Inversion of Fourier Transforms, Calculus with Complex-valued Functions, and Parametrisation and Contour Integration

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1 Inversion of Fourier Transforms

For a function f, we are able to perform a Fourier Transform on it to convert to F, and find solutions to the equations involving F. However, we ultimately want the solutions to the equations involving f, not to its Fourier Transform. Therefore, this approach will be useless if we cannot guarantee that we can convert F back to f. The Convergence Theorem for Fourier Transforms provides us with the guarantee that performing an inverse Fourier Transform on F will get us back exactly f, except at the jump discontuities of f.

Definition 1 (Fourier Transform)**.** The Fourier Transform of function f, denoted F, is defined as

$$F(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{-i\mu\xi}\mathrm{d}\xi.$$

Theorem 2 (Convergence Theorem for Fourier Transforms)**.** *Let $f(x), -\infty < x < \infty$ be a real or complex-valued function that is piecewise smooth over each finite interval. Suppose that $\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x$ is convergent, and $F(\mu)$ is the Fourier Transform of $f(x)$. Then for each point x,*

$$\lim_{M\to\infty}\int_{-M}^{M} F(\mu)e^{i\mu x}\mathrm{d}\mu = \frac{1}{2}f(x+0) + \frac{1}{2}f(x-0)$$

where $x+0$ is the right limit and $x-0$ is the left limit for each x.

This theorem tells us that the inverse Fourier Transform converges to the original function f for all x at which f is continuous, since in that case the left limit and right limit are equal. If f has a jump discontinuity at point x, then the inverse Fourier Transform converges to the average of the left and right limits: $\frac{1}{2}f(x+0)+\frac{1}{2}f(x-0)$, of f at that point instead.

The following example illustrates Theorem 2. Suppose we are given a "square wave" (Figure 1),

$$f(x) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } a \leq x \leq b \\ 0 & \text{if } x > b. \end{cases}$$

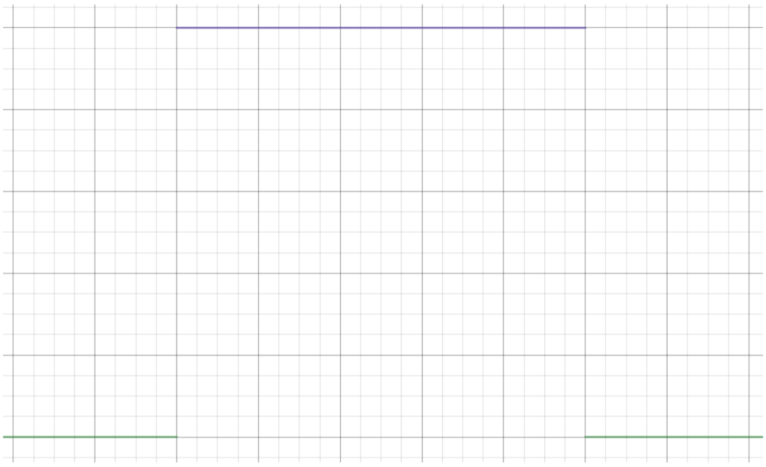

Figure 1: Graph of the square wave

If we find the Fourier Transform of the square wave and then inverse the Fourier Transform, we will get

$$\lim_{M\to\infty} \int_{-M}^{M} F(\mu)e^{i\mu x}\mathrm{d}\mu = \begin{cases} 0 & \text{if } x < a \\ \mathbf{0.5} & \mathbf{if\ x = a} \\ 1 & \text{if } a < x < b \\ \mathbf{0.5} & \mathbf{if\ x = b} \\ 0 & \text{if } x > b \end{cases}$$

which is the original function except for at the jump discontinuities $x = a$ and $x = b$. See figure 2 for an illustration of what happens at a jump discontinuity of the square wave.

In order to prove the Convergence Theorem for Fourier Transforms (Theorem 2) we will require Riemann's Lemma.

1.1 Riemann's Lemma

Lemma 3 (Riemann's Lemma)**.** *If f and f' are piecewise continuous on (a, b), then*

$$\lim_{\lambda\to\infty} \int_{a}^{b} f(x)sin(\lambda x)\mathrm{d}x = 0.$$

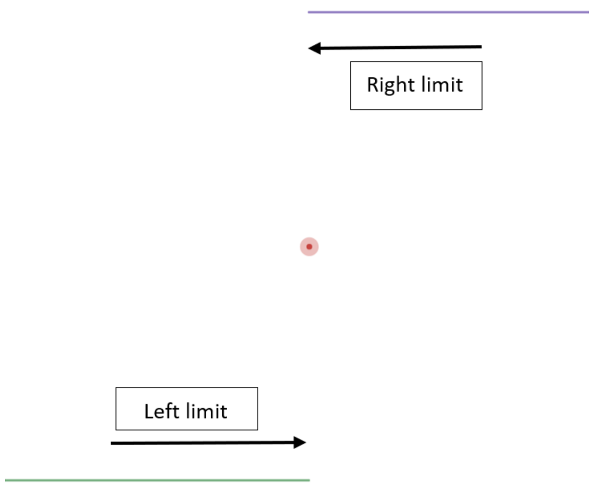

Figure 2: At jump discontinuities, the inverse Fourier Transform produces a point that is the average of the left and right limits of the original function at that point.

Note that another way to say that f and f' are piecewise continuous on (a,b) is to say that f is **piecewise continuously differentiable**. This is a weaker condition than **piecewise smooth** which means that f and all of its derivatives are piecewise continuous.

Riemann's Lemma essentially states that as $\lambda \to \infty$, the area of $f(x)\sin(\lambda x)$ above the x-axis and the area of $f(x)\sin(\lambda x)$ below the x-axis will cancel out completely. Click here for a graphical illustration of Riemann's Lemma.

Proof.

$$\int_a^b f(x)\sin(\lambda x)\mathrm{d}x = \sum_{i=0}^{p} \int_{x_i}^{x_i+1} f(x)\sin(\lambda x)\mathrm{d}x$$

which is the sum of the continuous differentiable pieces of the function. See figure 3 for an illustration of such a summation.

We want to show that each continuous piece converges to 0, i.e.

$$\lim_{\lambda\to\infty} \int_{x_i}^{x_{i+1}} f(x)\sin(\lambda x)\mathrm{d}x = 0.$$

Perform integration by parts with $u = f(x)$ and $v' = \sin(\lambda x)$. Note that since we are differentiating f at this step, we require that f be differentiable on (x_i, x_{i+1}).

Then,

$$\int_{x_i}^{x_{i+1}} f(x)\sin(\lambda x)\mathrm{d}x = \left[\frac{-f(x)\cos(\lambda x)}{\lambda}\right]_{x_i}^{x_{i+1}} + \frac{1}{\lambda}\int_{x_i}^{x_{i+1}} f'(x)\cos(\lambda x)\mathrm{d}x.$$

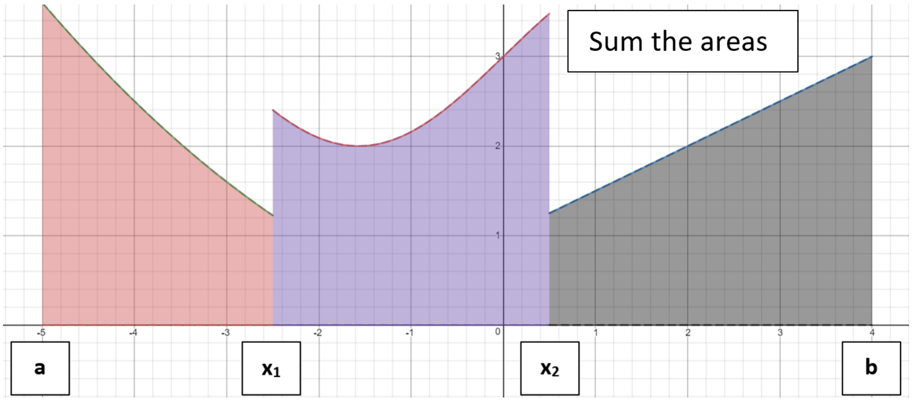

Figure 3: The total area of a piecewise continuous differentiable function is the sum of the area of each of its continuous differentiable pieces.

If $\forall x_i \in [a, b]$, $f(x_i)$ is finite, then

$$\lim_{\lambda\to\infty}\left[\frac{-f(x)\cos(\lambda x)}{\lambda}\right]_{x_i}^{x_{i+1}} = 0$$

since the numerator is always finite but the denominator grows to infinity. Moreover,

$$\int_{x_i}^{x_{i+1}} f'(x)\mathrm{d}x \geq \int_{x_i}^{x_{i+1}} f'(x)\cos(\lambda x)\mathrm{d}x.$$

Therefore,

$$\lim_{\lambda\to\infty}\frac{[f(x)]_{x_i}^{x_{i+1}}}{\lambda} = 0 \implies \lim_{\lambda\to\infty}\frac{\int_{x_i}^{x_{i+1}} f'(x)\cos(\lambda x)\mathrm{d}x}{\lambda} = 0.$$

Note that since this last step involves the integration of f', it requires that f' be continuous on (x_i, x_{i+1}). $\square$

Now we are ready to prove the Convergence Theorem for Fourier Transforms (Theorem 2).

Proof.

$$2\pi\int_{-M}^{M} F(\mu)e^{i\mu x}\mathrm{d}\mu = \int_{-M}^{M}\left(\int_{-\infty}^{\infty} f(\xi)e^{-i\mu\xi}\mathrm{d}\xi\right)e^{i\mu x}\mathrm{d}\mu$$

by substitution of $F(\mu)$ (see Definition 1).

$$\int_{-M}^{M}\left(\int_{-\infty}^{\infty} f(\xi)e^{-i\mu\xi}\mathrm{d}\xi\right)e^{i\mu x}\mathrm{d}\mu = \int_{-\infty}^{\infty}\left(\int_{-M}^{M} f(\xi)e^{-i\mu\xi}\mathrm{d}\mu\right)e^{i\mu x}\mathrm{d}\xi. \quad (1)$$

The order of integration was switched. This requires a great deal of justification, but we will deal with it later in §1.2. Proceeding, we have

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\int_{-M}^{M} f(\xi)e^{-i\mu\xi}\mathrm{d}\mu\right)e^{i\mu x}\mathrm{d}\xi &= \int_{-\infty}^{\infty}\left(\int_{-M}^{M} f(\xi)e^{i\mu(x-\xi)}\right)\mathrm{d}\mu\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty}\left[f(\xi)\frac{e^{i\mu(x-\xi)}}{i(x-\xi)}\right]_{-M}^{M}\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} f(\xi)\left(\frac{e^{iM(x-\xi)}-e^{i(-M)(x-\xi)}}{i(x-\xi)}\right)\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} f(\xi)\left(\frac{2i\sin(M(x-\xi))}{i(x-\xi)}\right)\mathrm{d}\xi \text{ (using a } \mathbb{C} \text{ identity)} \\ &= 2\int_{-\infty}^{\infty} f(\xi)\left(\frac{\sin(M(x-\xi))}{x-\xi}\right)\mathrm{d}\xi.\end{aligned}$$

Substitute $\eta = \xi - x$ to get

$$2\int_{-\infty}^{\infty} f(x+\eta)\frac{\sin M\eta}{\eta}\mathrm{d}\eta = 2\int_{-\infty}^{\infty}\frac{f(x+\eta)}{\eta}\sin(M\eta)\mathrm{d}\eta.$$

However, what if $\eta = 0$? Do we get a division by zero? If $\eta = 0$, then $x = \xi$. This implies that

$$\int_{-M}^{M} e^{i\mu(x-\xi)}\mathrm{d}\mu = \int_{-M}^{M} e^{i\mu(0)}\mathrm{d}\mu = \int_{-M}^{M} 1\mathrm{d}\mu = 0.$$

Therefore, we can effectively ignore the case of $\eta = 0$ in the integration interval.

Using Riemann's Lemma (Lemma 3), for some $\delta \in \mathbb{C}$,

$$\begin{aligned}&\lim_{M\to\infty}\int_{\delta}^{\infty}\frac{f(x+\eta)}{\eta}\sin(M\eta)\mathrm{d}\eta = 0 \text{ and} \\ &\lim_{M\to\infty}\int_{-\infty}^{-\delta}\frac{f(x+\eta)}{\eta}\sin(M\eta)\mathrm{d}\eta = 0.\end{aligned}$$

Therefore, we only have to analyse the integral for $-\delta \le \eta \le \delta$.

Riemann's Lemma (Lemma 3) also tells us that these two other integrals,

$$\begin{aligned}&2\int_{0}^{\delta}\frac{\sin M\eta}{\eta}(f(x+\eta)-f(x+0))\mathrm{d}\eta \quad \text{and} \\ &2\int_{-\delta}^{0}\frac{\sin M\eta}{\eta}(f(x+\eta)-f(x-0))\mathrm{d}\eta\end{aligned}$$

tend to 0 as $M \to \infty$. We will use these two integrals to introduce the left and right limits to our proof.

Since

$$
\begin{aligned}
&2\lim_{M\to\infty}\int_0^\delta \frac{\sin M\eta}{\eta}(f(x+\eta)-f(x+0))\mathrm{d}\eta \\
&=\lim_{M\to\infty}\left(2\int_0^\delta \frac{\sin M\eta}{\eta}f(x+\eta)\mathrm{d}\eta - 2\int_0^\delta \frac{\sin M\eta}{\eta}f(x+0)\mathrm{d}\eta\right)
\end{aligned}
$$

converges (to 0) and

$$2\int_0^\delta \frac{\sin M\eta}{\eta}f(x+0)\mathrm{d}\eta$$

converges, specifically to $\pi f(x+0)$ (see "Dirichlet Integral" and "Sinc function"), we can split the limit into two terms, i.e.

$$
\begin{aligned}
&2\lim_{M\to\infty}\int_0^\delta \frac{\sin M\eta}{\eta}(f(x+\eta)-f(x+0))\mathrm{d}\eta \\
&=\lim_{M\to\infty}\left(2\int_0^\delta \frac{\sin M\eta}{\eta}f(x+\eta)\mathrm{d}\eta\right) - \lim_{M\to\infty}\left(2\int_0^\delta \frac{\sin M\eta}{\eta}f(x+0)\mathrm{d}\eta\right) = 0 \\
&=\lim_{M\to\infty}\left(2\int_0^\delta \frac{\sin M\eta}{\eta}f(x+\eta)\mathrm{d}\eta\right) - \pi f(x+0) = 0. \qquad (2)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&2\lim_{M\to\infty}\int_{-\delta}^0 \frac{\sin M\eta}{\eta}(f(x+\eta)-f(x-0))\mathrm{d}\eta \\
&=\lim_{M\to\infty}\left(2\int_{-\delta}^0 \frac{\sin M\eta}{\eta}f(x+\eta)\mathrm{d}\eta\right) - \lim_{M\to\infty}\left(2\int_{-\delta}^0 \frac{\sin M\eta}{\eta}f(x-0)\mathrm{d}\eta\right) = 0 \\
&=\lim_{M\to\infty}\left(2\int_{-\delta}^0 \frac{\sin M\eta}{\eta}f(x+\eta)\mathrm{d}\eta\right) - \pi f(x-0) = 0 \qquad (3)
\end{aligned}
$$

since

$$2\lim_{M\to\infty}\int_{-\delta}^0 \frac{\sin M\eta}{\eta}f(x-0)\mathrm{d}\eta = \pi f(x-0).$$

Note that the sinc function, $\frac{\sin\eta}{\eta}$, is symmetric about the y-axis (see figure 4).

Summing equations (2) and (3), we get

$$
\begin{aligned}
&2\lim_{M\to\infty}\int_0^\delta \frac{\sin M\eta}{\eta}(f(x+\eta)-f(x+0))\mathrm{d}\eta + 2\lim_{M\to\infty}\int_{-\delta}^0 \frac{\sin M\eta}{\eta}(f(x+\eta)-f(x-0))\mathrm{d}\eta \\
&= 2\lim_{M\to\infty}\left(\int_0^\delta \frac{\sin M\eta}{\eta}f(x+\eta)\mathrm{d}\eta\right) - \pi f(x+0) \\
&\quad + 2\lim_{M\to\infty}\left(\int_{-\delta}^0 \frac{\sin M\eta}{\eta}f(x+\eta)\mathrm{d}\eta\right) - \pi f(x-0) = 0.
\end{aligned}
$$

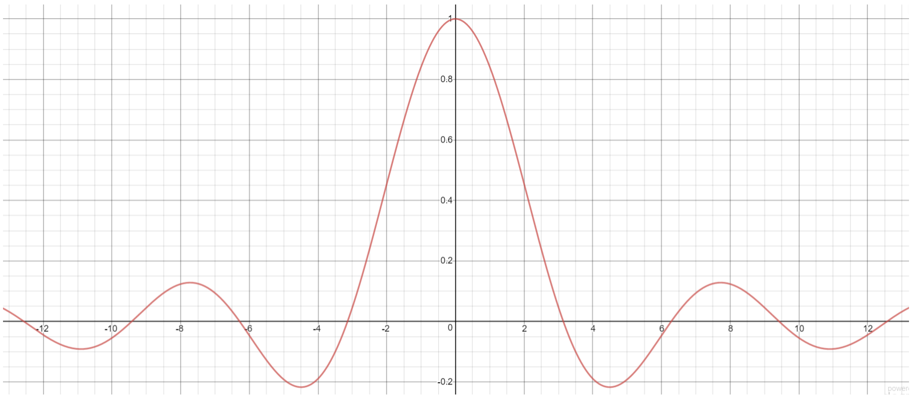

Figure 4: Graph of $\frac{\sin \eta}{\eta}$. The graph is symmetric about the y-axis.

This implies that

$$2\pi \lim_{M\to\infty} \int_{-M}^{M} F(\mu)e^{i\mu x}\mathrm{d}\mu = 2 \lim_{M\to\infty} \int_{-\delta}^{\delta} \frac{\sin M\eta}{\eta} f(x+\eta)\mathrm{d}\eta = \pi(f(x+0)+f(x-0))$$

$$\implies \lim_{M\to\infty} \int_{-M}^{M} F(\mu)e^{i\mu x}\mathrm{d}\mu = \frac{1}{2}(f(x+0)+f(x-0)).$$

□

1.2 Justification for the Interchange of Integration Order

Back during the step (1), we interchanged the order of integration. For this step to be valid, we require that

$$\int_{-\infty}^{\infty} f(\xi)e^{-i\mu\xi}\mathrm{d}\xi$$

converges absolutely and converges uniformly.

We are given that

$$\int_{\infty}^{-\infty} |f(x)|\mathrm{d}x$$

is convergent. Therefore,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(\xi)e^{-i\mu\xi}\mathrm{d}\xi\right| &= \int_{-\infty}^{\infty} |f(\xi)||e^{-i\mu\xi}|\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} |f(\xi)|\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} |f(x+\eta)|\mathrm{d}x\end{aligned}$$

must also be convergent. This implies absolute convergence. Moreover, observe that $\left|\int_{-\infty}^{\infty} f(\xi)e^{-i\mu\xi}\mathrm{d}\xi\right|$ is independent of μ, which implies that it converges uniformly.

This in turn implies that $\int_{-\infty}^{\infty} f(\xi)e^{-i\mu\xi}\mathrm{d}\xi$ converges uniformly (in fact, we say that the integral demonstrates **uniform absolute-convergence**).

For more information on the interchange of integration order, you may refer to `https://en.wikipedia.org/wiki/Order_of_integration_(calculus)`.

Now that we have proven the Convergence Theorem for Fourier Transforms (Theorem 2), we can formally define the Inverse Fourier Transform.

Definition 4 (Inverse Fourier Transform)**.** The Inverse Fourier Transform of function F to get back f is defined as

$$f(x) = \int_{-\infty}^{\infty} F(\mu)e^{i\mu x}\mathrm{d}\mu.$$

2 Calculus with Complex-valued Functions

A complex-valued function is a function from any set D to the set of complex numbers $\mathbb{C}$, i.e.

$$f : D \to \mathbb{C}.$$

In summary, almost every property of calculus with real-valued functions applies to calculus with complex-valued functions, with the notable exception of the mid-point theorems. Mid-point theorems do not work because we are now dealing with at least 3-Dimensions.

An example of a midpoint theorem is Lagrange's Mean Value Theorem which states that: if f is a continuous real valued function on $[a, b]$ which is differentiable on (a, b), then there exists $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

You can read more about the various midpoint (or mean value) theorems here: `https://en.wikipedia.org/wiki/Mean_value_theorem` (else you can take Real Analysis).

2.1 Complex Differentiation

Definition 5. Let f be a complex-valued function defined on an open interval (a, b). We define the derivative of f in the usual way by

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}f(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

Note that

$$f'(t) = (\mathrm{Re}\ f)'(t) + i(\mathrm{Im}\ f)'(t),$$

and $f'(t)$ exists if and only if $(\mathrm{Re}\ f)'(t)$ and $(\mathrm{Im}\ f)'(t)$ both exist.

Properties of Complex Differentiation

1. $(\alpha f+\beta g)'(t)=\alpha f'(t)+\beta g'(t)$.
2. $(fg)'(t)=f'(t)g(t)+f(t)g'(t)$ (Product rule).
3. $(f\circ g)'(t)=f(g(t))g'(t)$ (Chain rule).

2.2 Complex Integration

Definition 6. Let f be a piecewise continuous function defined on an interval $[a,b]$ and taking values in the complex plane. We define the **Riemann** integral of f as follows:

$$\int_a^b f(t)\mathrm{d}t=\int_a^b \mathrm{Re}\ f(t)\mathrm{d}t+i\int_a^b \mathrm{Im}\ f(t)\mathrm{d}t.$$

As a direct consequence of this definition, we have

$$\mathrm{Re}\int_a^b f(t)\mathrm{d}t=\int_a^b \mathrm{Re}\ f(t)\mathrm{d}t \text{ and } \mathrm{Im}\int_a^b f(t)\mathrm{d}t=\int_a^b \mathrm{Im}\ f(t)\mathrm{d}t$$

Properties of Complex Integration

1. $\displaystyle\int_a^b (f(t)\pm g(t))\mathrm{d}t=\int_a^b f(t)\mathrm{d}t\pm\int_a^b g(t)\mathrm{d}t$.
2. $\displaystyle\int_a^b \beta f(t)\mathrm{d}t=\beta\int_a^b f(t)\mathrm{d}t$.
3. $\displaystyle\left|\int_a^b f(t)\mathrm{d}t\right|\leq\int_a^b |f(t)|\mathrm{d}t$ (Triangle Inequality for Integrals).
4. If c lies between a and b on the integration path, then
$$\int_a^c f(t)\mathrm{d}t+\int_c^b f(t)\mathrm{d}t=\int_a^b f(t)\mathrm{d}t.$$
5. If f and g are differentiable on (a,b) and continuous on $[a,b]$, then integration by parts is valid, i.e.
$$\int_a^b f(t)g'(t)\mathrm{d}t=[f(t)g(t)]_a^b-\int_a^b f'(t)g(t)\mathrm{d}t.$$

Definition 7 (Antiderivative)**.** If f is a piecewise continuous complex-valued function on $[a,b]$, we say that F is an antiderivative of f if $F'=f$ at all the points of continuity of f on (a,b).

This definition is based on the definition of differentiation.

Theorem 8 (Fundamental Theorem of Calculus). *Suppose that f is a piecewise continuous complex-valued function on the interval $[a,b]$ and let F be a continuous antiderivative of f in $[a,b]$. Then*

$$\int_a^b f(t)\mathrm{d}t = F(a) - F(b).$$

This theorem links the previously independent concepts of differentiation and integration.

3 Parametrisation and Contour Integration

Definition 9 (Parametrisation). A parametric form of a curve is a representation of the curve by a pair of equations $x = x(t)$ and $y = y(t)$, where t ranges over a set of real numbers, usually a closed interval $[a,b]$. Each value of t determines a point $\gamma(t) = (x(t), y(t))$, which traces the curve as t moves from a to b.

For example, we can parametrise a circle with equation $x^2 + y^2 = r^2$ as

$$x = x(t) = r\cos(t), \quad y = y(t) = r\sin(t), \quad 0 \leq t \leq 2\pi, r \in \mathbb{R}.$$

A circle or circular arc has **positive orientation** if traversed in the counterclockwise orientation, and **negative orientation** if traversed in the clockwise direction. The circle defined above has positive orientation.

Parametrisation is especially useful when dealing with complex-valued functions. Since $z = x + iy$, it makes sense to adopt the notation

$$z(t) = x(t) + iy(t).$$

In particular, we can write the parametric form of a curve γ using complex notation as

$$\gamma(t) = x(t) + iy(t), a \leq t \leq b,$$

and think of the curve as the graph of a complex-valued function of a real variable t, i.e.

$$\gamma : \mathbb{R} \to \mathbb{C}.$$

In this way, we can go back to dealing with simpler and more familiar single variable real-valued functions.

Definition 10 (Paths \ Contours). A path or a contour is a curve γ defined on a closed interval $[a,b]$ which is continuously differentiable or piecewise continuously differentiable. The path γ is called **closed** if $\gamma(a) = \gamma(b)$.

Definition 11 (Contour Integration)**.** Suppose that γ is a path over a closed interval $[a, b]$ and that f is a continuous complex-valued function defined on the graph of γ. The path or contour integral of f on γ is defined as:

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

where γ is parametrised in terms of t.

Contour integration is akin to integration by substitution (with $z = \gamma(t), \mathrm{d}z = \gamma'(t)\mathrm{d}t$). See figure 5 for an illustration of contour integration.

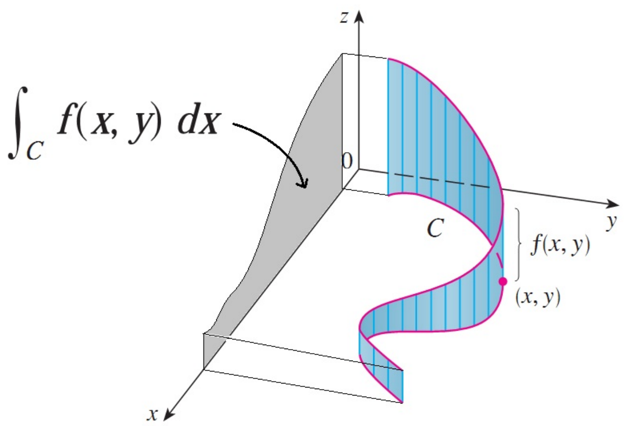

Figure 5: An illustration of contour integration with the contour labeled as C. Image taken from `https://www.wikihow.com/Calculate-Line-Integrals`.

Properties of Contour Integration

1. $\displaystyle\int_\gamma (\alpha f(z) \pm \beta g(z))\mathrm{d}z = \alpha \int_\gamma f(z)\mathrm{d}z \pm \beta \int_\gamma g(z)\mathrm{d}z.$

2. Let γ^* denote the reverse of γ (for circular paths this means the reverse orientation), then
$$\int_{\gamma^*} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z.$$

3. For $k = 1, \ldots, m$, let γ_k be a path defined on $[k-1, k]$ such that $\gamma_k(k) = \gamma_{k+1}(k)$, for all $k \leq m-1$. If f is a continuous function on the combined path $\Gamma = [\gamma_1, \gamma_2, \ldots, \gamma_m]$, then
$$\int_\Gamma f(z)\mathrm{d}z = \sum_{k=1}^m \int_{\gamma_k} f(z)\mathrm{d}z.$$

There are many different classes of parametrisations. We will only cover one specific class: polygonal paths.

Definition 12 (Polygonal path)**.** A polygonal path $\gamma = [z_1, z_2, \ldots, z_n]$ is the union of the line segments $[z_1, z_2], [z_2, z_3], \ldots, [z_{n-1}, z_n]$ for $z \in \mathbb{C}$. This is a piecewise linear path with initial point z_1 and terminal point z_n and the path may have self intersections.

A polygonal path is called **simple** if it does not have self intersections, except possibly at the endpoints, that is, z_1 and z_n may coincide. The polygonal path is called **closed** if $z_1 = z_n$.

Figures 6 and 7 show some of the different types of polygonal paths. One notable use of polygonal paths is to approximate a more complicated contour, see figure 8 for an example.

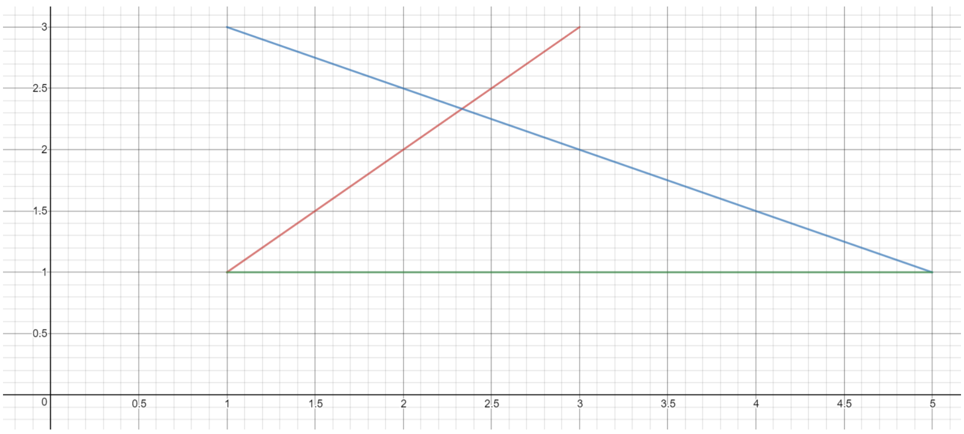

Figure 6: This is a simple, closed, polygonal path.

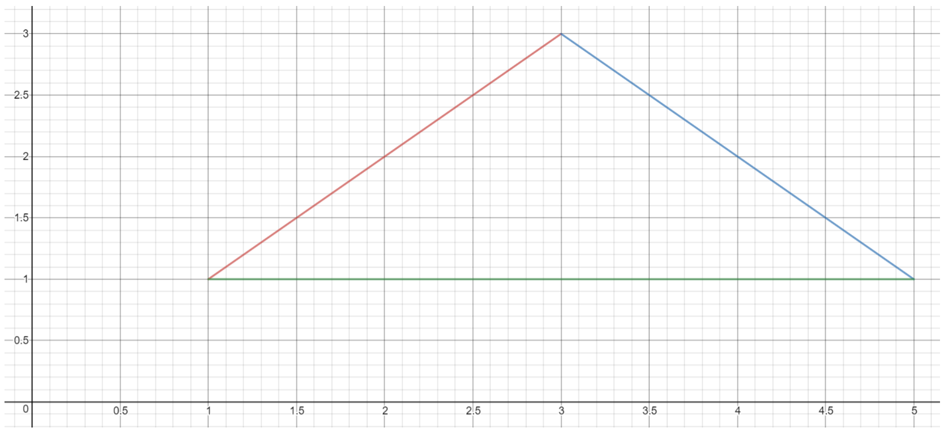

Figure 7: This is a self-intersecting, open, polygonal path.

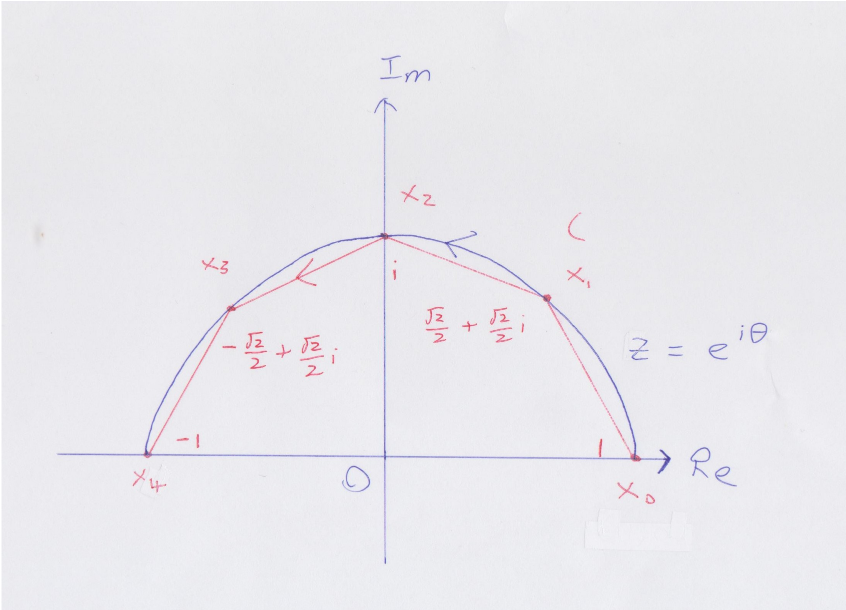

Figure 8: In this picture, a polygonal path is used to approximate a semicircle complex contour with positive orientation.

3.1 Equivalent Parametrisations

There are many different parametrisations, and they not unique to a function. However, different parametrisations of the same function may be equivalent.

Definition 13 (Equivalent Parametrisations)**.** We say that the paths $\gamma_1(t), a \leq t \leq b$, and $\gamma_2(s), c \leq s \leq d$, have equivalent parametrisations if there is a strictly increasing continuously differentiable function ϕ from $[c, d]$ onto $[a, b]$ such that $\phi(c) = a$ and $\phi(d) = b$ and $\gamma_2(s) = (\gamma_1 \circ \phi)(s)$ for all $s \in [c, d]$, or equivalently $\gamma_1(t) = (\gamma_2 \circ \phi^{-1})(t)$ for all $t \in [a, b]$, where ϕ^{-1} is the unique inverse of ϕ.

The term **strictly increasing** means that for any $s_1, s_2 \in [c, d]$, if $s_2 > s_1$, then $\phi(s_2) = t_2 > t_1 = \phi(s_1)$ for $t_1, t_2 \in [a, b]$.

The following is a non-comprehensive list of rules regarding equivalent parametrisations:

1. If two paths do not have the same range, they are not equivalent.

2. For two paths γ_1 with endpoints a, b, and γ_2 with endpoints c, d, if $\gamma_1(a) \neq \gamma_2(c)$ or $\gamma_1(b) \neq \gamma_2(d)$, then the two paths are not equivalent.

3. If the trajectories of two paths are not the same, then they are not equivalent. This is a corollary of rule 2. Though one can reverse the orientation of one path and recheck for equivalence.

4. For contours in the xy-plane, if two paths do not have the same number of turning points (both with respect to the x-axis, and w.r.t the y-axis), then they are not equivalent. Note that stationary points (derivative = 0) are not necessarily turning points.

5. Open paths are not equivalent to closed paths. This is a corollary of rule 2.

6. If two paths are equivalent, then a bijection exists between their domains. That bijection is in fact the function ϕ, which is why ϕ^{-1} exists.

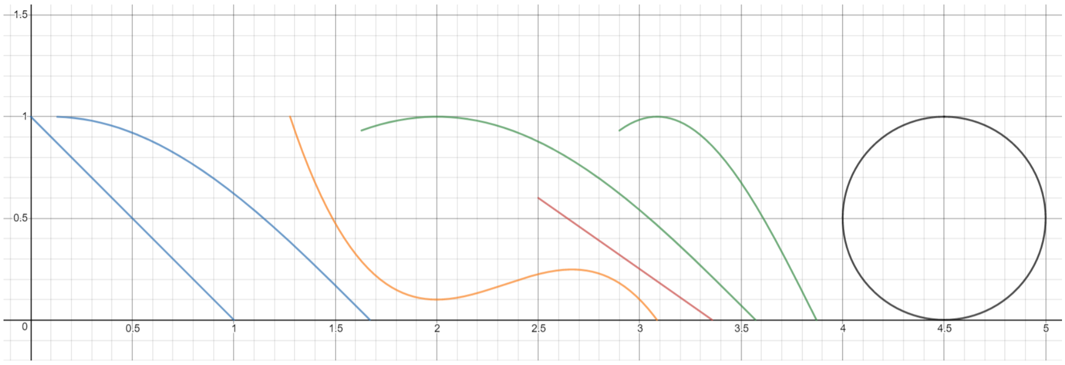

Figure 9: The paths in this picture have equivalent parametrisations if and only if they have the same colour. Assume the trajectories of the paths all have a left to right orientation. The graphs can be accessed here: `https://www.desmos.com/calculator/aaswr1xwcg`.

In figure 9, the red line is not equivalent to any other paths because its range is different from theirs. The black circle is not equivalent to any other paths because it is a closed path whereas all the other paths are open. Also, it is the only path with turning points with respect to the y-axis. The green and blue paths are not equivalent because of the endpoint rule (rule 2). Less evident is the fact that the orange path is not equivalent to the blue paths. This we will prove.

Proof. Suppose that the orange path, γ_2, and a blue path, γ_1, are equivalent. Choose a point a small distance to the left of the leftmost turning point of the orange path. Denote this point $(c_1, \gamma_2(c_1))$. Then, there exists $\phi(c_1) = t_1$ such that $\gamma_1(t_1) = \gamma_2(c_1)$. Since there is a turning point to the right of $(c_1, \gamma_2(c_1))$, there also exists c_2 such that $\gamma_2(c_2) = \gamma_1(t_1)$. Then, $\phi^{-1}(t_1) = c_1$ and $\phi^{-1}(t_1) = c_2$. Since ϕ is bijective, this implies that $c_1 = c_2$. However, $c_1 < c_2$, which is a contradiction. $\square$

Equivalent parametrisations are extremely important because of the following theorem:

Theorem 14 (Independence of Parametrisation)**.** *Suppose that the paths $\gamma_1(t), a \leq t \leq b$, and $\gamma_2(s), c \leq s \leq d$, have equivalent parametrisations and let f be a continuous function on this path. Then*

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z.$$

In one sentence, this theorem tells us that integrals over equivalent parametrisations produce identical results. This theorem accords us with a great deal of flexibility when choosing parametrisations.

Proof. We have $\gamma_2 = \gamma_1 \circ \phi$, where ϕ is a strictly increasing continuously differentiable function from $[c, d]$ onto $[a, b]$. Assume first that γ_1 is differentiable at every point on (a, b). Then $\gamma_2 = \gamma_1 \circ \phi$ is also differentiable at every point on (c, d). Using $\gamma_2(s) = \gamma_1(\phi(s))$ and $\gamma_2'(s) = \gamma_1'(\phi(s))\phi'(s)$, we perform a change of variable with $t = \phi(s)$, $\mathrm{d}t = \phi'(s)\mathrm{d}s$, and obtain

$$\begin{aligned}\int_c^d f(\gamma_2(s))\gamma_2'(s)\mathrm{d}s &= \int_c^d f(\gamma_1(\phi(s)))\gamma_1'(\phi(s))\phi'(s)\mathrm{d}s \\ &= \int_a^b f(\gamma_1(t))\gamma_1'(t)\mathrm{d}t.\end{aligned}$$

Therefore, parametrising with $z = \gamma_1(t)$ or $z = \gamma_2(s)$ produces equal integrals. The same result will be obtained if the change of variable was performed with $s = \phi^{-1}(t)$ instead. It remains to consider the case where γ_1 is not differentiable at some points $a_1 < \ldots < a_{m-1}$ in (a, b). Set $a_0 = a$ and $a_m = b$. Then γ_1 is differentiable at every point in the interval (a_j, a_{j+1}) for $j = 0, \ldots, m-1$. The first part of our proof tells us that

$$\int_{\phi^{-1}(a_j)}^{\phi^{-1}(a_{j+1})} f(\gamma_2(s))\gamma_2'(s)\mathrm{d}s = \int_{a_j}^{a_{j+1}} f(\gamma_1(t))\gamma_1'(t)\mathrm{d}t.$$

Sum the integrals over $0 \leq j \leq m-1$. Since $c = \phi^{-1}(a_0) = \phi^{-1}(a)$ and $d = \phi^{-1}(a_m) = \phi^{-1}(b)$, we get

$$\begin{aligned}\int_c^d f(\gamma_2(s))\gamma_2'(s)\mathrm{d}s &= \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\gamma_2(s))\gamma_2'(s)\mathrm{d}s \\ &= \int_a^b f(\gamma_1(t))\gamma_1'(t)\mathrm{d}t.\end{aligned}$$

□