itself. By suitable conventions, we can make this "*non-euclidean geometry*" obey all of the axioms of our system except the axiom of Euclid (group III). Since the possibility of the ordinary geometry has already been established, that of the non-euclidean geometry is now an immediate consequence of the above considerations.

§ 11. INDEPENDENCE OF THE AXIOMS OF CONGRUENCE.

We shall show the independence of the axioms of congruence by demonstrating that axiom IV, 6, or what amounts to the same thing, that the first theorem of congruence for triangles (theorem 10) cannot be deduced from the remaining axioms I, II, III, IV 1–5, V by any logical process of reasoning.

Select, as the points, straight lines, and planes of our new geometry of space, the points, straight lines, and planes of ordinary geometry, and define the laying off of an angle as in ordinary geometry, for example, as explained in § 9. We will, however, define the laying off of segments in another manner. Let A_1, A_2 be two points which, in ordinary geometry, have the co-ordinates x_1, y_1, z_1 and x_2, y_2, z_2, respectively. We will now define the length of the segment A_1A_2 as the positive value of the expression

$$\sqrt{(x_1 - x_2 + y_1 - y_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

and call the two segments A_1A_2 and $A'_1A'_2$ congruent when they have equal lengths in the sense just defined.

It is at once evident that, in the geometry of space thus defined, the axioms I, II, III, IV 1–2, 4–5, V are all fulfilled.

In order to show that axiom IV, 3 also holds, we select an arbitrary straight line a and upon it three points A_1, A_2, A_3 so that A_2 shall lie between A_1 and A_3. Let the points x, y, z of the straight line a be given by means of the equations

$$\begin{aligned} x &= \lambda t + \lambda', \\ y &= \mu t + \mu', \\ z &= \nu t + \nu', \end{aligned}$$

where λ, λ', μ, μ', ν, ν' represent certain constants and T is a parameter. If t_1, t_2 $(< t_1)$, t_3 $(< t_2)$ are the values of the parameter corresponding to the points A_1, A_2, A_3 we have as the lengths of the three segments A_1A_2 A_2A_3 and A_1A_3 respectively, the following values:

$$(t_1 - t_2)\left|\sqrt{(\lambda + \mu)^2 + \mu^2 + \nu^2}\right|$$

$$(t_2 - t_3)\left|\sqrt{(\lambda + \mu)^2 + \mu^2 + \nu^2}\right|$$