

FEM and Inverse notes

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July 2024

1 1D Variable Coefficient Diffusion

$$s(x, t) \frac{\partial u}{\partial t} = \left(t(x, t) \frac{\partial u}{\partial x} \right)_x + q(x, t), \quad 0 \leq x \leq L, t \leq T$$

$$u(0, t) = \alpha(t)$$

$$u(L, t) = \beta(t)$$

Assuming an approximate solution u^h , residual $u^h - u$ can be calculated by moving everything to the LHS and plugging in the approximation:

$$u^h - u = s(x, t) \frac{\partial u^h}{\partial t} - \left(t(x, t) \frac{\partial u^h}{\partial x} \right)_x - q(x, t)$$

enforce that error is not in test function space:

$$\int_0^L s(x, t) \frac{\partial u^h}{\partial t} V(x) dx - \int_0^L \left(t(x, t) \frac{\partial u^h}{\partial x} \right)_x V(x) dx - \int_0^L q(x, t) V(x) dx = 0$$

Integration by parts:

$$\int_0^L s(x, t) \frac{\partial u^h}{\partial t} V(x) dx + \int_0^L t(x, t) \frac{\partial u^h}{\partial x} \frac{\partial V(x)}{\partial x} dx - \left[t(x, t) \frac{\partial u^h}{\partial x} V(x) \right]_0^L - \int_0^L q(x, t) V(x) dx = 0$$

make space of $V(x)$ finite and expand $u^h = \sum_{j=0}^N \hat{u}_j \psi_j$ in the ψ basis and make N equations for N basis functions:

$$\int_0^L s(x, t) \sum_{j=0}^N \frac{\partial \hat{u}_j}{\partial t} \psi_j \psi_i dx - \int_0^L t(x, t) \sum_{j=0}^N \hat{u}_j \frac{\partial \psi_j}{\partial x} \frac{\partial \psi_i}{\partial x} dx + = t(L, t) \beta(t) \psi_{pN}(L) - t(0, t) \alpha(t) \psi_0(0) + \int_0^L q(x, t) \psi_i dx$$

this is for a single local element k :

$$M^k \hat{u}_t - S^k \hat{u} = F^k$$

$$M_{ij}^k = \int_k s(x, t) \psi_j \psi_i \quad S_{ij}^k = \int_k t(x, t) \frac{\partial \psi_j}{\partial x} \frac{\partial \psi_i}{\partial x} dx$$

2 2D Variable Coefficient Diffusion

$$s(x, y, t) \frac{\partial u}{\partial t} = \left(t(x, y, t) \frac{\partial u}{\partial x} \right)_x + \left(t(x, y, t) \frac{\partial u}{\partial y} \right)_y + q(x, y, t), \quad 0 \leq x \leq L_x, 0 \leq y \leq L_y, t \leq F$$

$$u(0, y, t) = \alpha(t)$$

$$u(L, t) = \beta(t)$$

Assuming an approximate solution u^h , residual $u^h - u$ can be calculated by moving everything to the LHS and plugging in the approximation:

$$u^h - u = s(x, y, t) \frac{\partial u^h}{\partial t} - \left(T(x, y, t) \frac{\partial u^h}{\partial x} \right)_x - \left(t(x, y, t) \frac{\partial u}{\partial y} \right)_y - q(x, y, t)$$

enforce that error is not in test function space:

$$\int_{\Omega} s(x, y, t) \frac{\partial u^h}{\partial t} V(x) dx - \int_{\Omega} \left(T(x, y, t) \frac{\partial u^h}{\partial x} \right)_x V(x) dx - \int_{\Omega} q(x, t) V(x) dx = 0$$

Integration by parts (greens identity):

$$\begin{aligned} & \int_{\Omega} s(x, y, t) \frac{\partial u^h}{\partial t} V(x) dx + \int_{\Omega} T(x, y, t) \frac{\partial u^h}{\partial x} \frac{\partial V(x)}{\partial x} dx - \int_{\Gamma} (\mathbf{n}_x \cdot \mathbf{n}_k) T(x, y, t) \frac{\partial u^h}{\partial x} V(x) + \\ & \int_{\Omega} T(x, y, t) \frac{\partial u^h}{\partial y} \frac{\partial V(x)}{\partial y} dx - \int_{\Gamma} (\mathbf{n}_y \cdot \mathbf{n}_k) T(x, y, t) \frac{\partial u^h}{\partial y} V(x) - \\ & \int_{\Omega} q(x, t) V(x) dx = 0 \end{aligned}$$

make space of $V(x)$ finite and expand $u^h = \sum_{j=0}^N \hat{u}_j \psi_j$ in the ψ basis and make N equations for N basis functions:

$$\int_{\Omega} s(x, y, t) \sum_{j=0}^N \frac{\partial \hat{u}_j}{\partial t} \psi_j \psi_i dx + \int_{\Omega} T(x, y, t) \sum_{j=0}^N \hat{u}_j \frac{\partial \psi_j}{\partial x} \frac{\partial \psi_i}{\partial x} dx + \int_{\Omega} T(x, y, t) \sum_{j=0}^N \hat{u}_j \frac{\partial \psi_j}{\partial y} \frac{\partial \psi_i}{\partial y} dx =$$

$$\text{boundary terms} + \int_{\Omega} q(x, t) \psi_i dx$$

this is for a single local element k :

$$M^k \hat{u}_t + S_x^k \hat{u} + S_y^k \hat{u} = F^k$$

$$M_{ij}^k = \int_k s(x, y, t) \psi_j \psi_i \quad S_{xij}^k = \int_k T(x, y, t) \frac{\partial \psi_j}{\partial x} \frac{\partial \psi_i}{\partial x} dx \quad S_{yij}^k = \int_k T(x, y, t) \frac{\partial \psi_j}{\partial y} \frac{\partial \psi_i}{\partial y} dx$$

3 heat 1st order system?

Alternatively split heat equation into first order system so that:

$$s(x, y, t) \frac{\partial u}{\partial t} = \left(t(x, y, t) \frac{\partial u}{\partial x} \right)_x + \left(t(x, y, t) \frac{\partial u}{\partial y} \right)_y + q(x, y, t), \quad 0 \leq x \leq L_x, 0 \leq y \leq L_y, t \leq F$$

turns into:

$$s(x, y, t) \frac{\partial u}{\partial t} = \left(\sqrt{t} q \right)_x + \left(t(x, y, t) \frac{\partial u}{\partial y} \right)_y + q(x, y, t)$$

4 basis and matrix construction

Reference element: $(-1, -1), (1, -1), (-1, 1)$ in the r, s plane. Transformation from the r, s plane to the x, y plane is:

$$T(r, s) = \frac{r+s}{2}v^1 + \frac{r+1}{2}v^2 + \frac{s+1}{2}v^3$$

where v^1, v^2, v^3 are the vertices of the element in the x, y plane.

jacobian of transformation is:

$$J = \frac{(v_x^2 - v_x^1)(v_y^3 - v_y^1)}{4} - \frac{(v_y^2 - v_y^1)(v_x^3 - v_x^1)}{4}$$

Assuming linear basis functions (lame) they are on the reference element:

$$\begin{aligned} \psi_1 &= \frac{1}{2}s + \frac{1}{2} \\ \psi_2 &= \frac{1}{2}r + \frac{1}{2} \\ \psi_3 &= -\frac{1}{2}r - \frac{1}{2}s \end{aligned}$$

derivatives are:

$$\begin{aligned} \frac{\partial \psi_1}{\partial r} &= 0 & \frac{\partial \psi_1}{\partial s} &= \frac{1}{2} \\ \frac{\partial \psi_2}{\partial r} &= \frac{1}{2} & \frac{\partial \psi_2}{\partial s} &= 0 \\ \frac{\partial \psi_3}{\partial r} &= -\frac{1}{2} & \frac{\partial \psi_3}{\partial s} &= -\frac{1}{2} \end{aligned}$$

Matrices need to be computed with numerical integration because of variable coefficients:

gaussian quad weights:

r_i	s_i	w_i
1/3	1/3	9/32
3/5	1/5	25/96
1/5	3/5	25/96
1/5	1/5	25/96