Optimization Hw2

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November 5, 2019

Problem 1

Let λ_1 and λ_2 be the dual variables of the first two constraints, repsectively, let θ_1 be the dual variable for the equality constraint and let $\lambda_3, \lambda_4 \lambda_5$ be the dual variables for the constraints on the sign of the x's. The the Langragian is:

$$\begin{gathered}
L(x, \lambda, \theta, \delta) = -a_1x_1 - a_2x_2 - a_3x_3 \\
+\lambda_1(b_1x_1 + b_2x_2 + b_3x_3 - e_1) \\
+\lambda_2(-c_1x_1 - c_2x_2 + e_2) \\
+\theta_1(d_3x_3 - e_3) \\
+\lambda_3(-x_1) + \lambda_4(-x_2) + \lambda_5(x_3)
\end{gathered}$$

Which when reararanged gives:

$$\begin{gathered}
L(x, \lambda, \theta, \delta) = (-a_1 + b_1\lambda_1 - c_1\lambda_2 - \lambda_3)x_1 \\
+(-a_2 + b_2\lambda_1 - c_2\lambda_2 - \lambda_4)x_2 \\
+(-a_3 + b_3\lambda_1 + d_3\theta_1 + \lambda_5)x_3 \\
+(-e_1\lambda_1 + e_2\lambda_2 - e_3\theta)
\end{gathered}$$

Minimizing the Lagrangian over x, we find that since it is linear, the only instance where L has a minimum are when x_1, x_2, and x_3 are all 0.

Which means that we can write the dual problem as:

$$\begin{gathered}
\max \quad -e_1\lambda_1 + e_2\lambda_2 - e_3\theta_1 \\
\text{st.} \; - a_1 + b_1\lambda_1 - c_1\lambda_2 - \lambda_3 = 0 \\
-a_2 + b_2\lambda_1 - c_2\lambda 2 - \lambda_4 = 0 \\
-a_3 + b_3\lambda_1 + d_3\theta_1 + \lambda_5 = 0 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \geq 0
\end{gathered}$$

Before formulating the Dual KKT conditions we can eliminate the dual problems dependence on λ_3, λ_4, and λ_5, because none appear in the objective function. This can be done 3 times. The first constraint paired with the fact that λ_3 is non-negative gives:

$$-a_1+b_1\lambda_1-c_1\lambda_2-\lambda_3=0 \implies -a_1+b_1\lambda_1-c_1\lambda_2 \geq 0 \implies a_1-b_1\lambda_1+c_1\lambda_2 \leq 0$$

The same goes almost identically for the second constraint, and the third becomes less than instead of greater than:

$$-a_3+b_3\lambda_1+d_3\theta_1+\lambda_5=0 \implies -a_3+b_3\lambda_1+d_3\theta_1 \leq 0$$

The dual problem rewritten is then:

$$\begin{gathered}\min \quad e_1\lambda_1-e_2\lambda_2+e_3\theta_1 \\ \text{st. } a_1-b_1\lambda_1+c_1\lambda_2 \leq 0 \\ a_2-b_2\lambda_1+c_2\lambda_2 \leq 0 \\ -a_3+b_3\lambda_1+d_3\theta_1 \leq 0 \\ \lambda_1, \lambda_2 \geq 0\end{gathered}$$

Letting δ_1, δ_2 and δ_3 be the dual variables for the first 3 constraints, and α_1 and α_2 be the dual variables for the sign constraints. The Lagrangian is then:

$$\begin{gathered}L(\lambda,\theta,\delta,\alpha)=e_1\lambda_1-e_2\lambda_2+e_3\theta_1 \\ +\delta_1(a_1-b_1\lambda_1+c_1\lambda_2) \\ +\delta_2(a_2-b_2\lambda_1+c_2\lambda_2) \\ +\delta_3(-a_3+b_3\lambda_1+d_3\theta_1) \\ -\alpha_1\lambda_1-\alpha_2\lambda_2\end{gathered}$$

To get the stationary condition we take the gradient of the Lagrangian (in this case with respect to λ_1, λ_2 and θ_1) and set it equal to 0. First rearraging gives:

$$\begin{gathered}L(\lambda,\theta,\delta,\alpha)=\lambda_1(e_1-b_1\delta_1-b_2\delta_2-\alpha) \\ +\lambda_2(-e_2+c_1\delta_1+c_2\delta_2-\alpha_2) \\ +\theta_1(e_3+d_3\delta_3) \\ +(a_1\delta_1+a_2\delta_2-a_3\delta)\end{gathered}$$

Taking the gradient and setting it equal to 0 gives us the 3 stationary conditions:

$$
\begin{gathered}
(e_1 - b_1\delta_1 - b_2\delta_2 - \alpha) = 0 \\
(-e_2 + c_1\delta_1 + c_2\delta_2 - \alpha_2) = 0 \\
(e_3 + d_3\delta_3) = 0
\end{gathered}
$$

For complementary slackness we have:

$$
\begin{gathered}
\delta_1(a_1 - b_1\lambda_1 + c_1\lambda_2) = 0 \\
\delta_2(a_2 - b_2\lambda_1 + c_2\lambda_2) = 0 \\
\delta_3(-a_3 + b_3\lambda_1 + d_3\theta_1) = 0 \\
\alpha_1\lambda_1 = 0 \\
\alpha_1\lambda_2 = 0
\end{gathered}
$$

And for Feasibility we have the constraints in the dual problem above.

Problem 2

Letting λ_j be for the dual variable for the first n constraints, θ_i be the dual variables for the next m constraints, and δ_{ij} be the constraints for the last sign constraints, the Lagragian then is:

$$
\begin{gathered}
L(x, \lambda, \theta, \delta) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij} \\
+ \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}((x_{ij}+1)\ln(x_{ij}+1) - x_{ij} \\
+ \sum_{j=1}^{n}\lambda_j(c_j - \sum_{i=1}^{m} x_{ij}) + \sum_{i=1}^{m}\theta_i(\sum_{j=1}^{n}(x_{ij}+c_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} + d_j) \\
+ \sum_{i=1}^{m}\sum_{j=1}^{n}\delta_{ij}x_{ij}
\end{gathered}
$$

If we take the gradient of the Lagragian with respect to x_{ij} we will get $n \times m$ constraints each for a particular i and j. Differentiating each term by each x_{ij} gives:

$$
a_{ij} + b_{ij}\ln(x_{ij}+1) - \lambda_j + \theta_i - m\theta_i + \delta_{ij} \quad \forall i, j
$$

Where the derivative of the log term comes from last homework. To fulfill the stationary condition the above must be equal to 0. For complementary slackness we have:

$$\lambda_j(c_j - \sum_{i=1}^{m} x_{ij}) = 0 \quad \forall j$$
$$\theta_i(\sum_{j=1}^{n}(x_{ij} + c_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} + d_j) = 0 \quad \forall i$$
$$\delta_{ij} x_{ij} = 0 \quad \forall i, j$$

Primal feasability are the constraints of the orginal problem, and dual feasability gives:

$$\lambda_j \geq 0 \quad \forall j$$
$$\theta_i \geq 0 \quad \forall i$$
$$\delta_{ij} \geq 0 \quad \forall i, j$$

Problem 3

$$\min x$$
$$\text{st. } x^2 \leq 0$$

Is a case where this happens. Slaters condition is not satisfied because the only feasible point is 0, which is "up against" the inequality. Finding the stationary conditions to this problem, by letting λ be the dual variable of the only constraint we have:

$$L(x, \lambda) = x + \lambda x^2$$
$$\text{Stationary: } x = -\frac{1}{2\lambda}$$

But a solution must be both primal and dual feasible i.e. $x^2 \leq 0$, and $\lambda > 0$. If $\lambda > 0$ then we know that the right have side of the stationary constraint is negative, meaning that x is negative, but the only feasible point for x is 0 by the constraint $x^2 \leq 0$. Therefore the KKT conditions do not hold in this case.