Evaluating

"Equivalent" complex integrals.

Suppose $-\infty < a \le b \le c < \infty$ and $f : [a,c] \to \mathbb{R}$ continuous. Then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \left\{\int_a^b + \int_b^c\right\} f(x)\,dx$$

These integrals are "equivalent" in that they evaluate to the same number.

Similarly, $\int_a^b f(x)\,dx$, $\left\{\int_a^c - \int_b^c\right\} f(x)\,dx$ are equivalent

$\int_a^b f(x)\,dx$, $-\int_b^a f(x)\,dx$ are equivalent.

- There are not many options for integrals of real variables, but more for complex integrals.
- Note that (Dion's §2.2) ~~this~~ the above all works for $f : [a,c] \to \mathbb{C}$ too.

~~What if $f : \mathbb{C} \to \mathbb{C}$?~~

We also already know FTC (Dion's theorem 8): If ~~F continuous and~~ $F : [a,c] \to \mathbb{C}$ continuous and $F' = f$ with $f : [a,c] \to \mathbb{C}$ continuous then

$$\int_a^c f(x)\,dx = F(c) - F(a) = F(c) - F(b) + F(b) - F(a)$$

$$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \left\{\int_a^b + \int_b^c\right\} f(x)\,dx$$

(equivalent integrals)

Consider a parametrised integral with $a=-1$, $c=1$, $f(z)=z^2$

We guess, because maybe these are equivalent?

$$\frac{2}{3} = \int_{-1}^{1} z^2\,dz \overset{\checkmark}{=} \left\{ \int_{-1}^{i} + \int_{i}^{1} \right\} z^2\,dz$$

Recall $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$

$$= \int_{-1}^{i} z^2\,dz \quad + \quad \int_{i}^{1} z^2\,dz$$

$z=\gamma_1(t)=-1+t(1+i)$, $\qquad z=\gamma_2(t)=i+t(1-i)$,

$\gamma_1'(t)=1+i \qquad \gamma_2'(t)=1-i$

$$= \int_0^1 [-1+t(1+i)]^2(1+i)\,dt \quad + \quad \int_0^1 [i+t(1-i)]^2(1-i)\,dt$$

$$= \dots [\text{exercise}] \dots = \frac{2}{3}$$

So $\boxed{\text{yes}}$ these integrals ~~are~~ seen to be equivalent

Note $$\int_{-1}^{1} z^2\,dz = \left.\frac{z^3}{3}\right|_{z=1} - \left.\frac{z^3}{3}\right|_{z=-1} = \frac{2}{3}$$

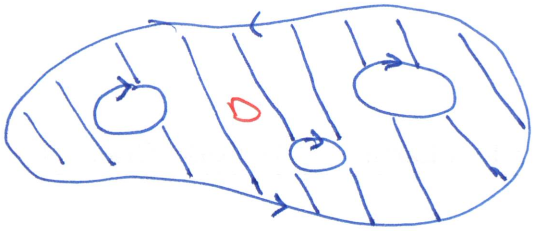
$$= F(1) - F(-1) \text{ using } F(z) = \frac{z^3}{3} \text{ so that } F'(z) = z^2.$$

$$\int_{-1}^{i} z^2\,dz + \int_{i}^{1} z^2\,dz = \cancel{F(i)} - F(-1) + F(1) - \cancel{F(i)} \quad \text{cancellation}$$

$$= F(1) - F(-1)$$

So FTC seems to work here too.

Green's theorem → Cauchy's theorem



∂D

Recall from FoAM module:

Theorem If D is a region with finitely many ~~reg~~ (pairwise disjoint) regions removed from it, ∂D is the positively-oriented boundary of D ~~then~~ and $P, Q : \mathrm{clos}(D) \to \mathbb{C}$ continuous, ~~then~~ with continuous partial derivatives, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Lemma (The Cauchy–Riemann Equations)

If f complex-valued ~~is analytic~~ and $f(x+iy) = u(x,y) + i v(x,y)$ for u, v real-valued and x, y real variables,

then $u_x = v_y$ and $u_y = -v_x$ iff f analytic.

Proof See Complex Analysis course, (or check directly for any f you care about).

Idea:

$$\int_\gamma f(z)\,dz = \int_a^b \left[u(x(t), y(t)) + i\,v(x(t), y(t)) \right] \left(x'(t) + i\,y'(t) \right) dt$$

$$= \ldots [\text{exercise}] \ldots = \int_\gamma (u\,dx - v\,dy) + i \int_\gamma (v\,dx + u\,dy)$$

Green's theorem on each integral

$Q = -v$, $P = u$ $\qquad$ $Q = u$, $P = v$

$$\overset{\text{Green's theorem}}{=} \underbrace{\iint_D \frac{\partial}{\partial x}(-v) - \frac{\partial}{\partial y}(u)\,dx\,dy}_{=0 \text{ because } u_y = -v_x} + i \underbrace{\iint_D \frac{\partial}{\partial x}(u) - \frac{\partial}{\partial y}(v)\,dx\,dy}_{=0 \text{ because } u_x = v_y} \overset{\text{Cauchy-Riemann equations}}{=} 0$$

So: Theorem (Cauchy's Theorem)

So, Provided • $\gamma = \partial D$ for D a region with finitely many regions removed

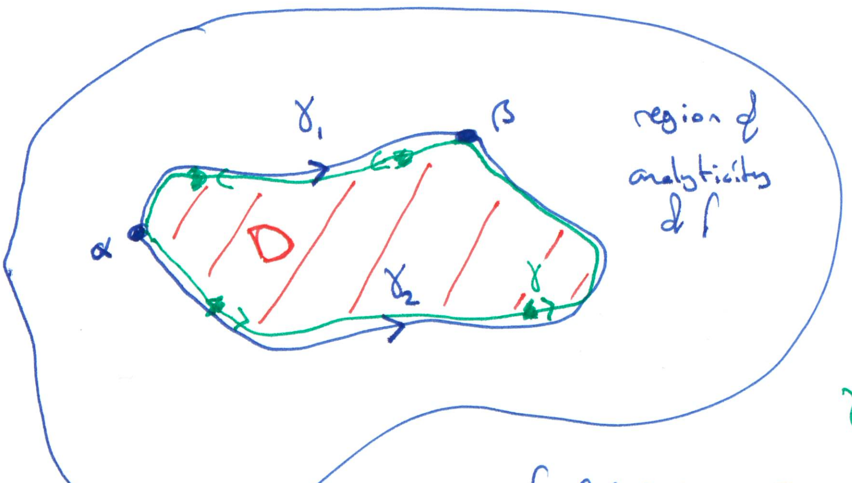
• f analytic on an open set containing $\mathrm{clos}(D)$

$$\int_\gamma f(z)\,dz = 0$$

Note: It seems we dropped the criterion f has continuous derivative, but expanding analyticity of f to an open set including $\mathrm{clos}(D)$ is enough to force f to have (infinitely many) continuous derivatives.

Example 1

$\gamma = \partial D = [-\gamma_1] \cup [\gamma_2]$



$$\int_\gamma f(z)\,dz = 0$$

So, because γ is the same path as [γ_1 in the opposite direction] followed by γ_2, equivalence of integrals tells us

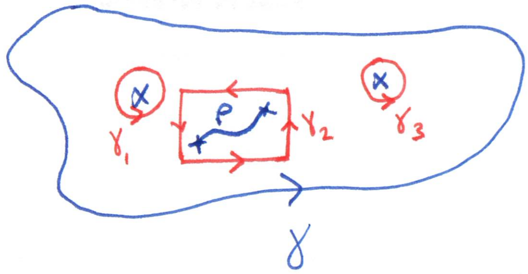
$$\left\{ -\int_{\gamma_1} + \int_{\gamma_2} \right\} f(z)\,dz = 0$$

So $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ ie. these integrals are equivalent despite lying on different paths!!

So if $F' = f$ and F cts then $\displaystyle\int_{\gamma_1} f(z)\,dz = F(\beta) - F(\alpha)$.

Example 2

f analytic except at x and except on p.



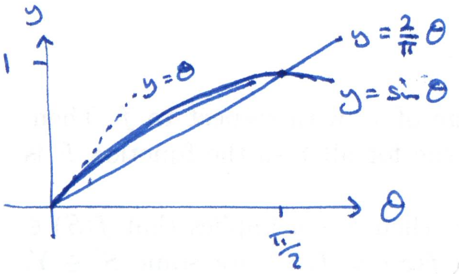
$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz$$

And γ_2, these integrals are easier to evaluate (at least numerically) because they can be parametrized using simple circular / polygonal paths.

Complex integrals to ∞ (improper complex integrals)

Sublemma $\forall R > 0, \forall x > 0, \displaystyle\int_0^\pi e^{-Rx\sin\theta}\,d\theta \leq \frac{\pi}{Rx}$

Proof



$\Rightarrow \forall \theta \in [0, \frac{\pi}{2}],\ \sin\theta \geq \frac{2}{\pi}\theta \geq 0$

$\Rightarrow -Rx\sin\theta \leq -\frac{2}{\pi}Rx\theta \leq 0$

$\Rightarrow 0 \leq e^{-Rx\sin\theta} \leq e^{-\frac{2x}{\pi}R\theta} \leq 1.$

$$\int_0^\pi e^{-Rx\sin\theta}\,d\theta = \int_0^{\frac{\pi}{2}} e^{-Rx\sin\theta}\,d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-Rx\sin\theta}\,d\theta$$

change variables $t = \pi - \theta$ in second integral

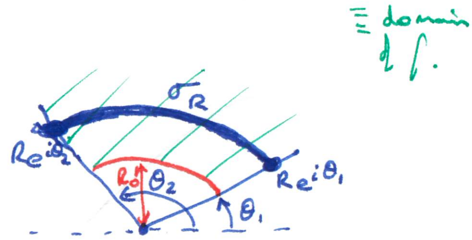
$$= \ldots [\text{exercise}] \ldots = 2\int_0^{\frac{\pi}{2}} e^{-Rx\sin\theta}\,d\theta$$

which I still can't integrate, but now I can use my inequality. Note the integrand is everywhere positive.

$$\Rightarrow \int_0^\pi e^{-Rx\sin\theta}\,d\theta \leq 2\int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}Rx\theta}\,d\theta = 2\left(1 - e^{-Rx}\right)\frac{\pi}{2Rx} \leq \frac{\pi}{Rx}. \quad \square$$

$t = \frac{2x}{\pi}R\theta$

Lemma (~~[illegible]~~ Jordan's lemma)



Suppose $R_0 \geq 0$, $0 \leq \theta_1 < \theta_2 \leq \pi$.

Let σ_R = circular arc $\{Re^{i\theta} : \theta \in [\theta_1, \theta_2]\}$

Suppose f continuous, complex-valued, ~~defined~~ on $\{Re^{i\theta} : \theta \in [\theta_1, \theta_2], R \geq R_0\}$.

Let ~~$M(R) = \max\{f(Re^{i\theta}) : \theta \in [\theta$, $\max\{|f(R)| : R$~~

$$M(R) = \max\{|f(z)| : z \in \sigma_R\}.$$

Suppose $\lim_{R\to\infty} M(R) = 0$.

Then, $\forall x > 0$,
$$\lim_{R\to\infty} \int_{\sigma_R} e^{ixz} f(z)\,dz = 0$$

"Even though the arc σ_R gets ~~longer and longer~~ infinitely long, the decay of the maximum of $|f(z)|$ on the arc, together with the oscillatory nature of e^{ixz}, is enough to make the integral tend to 0".

This is the same kind of result as the Riemann (-Lebesgue) lemma Dion presented.

Proof Parametrise σ_R by $\gamma(\theta) = Re^{i\theta}$ for $\theta \in [\theta_1, \theta_2]$. Then

WTS integral has limit zero, so it is equivalent to show its ~~absolute~~ modulus has limit zero.

$$\left| \int_{\sigma_R} e^{ixz} f(z)\,dz \right| = \left| \int_{\theta_1}^{\theta_2} e^{ix(Re^{i\theta})} f(Re^{i\theta})\, Rie^{i\theta}\,d\theta \right| \leq R\int_{\theta_1}^{\theta_2} |e^{ixRe^{i\theta}}| \cdot |f(Re^{i\theta})| \cdot |ie^{i\theta}|\,d\theta$$

$$\leq R \int_{\theta_1}^{\theta_2} e^{-xR\sin\theta} M(R)\,d\theta = RM(R) \int_{\theta_1}^{\theta_2} e^{-xR\sin\theta}\,d\theta$$

$$\leq R\,M(R)\int_{0}^{\pi} e^{-xR\sin\theta}\,d\theta$$

~~integrand positive, so can extend the domain of integration without loss with inequality~~

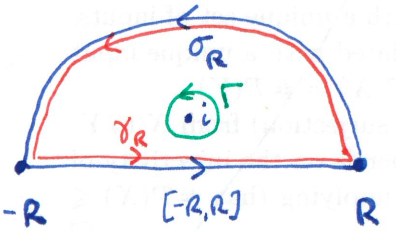
integral positive on $[0,\pi]$, so integral over $[0,\pi]$ bounds integral over subinterval $[\theta_1, \theta_2]$.

$$\leq R\,M(R)\,\frac{\pi}{xR} = \frac{\pi}{x}M(R) \to 0 \text{ as } R \to \infty. \qquad \square$$

Example 3

$f(x) = \dfrac{x}{x^2+1}$. Evaluate $I = \displaystyle\int_{-\infty}^{\infty} f(x)\cos(x)\,dx$

Note $I = \operatorname{Re}\left(\displaystyle\int_{-\infty}^{\infty} f(x)e^{ix}\,dx\right) = \operatorname{Re}\left(\lim_{R\to\infty}\displaystyle\int_{-R}^{R} f(x)e^{ix}\,dx\right)$



Extend f to $\mathbb{C}$ (or at least $\operatorname{clos}(\mathbb{C}^+)$) except f is not defined at $x = i$ (or $-i$; don't care).

Let $R \geq 2 = R_0$ (so we enclose i) and $\gamma_R = \sigma_R \cup [-R,R]$

Note f is analytic in all $\mathbb{C}$ except at $\pm i$.

So, $\forall R \geq 2$,

$$\int_{\Gamma} f(z)e^{iz}\,dz \overset{\text{Cauchy's Theorem}}{=} \int_{\gamma_2} f(z)e^{iz}\,dz = \int_{\gamma_R} f(z)e^{iz}\,dz = \int_{-R}^{R} f(z)e^{iz}\,dz + \int_{\sigma_R} f(z)e^{iz}\,dz$$

But $\left|f(Re^{i\theta})\right| = \dfrac{R\,|e^{i\theta}|}{|R^2e^{2i\theta}+1|} \leq \dfrac{R}{R^2-1} \to 0$ as $R \to \infty$

($\forall R \geq 2$)

So $\displaystyle\int_{\sigma_R} f(z)e^{iz}\,dz = 0$ by Jordan's lemma.

Therefore, $\forall R \geq 2$,

$$\int_{-R}^{R} f(z)e^{iz}\,dz = \int_\Gamma f(z)e^{iz}\,dz$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\cos(x)\,dx = \mathrm{Re}\left(\int_\Gamma f(z)e^{iz}\,dz\right).$$

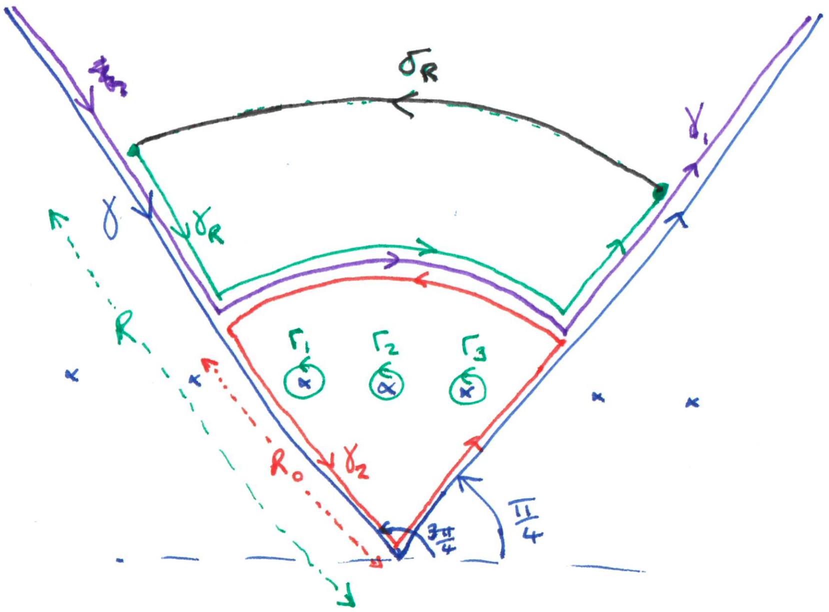
We replaced an improper integral with an integral about a circular contour.

~~Using~~ This would be easy to calculate numerically.

Even better, using Cauchy's residue calculus (see Complex Analysis ~~&~~ or FoAM), this integral can be calculated very easily.

Example 4

f analytic on $\mathbb{C}$ except at each $\times$ where it is undefined.
~~Not a removable~~ $|f(z)| \to 0$, uniformly in $\arg(z)$, as $z \to \infty$ within $\{Re^{i\theta} : R > R_0, \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}]\}$.



$$\int_\gamma f(z)e^{izx}\,dz = \int_{\gamma_1} f(z)e^{izx}\,dz + \int_{\gamma_2} f(z)e^{izx}\,dz$$

$$= \lim_{R\to\infty} \int_{\gamma_R} f(z)e^{ixz}\,dz + \left\{\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3}\right\} f(z)e^{ixz}\,dz$$

but $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} f(z)e^{ixz}\,dz = \lim_{R\to\infty}\left[\left\{\int_{\gamma_R} + \int_{\sigma_R}\right\} f(z)e^{ixz}\,dz - \int_{\sigma_R} f(z)e^{ixz}\,dz\right]$

by Cauchy's theorem

$$= \lim_{R \to \infty} \left[0 - \int_{\sigma_R} f(z) e^{ixz} \, dz \right]$$

$$= 0 .$$

So $$\int_{\gamma} f(z) e^{izx} \, dz = \left\{ \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} \right\} f(z) e^{ixz} \, dz .$$

infinite contour (difficult)

finite circular contours (easy, and even easier if you know residue calculus).

We will see many examples like this one in the coming lectures.