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The Fokas transform method for linear evolution equations in 1+1

1. ~~The~~ Introduction

1.1 Review of Fourier transform methods for linear evolution equations

Consider full line heat problem:

$$[\partial_t - \partial_{xx}]\, q(x,t) = 0 \qquad (x,t) \in \mathbb{R} \times (0,T) \qquad \text{(PDE)}$$

$$q(x,0) = q_0(x) \qquad x \in \mathbb{R} \qquad \text{(IC)}$$

$$q(\cdot,t) \in \mathcal{S}(\mathbb{R}) \qquad t \in [0,T] \qquad \text{(BC)}$$

where $q_0 \in \mathcal{S}(\mathbb{R})$ known and $\mathcal{S}(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) : \forall j,k \in \mathbb{N}_0,\ \lim_{|x|\to\infty} x^k \varphi^{(j)}(x) = 0$

ie φ and all derivatives decay faster than all polynomials, the Schwartz space, the space of rapidly decaying functions.

Take Fourier transform in space:

$$\widehat{\partial_t q}(\lambda;t) - \widehat{\partial_{xx} q}(\lambda;t) = 0,$$ where $\hat{\ }$ is defined as

$$\hat{\varphi}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} \varphi(x)\, dx$$

Note that the smoothness of q ensures Fourier transform is a linear operator

But "the Fourier transform turns differentiation into multiplication"

"the Fourier transform diagonalises the derivative operator"

Precisely,

$$\widehat{\frac{d^2}{dx^2}\varphi}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\varphi''(x)\,dx$$

$$= \Big[e^{-i\lambda x}\big(\varphi'(x) + i\lambda\varphi(x)\big)\Big]_{x=-\infty}^{x=\infty} - \lambda^2\int_{-\infty}^{\infty} e^{-i\lambda x}\varphi(x)\,dx$$

oscillatory

$$= \lim_{x\to\infty}\Big[e^{-i\lambda x}\big(\varphi'(x)+i\lambda\varphi(x)\big)\Big] - \lim_{y\to-\infty}\Big[e^{-i\lambda y}\big(\varphi'(y)+i\lambda\varphi(y)\big)\Big]$$

decaying

$$- \lambda^2\hat{\varphi}(\lambda)$$

$$= 0 - 0 - \lambda^2\hat{\varphi}(\lambda),$$ ~~because~~ for all $\varphi \in \mathcal{S}(\mathbb{R})$.

So $\frac{d}{dt}\hat{q}(\lambda;t) + \lambda^2\hat{q}(\lambda;t) = 0$. For each $\lambda \in \mathbb{R}$, this is an ODE ~~for $\hat{q}$~~ for $\hat{q}(\lambda;\cdot)$.

interchange of limits requires some assumption of smoothness in t. Let's just suppose it works.

Solve ODE: $\Rightarrow \hat{q}(\lambda;t) = e^{-\lambda^2 t}\hat{q}(\lambda;0) = e^{-\lambda^2 t}\int_{-\infty}^{\infty} e^{-i\lambda x}q(x,0)\,dx$

By initial condition

$$\stackrel{\downarrow}{=} e^{-\lambda^2 t}\int_{-\infty}^{\infty} e^{-i\lambda x}q_0(x)\,dx = e^{-\lambda^2 t}\hat{q}_0(\lambda)$$

$$\Rightarrow q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{q}_0(\lambda)\,d\lambda.$$

By inverse Fourier transform.

All done under assumption $\exists$ such q. But under that assumption, we have

- Solution formula
- Uniqueness of such q

Now, to justify existence, let $u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{q}_0(\lambda)\, d\lambda$ the solution formula we derived

and show u satisfies the problem for q.

IC: $u(x,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \hat{q}_0(\lambda)\, d\lambda = q_0(x)$ by validity of inverse Fourier transform formula on $\mathcal{S}(\mathbb{R})$

BC: $q_0 \in \mathcal{S}(\mathbb{R}) \Rightarrow \hat{q}_0 \in \mathcal{S}(\mathbb{R})$ as Fourier transform is an automorphism on Schwartz space.

$\Rightarrow \forall t \in [0,T]\; e^{-\lambda^2 t} \hat{q}_0(\lambda) \in \mathcal{S}(\mathbb{R})$

$\Rightarrow \forall t \in [0,T]\; \widehat{e^{-\lambda^2 t} \hat{q}_0(\lambda)} \in \mathcal{S}(\mathbb{R})$ as Fourier transform is an automorphism on Schwartz space.

$\Rightarrow \forall t \in [0,T]\; 2\pi u(-\cdot, t) \in \mathcal{S}(\mathbb{R})$

$\Rightarrow \forall t \in [0,T]\; u(\cdot, t) \in \mathcal{S}(\mathbb{R})$

PDE: $u_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\lambda^2 e^{i\lambda x - \lambda^2 t} \hat{q}_0(\lambda)\, d\lambda = u_{xx}$ as integral converges uniformly by $\hat{q}_0 \in \mathcal{S}(\mathbb{R})$.

u satisfies the problem, so it is unique by earlier argument, and we have explicit (double integral) expression for u.

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Now ~~lets beg~~ consider half line ~~Heat Dirichl~~ homogeneous Dirichlet heat problem.

$$[\partial_t - \partial_{xx}]\, q(x,t) = 0 \qquad (x,t) \in (0,\infty)\times(0,T) \qquad \text{(PDE)}$$

$$q(x,0) = q_0(x) \qquad x \in [0,\infty) \qquad \text{(IC)}$$

$$q(0,t) = 0, \quad q(\cdot,t) \in \mathcal{S}[0,\infty) \qquad t \in [0,T] \qquad \text{(BC)}$$

where $q_0 \in \mathcal{S}[0,\infty)$ known and ~~$\mathcal{S}(\mathbb{R}) = \{\varphi : \exists \psi \in \mathcal{S}(\mathbb{R})$~~

$$\mathcal{S}[0,\infty) = \{\varphi = \psi|_{[0,\infty)} : \psi \in \mathcal{S}(\mathbb{R})\}$$

i.e smooth functions on the half line rapidly decaying at $+\infty$.

Interaction of half line Fourier transform with ~~deriv~~ Dirichlet heat half line derivative operator:

$$\widehat{\frac{d^2}{dx^2}\varphi}(\lambda) = \int_0^\infty e^{-i\lambda x}\varphi''(x)\,dx$$

$$= \lim_{x\to\infty}\left[e^{-i\lambda x}\left(\varphi'(x) + i\lambda\varphi(x)\right)\right] - \left(\varphi'(0) + i\lambda\varphi(0)\right) - \lambda^2\hat{\varphi}(\lambda)$$

$$= 0 - \varphi'(0) - i\lambda\, 0 - \lambda^2\hat{\varphi}(\lambda)$$

So the ODE from applying spatial Fourier transform to (PDE) is not so simple: ~~$\frac{d}{dt}\hat{q}(\lambda,t) + \lambda^2\hat{\varphi}(\lambda) + q(0,t)\, q_x(0,t)$~~

$$\frac{d}{dt}\hat{q}(\lambda;t) + \lambda^2\hat{q}(\lambda;t) + q_x(0,t) = 0$$

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Try a different idea: Fourier sine transform,

$$[\mathcal{F}_s \varphi](\lambda) := \int_0^\infty \sin(\lambda x)\, \varphi(x)\, dx,$$

has the property

$$[\mathcal{F}_s \varphi''](\lambda) = \int_0^\infty \sin(\lambda x)\, \varphi''(x)\, dx$$

$$= \Big[\sin(\lambda x)\, \varphi'(x) - \lambda \cos(\lambda x)\, \varphi(x)\Big]_{x=0}^{x=\infty} - \lambda^2 \int_0^\infty \sin(\lambda x) \varphi(x)\, dx$$

$$= 0 - \sin(0)\, \varphi'(0) + \lambda \cos(0)\, \varphi(0) - \lambda^2 [\mathcal{F}_s \varphi](\lambda)$$

$$= -\lambda^2 [\mathcal{F}_s \varphi](\lambda)$$

So we get a simple ODE in t for $[\mathcal{F}_s\, q(\cdot\,; t)](\lambda)$,

and proceed as before.

Similarly, for the half line ~~Neumann~~ homogeneous Neumann heat problem, we use the Fourier cosine transform, and h-transforms can be used for Robin problems.

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Conclusions

Fourier transform methods:

1. Choose the "right" version of the Fourier transform by studying how it interacts with the boundary conditions.

2. Assume the problem has a solution. Obtain an explicit formula representing that solution. This argument gives uniqueness for free.

3. Take the formula obtained above, and show that function so defined solves the problem.

1.2 Review of Fourier series methods for linear evolution equations

Finite interval homogeneous Dirichlet heat ~~eq~~ problem.

$$[\partial_t - \partial_{xx}]\, q(x,t) = 0 \qquad (x,t) \in (0,1)\times(0,T) \qquad \text{(PDE)}$$

$$q(x,0) = q_0(x) \qquad x \in [0,1] \qquad \text{(IC)}$$

$$q(0,t) = 0, \quad q(1,t) = 0 \qquad t \in [0,T] \qquad \text{(BC)}$$

Sketch of argument (partial)

- Separate variables
- Solve temporal ODE
- Solve spatial Sturm-Liouville Problem
- ~~Postulate~~ Assume the problem has a solution and that solution is expressible as an expansion in the solutions of the SLP.

ie the eigenfunctions of the SLP are a complete system.

...

But this assumption is not always reasonable. For problems of ~~order~~ spatial order 2, ~~the~~ classical Sturm-Liouville theory guarantees the completeness. But not for 3rd order (Jackson 1915, Hopkins 1919)

Conclusions

This argument relies on an extra completeness assumption (disadvantage)

- solving a SLP is algorithmic, while choosing the "right" kind of Fourier transform is difficult.

Aim

Devise a method, applicable to finite interval problems, but without the "requires completeness" requirement. Should work for problems of arbitrary spatial order with arbitrary linear boundary conditions.

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2. The ~~unified~~ Fokas unified transform method via ad-hoc derivation

2.1 ~~Sketch~~ A 3 stage method.

Here the method is viewed as an ad-hoc extension of the Fourier transform method. Because the "right" integral transform is not known in advance, it is not exactly analogous to the ~~linear~~ Fourier transform methods studied in §1.1.

Stage 1 Assume the problem has a solution. Obtain

(i) Ehrenpreis Form: a formula for the solution in terms of complex contour integrals of transforms the initial datum and boundary values.

(ii) Global Relation: an equation relating transforms of the solution on boundaries of the space-time domain.

Stage 2 Continue under the existence assumption. Data to uNknown Map (D to N map). Use the boundary conditions and global relation to ~~specify~~ obtain expressions for transforms of the boundary values and substitute into the Ehrenpreis form to obtain an effective integral representation of the solution. ~~So~~ We now have uniqueness & solution representation, under assumption of existence.