Assignment 3

CIS 410/510: Selected Topics on Optimization

Problem 1 (4 points) Consider $f:\mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|_2$. Show that, for $x \neq 0$, the subgradient is $\frac{x}{\|x\|_2}$; for $x = 0$, the subgradient is any element in the set of $\{y \mid \|y\|_2 \leq 1\}$.

Problem 2 (6 points) Consider $f(x) = \max\{f_1(x), f_2(x)\}$, where $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are convex and differentiable. Show that, for $f_1(x) > f_2(x)$, the subgradient is $\nabla f_1(x)$; for $f_1(x) < f_2(x)$, the subgradient is $\nabla f_2(x)$; for $f_1(x) = f_2(x)$, the subgradient is any point on the line segment between $\nabla f_1(x)$ and $\nabla f_2(x)$.

Problem 3 (4 points) Use the definition of subdifferentials to show that the following two functions are *not* subdifferentiable at $x = 0$: (1) $f(x) = -x^{\frac{1}{2}}$; (2) $f(x)$ such that $f(0) = 1$ and $f(x) = 0$ for $x > 0$.

Problem 4 (2 points) Consider the subgradient method $x^+ = x - \alpha g$, where $g \in \partial f(x)$. Show that if $\alpha < \frac{2(f(x) - f(x^*))}{\|g\|_2^2}$, then we have $\|x^+ - x^*\|_2 < \|x - x^*\|_2$, i.e., every iteration moves closer to the optimal solution x^*.

Problem 5 (4 points) Use "dual decomposition" to design a distributed algorithm to find the optimal value of the objective function of the following problem, and describe your distributed algorithm elaborately. $a_{s,u}$, $b_{s,u}$, c_s, $\forall s \in \mathcal{S}$, $\forall u \in \mathcal{U}$ are all nonnegative constants. (Hint: There are two "dimensions", $\mathcal{S}$ and $\mathcal{U}$. You can use dual decomposition to decompose the problem over any of the two dimensions. For example, if you decompose it over $\mathcal{S}$, then each of the $|\mathcal{S}|$ subproblems needs to be solved separately.)

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\begin{aligned}
\min \quad & \sum_{s \in \mathcal{S}} \sum_{u \in \mathcal{U}} a_{s,u} x_{s,u} + \sum_{s \in \mathcal{S}} \sum_{u \in \mathcal{U}} b_{s,u} \left((x_{s,u} + 1) \ln(x_{s,u} + 1) - x_{s,u} \right) \\
s.t. \quad & \sum_{u \in \mathcal{U}} x_{s,u} \geq c_s, \ \forall s \in \mathcal{S}\,, \\
& \sum_{s \in \mathcal{S}} x_{s,u} \geq c_u, \ \forall u \in \mathcal{U}\,, \\
& x_{s,u} \geq 0, \ \forall s \in \mathcal{S}, \ \forall u \in \mathcal{U}\,.
\end{aligned}
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