FEM and Inverse notes

Tobias Harvey

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1 1D Variable Coefficent Diffusion

$$s(x,t)\frac{\partial u}{\partial t} = \left(t(x,t)\frac{\partial u}{\partial x}\right)_x + q(x,t), \qquad 0 \le x \le L, t \le T$$

$$u(0,t) = \alpha(t)$$

$$u(L,t) = \beta(t)$$

Assuming an approximate solution u^h, residual $u^h - u$ can be calculated by moving everything to the LHS and plugging in the approximation:

$$u^h - u = s(x,t)\frac{\partial u^h}{\partial t} - \left(t(x,t)\frac{\partial u^h}{\partial x}\right)_x - q(x,t)$$

enforce that error is not in test function space:

$$\int_0^L s(x,t)\frac{\partial u^h}{\partial t}V(x)dx - \int_0^L \left(t(x,t)\frac{\partial u^h}{\partial x}\right)_x V(x)dx - \int_0^L q(x,t)V(x)dx = 0$$

Integration by parts:

$$\int_0^L s(x,t)\frac{\partial u^h}{\partial t}V(x)dx + \int_0^L t(x,t)\frac{\partial u^h}{\partial x}\frac{\partial V(x)}{\partial x}dx - nt(x,t)\frac{\partial u^h}{\partial x}V(x)\bigg|_0^L - \int_0^L q(x,t)V(x)dx = 0$$

make space of $V(x)$ finite and expand $u^h = \sum_{j=0}^N \hat{u}_j\psi_j$ in the ψ basis and make N equations for N basis functions:

$$\int_0^L s(x,t)\sum_{j=0}^N \frac{\partial \hat{u}_j}{\partial t}\psi_j\psi_i dx - \int_0^L t(x,t)\sum_{j=0}^N \hat{u}_j\frac{\partial \psi_j}{\partial x}\frac{\partial \psi_i}{\partial x}dx + = t(L,t)\beta(t)\psi_{pN}(L) - t(0,t)\alpha(t)\psi_0(0) + \int_0^L q(x,t)\psi_i dx$$

this is for a single local element k:

$$M^k\hat{u}_t - S^k\hat{u} = F^k$$

$$M_{ij}^k = \int_k s(x,t)\psi_j\psi_i \quad S_{ij}^k = \int_k t(x,t)\frac{\partial \psi_j}{\partial x}\frac{\partial \psi_i}{\partial x}dx$$

2 2D Variable Coefficent Diffusion

$$s(x,y,t)\frac{\partial u}{\partial t} = \left(t(x,y,t)\frac{\partial u}{\partial x}\right)_x + \left(t(x,y,t)\frac{\partial u}{\partial y}\right)_y + q(x,y,t), \qquad 0 \le x \le L_x, 0 \le y \le L_y, t \le F$$

$$u(0,y,t) = \alpha(t)$$

$$u(L,t) = \beta(t)$$

Assuming an approximate solution u^h, residual $u^h - u$ can be calculated by moving everything to the LHS and plugging in the approximation:

$$u^h - u = s(x,y,t)\frac{\partial u^h}{\partial t} - \left(T(x,y,t)\frac{\partial u^h}{\partial x}\right)_x - \left(t(x,y,t)\frac{\partial u}{\partial y}\right)_y - q(x,y,t)$$

enforce that error is not in test function space:

$$\int_\Omega s(x,y,t)\frac{\partial u^h}{\partial t}V(x)dx - \int_\Omega \left(T(x,y,t)\frac{\partial u^h}{\partial x}\right)_x V(x)dx - \int_\Omega q(x,t)V(x)dx = 0$$

Integration by parts (greens identity):

$$\int_\Omega s(x,y,t)\frac{\partial u^h}{\partial t}V(x)dx + \int_\Omega T(x,y,t)\frac{\partial u^h}{\partial x}\frac{\partial V(x)}{\partial x}dx - \int_\Gamma (\boldsymbol{n_x}\cdot\boldsymbol{n_k})T(x,y,t)\frac{\partial u^h}{\partial x}V(x) +$$
$$\int_\Omega T(x,y,t)\frac{\partial u^h}{\partial y}\frac{\partial V(x)}{\partial y}dx - \int_\Gamma (\boldsymbol{n_y}\cdot\boldsymbol{n_k})T(x,y,t)\frac{\partial u^h}{\partial y}V(x) -$$
$$\int_\Omega q(x,t)V(x)dx = 0$$

make space of $V(x)$ finite and expand $u^h = \sum_{j=0}^N \hat{u}_j\psi_j$ in the ψ basis and make N equations for N basis functions:

$$\int_\Omega s(x,y,t)\sum_{j=0}^N \frac{\partial \hat{u}_j}{\partial t}\psi_j\psi_i dx + \int_\Omega T(x,y,t)\sum_{j=0}^N \hat{u}_j\frac{\partial \psi_j}{\partial x}\frac{\partial \psi_i}{\partial x}dx + \int_\Omega T(x,y,t)\sum_{j=0}^N \hat{u}_j\frac{\partial \psi_j}{\partial y}\frac{\partial \psi_i}{\partial y}dx =$$
$$\text{boundary terms } + \int_\Omega q(x,t)\psi_i dx$$

this is for a single local element k:

$$M^k\hat{u}_t + S_x^k\hat{u} + S_y^k\hat{u} = F^k$$

$$M_{ij}^k = \int_k s(x,y,t)\psi_j\psi_i \quad S_x^k{}_{ij} = \int_k T(x,y,t)\frac{\partial \psi_j}{\partial x}\frac{\partial \psi_i}{\partial x}dx \quad S_y^k{}_{ij} = \int_k T(x,y,t)\frac{\partial \psi_j}{\partial y}\frac{\partial \psi_i}{\partial y}dx$$

3 heat 1st order system?

Alternatively split heat equation into first order system so that:

$$s(x,y,t)\frac{\partial u}{\partial t} = \left(t(x,y,t)\frac{\partial u}{\partial x}\right)_x + \left(t(x,y,t)\frac{\partial u}{\partial y}\right)_y + q(x,y,t), \qquad 0 \le x \le L_x, 0 \le y \le L_y, t \le F$$

turns into:

$$s(x,y,t)\frac{\partial u}{\partial t} = \left(\sqrt{t}q\right)_x + \left(t(x,y,t)\frac{\partial u}{\partial y}\right)_y + q(x,y,t)$$

4 basis and matrix construction

Reference element: (-1, -1), (1, -1), (-1, 1) in the r, s plane. Transformation from the r, s plane to the x, y plane is:

$$T(r,s) = \frac{r+s}{2}v^1 + \frac{r+1}{2}v^2 + \frac{s+1}{2}v^3$$

where v^1, v^2, v^3 are the vertices of the element in the x, y plane.

jacobian of transformation is:

$$J = \frac{(v_x^2 - v_x^1)(v_y^3 - v_y^1)}{4} - \frac{(v_y^2 - v_y^1)(v_x^3 - v_x^1)}{4}$$

Assuming linear basis functions (lame) they are on the reference element:

$$\begin{aligned} \psi_1 &= \frac{1}{2}s + \frac{1}{2} \\ \psi_2 &= \frac{1}{2}r + \frac{1}{2} \\ \psi_3 &= -\frac{1}{2}r - \frac{1}{2}s \end{aligned}$$

derivaties are:

$$\begin{aligned} \frac{\partial \psi_1}{\partial r} &= 0 & \frac{\partial \psi_1}{\partial s} &= \frac{1}{2} \\ \frac{\partial \psi_2}{\partial r} &= \frac{1}{2} & \frac{\partial \psi_2}{\partial s} &= 0 \\ \frac{\partial \psi_3}{\partial r} &= -\frac{1}{2} & \frac{\partial \psi_3}{\partial s} &= -\frac{1}{2} \end{aligned}$$

Matrices need to be computed with numerical integration because of variable coefficents:
gaussian quad weights:

r$_i$	s$_i$	w$_i$
1/3	1/3	9/32
3/5	1/5	25/96
1/5	3/5	25/96
1/5	1/5	25/96

4.1 linear local mass computation

assume ψ_i is linear how to compute:

$$\begin{bmatrix} \int_k \psi_1 s \psi_1 d\Omega_k & \int_k \psi_1 s \psi_2 d\Omega_k & \int_k \psi_1 s \psi_3 d\Omega_k \\ \int_k \psi_2 s \psi_1 d\Omega_k & \int_k \psi_2 s \psi_2 d\Omega_k & \int_k \psi_2 s \psi_3 d\Omega_k \\ \int_k \psi_3 s \psi_1 d\Omega_k & \int_k \psi_3 s \psi_2 d\Omega_k & \int_k \psi_3 s \psi_3 d\Omega_k \end{bmatrix} =$$

$$\begin{bmatrix} \sum_q \psi_1(x_q,y_q)s(x_q,y_q)\psi_1(x_q,y_q)w_q & \sum_q \psi_1(x_q,y_q)s(x_q,y_q)\psi_2(x_q,y_q)w_q & \sum_q \psi_1(x_q,y_q)s(x_q,y_q)\psi_3(x_q,y_q)w_q \\ \sum_q \psi_2(x_q,y_q)s(x_q,y_q)\psi_1(x_q,y_q)w_q & \sum_q \psi_2(x_q,y_q)s(x_q,y_q)\psi_2(x_q,y_q)w_q & \sum_q \psi_2(x_q,y_q)s(x_q,y_q)\psi_3(x_q,y_q)w_q \\ \sum_q \psi_3(x_q,y_q)s(x_q,y_q)\psi_1(x_q,y_q)w_q & \sum_q \psi_3(x_q,y_q)s(x_q,y_q)\psi_2(x_q,y_q)w_q & \sum_q \psi_3(x_q,y_q)s(x_q,y_q)\psi_3(x_q,y_q)w_q \end{bmatrix} =$$

4.2 timestepping

everything boils down to:

$$M\hat{u}_t = S\hat{u} + F$$

doing crank:

$$M\frac{\hat{u}_{t+1} - \hat{u}_t}{\Delta t} = \frac{1}{2}(S\hat{u}_{t+1} + S\hat{u}_t) + \frac{1}{2}(F_{t+1} + F_t)$$
$$M\hat{u}_{t+1} - \frac{\Delta t}{2}S\hat{u}_{t+1} = M\hat{u}_t + \frac{\Delta t}{2}S\hat{u}_t + \frac{\Delta t}{2}F_{t+1} + \frac{\Delta t}{2}F_t$$