Dimensional Analysis

Fundamental dimensions include: length (L), mass (M), time (T), + temperature (θ)

describe with "units" (meters, kilograms, seconds + kelvin in S.I.)

e.x. force, $F \overset{\text{"has dimensions of"}}{\{=\}} ML/T^2$

$F \overset{\text{"has units of"}}{[=]} kg \cdot m/s^2$ or Newtons, N

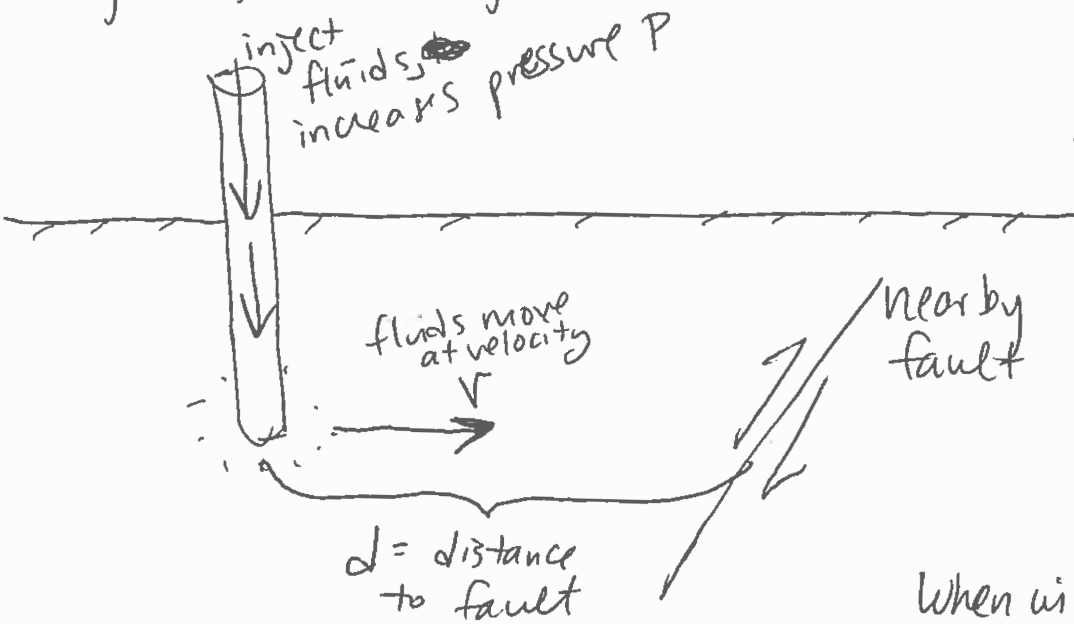
stress σ is force per area, so $\sigma \{=\} \frac{ML}{T^2} \cdot \frac{1}{L^2} = \frac{M}{LT^2}$

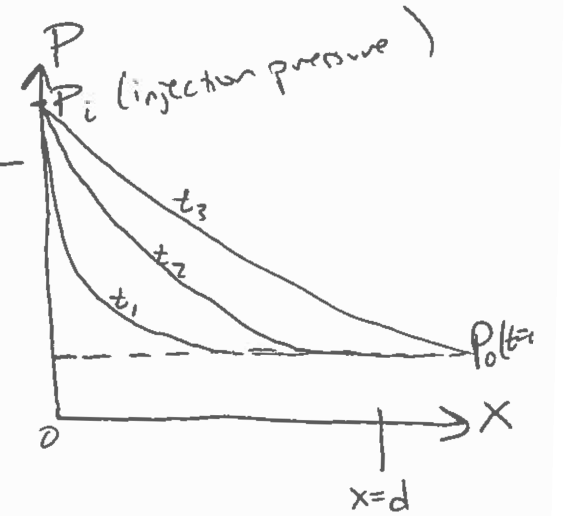
$\sigma [=] N \cdot m^{-2}$ or $\frac{kg}{m \cdot s^2}$

Dimensional analysis is useful when confronted with new equations and/or <u>before</u> trying to solve (can illuminate errors, may allow to throw out terms + make simpler)

<u>Example</u>: 1D advection-diffusion equation for groundwater flow

e.g. Wastewater injection + induced seismicity





When will pressure pulse reach fault?

1D advection - diffusion: $\dfrac{\partial P}{\partial t} = \omega \dfrac{\partial^2 P}{\partial x^2} - v \dfrac{\partial P}{\partial x}$

P = pressure } dep. variable

t = time, x = distance } ind. variables

ω = hydraulic diffusivity, v = fluid velocity } material / fluid characteristics

① Check for "dimensional homogeneity" (every term has same dimensions)

$t \{=\} T, \quad x \{=\} L, \quad P \{=\} \dfrac{M}{LT^2}, \quad \omega \{=\} \dfrac{L^2}{T}, \quad v \{=\} \dfrac{L}{T}$

$$\frac{\partial P}{\partial t} \{=\} \frac{1}{T} \cdot \frac{M}{LT^2} = \frac{M}{LT^3}$$

$$\omega \frac{\partial^2 P}{\partial x^2} \{=\} \frac{L^2}{T} \cdot \frac{1}{L^2} \cdot \frac{M}{LT^2} = \frac{M}{LT^3}$$

$$v \frac{\partial P}{\partial x} \{=\} \frac{L}{T} \cdot \frac{1}{L} \cdot \frac{M}{LT^2} = \frac{M}{LT^3}$$

all have $\dfrac{M}{LT^3}$ ✓

② Normalize the ind. variables, dep. vars & differential operators

(2a) Choose characteristic variables

$x^* = d, \quad t^* = d/v, \quad P^* = P_o$ or $P_i - P_o$ or P_i

(2b) Nondimensionalize the variables & operators

$$\tilde{x} = \frac{x}{x^*} = \frac{x}{d}, \quad \tilde{t} = \frac{tv}{d}, \quad \tilde{P} = \frac{P}{P_o},$$

$$\frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t} \cdot \frac{d}{v}, \quad \frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x} d, \quad \frac{\partial^2}{\partial \tilde{x}^2} = \frac{\partial^2}{\partial x^2} d^2$$

(2c) Substitute these back in:

$$\frac{v P_i}{d} \frac{\partial \tilde{P}}{\partial \tilde{t}} = \frac{\omega P_i}{d^2} \frac{\partial^2 \tilde{P}}{\partial \tilde{x}^2} - \frac{v P_i}{d} \frac{\partial \tilde{P}}{\partial \tilde{x}}$$

(2d) Divide through to nondimensionalize equation:

$$\frac{\partial \tilde{P}}{\partial \tilde{t}} = \frac{\omega}{vd} \frac{\partial^2 \tilde{P}}{\partial \tilde{x}^2} - \frac{\partial \tilde{P}}{\partial \tilde{x}}$$

Peclet # $Pe \equiv \dfrac{\omega}{vd}$ > 1 diffusion-dominated; < 1 advection-dominated