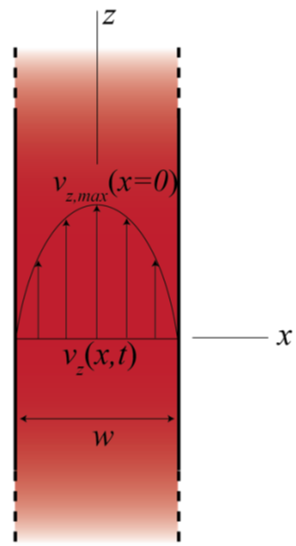
Mechanical Earth Homework 1, due in class Wednesday January 15th

1. Dimensional analysis and dimensionless groups

Volcanic eruptions commonly initiate through fissures fed by dikes (magma-filled planar cracks):



Fissure eruption in Hawaii 2018 (Terray Sylvester/Reuters)   Schematic of magma flow in a dike

The upward flow of magma through a vertical dike may be described by the Navier-Stokes equation for the special case of viscous flow of a constant-property fluid (magma density ρ and viscosity η remain constant in time and space):

$$\rho \frac{\partial v_z}{\partial t} = -\frac{\partial P}{\partial z} + \eta \frac{\partial^2 v_z}{\partial x^2} + \rho g \qquad (1)$$

Here the x-axis is perpendicular to the dike plane, and the flow direction (up) is z. g is the acceleration due to gravity. The pressure, $P(z, t)$, can vary with position along the direction of flow and it can vary with time. The only velocity component, $vz(x, t)$, can vary with position across the direction of flow and it can vary with time.

You may have never seen this equation before, but you should be able to describe the dimensions of each physical quantity (refer to the table I handed out in class). The left-hand side is essentially the mass times the acceleration of the magma, and the terms on the right-hand side are essentially the forces per unit volume acting on any element of magma.

a) Show that this equation is dimensionally homogeneous

b) Identify the independent variables, the dependent variables, and the material properties

c) Define a set of normalized independent and dependent variables and dimensionless differential operators, and substitute them into the equation (Hint for normalizing pressure: use the "dynamic pressure" ρv^2)

d) Rearrange the groups of constants in the normalized equation so that all the terms are dimensionless and, in doing so, identify any dimensionless groups you find. Explain what these dimensionless numbers describe (i.e. what forces are they comparing?)

2. Using dimensionless numbers to determine magma flow regimes

As it turns out, the inverse of one of the dimensionless numbers you should find is referred to as the "Reynold's number" and is defined as Re = $\frac{w\rho v_{max}}{\eta}$. In 1883, Osborne Reynolds studied the conditions in which flow through a channel transitioned from "laminar" to "turbulent." He found that for Re < 2000, flow was laminar and the particles of fluid all traveled along constant planes, but once Re > 2000 flow became turbulent (having irregular fluctuations).

a) We happen to know a decent amount about dikes in Hawaii because of how frequently they occur (nearly once a year during the last 50 years!). As they propagate, swarms of earthquakes tend to track the leading edge of the dike. From this, we can estimate the velocity of dike propagation, which tends to vary between ~0.1 – 1 m/s. When the dike reaches the surface, we can observe the width of the fissure, which tends to vary between ~0.5 – 2 m. In addition, the magma tends to be erupted is basaltic in composition, with values of bulk mass density around $\sim\rho$ = 2400 - 2900 kg/m3 (depending on number of gas bubbles in the magma) and viscosity that may vary between ~ η = 10 – 1000 Pa·s (depending on temperature, water content, and exact composition). Given these ranges of values, is the flow of basaltic magma through Hawaiian dikes laminar, turbulent, or either one depending on certain parameters? Show how you arrived at your answer.

b) On Earth we observe ancient "giant dikes" that can be nearly 100 meters across and 1000 kilometers long! Unfortunately, we know less about how fast these dikes may have propagated, but they do typically tend to contain basalt that may have behaved similarly to the type of magma erupted in Hawaii. Assuming everything else is the same aside from the dike width, do you think the flow of basaltic magma through these "giant" dikes was likely to be laminar, turbulent, or either one depending on certain parameters? Show how you arrived at your answer.

3. Practice with MATLAB

a) Open the script "flow_regime.m". Describe what the following lines of code do:

```
clear all
close all
clc
% Vary magma viscosity
p   = linspace(1,3,100);   % exponent for magma viscosity
eta = 10.^p;     % magma viscosity (Pa s)
% Vary dike width
w   = linspace(1,100,100);
[ETA,W] = meshgrid(eta,w);
% Make plots
Figure
contourf(log10(ETA),W,log10(Re))
xlabel('log viscosity (Pa s)')
```

```
ylabel('Dike width (m)')
set(gca, 'FontSize', 14)
colorbar
title(['log Re, v = ' num2str(v) ' m/s'])
caxis([0.5 4])
```

b) Now run the script “flow_regime.m”. Print out a contour plot of Reynold’s number as a function of the log magma viscosity and dike width. Draw the boundary for Re = 2000 (log(2000) = 3.3). Do this twice, one plot for velocity = 0.1 m/s and another for velocity = 1 m/s.

4. Practice with index and vector notation

a) In 3 dimensions, the Navier-Stokes equation for the flow of a linear, isotropic, and incompressible viscous fluid with constant material properties (eq 1) can be written compactly in index notation:

$$\rho \frac{\partial v_i}{\partial t} = -\frac{\partial P}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_k \partial x_k} + \rho g_i \qquad (2)$$

Write out in component form (using *x*,*y*,*z*) all 3 equations represented by the indicial form of Navier-Stokes (eq 2). Remember that when an index appears twice in a term, that index is understood to take on all the values of its range, so the resulting terms are summed.

b) Recall the definition of the del operator $\nabla = \hat{e}_i \frac{\partial}{\partial x_i}$ (also known as the gradient when applied to a scalar value) and the divergence of a vector $\nabla \cdot \bar{v} = \frac{\partial v_i}{\partial x_i}$.

Write out ∇P in component form.

c) The “Laplacian” operator $\nabla^2 f$ is the divergence of the gradient, i.e. $\nabla \cdot \nabla f$ where *f* is a scalar.

For the x-component of velocity v_x (or v_1), compute the Laplacian $\nabla^2 v_x = \nabla \cdot \nabla v_x$

d) Using your results from b) and c), rewrite the Navier-Stokes equation above (eq 2) using vector notation (i.e. $\rho \frac{\partial \bar{v}}{\partial t} = \cdots$)