Q1. Need to show closure under addition, scalar multiplication and that $\vec{0} \in W$ is also in $U+V$.

$\vec{0} \in W$ is in $U+V$:

Since U is a subspace and V is a subspace both contain $\vec{0} \in W$ so $\vec{0}+\vec{0} = \vec{0} \in U+V$

closure under addition:

Let $P_1, P_2 \in U+V$

We can write:

$P_1 = u_1 + v_1 \quad u_1 \in U \quad v_1 \in V$

and $P_2 = u_2 + v_2 \quad u_2 \in U, v_2 \in V$

So that:

$$P_1 + P_2 = u_1 + v_1 + u_2 + v_2 = (u_1 + u_2) + (v_1 + v_2)$$

Since U is a subspace it is closed under addition so $u_1 + u_2 \in U$ and V is a subspace so similarly $v_1 + v_2 \in V$

Since $U+V = \{u+v$ where $u \in U$ and $v \in V\}$ we have that $P_1 + P_2 \in U+V$

closure under scalar mult:

Let $\alpha \in F$ $P_1 \in U+V$ where $P_1 = u_1 + v_1$, $u_1 \in U$, $v_1 \in V$

$$\alpha P_1 = \alpha(u_1 + v_1) = \alpha u_1 + \alpha v_1$$

Since U and V are closed under scalar mult $\alpha u_1 \in U$, $\alpha v_1 \in V$

so that $\alpha P_1 \in U+V$.

I remember from reading our book that

$$\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

I am assuming I do not need to prove this as the question does not ask.

Since we know that a spanning set with length $\dim(U+V)$ is a basis for $U+V$, and we have $\dim(U)$, $\dim(V)$, and could compute $\dim(U \cap V)$, then we have a basis if we can find a spanning set of length $\dim(U+V)$. If $u_1, u_2, u_3 \ldots$ is a basis for U and $v_1, v_2, v_3 \ldots$ is a basis for V we know that $u_1, u_2, u_3 \ldots v_1, v_2, v_3 \ldots$ has a length longer than $\dim(U+V)$ because $u_1, u_2, u_3 \ldots$'s length is $\dim(U)$ and $v_1, v_2, v_3 \ldots$'s length is $\dim(V)$, and $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$

so our algorthim can start throwing away vectors from $u_1, u_2, u_3 \ldots v_1, v_2, v_3 \ldots$ check if the new list still spans $U+V$, if not add the vector back, if so try deleteing another vector. Do this till the length of the list is $\dim(U+V)$

Q2. Let v_1, v_2, v_3 be linearly independent vectors in V

so that we know that:

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \vec{0}$$

only when $a_1 = a_2 = a_3 = 0$

If $v_1, v_1 - v_2, v_1 - v_2 + 5v_3$ are linearly independent then:

$$b_1 v_1 + b_2 (v_1 - v_2) + b_3 (v_1 - v_2 + 5v_3) = \vec{0} \quad (1)$$

only when $b_1 = b_2 = b_3 = 0$. Is this the case?

rearranging (1) we get:

$$(b_1 + b_2 + b_3) v_1 + (-b_2 - b_3) v_2 + (5 b_3) v_3 = \vec{0}$$

Since v_1, v_2, v_3 are linearly independent we know:

$$b_1 + b_2 + b_3 = 0$$
$$-b_2 - b_3 = 0$$
$$5 b_3 = 0$$

solving for b_3 we see that $b_3 = 0$ and plugging this into other 2 eqs. we get:

$$b_1 + b_2 = 0$$
$$-b_2 = 0$$

so $b_2 = 0$, and plugging that in gives $b_1 = 0$.

Therefore $v_1, v_1 - v_2, v_1 - v_2 + 5v_3$ are linearly independent vectors.

Is the $\operatorname{span}\{v_1, v_2, v_3\} = \operatorname{span}\{v_1, v_1 - v_2, v_1 - v_2 + 5v_3\}$?

By way of double inclusion we need to show

$$\operatorname{span}\{v_1, v_2, v_3\} \subseteq \operatorname{span}\{v_1, v_1 - v_2, v_1 - v_2 + 5v_3\} \quad (2)$$

and

$$\operatorname{span}\{v_1, v_2, v_3\} \supseteq \operatorname{span}\{v_1, v_1 - v_2, v_1 - v_2 + 5v_3\} \quad (3)$$

First we show (2):

Let $v \in \operatorname{span}\{v_1, v_1 - v_2, v_1 - v_2 + 5v_3\}$ then we can write v as:

$$v = a_1(v_1) + a_2(v_1 - v_2) + a_3(v_1 - v_2 + 5v_3)$$

If we can show $v_1, v_2, v_3 \in \operatorname{span}\{v_1, v_1 - v_2, v_1 - v_2 + 5v_3\}$ then we know (2) because if a vector space contains a list of vectors it also contains its span.

We can write v_1 as:

$$v = 1(v_1) + 0(v_1 - v_2) + 0(v_1 - v_2 + 5v_3)$$

v_2 as:

$$v = 1(v_1) + (-1)(v_1 - v_2) + 0(v_1 - v_2 + 5v_3)$$

v_3 as:

$$v = 0(v_1) + \left(-\tfrac{1}{5}\right)(v_1 - v_2) + \tfrac{1}{5}(v_1 - v_2 + 5v_3)$$

Now we follow the same strategy for (2):

Letting $v \in \operatorname{span}\{v_1, v_2, v_3\}$ we have that

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3$$

Similar to the above if $v_1, v_1 - v_2, v_1 - v_2 + 5v_3 \in \operatorname{span}\{v_1, v_2, v_3\}$ then we have (3). We can write v_1 as:

$$v = 1v_1 + 0v_2 + 0v_3$$

$v_1 - v_2$ as: $v = 1v_1 + (-1)v_2 + 0v_3$

and $v_1 - v_2 + 5v_3$ as: $v = 1v_1 + (-1)v_2 + (5)v_3$

So by double inclusion we have $\operatorname{span}\{v_1, v_2, v_3\} = \operatorname{span}\{v_1, v_1 - v_2, v_1 - v_2 + 5v_3\}$

Q3. Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ as $T\vec{v} = (1,1)$

This is is not surjective because the range only contains one vector in $\mathbb{R}^2$, i.e. not every vector in $\mathbb{R}^2$ is mapped to. This is not injective because every vector in the domain maps to the same vector $(1,1)$, meaning there is a vector in the range that is mapped to by more than one element in the domain.

Q4.

Let $T: V \to W$ be an injective linear transformation so that $Ta = Tb : Ta, Tb \in W$ implies $a = b \;\forall a, b \in V$. Now Let $\vec{c} \in \text{Null}\,T$ so that $T\vec{c} = 0$, But since a linear map alway maps $0 \in V$ to $0 \in W$ we have $T\vec{c} = 0 = T\vec{0}$, and since our map is injective $T\vec{c} = T\vec{0} \Rightarrow \vec{c} = \vec{0}$. Therefore $\text{Null}\,T = 0$

Q5. Let $T: P(x) \to P(x)$ be defined as $T(v(x)) = x\,v(x)$

To show this is a linear transform we need for $v(x), u(x) \in P(x)$

to satisfy $T(v(x)+u(x)) = T(v(x)) + T(u(x))$

To show this let $v(x) = v_0 + v_1 x + v_2 x^2 + \dots$

and $u(x) = u_0 + u_1 x + u_2 x^2 + \dots$ so we have:

$T(v_0 + v_1 x + v_2 x^2 \dots + u_0 + u_1 x + u_2 x^2 + \dots) =$

$x(v_0 + v_1 x + v_2 x^2 \dots + u_0 + u_1 x + u_2 x^2 \dots)$

$= x v_0 + v_1 x^2 + v_2 x^3 + \dots + u_0 x + u_1 x^2 + u_2 x^3 + \dots$

$= x(v_0 + v_1 x + v_2 x \dots) + x(u_0 + u_1 x + u_2 x^2 + \dots)$

$= T(v(x)) + T(u(x))$

We also need to show (assuming $a \in \mathbb{R}$) — It seems unclear what the field is from the question ???

that $aT(v(x)) = T(a v(x))$

taking the same $v(x)$ as above we have:

$aT(v_0 + v_1 x + v_2 x^2 \dots) = a(v_0 x + v_1 x^2 + v_2 x^3 \dots)$

$= a v_0 x + a v_1 x^2 + a v_2 x^3 = x(a v_0 + a v_1 x + a v_2 x^2)$

$= T(a v(x))$

The null space of T is $\{\vec{0}\}$ we can see this by using the proposition from Q4.

T is injective because if $g(x), f(x) \in P(x)$ then if we have $T(g(x)) = T(f(x))$

Q5. Let $T: P(x) \to P(x)$ be defined as $T(v(x)) = x\,v(x)$

or $x(g_0 + g_1 x + g_2 x^2 \ldots) = x(f_0 + f_1 x + f_2 x^2 \ldots)$

$x g_0 + g_1 x^2 + g_2 x^3 \ldots = x f_0 + f_1 x^2 + f_2 x^3 \ldots$

The only way this is possible is if $g_0 = f_0,\ g_1 = f_1,\ g_2 = f_2 \ldots$ which would mean $f = g$, and thus T is injective meaning its null space is $\{\vec{0}\}$.

The range of T is $P(x) / \{ f(x) \in P(x) : P(x) = p_0 x, p_0 \in \mathbb{R} \}$

Tc

an

T

x

= x

Q6. If we let V be a vector space over the complex numbers

Then scalar multiplication for $v \in V$ and $a \in \mathbb{C}$ so that

$av \in V$ is defined
is we only consider $a = \alpha + i\beta$, $\beta = 0$ then
$a \in \mathbb{R}$ and V can instead be over the real numbers, so
if $\beta = 0$ then yes, to the first question.

To the second question, yes.

If we have a subset S of $\mathbb{C}^n$ where $\lambda = a + 0i \in \mathbb{R}$

$v \in S$ and scalar multiplication is defined as

$$\lambda v = a + 0i(b + ci, d + ei, \dots)$$

$$(ab + aci, ad + aei, \dots)$$

even though $\lambda = a + 0i$ in this subset we can make
every vector in $\mathbb{C}^n$ from vector addition and scalar multiplication
So S is a subspace of $\mathbb{C}^n$ because it is $\mathbb{C}^n$ itself.