1\lambda_2 - \lambda_3)x_1 \\ & +(-a_2 + b_2\lambda_1 - c_2\lambda_2 - \lambda_4)x_2 \\ & +(-a_3 + b_3\lambda_1 + d_3\theta_1 + \lambda_5)x_3 \\ & +(-e_1\lambda_1 + e_2\lambda_2 - e_3\theta) \end{aligned}$$

Minimizing the Lagrangian over  $x$ , we find that since it is linear, the only instance where  $L$  has a minimum are when  $x_1, x_2$ , and  $x_3$  are all 0.

Which means that we can write the dual problem as:

$$\begin{aligned} \max \quad & -e_1\lambda_1 + e_2\lambda_2 - e_3\theta_1 \\ \text{st.} \quad & -a_1 + b_1\lambda_1 - c_1\lambda_2 - \lambda_3 = 0 \\ & -a_2 + b_2\lambda_1 - c_2\lambda_2 - \lambda_4 = 0 \\ & -a_3 + b_3\lambda_1 + d_3\theta_1 + \lambda_5 = 0 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \geq 0 \end{aligned}$$

Before formulating the Dual KKT conditions we can eliminate the dual problems dependence on  $\lambda_3, \lambda_4$ , and  $\lambda_5$ , because none appear in the objective function. This can be done 3 times. The first constraint paired with the fact that  $\lambda_3$  is non-negative gives:

$$-a_1 + b_1\lambda_1 - c_1\lambda_2 - \lambda_3 = 0 \implies -a_1 + b_1\lambda_1 - c_1\lambda_2 \geq 0 \implies a_1 - b_1\lambda_1 + c_1\lambda_2 \leq 0$$

The same goes almost identically for the second constraint, and the third becomes less than instead of greater than:

$$-a_3 + b_3\lambda_1 + d_3\theta_1 + \lambda_5 = 0 \implies -a_3 + b_3\lambda_1 + d_3\theta_1 \leq 0$$

The dual problem rewritten is then:

$$\begin{aligned} \min \quad & e_1\lambda_1 - e_2\lambda_2 + e_3\theta_1 \\ \text{st.} \quad & a_1 - b_1\lambda_1 + c_1\lambda_2 \leq 0 \\ & a_2 - b_2\lambda_1 + c_2\lambda_2 \leq 0 \\ & -a_3 + b_3\lambda_1 + d_3\theta_1 \leq 0 \\ & \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

Letting  $\delta_1, \delta_2$  and  $\delta_3$  be the dual variables for the first 3 constraints, and  $\alpha_1$  and  $\alpha_2$  be the dual variables for the sign constraints. The Lagrangian is then:

$$\begin{aligned} L(\lambda, \theta, \delta, \alpha) = & e_1\lambda_1 - e_2\lambda_2 + e_3\theta_1 \\ & + \delta_1(a_1 - b_1\lambda_1 + c_1\lambda_2) \\ & + \delta_2(a_2 - b_2\lambda_1 + c_2\lambda_2) \\ & + \delta_3(-a_3 + b_3\lambda_1 + d_3\theta_1) \\ & - \alpha_1\lambda_1 - \alpha_2\lambda_2 \end{aligned}$$

To get the stationary condition we take the gradient of the Lagrangian (in this case with respect to  $\lambda_1, \lambda_2$  and  $\theta_1$ ) and set it equal to 0. First rearranging gives:

$$\begin{aligned} L(\lambda, \theta, \delta, \alpha) = & \lambda_1(e_1 - b_1\delta_1 - b_2\delta_2 - \alpha) \\ & + \lambda_2(-e_2 + c_1\delta_1 + c_2\delta_2 - \alpha_2) \\ & + \theta_1(e_3 + d_3\delta_3) \\ & + (a_1\delta_1 + a_2\delta_2 - a_3\delta) \end{aligned}$$

Taking the gradient and setting it equal to 0 gives us the 3 stationary conditions:

$$\begin{aligned}(e_1 - b_1\delta_1 - b_2\delta_2 - \alpha) &= 0 \\ (-e_2 + c_1\delta_1 + c_2\delta_2 - \alpha_2) &= 0 \\ (e_3 + d_3\delta_3) &= 0\end{aligned}$$

For complementary slackness we have:

$$\begin{aligned}\delta_1(a_1 - b_1\lambda_1 + c_1\lambda_2) &= 0 \\ \delta_2(a_2 - b_2\lambda_1 + c_2\lambda_2) &= 0 \\ \delta_3(-a_3 + b_3\lambda_1 + d_3\theta_1) &= 0 \\ \alpha_1\lambda_1 &= 0 \\ \alpha_1\lambda_2 &= 0\end{aligned}$$

And for Feasibility we have the constraints in the dual problem above.

### Problem 2

Letting  $\lambda_j$  be for the dual variable for the first  $n$  constraints,  $\theta_i$  be the dual variables for the next  $m$  constraints, and  $\delta_{ij}$  be the constraints for the last sign constraints, the Lagragian then is:

$$\begin{aligned}L(x, \lambda, \theta, \delta) &= \sum_{i=1}^m \sum_{j=1}^n a_{ij}x_{ij} \\ &+ \sum_{i=1}^m \sum_{j=1}^n b_{ij}((x_{ij} + 1) \ln(x_{ij} + 1) - x_{ij}) \\ &+ \sum_{j=1}^n \lambda_j(c_j - \sum_{i=1}^m x_{ij}) + \sum_{i=1}^m \theta_i(\sum_{j=1}^n (x_{ij} + c_j) - \sum_{i=1}^m \sum_{j=1}^n x_{ij} + d_j) \\ &+ \sum_{i=1}^m \sum_{j=1}^n \delta_{ij}x_{ij}\end{aligned}$$

If we take the gradient of the Lagragian with respect to  $x_{ij}$  we will get  $n \times m$  constraints each for a particular  $i$  and  $j$ . Differentiating each term by each  $x_{ij}$  gives:

$$a_{ij} + b_{ij} \ln(x_{ij} + 1) - \lambda_j + \theta_i - m\theta_i + \delta_{ij} \quad \forall i, j$$

Where the derivative of the log term comes from last homework. To fulfill the stationary condition the above must be equal to 0. For complementary slackness we have:

$$\begin{aligned}\lambda_j(c_j - \sum_{i=1}^m x_{ij}) &= 0 \quad \forall j \\ \theta_i(\sum_{j=1}^n (x_{ij} + c_j) - \sum_{i=1}^m \sum_{j=1}^n x_{ij} + d_j) &= 0 \quad \forall i \\ \delta_{ij}x_{ij} &= 0 \quad \forall i, j\end{aligned}$$

Primal feasibility are the constraints of the original problem, and dual feasibility gives:

$$\begin{aligned}\lambda_j &\geq 0 \quad \forall j \\ \theta_i &\geq 0 \quad \forall i \\ \delta_{ij} &\geq 0 \quad \forall i, j\end{aligned}$$

### Problem 3

$$\begin{aligned}\min x \\ \text{st. } x^2 &\leq 0\end{aligned}$$

Is a case where this happens. Slaters condition is not satisfied because the only feasible point is 0, which is “up against” the inequality. Finding the stationary conditions to this problem, by letting  $\lambda$  be the dual variable of the only constraint we have:

$$\begin{aligned}L(x, \lambda) &= x + \lambda x^2 \\ \text{Stationary: } x &= -\frac{1}{2\lambda}\end{aligned}$$

But a solution must be both primal and dual feasible i.e.  $x^2 \leq 0$ , and  $\lambda > 0$ . If  $\lambda > 0$  then we know that the right hand side of the stationary constraint is negative, meaning that  $x$  is negative, but the only feasible point for  $x$  is 0 by the constraint  $x^2 \leq 0$ . Therefore the KKT conditions do not hold in this case.