Linear Algebra Hw1

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April 6, 2020

(1a)

$$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right) = \left(\frac{1}{4}-\frac{3}{4}-\frac{\sqrt{3}i}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)$$
$$= \left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right) = \frac{1}{4}+\frac{3}{4} = 1$$

(2a)

Let $\alpha, \beta \in \mathbb{C}$ so that $\alpha = a+bi$, $\beta = c+di$. If $\alpha\beta = 1$, then by the definition of multiplication in $\mathbb{C}$ we have:

$$(ac-bd)+(ad+bc)i = 1+0i \implies ac-bd = 1 \text{ and } ad+bc = 0$$

This is a linear system of two equations in c and d. Eliminating the first term in each equation we get:

$$(-abc+b^2d = -b)+(abc+a^2d = 0) \implies b^2d+a^2d = -b \implies d = -\frac{b}{b^2+a^2}$$

pluggin this back to $ad+bc = 0$ we get:

$$a\left(\frac{-b}{b^2+a^2}\right)+bc = 0 \implies c = \frac{a}{b^2+a^2}$$

Therefore the multplicative identity in $\mathbb{C}$ is $\frac{a}{b^2+a^2} - \frac{b}{b^2+a^2}i$

(11a)

By way of contradiction, assume there exists $\lambda \in \mathbb{C}, \lambda = a + bi$ so that:

$$\begin{aligned}\lambda(2-3i) &= 12-5i\\ \lambda(5+4i) &= 7+22i\\ \lambda(-6+7i) &= -32-9i\end{aligned}$$

expanding this out:

$$\begin{aligned}2a+3b+(-3a+2b)i &= 12-5i\\ 5a-4b+(4a+5b)i &= 7+22i\\ -6a-7b+(7a-6b)i &= -32-9i\end{aligned}$$

or:

$$\begin{aligned}2a+3b &= 12\\ -3a+2b &= -5\\ 5a-4b &= 7\\ 4a+5b &= 22\\ -6a-7b &= -32\\ 7a-6b &= -9\end{aligned}$$

This is a system of 6 equations with 2 unknowns so it is very over determined. We can see it has no solution if we find the a solution to only the first two equations:

$$\left(\frac{3}{2}(2a+3b) = \frac{3}{2}(12)\right) + (-3a+2b=-5) \implies \frac{13}{2}b = 13 \implies b = 2$$

so that $-3a+4=-5 \implies a=3$
Plugging these into the sixth equation we get:

$$7(3)-6(2) = -9 \implies 9 = -9$$

Which is a contradiction.

(16a)

$$
\begin{gathered}
(a+b)x = \\
((a+b)x_1, (a+b)x_2, ..., (a+b)x_n) = \\
(ax_1 + bx_1, ax_2 + bx_2, ..., ax_n + bx_n) = \\
(ax_1, ax_2, ...ax_n) + bx_1, bx_2, ...bx_n) = \\
a(x_1, x_2, ..., x_n) + b(x_1, x_2, ..., x_n) = ax + bx
\end{gathered}
$$

(4b)

There are no elements in the empty set so the there is no existence of the additive identity.

(5b)

We need to show that the condition $0\vec{v} = \vec{0}$ implies that there exists $\vec{w} = -\vec{v}$ such that $\vec{v} + (-\vec{v}) = 0$:

$$\vec{0} = 0\vec{v} = (1 + (-1))\vec{v} = 1\vec{v} + (-1)\vec{v} = \vec{v} + (-\vec{v})$$

Where in the last equality we have let $-1\vec{v} = (-\vec{v}) = \vec{w}$.

(1c)

(a) First the additive identity is in this subspace because $0 + 2(0) = 3(0) \implies 0 = 0$.

Addition is closed: Let W denote the subspace in question. Let $x = (x_1, x_2, x_3) \in W$, and $y = (y_1, y_2, y_3) \in W$. We need to show $x + y$ is still in the subspace:

$$(x_1+y_1)+2(x_2+y_2)+3(x_3+y_3) = (x_1+2x_2+3x_3)+(y_1+2y_1+3x_3) = 0+0 = 0$$

Multiplication is closed: Let $a \in \mathbb{F}$. We must show that $ax \in W$:

$$ax_1 + 2ax_2 + 3ax_3 = a(x_1 + 2x_2 + 3x_3) = a(0) = 0$$

So this is a subspace.

(b) This is not a subspace of $\mathbb{F}^3$ because it does not contain the additive identity of $\mathbb{F}^3$, $\vec{0}$.

(c) This is not a subspace. Example of it not being closed under addition:

$$(1,0,0),(0,1,1) \in W \qquad (1,0,0) + (0,1,1) = (1,1,1)$$

$$1(1)(1) = 1 \implies (1,1,1) \notin W$$

(d) W does contain the addative identity since $0 = 5(0) \implies 0 = 0$.
Addition is closed:

$$(x_1 + y_1) - 5(x_2 + y_2) = (x_1 - 5x_2) + (y_1 - 5y_2) = 0 + 0 = 0$$

Multiplication is closed:

$$ax_1 - 5ax_3 = a(x_1 - 5x_3) = a(0) = 0$$