Stage 3 Begin with the "solution representation" obtained in stage 2, but treat it simply as a function q defined by a formula of irrelevant providence. Show that q satisfies the problem. This proves existence.

End of lecture 1

2.2 Finite interval, inhomogeneous Dirichlet heat problem

Consider

$$[\partial_t - \partial_{xx}]\, q(x,t) = 0 \qquad (x,t) \in (0,1) \times (0,T) \qquad \text{(PDE)}$$

$$q(x,0) = q_0(x) \qquad x \in [0,1] \qquad \text{(IC)}$$

$$q(0,t) = g_0(t), \quad q(1,t) = g_1(t) \qquad t \in [0,T] \qquad \text{(BC)}$$

where q_0, g_j appropriately smooth, (i.e. C^∞).

We use $\hat{\varphi}$ to represent the (spatial) Fourier transform of $\varphi \in C^\infty[0,1]$.

Preliminary work: How does $\hat{\ }$ interact with the second derivative operator on $C^\infty[0,1]$?

$$\widehat{-\frac{d^2}{dx^2}\varphi}(\lambda) = -\int_0^1 e^{-i\lambda x}\varphi''(x)\,dx = -\Big[e^{-i\lambda x}\big(\varphi'(x) + i\lambda\varphi(x)\big)\Big]_{x=0}^{x=1} + \lambda^2 \int_0^1 e^{-i\lambda x}\varphi(x)\,dx$$

$$= \big[\varphi'(0) + i\lambda\varphi(0)\big] - e^{-i\lambda}\big[\varphi'(1) + i\lambda\varphi(1)\big] + \lambda^2\hat{\varphi}(\lambda).$$

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2.2.1 Stage 1

Assume $\exists\ q : [0,1] \times [0,T] \to \mathbb{C}$, as smooth as we need, satisfying (PDE) & (IC).

Apply Fourier transform to (PDE):

$$0 = \widehat{\left[\partial_t - \partial_{xx}\right] q}(\lambda; t) \cancel{= \left[\frac{d}{dt} + \lambda^2\right] \hat{q}(\lambda;t) + \left(\varphi'(0) + i\lambda\varphi(0)\right) - e^{-i\lambda}\left(\varphi'(1) + i\lambda\varphi(1)\right)}$$

$$= \left[\frac{d}{dt} + \lambda^2\right] \hat{q}(\lambda;t) + \left(\partial_x q(0,t) + i\lambda q(0,t)\right) - e^{-i\lambda}\left(\partial_x q(1,t) + i\lambda q(1,t)\right)$$

$$\Rightarrow\ 0 = \frac{d}{dt}\left(e^{\lambda^2 t}\hat{q}(\lambda;t)\right) + e^{\lambda^2 t}\left(\partial_x q(0,t) + i\lambda q(0,t)\right) - e^{-i\lambda + \lambda^2 t}\left(\partial_x q(1,t) + i\lambda q(1,t)\right).$$

Integrate in time to solve the ODE for $\hat{q}(\lambda;\cdot)$.

$$\Rightarrow\ 0 = e^{\lambda^2 t}\hat{q}(\lambda;t) - \hat{q}(\lambda;0) + \int_0^t e^{\lambda^2 s}\left(\partial_x q(0,s) + i\lambda q(0,s)\right) ds$$
$$- e^{-i\lambda}\int_0^t e^{\lambda^2 s}\left(\partial_x q(1,s) + i\lambda q(1,s)\right) ds$$

Note that (IC) $\Rightarrow$ $\hat{q}(\lambda;0) = \hat{q}_0(\lambda)$. Also introduce notation

$$f_j(\lambda; X, t) := \int_0^t e^{\lambda^2 s}\, \partial_x^j q(X,s)\, ds.$$

Then

$$\hat{q}_0(\lambda) - e^{\lambda^2 t}\hat{q}(\lambda;t) = i\lambda f_0(\lambda;0,t) + f_1(\lambda;0,t)$$
$$- e^{-i\lambda}\left(i\lambda f_0(\lambda;1,t) + f_1(\lambda;1,t)\right), \qquad \text{(GR)}$$

valid $\forall\ \lambda \in \mathbb{C}$, $\forall\ t \in [0,T]$.

(GR) is the global relation. It relates the ~~solution to~~ Fourier transform of the solution to the Fourier transform of the initial datum and temporal transforms of the Dirichlet & Neumann boundary values.

Now solve for $q(x,t)$. Rearrange:

$$\hat{q}(\lambda;t) = e^{-\lambda^2 t}\hat{q}_0(\lambda) - e^{-\lambda^2 t}\left(i\lambda f_0(\lambda;0,t) + f_1(\lambda;0,t)\right)$$
$$+ e^{-i\lambda-\lambda^2 t}\left(i\lambda f_0(\lambda;1,t) + f_1(\lambda;1,t)\right)$$

Apply inverse ~~FT~~ Fourier transform:

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{q}_0(\lambda)\,d\lambda - \int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\left(i\lambda f_0(\lambda;0,t)+f_1(\lambda;0,t)\right)d\lambda$$
$$+ \int_{-\infty}^{\infty} e^{i\lambda(x-1)-\lambda^2 t}\left(i\lambda f_0(\lambda;1,t)+f_1(\lambda;1,t)\right)d\lambda$$

Notes.
- Properly these integrals should be considered together as a single Cauchy Principal Value Integral. This splitting & removal of VP can be rigourously justified later in the argument, at least for most problems for most equations, for most x and most t.

- We cannot reasonably expect this formula to hold at $x = ~~1~~\ 0$, or $x=1$ because (the full line extension of) q_0 is discontinuous at 0 and 1, so we use the above formula to evaluate $q(0,t)$, $q(1,t)$ only via interior limits.

We aim to deform the latter two contours of integration away from $\mathbb{R}$.

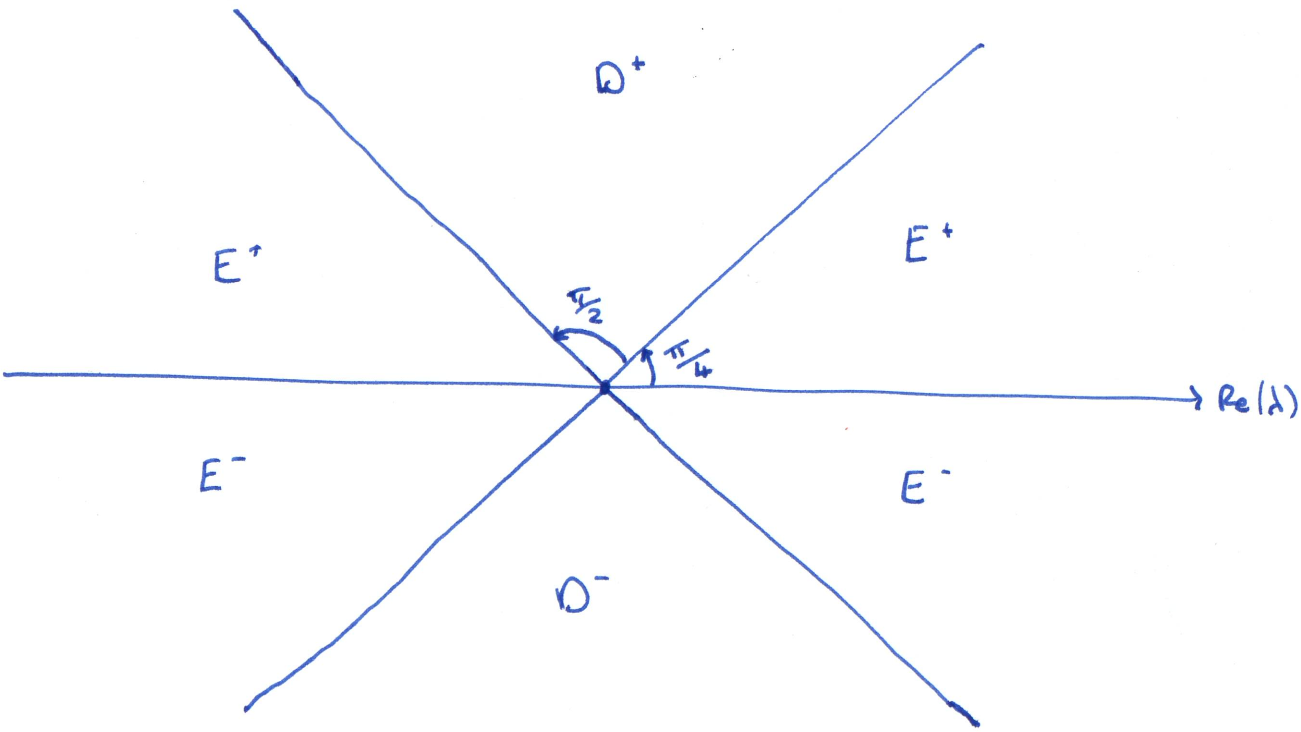
We need:

Definitions $\quad \mathbb{C}^\pm := \{\lambda \in \mathbb{C} : \pm \mathrm{Im}(\lambda) > 0\}$,

$D := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda^2) < 0\}$, $\qquad E := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda^2) > 0\}$,

$D^\pm := D \cap \mathbb{C}^\pm$, $\qquad E^\pm := E \cap \mathbb{C}^\pm$.

Orient the boundaries of these (unions of) sectors in the positive sense; the sector lies to the left of its boundary.



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Tools

- Cauchy's integral theorem (see earlier class)

- Jordan's lemma (see earlier class)

In particular, we will do lots of contour deformation from ~~the~~ infinite contours to much simpler contours, and showing that various contour integrals evaluate to 0. See examples 3 & 4 from earlier class.

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Integrate by parts in s:

$$e^{-\lambda^2 t} f_j(\lambda; X, t) = \int_0^t e^{-\lambda^2(t-s)} \partial_x^j q(X,s)\, ds$$

$$= \lambda^{-2}\left[e^{-\lambda^2(t-s)} \partial_x^j q(X,s)\right]_{s=0}^{s=t} - \lambda^{-2} \underbrace{\int_0^t e^{-\lambda^2(t-s)} \partial_t \partial_x^j q(X,s)\, ds}_{\mathcal{O}(1),\text{ uniformly in } \arg(\lambda) \text{ by Riemann-Lebesgue}}$$

$$= \mathcal{O}(|\lambda|^{-2}), \text{ uniformly in } \arg(\lambda) \text{ as } \lambda \to \infty \text{ within } \operatorname{clos}(E).$$

Note that E is chosen so that $e^{-\lambda^2(t-s)} = \mathcal{O}(1)$ for $s \in [0,t]$.

$$\Rightarrow e^{-\lambda^2 t}\left(i\lambda f_0(\lambda; X, t) + f_1(\lambda; X, t)\right) = \mathcal{O}(|\lambda|^{-1}), \text{ uniformly in } \arg(\lambda) \text{ as } \lambda \to \infty \text{ within } \operatorname{clos}(E).$$

Also, $e^{-\lambda^2 t}\left(i\lambda f_0(\lambda; X, t) + f_1(\lambda; X, t)\right)$ is entire as $\partial_x^j q(X, \cdot) \in L^1[0,T]$.

Hence, by ~~our corollary~~ Jordan's lemma, $\displaystyle\int_{\partial E^+} e^{i\lambda x - \lambda^2 t}\left(i\lambda f_0(\lambda; 0, t) + f_1(\lambda; 0, t)\right) d\lambda = 0,$ for all $x > 0$.

$$\text{so } \int_{-\infty}^{\infty} \cdots\, d\lambda = \left\{\int_{-\infty}^{\infty} - \int_{\partial E^+}\right\} \cdots\, d\lambda = \int_{\partial D^+} \cdots\, d\lambda.$$

Similarly, for the third integrand,

$$\int_{-\infty}^{\infty} \cdots\, d\lambda = -\int_{+\infty}^{-\infty} \cdots\, d\lambda = -\left\{\int_{+\infty}^{-\infty} - \int_{\partial E^-}\right\} \cdots\, d\lambda = \int_{\partial D^-} \cdots\, d\lambda.$$

Cauchy's integral theorem allows deformation of contours of integration over finite regions in which the integrand is analytic. Using ~~our~~ Jordan's lemma ~~corollary~~, the same can be done over appropriately chosen infinite sectors, with the additional decay criterion. We refer to such as an "infinite contour deformation" and ~~describe~~ summarise the argument as "by Jordan's lemma".

We have arrived at the Ehrenpreis form:

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\,\hat{q}_0(\lambda)\,d\lambda - \int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(i\lambda f_0(\lambda;0,t) + f_1(\lambda;0,t)\right)d\lambda$$

$$- \int_{\partial D^-} e^{i\lambda(x-1)-\lambda^2 t}\left(i\lambda f_0(\lambda;1,t) + f_1(\lambda;1,t)\right)d\lambda, \qquad (EFt)$$

valid for $(x,t)\in(0,1)\times[0,T]$.

By a similar argument, $\forall\, \tau \in [t,T]$,

$$e^{-\lambda^2 t}\left(i\lambda\int_t^\tau e^{\lambda^2 s}q(x,s)\,ds + \int_t^\tau e^{\lambda^2 s}\partial_x q(x,s)\,ds\right) = \mathcal{O}(|\lambda|^{-1}) \text{ uniformly in } \arg(\lambda)$$

as $\lambda\to\infty$ within $\operatorname{clos}(\underline{D})$, so

$$\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(i\lambda\left[f_0(\lambda;0,\tau)-f_0(\lambda;0,t)\right] + \left[f_1(\lambda;0,\tau)-f_0(\lambda,0,t)\right]\right)d\lambda = 0,$$

similarly for $\int_{\partial D^-} \ldots\, d\lambda = 0$. This yields

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$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\,\hat{q}_0(\lambda)\,d\lambda - \int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left(i\lambda f_0(\lambda;0,\tau) + f_1(\lambda;0,\tau)\right)d\lambda$$

$$- \int_{\partial D^-} e^{i\lambda(x-1) - \lambda^2 t}\left(i\lambda f_0(\lambda;1,\tau) + f_1(\lambda;1,\tau)\right)d\lambda, \qquad (EF\tau)$$

valid for $(x,t) \in (0,1) \times [0,\tau]$, $\tau \in [0,T]$.

This formula has the advantage of very simple (x,t) dependence.

Summary ~~Analysis~~ of progress

We started by assuming that a solution exists and showed that ~~for~~ any solution must satisfy $(EF\tau)$ and (GR). The value of (GR) is not clear yet, except in deriving $(EF\tau)$:

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\,\hat{q}_0(\lambda)\,d\lambda - \int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left(i\lambda f_0(\lambda;0,\tau) + f_1(\lambda;0,\tau)\right)d\lambda$$

$$- \int_{\partial D^-} e^{i\lambda(x-1) - \lambda^2 t}\left(i\lambda f_0(\lambda;1,\tau) + f_1(\lambda;1,\tau)\right)d\lambda$$

where $\hat{q}_0(\lambda) = \int_0^1 e^{-i\lambda y} q_0(y)\,dy$ known

$f_j(\lambda; X, \tau) = \int_0^{\tau} e^{\lambda^2 s}\,\partial_x^j q(X,s)\,ds$ unknown

So ~~are~~ $(EF\tau)$ is not an explicit representation of the solution; we still have work to do.