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$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\,\hat{q}_0(\lambda)\,d\lambda - \int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left(i\lambda f_0(\lambda;0,\tau) + f_1(\lambda;0,\tau)\right)d\lambda$$

$$- \int_{\partial D^-} e^{i\lambda(x-1) - \lambda^2 t}\left(i\lambda f_0(\lambda;1,\tau) + f_1(\lambda;1,\tau)\right)d\lambda, \qquad (EF\tau)$$

valid for $(x,t) \in (0,1) \times [0,\tau]$, $\tau \in [0,T]$.

This formula has the advantage of very simple (x,t) dependence.

Summary
~~Analysis~~ of progress

We started by assuming that a solution exists and showed that ~~4~~ any solution must satisfy $(EF\tau)$ and (GR). The value of (GR) is not clear

$\boxed{\text{End of lecture 2}}$

yet, except in deriving $(EF\tau)$:

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\,\hat{q}_0(\lambda)\,d\lambda - \int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left(i\lambda f_0(\lambda;0,\tau) + f_1(\lambda;0,\tau)\right)d\lambda$$

$$- \int_{\partial D^-} e^{i\lambda(x-1) - \lambda^2 t}\left(i\lambda f_0(\lambda;1,\tau) + f_1(\lambda;1,\tau)\right)d\lambda$$

where $\hat{q}_0(\lambda) = \int_0^1 e^{-i\lambda y} q_0(y)\,dy$ known

$$f_j(\lambda;X,\tau) = \int_0^\tau e^{\lambda^2 s}\,\partial_x^j q(X,s)\,ds \qquad \text{unknown}$$

So ~~we~~ $(EF\tau)$ is not an explicit representation of the solution; we still have work to do.

2.2.2 Stage 2

Assume that q satisfies not only (PDE) & (IC) but also (BC).

Then

$$f_0(\lambda;0,\tau) = \int_0^\tau e^{\lambda^2 s}\, q(0,s)\,ds = \int_0^\tau e^{\lambda^2 s} g_0(s)\,ds =: h_0(\lambda;\tau),$$

$$f_0(\lambda;1,\tau) = \int_0^\tau e^{\lambda^2 s}\, q(1,s)\,ds = \int_0^\tau e^{\lambda^2 s} g_1(s)\,ds =: h_1(\lambda;\tau)$$

are both known. But $f_1(\lambda;0,\tau)$, $f_1(\lambda;1,\tau)$ are both unknown. Note that if we <u>did</u> know these then the problem would be overspecified. In general, at most half of the $f_j(\lambda;x,\tau)$ may be explicitly specified by the boundary conditions.

We need equations involving $f_1(\lambda;0,\tau)$, $f_1(\lambda;1,\tau)$. We have (GR). After applying the boundary conditions, (GR) becomes

$$\underbrace{f_1(\lambda;0,\tau) - e^{-i\lambda} f_1(\lambda;1,\tau)}_{\text{unknown}} = \overbrace{-i\lambda h_0(\lambda;\tau) + i\lambda e^{-i\lambda} h_1(\lambda;\tau) + \hat{q}_0(\lambda)}^{\text{data}} - e^{\lambda^2\tau}\underbrace{\hat{q}(\lambda;\tau)}_{\text{also unknown!}}.$$

What good is this? We had 2 unknowns

(i) We only introduced 1 equation; surely we need 2 linearly independent equations.

(ii) We also introduced an extra unknown.

It appears we have achieved nothing!

Let us, temporarily, ignore issue (ii).

Consider applying the maps $\lambda \mapsto \lambda$, $\lambda \mapsto -\lambda$ to (GR). Observe that

$$f_j(-\lambda; x, \tau) = f_j(\lambda; x, \tau)$$

because f_j depends on λ only through λ^2, not through λ directly. But ~~the~~ the coefficient $e^{-i\lambda}$ is not preserved by $\lambda \mapsto -\lambda$.
So $(GR)|_{\lambda \mapsto \lambda}$ and $(GR)|_{\lambda \mapsto -\lambda}$ yield a system of two linearly independent equations in the two unknowns:

$$\begin{pmatrix} 1 & -e^{-i\lambda} \\ 1 & -e^{i\lambda} \end{pmatrix} \begin{pmatrix} f_1(\lambda; 0, \tau) \\ f_1(\lambda; 1, \tau) \end{pmatrix} = \underbrace{\begin{pmatrix} M(\lambda) \\ M(-\lambda) \end{pmatrix} + \begin{pmatrix} \hat{q}_0(\lambda) \\ \hat{q}_0(-\lambda) \end{pmatrix}}_{\text{data}} - e^{\lambda^2 \tau} \underbrace{\begin{pmatrix} \hat{q}(\lambda; \tau) \\ \hat{q}(-\lambda; \tau) \end{pmatrix}}_{\text{pretend these are data}},$$

where $M(\lambda) = -i\lambda h_0(\lambda; \tau) + i\lambda e^{-i\lambda} h_1(\lambda; \tau)$.

Note that the data vector takes the form $\begin{pmatrix} \varphi(\lambda) \\ \varphi(-\lambda) \end{pmatrix}$. Of course it does, because that is the origin of the system, but it makes for a notational simplification.

Solve the system using Cramer's rule. Let

$\Delta(\lambda) = -2i \sin(\lambda)$ Determinant of system,

$\zeta^+(\lambda; \varphi) = -\varphi(\lambda) e^{i\lambda} + \varphi(-\lambda) e^{-i\lambda}$ Determinant where 1st column has been replaced by $(\varphi(\lambda), \varphi(-\lambda))$,

$\zeta^-(\lambda; \varphi) = -\varphi(\lambda) + \varphi(-\lambda)$ Determinant where 2nd column has been replaced by $(\varphi(\lambda), \varphi(-\lambda))$.

Then

$$i\lambda f_0(\lambda;0,\tau) + f_1(\lambda;0,\tau) = i\lambda h_0(\lambda;\tau) + \frac{\zeta^+(\lambda;M(\cdot;\tau))}{\Delta(\lambda)} + \frac{\zeta^+(\lambda;\hat{q}_0)}{\Delta(\lambda)} - e^{\lambda^2\tau}\frac{\zeta^+(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}$$

$$i\lambda f_0(\lambda;1,\tau) + f_1(\lambda;1,\tau) = \underbrace{i\lambda h_1(\lambda;\tau) + \frac{\zeta^-(\lambda;M(\cdot;\tau))}{\Delta(\lambda)} + \frac{\zeta^-(\lambda;\hat{q}_0)}{\Delta(\lambda)}}_{\text{data}} - e^{\lambda^2\tau}\underbrace{\frac{\zeta^-(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}}_{\text{not data}}.$$

Substitute into (E F τ):

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{q}_0(\lambda)\,d\lambda - \int_{\partial D^+}\text{data}\,d\lambda - \int_{\partial D^-}\text{data}\,d\lambda$$

$$+ \int_{\partial D^+} e^{i\lambda x} e^{\lambda^2(\tau-t)} \frac{\zeta^+(\lambda;\hat{q}(\cdot;\tau))}{\Delta(\lambda)}\,d\lambda$$

$$+ \int_{\partial D^-} e^{i\lambda(x-1)} e^{\lambda^2(\tau-t)} \frac{\left(-\hat{q}(\lambda;\tau) + \hat{q}(-\lambda;\tau)\right)}{e^{-i\lambda} - e^{i\lambda}}\,d\lambda .$$

This requires some justification.

(i) The integral over ∂D^+ has been split into two parts with different integrands. The integrands are now meromorphic rather than analytic so we have to be sure we did not integrate over any zeros of Δ. In this case, the only zero of Δ we hit is at the origin and that turns out to be a removable singularity of the integrand. In general, some small circular contour deformations may be necessary to avoid such an issue.

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(ii) Although the original integral converged, each constituent part might not.

However it turns out this will be OK, as described below.

We aim to show the terms involving $\hat{q}(\lambda;\tau)$ evaluate to 0. This is essential to get an effective solution representation that depends only upon the data of the problem. It ~~also means~~ will also justify the above splitting of integrals.

$$\text{Ratio in } \int_{\partial D^-} \text{ is } \quad e^{\lambda^2(\tau-t)}\left(\frac{-\hat{q}(\lambda;\tau)+\hat{q}(-\lambda;\tau)}{e^{-i\lambda}-e^{i\lambda}}\right)$$

Note that

$$\left.\begin{array}{lll} e^{-i\lambda}, & \hat{q}(\lambda;\tau) & \text{decay} \\ e^{i\lambda}, & \hat{q}(-\lambda;\tau) & \text{blow up} \\ & e^{\lambda^2\underbrace{(\tau-t)}_{\geq 0}} & \text{decay or, at worst, oscillatory} \end{array}\right\} \text{ as } \lambda\to\infty \text{ from within } \operatorname{clos}(D^-)$$

$$\Rightarrow \text{Ratio}(\lambda) = e^{\lambda^2(\tau-t)}\left(-e^{-i\lambda}\hat{q}(-\lambda;\tau)+\mathcal{O}\left(e^{-|\lambda|/\sqrt{2}}\right)\right)$$

$$= e^{\lambda^2(\tau-t)}\left(-\int_0^1 e^{-i\lambda(1-y)}q(y;\tau)\,dy+\mathcal{O}\left(e^{-|\lambda|/\sqrt{2}}\right)\right)$$

integrate by parts

$$= \mathcal{O}\left(|\lambda|^{-1}\right), \quad \text{uniformly in } \arg(\lambda), \text{ as } \lambda\to\infty \text{ within } \operatorname{clos}(D^-).$$

Hence, by Jordan's lemma, $\int_{\partial D^-} e^{i\lambda(x-1)}\,\text{Ratio}(\lambda)\,d\lambda = 0$. Similarly for $\int_{\partial D^+} \dots d\lambda$

We have obtained the solution representation, ~~with the~~ in terms of contour integrals around D:

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\,\hat{q}_0(\lambda)\,d\lambda \;-\; \int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(i\lambda h_0(\lambda;\tau) + \frac{\zeta^+(\lambda; M(\cdot;\tau)+\hat{q}_0)}{\Delta(\lambda)}\right)d\lambda$$

$$- \int_{\partial D^-} e^{i\lambda(x-1)-\lambda^2 t}\left(i\lambda h_1(\lambda;\tau) + \frac{\zeta^-(\lambda; M(\cdot;\tau)+\hat{q}_0)}{\Delta(\lambda)}\right)d\lambda \qquad (\mathrm{SR}\overset{\tau}{\mathrm{D}})$$

where $h_j, M, \hat{q}_0, \Delta, \zeta^\pm$ are explicitly defined in terms of the data of the problem.

<u>Remark 1</u> Every solution that exists must satisfy equation $(\mathrm{SR}\overset{\tau}{\mathrm{D}})$. But (SRD) is explicit. So existence $\Rightarrow$ unicity

$\boxed{\text{End of lecture 3}}$

<u>Remark 2</u> $(\mathrm{SR}\overset{\tau}{\mathrm{D}})$ holds for all $\tau \geq t$. So we can use whichever formula is more convenient. (Typically $\tau = t$ or $\tau = T$.)

<u>Remark 3</u> Apparent dependence, via M_k & h_k, of $(\mathrm{SR}\overset{\tau}{\mathrm{D}})$ upon $g_k(s)$, for $s \in (t,\tau)$ is a mirage. Via contour deformation (Jordan's lemma),

$$\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(i\lambda\left[h_0(\lambda;\tau) - h_0(\lambda;t)\right] + \frac{\zeta^+(\lambda; M(\cdot;\tau) - M(\cdot;t))}{\Delta(\lambda)}\right)d\lambda = 0$$

similarly $\int_{\partial D^-} \cdots \, d\lambda$

~~Remark 4 You may have noticed that, for $\tau = t$~~