Assignment 4

CIS 410/510: Selected Topics on Optimization

Problem 1 (10 points) For the graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of edges, we define the following nonlinear integer program, where $w_{i,j} \geq 0$, $\forall (i,j) \in \mathcal{E}$ and k is a nonnegative integer:

$$\begin{aligned} \max \quad & \sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j) \\ s.t. \quad & \sum_{i\in\mathcal{U}} x_i = k\,, \\ & x_i \in \{0,1\},\ \forall i \in \mathcal{U}\,. \end{aligned}$$

- Show that the following linear program is a relaxation of the above problem:

$$\begin{aligned} \max \quad & \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \\ s.t. \quad & z_{i,j} \leq x_i + x_j,\ \forall (i,j) \in \mathcal{E}\,, \\ & z_{i,j} \leq 2 - x_i - x_j,\ \forall (i,j) \in \mathcal{E}\,, \\ & \sum_{i\in\mathcal{U}} x_i = k\,, \\ & 0 \leq x_i \leq 1,\ \forall i \in \mathcal{U}\,, \\ & 0 \leq z_{i,j} \leq 1,\ \forall (i,j) \in \mathcal{E}\,. \end{aligned}$$

- Let $F(x) = \sum_{(i,j)\in\mathcal{E}} w_{i,j}(x_i + x_j - 2x_i x_j)$ be the objective function of the nonlinear integer program. Show that for any (x, z) that is feasible to the linear program, $F(x) \geq \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j}$.
- Show that for a given fractional solution x, using "pipage rounding" to convert fractional variables x_i and x_j into integers does not decrease $F(x)$, and also, based on this and the above arguments, design a $\frac{1}{2}$-approximation algorithm for the nonlinear integer program.

Problem 2 (10 points) For the directed graph $G = (\mathcal{U}, \mathcal{E})$, where $\mathcal{U}$ is the set of vertices and $\mathcal{E}$ is the set of directed edges, we want to partition $\mathcal{U}$ into two sets $\mathcal{V}$ and $\mathcal{W} = \mathcal{U} \setminus \mathcal{V}$ in order to maximize the total weight of the edges going from $\mathcal{V}$ to $\mathcal{W}$ (i.e., the edges (i,j) with $i \in \mathcal{V}$ and $j \in \mathcal{W}$).

- Give a randomized $\frac{1}{4}$-approximation algorithm for this problem.
- Show that the following linear program is a relaxation of this problem:

$$\begin{aligned} \max \quad & \sum_{(i,j)\in\mathcal{E}} w_{i,j} z_{i,j} \\ s.t. \quad & z_{i,j} \leq x_i,\ \forall (i,j) \in \mathcal{E}\,, \\ & z_{i,j} \leq 1 - x_j,\ \forall (i,j) \in \mathcal{E}\,, \\ & 0 \leq x_i \leq 1,\ \forall i \in \mathcal{U}\,, \\ & 0 \leq z_{i,j} \leq 1,\ \forall (i,j) \in \mathcal{E}\,. \end{aligned}$$

- For the above linear program, give a randomized $\frac{1}{2}$-approximation algorithm based on rounding x_i, $\forall i \in \mathcal{U}$ to 1 with the probability of $\frac{1}{2}x_i + \frac{1}{4}$.