Mechanical Earth Hw4

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Problem 1

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$
$$E_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2$$

So that the error equals:

$$\left|\frac{\epsilon_{11} - E_{11}}{E_{11}}\right| = .1$$
$$\left|\frac{\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2}{\frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2}\right| = .1$$
$$\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 = \frac{1}{10}\left|\frac{\partial u_1}{\partial x_1}\right| + \frac{1}{20}\left(\frac{\partial u_1}{\partial x_1}\right)^2$$
$$\frac{9}{20}\left(\frac{\partial u_1}{\partial x_1}\right)^2 - \frac{1}{10}\left|\frac{\partial u_1}{\partial x_1}\right| = 0$$
$$\frac{\partial u_1}{\partial x_1}\left(\frac{9}{20}\frac{\partial u_1}{\partial x_1} \pm \frac{1}{10}\right) = 0$$
$$\frac{\partial u_1}{\partial x_1} = \pm\frac{2}{9}$$

Problem 2
a) We have these three equations:

$$\sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\epsilon_{11}+\epsilon_{22}) \tag{1}$$
$$\epsilon_{11} = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \tag{2}$$
$$\epsilon_{22} = \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) \tag{3}$$

Plugging (2) and (3) into one we get:

$$\sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}) + \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}+\sigma_{33}))$$
$$\sigma_{33} = \frac{\nu}{1-2\nu}\sigma_{11} - \frac{\nu^2}{(1+\nu)(1-2\nu)}(\sigma_{11}+\sigma_{22}+\sigma_{33}) + \frac{\nu}{1-2\nu}\sigma_{22} - \frac{\nu^2}{(1+\nu)(1-2\nu)}(\sigma_{11}+\sigma_{22}+\sigma_{33})$$
$$(1+\nu)(1-2\nu)\sigma_{33} = (\nu+\nu^2)\sigma_{11} + (\nu+\nu^2)\sigma_{22} - 2\nu^2(\sigma_{11}+\sigma_{22}+\sigma_{33})$$
$$\sigma_{33} - \nu\sigma_{33} - 2\nu^2\sigma_{33} = \nu\sigma_{11} + \nu^2\sigma_{11} + \nu\sigma_{22} + \nu^2\sigma_{22} - 2\nu^2\sigma_{11} - 2\nu^2\sigma_{22} - 2\nu^2\sigma_{33}$$
$$(1-\nu)\sigma_{33} = \nu\sigma_{11} - \nu^2\sigma_{11} + \nu\sigma_{22} - \nu^2\sigma_{22}$$
$$(1-\nu)\sigma_{33} = (1-\nu)\nu\sigma_{11} + (1-\nu)\nu\sigma_{22}$$
$$\sigma_{33} = \nu(\sigma_{11}+\sigma_{22})$$

b) From the RHS of a) we know that $\lambda\epsilon_{kk}\delta_{ij} = \nu(\sigma_{11}+\sigma_{22})\delta_{ij}$ for plane strain, So:

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

becomes (since $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$) only the 4 equations:

$$2\mu\epsilon_{11} = \sigma_{11} - \nu(\sigma_{11}+\sigma_{22})$$
$$2\mu\epsilon_{12} = \sigma_{12}$$
$$2\mu\epsilon_{11} = \sigma_{21}$$
$$2\mu\epsilon_{22} = \sigma_{22} - \nu(\sigma_{11}+\sigma_{22})$$

Which compactly can be written as:

$$2\mu\epsilon_{\alpha\beta} = \sigma_{\alpha\beta} - \nu(\sigma_{11}+\sigma_{22})\delta_{\alpha\beta} \qquad \alpha,\beta = 1,2$$

c) we need to find $\lambda\epsilon_{kk}\delta_{ij}$ interms of stress assuming $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$:

$$\lambda\epsilon_{kk}\delta_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)}\epsilon_{kk}$$

$$= \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}) + \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}) - \frac{\nu}{E}(\sigma_{11}+\sigma_{22})\right)$$

$$= \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\frac{1+\nu}{E}(\sigma_{11}+\sigma_{22}) - \frac{3\nu}{E}(\sigma_{11}+\sigma_{22})\right)$$

$$= \frac{\nu}{1-2\nu}(\sigma_{11}+\sigma_{22}) - \frac{3\nu^2}{(1+\nu)(1-2\nu)}(\sigma_{11}+\sigma_{22})$$

$$= \frac{\nu\sigma_{11} - 2\nu^2\sigma_{11} + \nu\sigma_{22} - 2\nu^2\sigma_{22}}{(1+\nu)(1-2\nu)}$$

$$= \frac{\nu\sigma_{11}(1-2\nu) + \nu\sigma_{22}(1-2\nu)}{(1+\nu)(1-2\nu)}$$

$$= \frac{\nu}{1+\nu}(\sigma_{11}+\sigma_{22})$$

Problem 3

a) For strain:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

So taking into account all of the zero displacement gradients we have:

$$\epsilon_{31} = \frac{1}{2}\frac{\partial u_3}{\partial x_1} = \epsilon_{13}$$

$$\epsilon_{32} = \frac{1}{2}\frac{\partial u_3}{\partial x_2} = \epsilon_{23}$$

b) Using Hooke's law:

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

We get that:

$$\sigma_{31} = 2\mu\epsilon_{31} = \mu\frac{\partial u_3}{\partial x_1} = \sigma_{13}$$

$$\sigma_{32} = 2\mu\epsilon_{32} = \mu\frac{\partial u_3}{\partial x_2} = \sigma_{23}$$