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Greeks and Risk Management

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Contents

Terminology

Through this paper we will use some finance terms, that will be defined there. With these definitions in mind, the reader does not need to know anything about finance to understand this paper.

Option:
An **option** is a contract that gives its owner the right, but not the obligation, to buy or sell a certain quantity of an underlying asset at a specific price, during a specific time window. Along this paper, we will often talk about the options European call and European put, which are defined just below.

European call and **put**:

- A **European call** is an option depending on three parameters : S_t the underlying's price at time t, K the strike price, and T the expiration date. It gives its owner the right, but not the obligation, to buy a predetermined number of shares of the underlying at price K, and this has to happen at time T precisely.
- Similarly, a **European put** is an option on the three same parameters, that gives its owner the right, but not the obligation, to sell a predetermined, number of shares of the underlying at price K, at time T precisely.

Long and **Short**:

- A **long** position means that the investor buys an asset (e.g., stocks, bonds, or commodities) with the expectation that its price will rise. For instance if you believe a company's stock will increase in value, you buy shares. You are then "long" in that stock. If the stock rises, you can sell it at a higher price and make a profit.
- A **short** position means the investor sells an asset they don't own, typically borrowing it, with the expectation that its price will fall. If you expect a

company's stock to decrease in value, you borrow shares of that stock and sell them at the current market price. Later, when the stock falls, you buy the shares back at a lower price, return them to the lender, and pocket the difference.

ATM, **ITM** and **OTM**:

- An option is said to be **ATM** (At The Money) when $K \approx S_t$, i.e. the strike price is equal (or very close) to the price of the underlying.

- An option is said to be **ITM** when immediate exercise would yield a profit. For a Call, this means $K < S_t$. For a Put, this means $K > S_t$.

- An option is said to be **OTM** when exercising for the current price of the underlying would yield a loss (the option has no intrinsic value). For a Call, this means $K > S_t$. For a Put, this means $K < S_t$.

Portfolio:
A **Portfolio** is a collection of financial assets held by an investor or a financial institution. Examples of financial assets include (we will mostly talk about the two firsts along this paper) stocks, derivatives, bonds, commodities, real estate, currencies, and many others.

Chapter 1

Introduction

In trading, there is, in theory, no such thing as free money. Every potential profit comes together with a risk, often proportional to the potential gain. Effective trading, therefore, relies heavily on how well one manages these risks. This is where the **Greeks**, a set of financial metrics that measure an option's sensitivity to various market factors, become useful tools. This paper, that is structured into three main sections, is about the use of this tools related to risk management. We will begin by introducing the Black-Scholes model, a foundational framework in options pricing, along with its core assumptions. This model provides a basis for calculating option prices as well as for deriving the **Greeks**. Next, we will delve into the five most important **Greeks**, namely **Delta**, **Gamma**, **Theta**, **Vega**, and **Rho**. For each, we will compute closed-form solutions for simple European options and explore basic risk management strategies in an hopefully illustrative and intuitive manner. Finally, we will discuss the limitations of the Black-Scholes model, how these limitations impact the **Greeks** and, consequently, their effectiveness in real-world risk management.

Chapter 2

The Black-Scholes model

2.1 Introduction to the Black-Scholes Model

The Black-Scholes model (also called Black-Scholes-Merton model) is a framework in financial mathematics for options pricing. Introduced in the early 1970s, it revolutionized the world of finance by providing a systematic approach for derivatives valuation, in particular for pricing European options. At its core, the Black-Scholes model is based on several key assumptions:

1. **The underlying asset price follows a geometric Brownian motion**, described by the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

   where:

   - S_t is the price of the underlying asset at time t,
   - μ is the constant drift rate (expected return),
   - σ is the volatility of the asset,
   - dW_t is an increment of a Wiener process (standard Brownian motion), capturing the randomness in the price movement.

2. **The market is frictionless**: No transaction costs or taxes, and assets can be bought or sold continuously.

3. **No arbitrage opportunities**: The model assumes that there are no arbitrage possibilities (i.e. it is impossible to earn profits without risk).

4. **The risk-free interest rate r is constant**, and borrowing/lending at the risk-free rate is possible.

2.2 The Black-Scholes Partial Differential Equation & Formula

The fundamental output of this model is a partial differential equation that governs the evolution of the price of a European call or put option. The price $V(S,t)$ of an option as a function of the underlying asset price S and time t is derived from the stochastic process described above and results in the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where:

- $V = V(S,t)$ is the price of the option,
- σ is the volatility of the underlying asset,
- r is the risk-free interest rate.

For a European call option, it can be shown (the proof is outside the scope of this paper, so let us only state the important results) that the solution to the Black-Scholes partial differential equation leads to the well-known Black-Scholes formula, which gives the price of the call option C as:

$$C(S_t,t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{2.1}$$

where:

- S_t is the price of the underlying asset at time t,
- K is the strike price of the option,
- T is the time to maturity,
- $N(\cdot)$ is the cumulative distribution function of the standard normal distribution define by $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}\,dz$,

- d_1 and d_2 are intermediate variables defined as:

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

For a European put option, the corresponding Black-Scholes price P is:

$$P(S_t, t) = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1) \tag{2.2}$$

Along this paper, we will compute various derivatives of C and P, hence equations 2.1 and 2.2 are worth be kept in mind.

2.3 From Black-Scholes to the Greeks

The Black-Scholes model not only provides a method for option pricing but also serves as the foundation for understanding the main object of this paper : the **Greeks** - a set of risk management tools that measure the sensitivity of an option's price to various factors. The most important/used ones are:

- **Delta (Δ)**: Sensitivity of the option price to changes in the underlying asset price, defined as $\Delta = \frac{\partial V}{\partial S}$.
- **Gamma (Γ)**: Sensitivity of Delta to changes in the underlying price, defined as $\Gamma = \frac{\partial^2 V}{\partial S^2}$.
- **Theta (Θ)**: Sensitivity of the option price to the passage of time, defined as $\Theta = \frac{\partial V}{\partial t}$.
- **Vega ($\mathcal{V}$)**: Sensitivity of the option price to changes in the volatility of the underlying asset, defined as $\mathcal{V} = \frac{\partial V}{\partial \sigma}$.
- **Rho (ρ)**: Sensitivity of the option price to changes in the risk-free interest rate, defined as $\rho = \frac{\partial V}{\partial r}$.

These **Greeks** are critical for constructing hedging strategies, allowing traders to manage the risks associated with changes in market conditions.

Chapter 3

The Greeks & Hedging

3.1 Delta

At the core of any option is a particular (or an assemblage of) asset, with its own price S_t indexed by the time. This price is often subject to change, which will have an impact on the value V_t of the option. Therefore, it can be useful to know the rate of change of the latter with respect to a change in the underlying. To do so, we compute the quantity

$$\Delta = \frac{\partial V}{\partial S}$$

called **Delta**. Graphically, **Delta** can be seen as the slope of the tangent line to the graph of $V(S)$ at a specific value of S.

3.1.1 Delta of European Call & Put

Recall the formulas 2.1 and 2.2 in the previous part. Based on those, we are interested in computing the associated delta. It is important to notice that d_1 and d_2 defined in 2.2 are function of S_t. So to compute

$$\Delta_C = \frac{\partial C(S_t, t)}{\partial S_t}$$

we need to use the product rule, then the chain rule. The product rule gives us first:

$$\frac{\partial}{\partial S_t}\left(S_t N(d_1)\right) = N(d_1) + S_t \frac{\partial N(d_1)}{\partial S_t}$$

Recalling that d_1 is a function of S_t, we apply the chain rule to the left term:

$$\frac{\partial N(d_1)}{\partial S_t} = N'(d_1)\frac{\partial d_1}{\partial S_t}$$

for $N'(d_1)$ the probability density function (PDF) of the standard normal distribution defined as $N'(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$. Using that $\log'(x) = \frac{1}{x}$, we obtain:

$$\frac{\partial d_1}{\partial S_t} = \frac{1}{S_t\sigma\sqrt{T-t}}$$

So:

$$\frac{\partial}{\partial S_t}\left(S_t N(d_1)\right) = N(d_1) + \frac{N'(d_1)}{\sigma\sqrt{T-t}}$$

Now for the right term we apply the product rule:

$$\frac{\partial}{\partial S_t}\left(Ke^{-r(T-t)}N(d_2)\right) = Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S_t}$$

As d_1 and d_2 have the same derivatives with respect to S_t we obtain:

$$\frac{\partial}{\partial S_t}\left(Ke^{-r(T-t)}N(d_2)\right) = Ke^{-r(T-t)}N'(d_2)\frac{1}{S_t\sigma\sqrt{T-t}}$$

Thus summing gives the full expression of the partial derivative:

$$\Delta_C = N(d_1) + \frac{N'(d_1)}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\frac{N'(d_2)}{S_t\sigma\sqrt{T-t}}$$

We can simplify this expression since $d_2 = d_1 - \sigma\sqrt{T-t}$, which implies that:

$$N'(d_2) = \frac{1}{\sqrt{2\pi}}e^{\frac{-(d_1-\sigma\sqrt{T-t})^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{\frac{-(d_1^2-2d_1\sigma\sqrt{T-t}+(\sigma\sqrt{T-t})^2)}{2}} = N'(d_1)\frac{S_t}{K}e^{r(T-t)}$$

Because:

$$e^{\frac{2d_1\sigma\sqrt{T-t}-(\sigma\sqrt{T-t})^2)}{2}} = e^{\log(\frac{S_t}{K})+(r+\frac{1}{2}\sigma^2)(T-t)-\frac{\sigma^2(T-t)}{2}} = \frac{S_t}{K}e^{r(T-t)}$$

What we proved in the last two lines, namely the equation:

$$N'(d_2) = N'(d_1)\frac{S_t}{K}e^{r(T-t)} \tag{3.1}$$

is worth keeping in mind, as it will be useful for later computations. Going back to our initial computation, the second and third terms in Δ_C cancel each others to give:

$$\Delta_C = N(d_1) \tag{3.2}$$

We can use the computations we made to compute

$$\Delta_P = \frac{\partial C(S_t, t)}{\partial S_t}$$

For the second term in 2.2 we get:

$$\frac{\partial}{\partial S_t}\left(S_t N(-d_1)\right) = N(-d_1) - \frac{N'(-d_1)}{\sigma\sqrt{T-t}}$$

And for the first:

$$\frac{\partial}{\partial S_t}\left(Ke^{-r(T-t)}N(-d_2)\right) = -Ke^{-r(T-t)}N'(-d_2)\frac{1}{S_t\sigma\sqrt{T-t}}$$

So we sum the two terms to get:

$$\Delta_P = -N(-d_1) + \frac{N'(-d_1)}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}N'(-d_2)\frac{1}{S_t\sigma\sqrt{T-t}}$$

We notice that $N'(x) = f(x^2)$ for a function f, which implies that $N'(-x) = N'(x)$. So using this and the simplification made for Δ_C the two last terms cancel each others:

$$\frac{N'(-d_1)}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}N'(-d_2)\frac{1}{S_t\sigma\sqrt{T-t}} = \frac{N'(d_1)}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}N'(d_2)\frac{1}{S_t\sigma\sqrt{T-t}} = 0$$

Now, by the fact that the normal is symmetric around 0, and because the total probability is 1 we know that $N(-x) = 1 - N(x)$. Consequently, our final expression is:

$$\Delta_P = N(d_1) - 1 \tag{3.3}$$

Because the image set of the function $N(\cdot)$ is the segment $[0, 1]$, we notice the following inequality:

$$-1 \leq \Delta_P \leq 0 \leq \Delta_C \leq 1$$

Furthermore, we also have:

$$\Delta_C - \Delta_P = 1$$

We can observe this difference in the following plot figure 3.1:

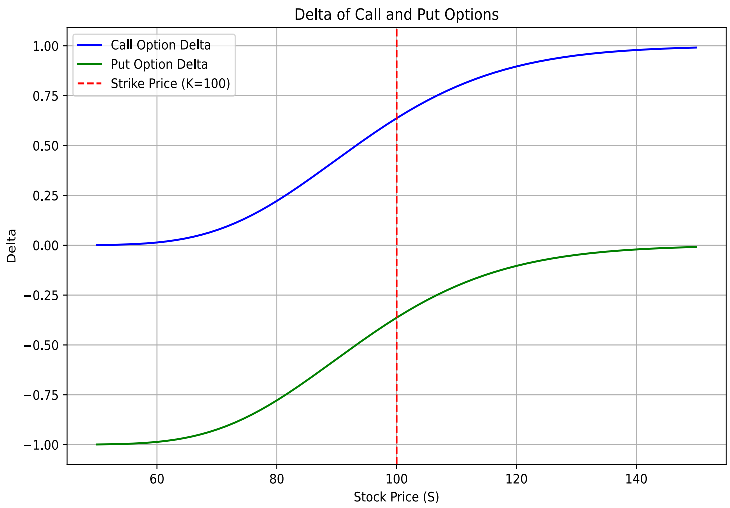

Figure 3.1: Δ of European Call and Put versus the stock price

Seeing this relationship from an hedging perspective, this means that simultaneously holding a long position in a call and a short position in a put on the same underlying (with same strike price and expiration) is strictly equivalent to holding one unit of the underlying. We say that holding those positions simultaneously is **replicating the underlying asset**.

3.1.2 Delta Hedging

First of all, if we assume that **Delta** is constant, it appears the information it contains is enough to nullify the risk of an option position. To hedge an option position, we can take a position in the underlying asset that offsets the **Delta** exposure of the option.

Let us assume we own 10 call options (we will use the usual assumption that an option is a contract over 100 shares of the underlying), with an associated delta of 0.7. We assume that the stock price is 100 CHF. We want to know how our position behaves when the price moves, so we compute 0.7×100 shares/option $\times 10$ options $= 700$ shares, i.e. our position behaves like the asset position "buying 700 shares" (or in finance language, a 700-shares long stock position). Thus to hedge this we sell

700 shares (or in finance language, we short 700 shares of stock) of the underlying. We can distinguish the two different scenarios:

- If the stock price increases by x CHF, our short stock position incurs a loss of 700 shares $\times x$ CHF $= x \times 700$ CHF. But our call option position brings a profit of $0.7 \times x$ CHF $\times 100$ shares/option $\times 10$ options $= x \times 700$ CHF. Thus a perfectly hedged position.

- If the stock price decreases by x CHF, our short stock position brings a profit of 700 shares $\times x$ CHF $= x \times 700$ CHF. But our call option position incurs a loss of $0.7 \times x$ CHF $\times 100$ shares/option $\times 10$ options $= x \times 700$ CHF. Thus once again, a perfectly hedged position.

The necessary hypothesis for our hedging strategy to work here is the fact that **Delta** remained constant. It is not the case in real life, where the value of an option is not linear in the price of the underlying. Thus such a strategy can only work for a very small interval of time, where delta stays close to its originally computed value. To make this work in the long term, we have to calculate the new delta and rebalance our portfolio to adjust our hedging accordingly. This is called **Dynamic Delta Hedging**.

3.2 Gamma

A natural question that may arises when considering **Dynamic Delta Hedging** strategies is 'How often should I compute **Delta** and rebalance my portfolio to hedge the risk successfully ?'. The answer of course depends on the specific situation, and is heavily influenced by **Delta**'s sensitivity to price change of the underlying. To assess this sensitivity, we can compute the derivative of **Delta** with respect to S_t, namely **Gamma**:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$$

From a hedging point of view, we would like our **Gamma** to be as close to zero as possible, as it means that **Delta** is very insensitive to changes of price, and consequently we do not need to update **Delta** frequently to hedge our portfolio. This is often the case with call options that are deep ITM, as in this situation **Delta** is close to 1 and almost constant because the options behaves almost like the underlying, and for deep OTM put options as then **Delta** is near 0 because the option is almost sure not to be exercised which makes it insensitive to small price changes. On the contrary, if **Gamma** is very big, **Delta** is very sensitive to any price's motion. This is frequent for ATM options, for then small movements can impact with great significance the position of the option at expiration [1]. In this case, we need to rebalance frequently to maintain our hedge as close as possible to **Delta**-neutrality. This is bad because this multiply our number of trading occurrences, which in real life comes with a greater trading cost. Similarly, a greatly negative **Gamma** is not good because it means that **Delta** is also very sensitive. However, a negative **Gamma** does not arise for simple options like calls and puts, but only for more complex like exotic options.

3.2.1 Gamma of European Call & Put

Let us compute the **Gamma** for an Europen call option, namely Γ_C. Based on equations 3.2 and 3.3, we see that the **Delta**s of a call and a put only differ by a constant. Hence we only need to compute

$$\Gamma_C = \frac{\partial \Delta_C}{\partial S_t}$$

as $\Gamma_C = \Gamma_P$. Let us start by using the chain rule:

$$\frac{\partial \Delta_C}{\partial S_t} = N'(d_1)\frac{\partial d_1}{\partial S_t}$$

Thanks to the results in 3.1.1, we know that:

$$N'(d_1) = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}$$

and

$$\frac{\partial d_1}{\partial S_t} = \frac{1}{S_t \sigma \sqrt{T-t}}$$

So that finally:

$$\Gamma_C = \frac{e^{-\frac{d_1^2}{2}}}{S_t \sigma \sqrt{2\pi(T-t)}}$$

Having this closed-form formula, we can easily obtain the graph of **Gamma** versus the price of the underlying. We notice that **Gamma** peaks close to the point when the call is ATM, namely when the underlying's price is close to the strike price

[1] here, position is intended in the sense of OTM, ITM or ATM

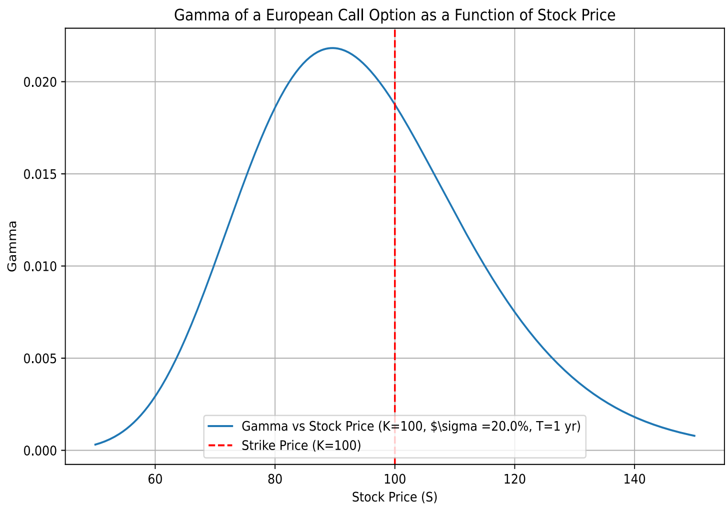

Figure 3.2: Γ versus the stock price

K. We also notice a decay in **Gamma** when moving further away from K in both directions. These observations are coherent with our explanation in the introductory paragraph of 3.2. Also, one may wonder why the peak is slightly before K. This is because of d_1, as we can see in the expression 2.2 that it is not 0 when $S = K$ because the positive term $(r + \frac{1}{2}\sigma^2)(T - t)$ needs to be compensated.

3.2.2 Gamma Hedging

Delta hedging, the first form of risk hedging we saw in section 3.1.2, is very efficient when it comes to protecting from small price changes. But when larger swings in the price of the underlying happen, **Delta** might become outdated too quickly. Hence the importance of **Gamma** hedging. When considering **Delta**, buying or selling shares of the underlying directly has proven itself to be an effective hedging strategy. Unfortunately, a position in the underlying is linear in the underlying and thus has a **Gamma** of 0 (one can clearly see that $\frac{\partial^2 S_t}{\partial S_t^2} = 0$). Consequently, to balance the **Gamma** of a portfolio we need to take a position in an option with an non-zero **Gamma**. Let us illustrate this through the following example. Let us possess a **Delta**-neutral portfolio with:

$$\Gamma_{portfolio} = 0.3$$

Moreover assume we have access to an option with **Gamma** non-zero of:

$$\Gamma_{opt} = 0.001$$

We want to buy/sell x options in order to obtain a new portfolio **Gamma**-neutral. Setting

$$x = -\frac{\Gamma_{portfolio}}{\Gamma_{opt}} = -300$$

gives us a total **Gamma** of:

$$\Gamma = -300\Gamma_{opt} + \Gamma_{portfolio} = 0$$

as desired. So shorting 300 options **Gamma**-hedges the portfolio. But one may argue that doing as such would break the **Delta**-neutrality of our portfolio. To solve this, one can simply treat the case as a **Delta**-hedging case and buy/sell some stock, as this will not affect our total **Gamma**. For instance, if we assume the traded option to have a **Delta** of $\Delta_{opt} = 0.4$, our intermediate portfolio has now a **Delta** of:

$$\Delta_{portfolio} = -300 \times 0.4 = -120$$

Hence to get our final portfolio we have to buy 120 shares of the underlying to retrieve **Delta**-neutrality, while maintaining **Gamma**-neutrality.

3.3 Theta

The first two **Greeks** we discussed, **Delta** and **Gamma**, are concerned with the impact of the underlying's price variation on the value of an option. Besides the price of the asset, other parameters such as the volatility, the risk-free interest rate, and the time, play an important role in determining an option's pric. **Theta**, our next member of the **Greeks**, measures the influence of the passage of time on the price of an option. Mathematically, **theta** is defined as:

$$\Theta = \frac{\partial V}{\partial t}$$

One can often find in the literature the alternative following definition of $\Theta = -\frac{\partial V}{\partial \tau}$, where τ is defined as the time to expiration $\tau = T - t$. It is straightforward to see that these two definitions coincide. **Theta** represents the rate of decline of an option's price as time passes, and is often referred to as the "time decay" of the associated option. This is due to the fact that the value of an option diminishes with the passage of time. Before going on and computing the value of **Theta** for European call and put as usual, let us briefly explain why. The price of an option is determined by two factors: its intrinsic value, i.e. the difference between the current price of the underlying and the strike price, and its time value, i.e. the possibility for the price of the underlying to move in a profitable direction for the option's holder. As time passes, expiration approaches and the second value decreases simply because there is less time for the underlying's price to move in a favorable way for the investor. In other words, with the time window closing there is less room for potential improving opportunities.

3.3.1 Theta of European Call & Put

Before going on with the calculation, let us clarify something. When looking at the expression for the prices of European call & put options in equations 2.1 and 2.2, one notices the variable S_t has the subscript t. This implies a dependency of the price over time, which can be seen in the closed form of S_t namely $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$. However, as **Theta** is a partial derivative, S_t is considered as a fixed variable in the incoming calculation. With this in mind, let us go through the derivation of:

$$\Theta_C = \frac{\partial C(S_t, t)}{\partial t}$$

As previously, we proceed term by term. We compute the partial derivative of the first term $S_t N(d_1)$ via the chain rule:

$$\frac{\partial}{\partial t}\left(S_t N(d_1)\right) = S_t \frac{\partial N(d_1)}{\partial t} = S_t N'(d_1) \frac{\partial d_1}{\partial t}$$

where $N'(\cdot)$ is the standard normal probability density function as usual. Let us now compute $\frac{\partial d_1}{\partial t}$. Recall that:

$$d_1 = \frac{\ln(S_t/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Using $\frac{\partial}{\partial t}(\frac{u(t)}{v(t)}) = \frac{u'(t)v(t) - v'(t)u(t)}{v^2(t)}$, we get:

$$\frac{\partial d_1}{\partial t} = \frac{(-r - \frac{1}{2}\sigma^2)(\sigma\sqrt{T-t}) + (\frac{\sigma}{2\sqrt{T-t}})(\ln(S_t/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t))}{\sigma^2(T-t)} = \frac{-r - \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} + \frac{d_1}{2(T-t)}$$

So that the partial derivative of the first is:

$$\frac{\partial}{\partial t}\left(S_t N(d_1)\right) = S_t N'(d_1)\left(\frac{-r - \frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} + \frac{d_1}{2(T-t)}\right)$$

Now for the second term $Ke^{-r(T-t)}N(d_2)$ we apply the product rule:

$$\frac{\partial}{\partial t}\left(Ke^{-r(T-t)}N(d_2)\right) = K\left(\frac{\partial}{\partial t}\left(e^{-r(T-t)}\right)N(d_2) + e^{-r(T-t)}\frac{\partial N(d_2)}{\partial t}\right)$$

We see two time derivative in this expression. The first is simply:

$$\frac{\partial}{\partial t}\left(e^{-r(T-t)}\right) = re^{-r(T-t)}$$

Now, we obtain the second proceeding in a similar way as before:

$$\frac{\partial N(d_2)}{\partial t} = N'(d_2)\frac{\partial d_2}{\partial t}$$

Since $d_2 = d_1 - \sigma\sqrt{T-t}$, we have:

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}$$

Thus, the derivative of the second term becomes:

$$\frac{\partial}{\partial t}\left(Ke^{-r(T-t)}N(d_2)\right) = Ke^{-r(T-t)}\left(rN(d_2) + N'(d_2)\left(\frac{-r-\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} + \frac{d_1}{2(T-t)} + \frac{\sigma}{2\sqrt{T-t}}\right)\right)$$

Recalling equation 3.1, this expression simplifies as:

$$\frac{\partial}{\partial t}\left(Ke^{-r(T-t)}N(d_2)\right) = rKe^{-r(T-t)}N(d_2)+S_tN'(d_1)\left(\frac{-r-\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} + \frac{d_1}{2(T-t)} + \frac{\sigma}{2\sqrt{T-t}}\right)$$

So the first two terms in the last parenthesis vanish when subtracting to $\frac{\partial}{\partial t}\left(S_tN(d_1)\right)$. This leaves us with the final expression for the **Theta** of a European call option:

$$\Theta_C = -\frac{S_t\sigma N'(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

Now let us do the same for an European put option:

$$\Theta_P = \frac{\partial C(S_t,t)}{\partial t}$$

Based on the formula 2.2, we proceed as before term by term. For the first the product rule gives:

$$\frac{\partial}{\partial t}\left(Ke^{-r(T-t)}N(-d_2)\right) = K\left(\frac{\partial}{\partial t}e^{-r(T-t)}N(-d_2) + e^{-r(T-t)}\frac{\partial N(-d_2)}{\partial t}\right)$$

This is very close the what we obtained for the call, hence following the same path (without forgetting that we presently have $-d_2$ instead of d_2) we obtain:

$$\frac{\partial}{\partial t}\left(Ke^{-r(T-t)}N(-d_2)\right) = Ke^{-r(T-t)}\left(rN(-d_2) + N'(-d_2)\left(\frac{r+\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} - \frac{d_1}{2(T-t)} - \frac{\sigma}{2\sqrt{T-t}}\right)\right)$$

And the same simplification together with the fact that $N'(-d_2) = N'(d_2)$ yields:

$$\frac{\partial}{\partial t}\left(Ke^{-r(T-t)}N(-d_2)\right) = rKe^{-r(T-t)}N(-d_2) + S_tN'(d_1)\left(\frac{r+\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} - \frac{d_1}{2(T-t)} - \frac{\sigma}{2\sqrt{T-t}}\right)$$

For the second term we obtain the expression:

$$\frac{\partial}{\partial t}\left(S_tN(-d_1)\right) = S_t\frac{\partial N(-d_1)}{\partial t} = S_tN'(-d_1)\frac{\partial(-d_1)}{\partial t} = -S_tN'(d_1)\frac{\partial(d_1)}{\partial t}$$

And can therefore use previous result to obtain:

$$\frac{\partial}{\partial t}\left(S_tN(-d_1)\right) = S_tN'(d_1)\left(\frac{r+\frac{1}{2}\sigma^2}{\sigma\sqrt{T-t}} - \frac{d_1}{2(T-t)}\right)$$

Subtracting the second from the first simplifies similarly to the the call, and we obtain the final form:

$$\Theta_P = -\frac{S_t\sigma N'(d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$$

We notice that Θ_C and Θ_P have very similar expressions, with first term capturing the impact of time decay and volatility, while the second term accounts for the discounted strike price. We can use the expressions we obtained to plot **Theta** of a call and a put versus the stock price in figure 3.3. We notice a very similar behaviour for the two **Thetas**, up to constant shift. Indeed, knowing that $1 - N(x) = N(-x)$ we obtain:

$$\Theta_P - \Theta_C = rKe^{-r(t-T)}$$

The reason of this positive difference in favor of the put is due to the fact that put options have less time decay pressure than call options. This is because for a put the potential gain is capped, as the stock price can not go below zero. It can however grow (theoretically) up to infinity, hence a call have unlimited upside. Thus a call can always potentially gain more value than a put in a similar time window, leading to a higher rate of decline as time passes.

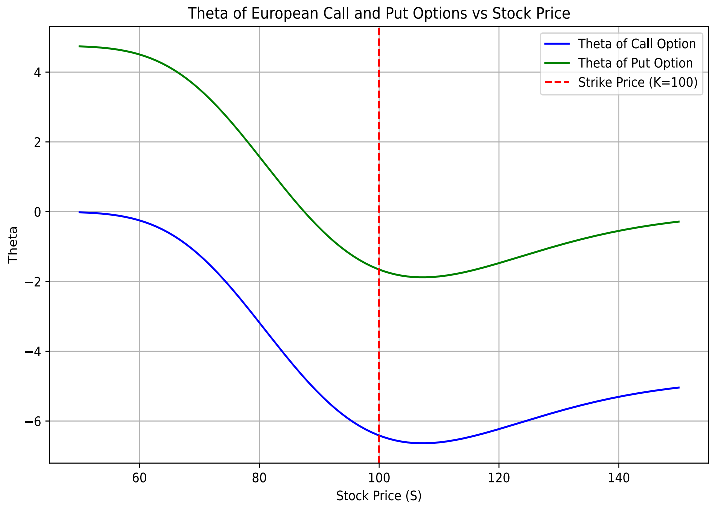

Figure 3.3: Θ_C and Θ_P versus the stock price

3.3.2 Theta Monitoring & Relation with Delta & Gamma

Let us recall the Black-Scholes PDE given in 2.2:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

When replacing the partial derivatives by the corresponding **Greeks**, we obtain the following equation:

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV = 0$$

This holds for an option, but since a portfolio Π is nothing but a linear combination of derivatives and stocks, this holds likewise for V_Π the value of the portfolio[2]:

$$\Theta_\Pi + \frac{1}{2}\sigma^2 S^2 \Gamma_\Pi + rS\Delta_\Pi - rV_\Pi = 0$$

Let our portfolio be **Delta**-neutral, which makes the third term vanish. Often, for example when rV_Π is very small or when we are close to expiration[3], we can disregard the fourth term and therefore be left with:

$$\Theta_\Pi \approx -\frac{1}{2}\sigma^2 S^2 \Gamma_\Pi$$

[2]The Π subscript below the **Greeks** indicates here that we took the partial derivatives of V_Π the portfolio's value

[3]Close to expiration, if the option is OTM then its intrinsic value shrinks whereas when it is ITM it becomes very small compared to **Gamma** and **Theta**. Furthermore, the time value shrinks too.

We see that a large negative **Theta** implies a high positive **Gamma** as they are inversely related by a negative factor. Hence it appears that for a **Delta**-hedged portfolio, **Theta** can be a good proxy for **Gamma**. Consequently, by observing and monitoring **Theta**, a trader can obtain good insights on their exposure to **Gamma** and therefore on the ideal frequency of rebalancing to maintain a good **Delta**-hedge.

3.4 Vega

In the Black-Scholes framework, the volatility σ represents the uncertainty associated with the underlying's price movements. The higher σ is, the more the price is expected to vary, and consequently the more large price swings are expected to happen. This has an impact on options' value because this affect the probability of options either to land ITM or OTM. Let us bring up two things to the eyes of the reader. Firstly, σ is assumed to be constant in the Black-Scholes model. Secondly, it is not observable, it needs to be estimated by using historical data (historical volatility) or market prices (implied volatility)[4]. The first assumption is a simplification of reality, where volatility can fluctuate over time (we will discuss the limitation of this simplification in section 4.2). It is useful to know the impact of these fluctuations on the option's price, hence we compute **Vega**:

$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$

As previously said, volatility is not a constant parameter in real life. Events that brings doubt and fear in people's mind like wars, geopolitical tensions, natural disasters, pandemics, etc ; are likely to increase volatility. Most of these events are unexpected, thus it is important to be insensitive to volatility variations, and that is why **Vega** hedging is crucial. Let us now compute the **Vega** of European call and put options as usual.

[4]To put it in a nutshell, historical volatility is calculated by analysis of the past behaviour of volatility (this assume that past volatility is a good indicator of past volatility, which is not always the case), and implied volatility estimated by adjusting the σ in the BS to obtain theoretical prices until enough theoretical prices coincide with the market prices.

3.4.1 Vega of European Call & Put

As previously, it is important to notice that **Vega** is a partial derivative, with S_t considered as a variable. We want to compute:

$$\mathcal{V}_C = \frac{\partial C(S_t, t)}{\partial \sigma}$$

With as usual $C(S_t, t)$ stemming from 2.1:

$$C(S_t, t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

And d_1 & d_2 from 2.2, where we can see the dependency in σ. Actually, we have nothing more to do than using the chain rule and computing the partial derivatives of d_1 and d_2 with respect to σ. Let us proceed as such. Using $\frac{\partial}{\partial \sigma}(\frac{u(\sigma)}{v(\sigma)}) = \frac{u'(\sigma)v(\sigma) - v'(\sigma)u(\sigma)}{v^2(\sigma)}$ together with the expression of d_1 we obtain:

$$\frac{\partial d_1}{\partial \sigma} = \frac{\sigma(T-t)\sigma\sqrt{T-t} - \sqrt{T-t}(\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t))}{\sigma^2(T-t)} = \sqrt{T-t} - \frac{d_1}{\sigma}$$

Next using $d_2 = d_1 - \sigma\sqrt{T-t}$ we have:

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T-t} = -\frac{d_1}{\sigma}$$

Hence we have the expression:

$$\mathcal{V}_C = S_t(\sqrt{T-t} - \frac{d_1}{\sigma})N'(d_1) + K e^{-r(T-t)} \frac{d_1}{\sigma} N'(d_2)$$

But recall that according to 3.1:

$$N'(d_2) = N'(d_1)\frac{S_t}{K}e^{r(T-t)}$$

So the last term of our expression cancels with the $-S_t \frac{d_1}{\sigma} N'(d_1)$ from the first, and we are left with:

$$\mathcal{V}_C = S_t\sqrt{T-t}N'(d_1)$$

For the put, as from 2.2:

$$P(S_t, t) = K e^{-r(T-t)} N(-d_2) - S_t N(-d_1)$$

We obtain:

$$\mathcal{V}_P = Ke^{-r(T-t)}\frac{d_1}{\sigma}N'(-d_2) - S_t(\frac{d_1}{\sigma} - \sqrt{T-t})N'(-d_1)$$

Using once again the symmetry of $N'(\cdot)$ around 0 together with 3.1, we get:

$$\mathcal{V}_P = S_t\sqrt{T-t}N'(d_1)$$

So that as in the **Gamma** case we have the same **Vega** for European call and put options. Thanks to this expression, we can plot **Vega** against the stock price in figure 3.4. The code of figure 3.4 can be found in the Appendix

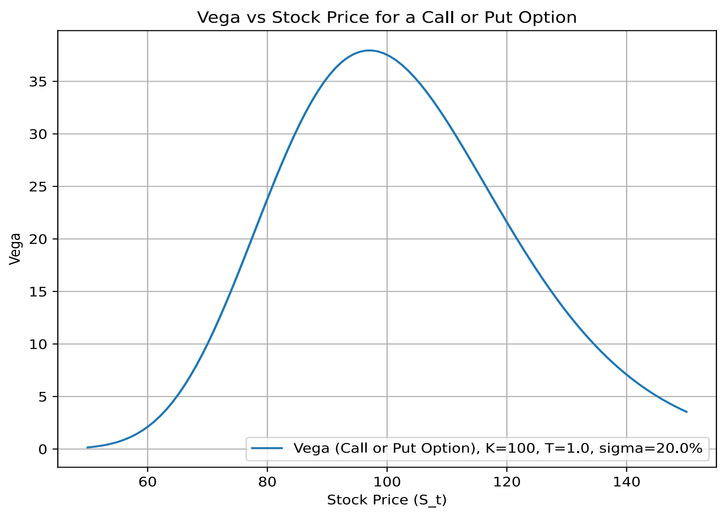

Figure 3.4: $\mathcal{V}$ versus the stock price

3.4.2 Vega Hedging

As explained previously, changes in volatility are often unpredictable, therefore there is a need for building robust portfolio. It is clear that a position in the underlying has 0 **Vega** (because $\frac{\partial S}{\partial \sigma} = 0$), therefore we need to combine different options with non-zero **Vega** to make our portfolio neutral. For instance, assume that we have a portfolio with zero **Gamma**, but a **Vega** of 3900. We also have two tradable options with the following characteristics:

- Option 1 has a **Vega** of 0.5 and a **Gamma** of 0.1.
- Option 2 has a **Vega** of 0.2 and a **Gamma** of 0.3.

Because **Delta** neutrality can be attained by trading stock (as we saw in the **Delta** hedging section) and since the stock has zero **Gamma** and zero **Vega**, we assume the options and the portfolio to be delta-neutral without loss of generality (WLOG). Let a represent the quantity of Option 1 that we buy, and b represent the quantity of Option 2. To hedge the **Vega** and **Gamma** risk, we need to solve the following linear system:

$$3900 - 0.5a - 0.2b = 0$$

$$0.1a + 0.3b = 0$$

From the second equation, we can solve for a in terms of b:

$$a = -3b$$

Substituting this into the first equation yields:

$$3900 - 0.5(-3b) - 0.2b = 0$$

$$3900 + 1.3b = 0$$

$$b = -3000$$

Now, substituting $b = -3000$ into $a = -3b$, we get:

$$a = -3(-3000) = 9000$$

Thus, to hedge the portfolio, we sell 3000 units of Option 2 and buy 9000 units of Option 1.

3.5 Rho

To finish this chapter, we will learn about the **Greek** that measures the sensitivity of an option's price with respect to changes ins the risk free interest rate. Namely, we define mathematically **Rho** as the partial derivative of an option's value with respect to r:

$$\rho = \frac{\partial V}{\partial r}$$

It is often stated in the literature that **Rho** is less important than the other **Greeks** we saw, that it is not of critical interest. This is due to the fact that r tends to move more slowly and in a more predictable fashion than other parameters such as volatility and asset prices. Nevertheless, during periods of troubles r can change substantially, making **Rho** a crucial indicator for risk management.

3.5.1 Rho of European Call & Put

As usual, we are now going to compute the **Rho** of a European call and of an European put, based on expressions 2.1 and 2.2. The computations are straightforwards, and yield:

$$\rho_C = K(T-t)e^{-r(T-t)}N(d_2)$$

$$\rho_P = -K(T-t)e^{-r(T-t)}N(-d_2)$$

On one hand, this shows that the **Rho** of a call is always positive, so that an increase in r makes the value of the call option higher. On the other hand, this shows that the **Rho** of a put is always negative, so that an increase in r makes the value of the put option smaller. To ensure why this makes sense, we can look into the meaning of r. It is clear that everyone would prefer to have x CHF today rather than having the same amount tomorrow, in 2 weeks, or any when in the future. This is because money has potential earning capacity, making its present value greater than in the future. But how much money in d days would be considered equivalent to x today ? The answer is $xe^{r\frac{d}{365}}$ where r is the risk free interest rate. Thus a higher r means that one needs a larger future amount to match the value of an amount x today, emphasizing the greater value of the present. Conversely, a smaller r implies that less future money is required to match x today, giving less weight to the present. With this in mind, the above results make sense : if r increases, the future has less weight and it is less of a burden to pay the strike price K for a call ; and for a put it is less attractive to sell at the strike price K. The key idea resides in the fact that K is a fixed quantity that does not depend on r or the time.

3.5.2 Rho Hedging Considerations

Compared to the other types of hedging we saw, hedging with respect to **Rho** is uncommon in practice. First of all, the interest rate is usually stable and if it changes it tends to happen slowly in an easy to predict motion. So options are less sensitive to fluctuations in **Rho** than in the other **Greeks**. Also, though it is clear that **Rho** can have an impact on an option's value if it is traded long before expiration, because it builds a cumulative effect over time, most of the traded actions are short or medium term, making **Rho**'s influence minimal. Nevertheless, **Rho** hedging can be relevant if we deal with long term options, or complex derivatives for which even a small change in r can impact strongly the option's value. To summarize, **Rho** hedging is rarely used, typically only in very specific contexts when the sensitivity to changes in the interest rate is unusually high.

Chapter 4

Limitations of the Black-Scholes Model and the Greeks

As we saw in section 2.1, the Black-Scholes model relies on several assumptions that are nothing but simplifications of reality. While making calculations easier, these assumptions are often violated in practice, which can diminish the efficiency and/or the applicability of the **Greeks**. In this chapter, we will discuss some limitations of the Black-Scholes model and their impact on the **Greeks**.

4.1 Lognormal Returns Assumption

The Black-Scholes assumption that the underlying asset price follows a geometric Brownian motion can be essentially translated by "the returns are log-normally distributed". Indeed, solving the stochastic differential equations in 2.1 gives:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

Knowing the law of a Browinan motion W_t is $\mathcal{N}(0, t)$, we have:

$$\ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$$

In other words, the log-returns are normally distributed, or the returns are log-normal. Because the tails of a Gaussian distribution tend to be thin, this assumption implies that extreme events, such as price jumps or crashes, have a very low probability. However, in real life extreme event happen more often than predicted by the distribution. We say the Black-Scholes model "underestimate fat tails". This has a direct impact on **Delta**, because under the Black-Scholes assumptions it tends to move rather smoothly, whereas in real life where extreme moves are more common we can observe bigger and quicker changes in **Delta**. Hence traders relying solely on Black-Scholes **Delta** can find themselves in very under-hedged positions if the market moves too sharply. The "fat tails" problem impacts **Gamma** in a similar fashion, meaning that Black-Scholes **Gamma** underestimates the real value of **Gamma**, which can lead to unexpected and expensive hedging adjustments. To account for "fat tails", traders look into the Merton Jump-diffusion model, that adds random jumps in the asset's price.

4.2 Constant Volatility Assumption

As evoked in section 3.4, volatility is not constant in reality. Implied volatility tends to be high for deep ITM otpions, then lower for ATM options, and high again for deep OTM options, drawing the shape of a smile[1]. This change is failed to be captured by the Black-Scholes model. With a smile we can observe a significant reduction in **Vega**'s drop ITM and OTM, and more irregularity. Let us model the volatility

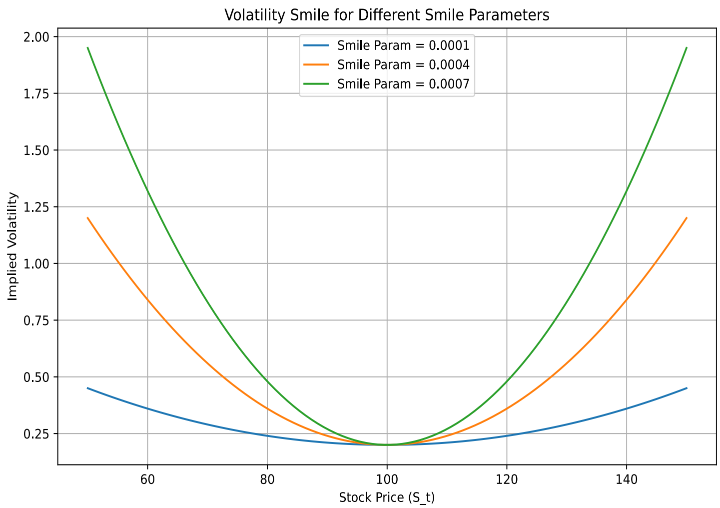

Figure 4.1: Volatility smiles

smile with a simple polynomial centered around the strike price, with more or less flatness in figure 4.1. Now if we plot **Vega** with this changing volatility instead of the constant one, we obtain a plot significantly different from the nice regularity of

[1]Hence the name volatility smile

figure 3.4, as can be seen in figure[2] 4.2:

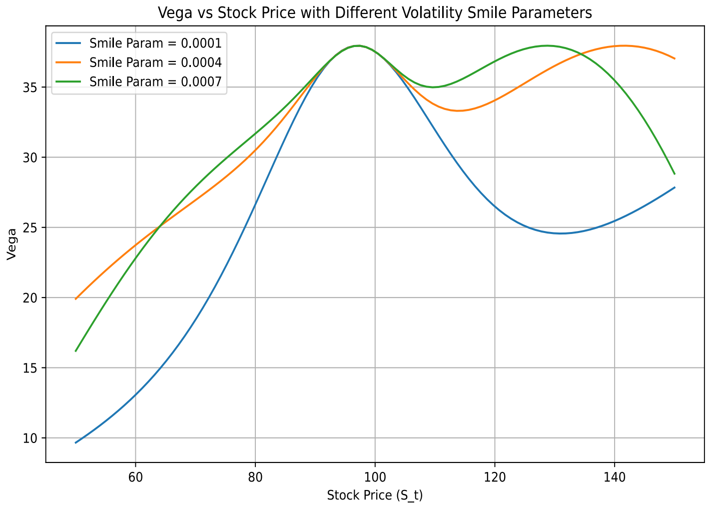

Figure 4.2: The impact of volatility smile on **Vega**

We see that compared to figure 3.4, when we move away from ATM, the option's value is far more sensitive to change in volatility in this situation, making hedging more complex. To mitigate this problem, one can consider using models with non-constant volatility, like the Heston Model or Local Volatility Models.

4.3 The Frictionless Market Assumption

A frictionless market, as assumed in the Black-Scholes model, supposes no liquidity constraint[3], no transaction costs, no taxes, no time delay ... However, this is not the case in real life, where frictions are numerous. For example, delay are common, and can lead to arbitrage opportunities: if someone is aware of a change before others, he can take advantage of it and create imbalances for his own profit. Perfect liquidity is also a flawed assumption, as it can happen that a default of buyers or sellers can prevent trades to happen at the optimal price. The no-transactions is also flawed, as in reality when trading a small price has to be paid to the intermediaries. In particular, **Delta** and **Gamma** hedging strategies have a need for frequent updates, so even small transaction costs can sum up to a substantial loss if too many transactions are done. Hence, when hedging one has to take these frictions into account

[2]The codes of both figures can be found in the Appendix
[3]Meaning we always have enough options on the market to buy and/or sell

because they can bring a non-substantial price to be paid, that is not captured by the Black-Scholes model.

Chapter 5

Conclusion

To conclude, this paper analyzed the **Greeks Delta, Gamma, Theta, Vega** and **Rho** and their use in managing the risks of trading. We first started with the Black-Scholes to establish a mathematical basis for pricing European options, and derived the **Greeks** within this framework.

For each **Greek** we derived a closed form solutions for European put and call options, that we used to obtain graphs. We also showcased example of hedging strategies for **Delta, Gamma, Theta** and **Vega**, and discussed the usefulness of hedging with respect to **Rho**.

Lastly, we examined some limitations of the infamous but simple Black-Scholes model. We illustrated how these limitations impact the accuracy of the **Greeks**, highlighting the need for adaptive and cautious hedging strategies.

Appendix A

Python code

In this section, you can find all of the codes used to plot the obtain the various graphs presented along this paper. They are independent, and each one can be used alone.

A.1 Delta

Here is the Python code to plot and save the graph 3.1 of **Delta** vs the stock price for a European Call and a European Put:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

# Define parameters for Black-Scholes Model
S = np.linspace(50, 150, 500)  # Range of stock prices
K = 100  # Strike price
T = 1.0  # Time to maturity (1 year)
r = 0.05  # Risk-free rate (5%)
sigma = 0.2  # Volatility (20%)

# Black-Scholes Delta for a Call and Put option
def d1(S, K, T, r, sigma):
    return (np.log(S/K)+(r+0.5*sigma**2)*T)/(sigma*np.sqrt(T))

def delta_call(S, K, T, r, sigma):
    return norm.cdf(d1(S, K, T, r, sigma))

```

```
def delta_put(S, K, T, r, sigma):
    return -norm.cdf(-d1(S, K, T, r, sigma))

# Calculate delta for call and put options
delta_c = delta_call(S, K, T, r, sigma)
delta_p = delta_put(S, K, T, r, sigma)

# Plotting the Delta of Call and Put options
plt.figure(figsize=(10, 6))
plt.plot(S, delta_c, label='Call Option Delta', color='blue')
plt.plot(S, delta_p, label='Put Option Delta', color='green')
plt.title('Delta of Call and Put Options')
plt.xlabel('Stock Price (S)')
plt.ylabel('Delta')
plt . axvline (x=K, color='r', linestyle='--', label=f'Strike
    Price (K={K})')
plt.legend()
plt.grid(True)
plt.savefig("delta_vs_stockprice.pdf")
plt.show()
```

A.2 Gamma

Here is the Python code to plot and save the graph 3.2 of **Gamma** vs the stock price:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

# Parameters
S = np.linspace(50, 150, 500)  # Range of stock prices
K = 100  # Strike price
r = 0.05  # Risk-free interest rate
T = 1  # Time to maturity (1 year)
sigma = 0.2  # Volatility (20%)
t = 0  # Current time

# Calculate d1
```

```
def d1(S, K, r, T, t, sigma):
    return (np.log(S/K) + (r+0.5*sigma**2)*(T-t))/(sigma*np.
        sqrt(T-t))

# Gamma formula
def gamma(S, K, r, T, t, sigma):
    d1_val = d1(S, K, r, T, t, sigma)
    return norm.pdf(d1_val)/(S*sigma*np.sqrt(T-t))

# Compute Gamma
gamma_values = gamma(S, K, r, T, t, sigma)

# Plot Gamma
plt.figure( figsize=(10, 6))
plt.plot(S, gamma_values, label=f'Gamma vs Stock Price (K={K},
    sigma ={sigma*100}%, T={T} yr)')
plt.title("Gamma of a European Call Option as a Function of
   Stock Price")
plt.xlabel("Stock Price (S)")
plt.ylabel("Gamma")
plt.axvline(x=K, color='r', linestyle='--', label=f'Strike
   Price (K={K})')
plt.legend()
plt.grid(True)
plt . savefig ( " gamma_vs_the_stock_price . pdf " )
plt.show
```

A.3 Theta

Here is the Python code to plot and save the graph 3.3 of **Theta** vs the stock price:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

# Parameters
S = np.linspace(50, 150, 500)  # Range of stock prices
K = 100  # Strike price
r = 0.05  # Risk-free interest rate
```

```
T = 1  # Time to maturity (1 year)
sigma = 0.2  # Volatility (20%)
t = 0  # Current time

# Calculate d1 and d2
def d1(S, K, r, T, t, sigma):
    return (np.log(S/K)+(r+0.5*sigma**2)*(T-t))/(sigma*np.sqrt
        (T-t))

def d2(S, K, r, T, t, sigma):
    return d1(S, K, r, T, t, sigma)-sigma*np.sqrt(T-t)

# Theta formula
def theta_call(S, K, r, T, t, sigma):
    d1_val = d1(S, K, r, T, t, sigma)
    d2_val = d2(S, K, r, T, t, sigma)

    term1 = -S*sigma*norm.pdf(d1_val)/(2*np.sqrt(T-t))
    term2 = r*K*np.exp(-r*(T-t))*norm.cdf(d2_val)

    return term1 - term2

# Compute Theta
theta_values = theta_call(S, K, r, T, t, sigma)

# Plot Theta similarly to the Gamma plot
plt.figure(figsize=(10, 6))
plt.plot(S, theta_values, label=f'Theta vs Stock Price (K={K},
    sigma={sigma*100}%, T={T} yr)')
plt.title("Theta of a European Call Option as a Function of
   Stock Price")
plt.xlabel("Stock Price (S)")
plt.ylabel("Theta")
plt.axvline(x=K, color='r', linestyle='--', label=f'Strike
   Price (K={K})')
plt.legend()
plt.grid(True)
plt.savefig("theta_vs_stock_price.pdf")
plt.show()
```

A.4 Vega

Here is the Python code to plot and save the graph 3.4 of **Vega** vs the stock price:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

# Define parameters
T = 1.0  # time to maturity (1 year)
t = 0.0  # current time
sigma = 0.2  # volatility
r = 0.05  # risk-free rate
K = 100  # strike price
S_t = np.linspace(50, 150, 100)  # stock prices range

# Define d1 function
def d1(S_t, K, T, t, r, sigma):
    return (np.log(S_t/K)+(r+0.5*sigma**2)*(T-t))/(sigma*np.
       sqrt(T-t))

# Vega calculation
vega_call = S_t*np.sqrt(T-t)*norm.pdf(d1(S_t, K, T, t, r,
   sigma))

# Plot Vega vs Stock Price with K, T, and sigma indicated in
   the legend
plt.figure(figsize=(8,6))
plt.plot(S_t, vega_call, label=f"Vega (Call or Put Option), K
   ={K}, T={T},sigma={sigma*100}%")
plt.title("Vega vs Stock Price for a Call or Put Option")
plt.xlabel("Stock Price (S_t)")
plt.ylabel("Vega")
plt.grid(True)

# Adding the legend with K, T, and sigma values
plt.legend()
```

```
plt.savefig("vega_of_european_option.pdf")
plt.show()
```

A.5 Volatility Smile

Here after you can find the code of figure 4.1, in which we used different arbitrary polynomials to illustrate the volatility smile:

```
    import numpy as np
import matplotlib.pyplot as plt

# Define parameters
S_t = np.linspace(50, 150, 100)  # stock prices range
atm_strike = 100  # at-the-money strike price

# Define the volatility smile plot function for different
   smile parameters
def plot_volatility_smile(S_t, atm_strike, smile_param_values)
   :
    plt.figure(figsize=(10, 6))

    for smile_param in smile_param_values:
        # Sigma function with the specified smile parameter
        sigma_values = 0.2 + smile_param * (S_t - atm_strike)
           **2

        # Plot volatility smile with the current smile
           parameter
        plt.plot(S_t, sigma_values, label=f"Smile Param = {
           smile_param:.4f}")

    plt.title("Volatility Smile for Different Smile Parameters
       ")
    plt.xlabel("Stock Price (S_t)")
    plt.ylabel("Implied Volatility")
    plt.grid(True)
    plt.legend()
    plt.savefig ("smile.pdf")
    plt.show()
```

```

# Parameters for the smile parameter
smile_param_values = [0.0001, 0.0004, 0.0007]  # Specified
    smile parameters

# Plot the volatility smile with the given smile parameters
plot_volatility_smile(S_t, atm_strike, smile_param_values)
```

A.6 Volatility Smile and Vega

Here you can find the code of figure 4.2. We adapted the code of A.4 and replace the constant volatility by the smiles in A.5.

```
    import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

# Define parameters
T = 1.0  # time to maturity (1 year)
t = 0.0  # current time
r = 0.05  # risk-free rate
K = 100  # strike price
S_t = np.linspace(50, 150, 100)  # stock prices range
atm_strike = 100  # at-the-money strike price for the
    volatility smile

# Define d1 function
def d1(S_t, K, T, t, r, sigma):
    return (np.log(S_t / K) + (r + 0.5 * sigma**2) * (T - t))
        / (sigma * np.sqrt(T - t))

# Vega plot function with tunable smile parameter
def plot_vega_with_smile(S_t, K, T, t, r, atm_strike,
    smile_param_values):
    plt.figure(figsize=(10, 6))

    for smile_param in smile_param_values:
        # Sigma function with tunable smile parameter
        def sigma_smile(S_t, atm_strike):
```

```
            return 0.2 + smile_param * (S_t - atm_strike)**2

        # Calculate sigma values with the current smile parameter
        sigma_values = sigma_smile(S_t, atm_strike)

        # Vega calculation using the current volatility smile
        vega_call = S_t * np.sqrt(T - t) * norm.pdf(d1(S_t, K, T, t, r, sigma_values))

        # Plot Vega with the current smile parameter
        plt.plot(S_t, vega_call, label=f"Smile Param = {smile_param:.4f}")

    plt.title("Vega vs Stock Price with Different Volatility Smile Parameters")
    plt.xlabel("Stock Price (S_t)")
    plt.ylabel("Vega")
    plt.grid(True)
    plt.legend()
    plt.savefig ("smileandvega.pdf")
    plt.show()

# Parameters for the smile parameter
smile_param_values = [0.0001, 0.0004, 0.0007]  # Different values for the smile parameter

# Plot Vega with different smile parameters
plot_vega_with_smile(S_t, K, T, t, r, atm_strike, smile_param_values)
```

Bibliography

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