Unified Transform Method for Multipoint Problems

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1 Adjoint of an ordinary differential operator

In this section, we extend the construction of an adjoint problem, mostly following a similar argument as given by Linda in [3].

1.1 Formulation of the problem

Consider a closed interval $[a, b]$. Fix $n \in \mathbb{N}$, and let the differential operator be defined as

$$L := \sum_{k=0}^{n} a_k(t) \left(\frac{d}{dt}\right)^k, \text{ where } a_k(t) \in C^\infty[a,b] \text{ and } a_n(t) \neq 0 \ \forall t \in [a,b].$$

Fix $k \in \mathbb{N}$, and let $\pi = \{a = x_0 < x_1 < \ldots < x_k = b\}$ be a partition of $[a, b]$. Let the domain of L be given by the function space

$$C_\pi^{n-1}[a,b] = \Big\{ f : [a,b] \to \mathbb{C} \text{ s.t. } \forall l \in \{1, 2, \ldots, k\}, \\ f_l := f\big|_{(\eta_{l-1},\eta_l)} \text{ admits an extension } g_l \text{ to } [\eta_{l-1}, \eta_l] \text{ s.t. } g_l \in C^{n-1}[\eta_{l-1}, \eta_l] \Big\}.$$

Consider a homogeneous multipoint BVP of rank m

$$\pi_m : Lq = 0, \qquad Uq = \vec{0},$$

where $U = (U_1, \ldots, U_m)$ is a multipoint boundary form with

$$U_i(q) = \sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l)], \qquad i \in \{1, \ldots, m\},$$

where $\alpha_{ijl}, \beta_{ijl} \in \mathbb{R}, q \in C_\pi^{n-1}[a,b]$. Our goal is to construct the adjoint multipoint value problem (MVP) to π_m

$$\pi^+_{2nk-m} : L^+ q = 0, \qquad U^+ q = \vec{0},$$

with

$$L^+ := \sum_{k=0}^{n} (-1)^k \overline{a_k}(t) \left(\frac{d}{dt}\right)^k, \text{ where } \overline{a_k}(t) \text{ is the complex conjugate of } a_k(t), \ k = 0, \ldots, n,$$

and U^+ is an appropriate multipoint boundary form.

1.2 Green's formula

For any $f, g \in C_{\pi}^{n-1}[a,b]$, application of Green's formula yields

$$\langle Lf, g\rangle - \langle f, L^{+}g\rangle = \sum_{l=1}^{k}\sum_{p,q=0}^{n-1}[F_{p\,q}(x_l)f_l^{(p)}(x_l)g_l^{(q)}(x_l) - F_{p\,q}(x_{l-1})f_l^{(p)}(x_{l-1})g_l^{(q)}(x_{l-1})],$$

where $F(t)$ denotes an $n\times n$ boundary matrix at the point $t \in [a,b]$. From [1, p. 1286], the entries of $F(t)$ are given by

$$\begin{aligned} F_{p\,q}(t) &= \sum_{k=j}^{n-p-1}(-1)^k\binom{k}{j}\left(\frac{d}{dt}\right)^{k-j}a_{p+k+1}(t), && p+q<n-1 \\ F_{p\,q}(t) &= (-1)^q a_n(t), && p+q=n-1 \\ F_{p\,q}(t) &= 0, && p+q>n-1. \end{aligned}$$

Observe that since $\det(F(t)) = (a_0(t))^n \neq 0$, the matrix $F(t)$ is non-singular.

Our goal is to rewrite the Green's formula as a *semibilinear* form $\mathcal{S}$. First, let $\vec{f_l} := (f_l, \ldots, f_l^{(n-1)})$, and observe that

$$\begin{aligned} [fg]_l(t) := \sum_{p,q=0}^{n-1} F_{p\,q}(t)f_l^{(p)}(t)g_l^{(q)}(t) &= \sum_{p,q=0}^{n-1}\left[F_{p\,q}f_l^{(p)}g_l^{(q)}\right](t) \\ &= \sum_{q=0}^{n-1}\left[\left(\sum_{p=0}^{n-1}F_{p\,q}f_l^{(p)}\right)g_l^{(q)}\right](t) \\ &= F(t)\vec{f_l}(t)\cdot\vec{g_l}(t), \end{aligned}$$

where $\cdot$ refers to dot product. The Green's formula can then be rewritten as

$$\langle Lf, g\rangle - \langle f, L^{+}g\rangle = \sum_{l=1}^{k}[fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k}F(x_l)\vec{f_l}(x_l)\cdot\vec{g_l}(x_l) - F(x_{l-1})\vec{f_l}(x_{l-1})\cdot\vec{g_l}(x_{l-1}). \quad (1)$$

Note that

$$F(x_l)\vec{f_l}(x_l)\cdot\vec{g_l}(x_l) - F(x_{l-1})\vec{f_l}(x_{l-1})\cdot\vec{g_l}(x_{l-1}) = \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l)\end{bmatrix}\begin{bmatrix}\vec{f_l}(x_{l-1}) \\ \vec{f_l}(x_l)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_l}(x_{l-1}) \\ \vec{g_l}(x_l)\end{bmatrix},$$

so that we obtain

$$\langle Lf, g\rangle - \langle f, L^{+}g\rangle = \sum_{l=1}^{k}[fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k}\begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l)\end{bmatrix}\begin{bmatrix}\vec{f_l}(x_{l-1}) \\ \vec{f_l}(x_l)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_l}(x_{l-1}) \\ \vec{g_l}(x_l)\end{bmatrix}.$$

Now, expansion of the sum yields

$$\begin{aligned} &\sum_{l=1}^{k}\begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l)\end{bmatrix}\begin{bmatrix}\vec{f_l}(x_{l-1}) \\ \vec{f_l}(x_l)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_l}(x_{l-1}) \\ \vec{g_l}(x_l)\end{bmatrix} \\ &= \begin{bmatrix} -F(x_0) & 0_{n\times n} \\ 0_{n\times n} & F(x_1)\end{bmatrix}\begin{bmatrix}\vec{f_1}(x_0) \\ \vec{f_1}(x_1)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_1}(x_0) \\ \vec{g_1}(x_1)\end{bmatrix} + \ldots + \begin{bmatrix} -F(x_{k-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_k)\end{bmatrix}\begin{bmatrix}\vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k)\end{bmatrix} \\ &= \underbrace{\begin{bmatrix} -F(x_0) & 0 & \ldots & 0 & 0 \\ 0 & F(x_1) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -F(x_{k-1}) & 0 \\ 0 & 0 & \ldots & 0 & F(x_k)\end{bmatrix}}_{2nk\times 2nk}\begin{bmatrix}\vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vec{f_2}(x_1) \\ \vec{f_2}(x_2) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vec{g_2}(x_1) \\ \vec{g_2}(x_2) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k)\end{bmatrix} \end{aligned}$$

$$=: S \begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix} \cdot \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix} = \mathcal{S}\left(\begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix} \right), \tag{2}$$

where the matrix S is associated with the semibilinear form $\mathcal{S}$ and S is a block matrix where each block is $n \times n$. Further, note that the form $\mathcal{S}$ is the action of applying matrix S to the first argument and taking dot product of this result and the second argument. Thus, we managed to express the Green's Formula as a semibilinear form $\mathcal{S}$.

1.3 Boundary-Form formula

We turn to characterising an adjoint multipoint boundary condition using an extension of boundary form formula that Linda derived in her work. First, recall that the multipoint boundary conditions are of the form

$$Uq = \begin{bmatrix} U_1(q) \\ \vdots \\ U_m(q) \end{bmatrix} = \vec{0},$$

with

$$U_i(q) = \sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l)], \qquad i \in \{1, \ldots, m\},\ \alpha_{ijl}, \beta_{ijl} \in \mathbb{R}.$$

Note that $U_1, \ldots, U_m$ are linearly independent when $\sum_{i=1}^{m} c_i U_i q = 0$ if and only if $c_i = 0$. When $U_1, \ldots, U_m$ are linearly independent, we say that U has full rank m. For now, suppose that U has full rank, and define

$$\vec{q_l} = \begin{bmatrix} q_l \\ q_l' \\ \vdots \\ q_l^{(n-1)} \end{bmatrix}, M_l = \begin{bmatrix} \alpha_{1\ 0\ l} & \alpha_{1\ 1\ l} & \cdots & \alpha_{1\ (n-1)\ l} \\ \alpha_{2\ 0\ l} & \alpha_{2\ 1\ l} & \cdots & \alpha_{2\ (n-1)\ l} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \alpha_{m\ 1\ l} & \cdots & \alpha_{m\ (n-1)\ l} \end{bmatrix}, N_l = \begin{bmatrix} \beta_{1\ 0\ l} & \beta_{1\ 1\ l} & \cdots & \beta_{1\ (n-1)\ l} \\ \beta_{2\ 0\ l} & \beta_{2\ 1\ l} & \cdots & \beta_{2\ (n-1)\ l} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m\ 0\ l} & \beta_{m\ 1\ l} & \cdots & \beta_{m\ (n-1)\ l} \end{bmatrix}$$

Then,

$$\begin{aligned} Uq &= \begin{bmatrix} U_1(q) \\ \vdots \\ U_m(q) \end{bmatrix} \\ &= \sum_{l=1}^{k} \sum_{j=0}^{n-1} \begin{bmatrix} \alpha_{1\ j\ l} \\ \vdots \\ \alpha_{m\ j\ l} \end{bmatrix} q_l^{(j)}(x_{l-1}) + \begin{bmatrix} \beta_{1\ j\ l} \\ \vdots \\ \beta_{m\ j\ l} \end{bmatrix} q_l^{(j)}(x_l) \\ &= \sum_{l=1}^{k} \begin{bmatrix} \alpha_{1\ 0\ l} & \cdots & \alpha_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \cdots & \alpha_{m\ (n-1)\ l} \end{bmatrix} \begin{bmatrix} q_l(x_{l-1}) \\ \vdots \\ q_l^{(n-1)}(x_{l-1}) \end{bmatrix} + \begin{bmatrix} \beta_{1\ 0\ l} & \cdots & \beta_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots \\ \beta_{m\ 0\ l} & \cdots & \beta_{m\ (n-1)\ l} \end{bmatrix} \begin{bmatrix} q_l(x_l) \\ \vdots \\ q_l^{(n-1)}(x_l) \end{bmatrix} \\ &= \sum_{l=1}^{k} M_l \vec{q_l}(x_{l-1}) + N_l \vec{q_l}(x_l), \end{aligned} \tag{†}$$

where M_l, N_l are $m \times n$ matrices. In addition, letting

$$[M_l : N_l] = \begin{bmatrix} \alpha_{1\ 0\ l} & \cdots & \alpha_{1\ (n-1)\ l} & \beta_{1\ 0\ l} & \cdots & \beta_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \cdots & \alpha_{m\ (n-1)\ l} & \beta_{m\ 0\ l} & \cdots & \beta_{m\ (n-1)\ l} \end{bmatrix},$$

we can write

$$Uq = \sum_{l=1}^{k} \begin{bmatrix} M_l : N_l \end{bmatrix} \begin{bmatrix} \vec{q_l}(x_{l-1}) \\ \vec{q_l}(x_l) \end{bmatrix} = \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \end{bmatrix} \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}. \tag{$\star$}$$

Thus we have found two compact ways to write the multipoint boundary forms, namely (†) and ($\star$). Next, we extend the notion of a complementary boundary form.

Definition 1. If $U = (U_1, \dots, U_m)$ is any multipoint boundary form with $\text{rank}(U) = m$, and $U_c = (U_{m+1}, \dots, U_{2nk})$ is a multipoint boundary form with $\text{rank}(U_c) = 2nk - m$ such that $\text{rank}(U_1, \dots, U_{2nk}) = 2nk$, then U and U_c are **complementary multipoint boundary forms**.

Note that extending $U_1, \dots, U_m$ to $U_1, \dots, U_{2nk}$ is equivalent to embedding the matrices M_l, N_l in a $2nk \times 2nk$ non-singular matrix, i.e. we can write

$$\begin{aligned}
\begin{bmatrix} Uq \\ U_c q \end{bmatrix} &= \sum_{l=1}^{k} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{q_l}(x_{l-1}) \\ \vec{q_l}(x_l) \end{bmatrix} \\
&= \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \end{bmatrix} + \begin{bmatrix} M_2 & N_2 \\ \overline{M}_2 & \overline{N}_2 \end{bmatrix} \begin{bmatrix} \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \end{bmatrix} + \dots + \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \begin{bmatrix} \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} M_1 & N_1 & M_2 & N_2 & \dots & M_k & N_k \\ \overline{M}_1 & \overline{N}_1 & \overline{M}_2 & \overline{N}_2 & \dots & \overline{M}_k & \overline{N}_k \end{bmatrix}}_{2nk \times 2nk} \underbrace{\begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}}_{2nk \times 1} \\
&=: H \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}.
\end{aligned} \tag{3}$$

where $\text{rank}(H) = 2nk$ and $\overline{M}_l, \overline{N}_l$ are $(2nk - m) \times n$ matrices. Just like the boundary form formula proven by Linda, the multipoint boundary form formula is motivated by the desire to express Green's formula as a combination of boundary forms U and U_c. Namely, we have:

Theorem 2 (Multipoint Boundary Form Formula)**.** *Given any boundary form U of rank m, and any complementary form U_c, there exist unique boundary forms U_c^+, U^+ of rank m and $2nk - m$, respectively, such that*

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g. \tag{4}$$

We will use the following proposition from Linda's capstone [3] in the proof of Theorem 2:

Proposition 3 (Prop. 2.12 in Linda's capstone)**.** *Let $\mathcal{S}$ be the semibilinear form associated with a nonsingular matrix S. Suppose $\vec{f} := Ff$ where F is a nonsingular matrix. Then, there exists a unique nonsingular matrix G such that if $\vec{g} = Gg$, then $\mathcal{S}(f, g) = \vec{f} \cdot \vec{g}$ for all f, g.*

Proof of Theorem 2. First, we have

$$\begin{bmatrix} Uf \\ U_cf \end{bmatrix} = H \begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix}.$$

From equation (2), we can write

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = \mathcal{S}\left(\begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix} \right).$$

By Proposition 3, there exists a unique $2nk \times 2nk$ nonsingular matrix J such that

$$\mathcal{S}\left(\begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix} \right) = H \begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix} \cdot J \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix}.$$

Note that if S is the matrix associated with $\mathcal{S}$, then by Proposition 3, $J = (SH^{-1})^*$, where A^* refers to the conjugate transpose of matrix A.

Let U^+, U_c^+ be such that

$$\begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} = J \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix}.$$

Now, we obtain

$$\begin{aligned} \sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = \mathcal{S}\left(\begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix} \right) &= H \begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix} \cdot J \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix} \\ &= \begin{bmatrix} Uf \\ U_cf \end{bmatrix} \cdot \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} \\ &= Uf \cdot U_c^+ g + U_cf \cdot U^+ g, \end{aligned}$$

which completes the proof. □

Theorem 2 allows us to define an adjoint multipoint boundary form. Namely,

Definition 4. Suppose $U = (U_1, \ldots, U_m)$ is a multipoint boundary form with $\operatorname{rank}(U) = m$, along with the condition that $Uq = \vec{0}$ for functions $q \in C_\pi^{n-1}[a, b]$. If U^+ is any boundary form with $\operatorname{rank}(U^+) = 2nk - m$, determined as in Theorem 2, then the equation

$$U^+ q = \vec{0}$$

is an **adjoint multipoint boundary form** to $Uq = \vec{0}$.

In turn, the above lets us define the adjoint multipoint problem:

Definition 5. Suppose $U = (U_1, \ldots, U_m)$ is a multipoint boundary form with $\mathrm{rank}(U) = m$. Then, the problem of solving

$$\pi_m : Lq = 0, \qquad Uq = \vec{0},$$

is called a homogeneous multipoint boundary value problem of rank m. The problem of solving

$$\pi^+_{2nk-m} : L^+q = 0, \qquad U^+q = \vec{0},$$

is an **adjoint multipoint boundary value problem** to π_m.

The preceding construction allows us to state the following:

Proposition 6. *Let $f, g \in C^{n-1}_\pi[a,b]$ with $Uf = \vec{0}$ and $U^+g = \vec{0}$. Then, $\langle Lf, g\rangle = \langle f, L^+g\rangle$.*

Kanıt. We apply Green's formula and multipoint boundary form formula:

$$\langle Lf, g\rangle - \langle f, L^+g\rangle = \sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g = \vec{0} \cdot U_c^+ g + U_c f \cdot \vec{0} = 0. \quad \square$$

1.4 Checking adjointness

Finally, we extend Theorem 2.19 on Linda's Capstone [3].

Theorem 7. *The boundary condition $U^+g = \vec{0}$ is adjoint to $Uf = \vec{0}$ if and only if*

$$\sum_{l=1}^{k} M_l F^{-1}(x_{l-1}) P_l = \sum_{l=1}^{k} N_l F^{-1}(x_l) Q_l,$$

where $F(t)$ is the $n \times n$ matrix as given in Green's formula subsection.

Recall that just how U is associated with a collection of $m \times n$ matrices M_l, N_l, such that

$$Uf = \sum_{l=1}^{k} M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l), \qquad \mathrm{rank}\left[M_1 : N_1 : \ \ldots \ : M_k : N_k\right] = m, \tag{5}$$

so is U^+ associated with $n \times (2nk - m)$ matrices P_l, Q_l, for $l = 1, \ldots, k$, such that

$$U^+g = \sum_{l=1}^{k} P_l^* \vec{g}_l(x_{l-1}) + Q_l^* \vec{g}_l(x_l), \qquad \mathrm{rank}\left[P_1^* : Q_1^* : \ \ldots \ : P_k^* : Q_k^*\right] = 2nk - m. \tag{6}$$

Proof of Theorem 7. Suppose that $U^+f = \vec{0}$ is adjoint to $Uf = \vec{0}$. By definition of adjoint multipoint boundary condition, U^+ is determined as in Theorem 2. Thus, in determining U^+, there exist multipoint boundary forms U_c, U_c^+ of rank $2nk - m$ and m respectively, such that the multipoint boundary form formula (4) holds. As such, let matrices $\overline{M}_l, \overline{N}_l, \overline{P}_l, \overline{Q}_l$ be such that

$$U_c f = \sum_{l=1}^{k} \overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l), \qquad \mathrm{rank}\left[\overline{M}_1 : \overline{N}_1 : \ \ldots \ : \overline{M}_k : \overline{N}_k\right] = 2nk - m \tag{7}$$

$$U_c^+ g = \sum_{l=1}^{k} \overline{P}_l^* \vec{g}_l(x_{l-1}) + \overline{Q}_l^* \vec{g}_l(x_l), \qquad \mathrm{rank}\left[\overline{P}_1^* : \overline{Q}_1^* : \ \ldots \ : \overline{P}_k^* : \overline{Q}_k^*\right] = m \tag{8}$$

First, note that in the context of semibilinear form, we have $\mathcal{S}(f,g) = Sf \cdot g = f \cdot S^*g$, as given in Proposition 2.11 of Linda's capstone [3, p.18]. We use this to rewrite the multipoint boundary form formula (4) as follows:

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g$$

$$\begin{aligned}
&= \left(\sum_{l=1}^{k} M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l)\right) \cdot \left(\sum_{i=1}^{k} (\overline{P}_i)^* \vec{g}_i(x_{i-1}) + (\overline{Q}_i)^* \vec{g}_i(x_i)\right) \\
&+ \left(\sum_{l=1}^{k} \overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l)\right) \cdot \left(\sum_{i=1}^{k} P_i^* \vec{g}_i(x_{i-1}) + Q_i^* \vec{g}_i(x_i)\right) \qquad \text{(by equations (5), (6), (7), (8))} \\
&= \sum_{l=1}^{k} \sum_{i=1}^{k} \Bigg(\left(M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l)\right) \cdot \left(\overline{P}_i^* \vec{g}_i(x_{i-1}) + \overline{Q}_i^* \vec{g}_i(x_i)\right) \\
&+ \left(\overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l)\right) \cdot \left(P_i^* \vec{g}_i(x_{i-1}) + Q_l^* \vec{g}_i(x_i)\right) \Bigg),
\end{aligned}$$

where taking out the sum upfront follows due to distributivity and associativity of inner product. Moreover, using additivity of inner product and that $Sf \cdot g = f \cdot S^* g$, we write the above as

$$\begin{aligned}
\sum_{l=1}^{k} \sum_{i=1}^{k} &(\overline{Q}_i N_l + Q_i \overline{N}_l) \vec{f}_l(x_l) \cdot \vec{g}_i(x_i) + (\overline{P}_i N_l + P_i \overline{N}_l) \vec{f}_l(x_l) \cdot \vec{g}_i(x_{i-1}) \\
&+ (\overline{Q}_i M_l + Q_i \overline{M}_l) \vec{f}_l(x_{l-1}) \cdot \vec{g}_i(x_i) + (\overline{P}_i M_l + P_i \overline{M}_l) \vec{f}_l(x_{l-1}) \cdot \vec{g}_i(x_{i-1}).
\end{aligned} \tag{9}$$

From Green's formula (1), we have

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k} F(x_l) \vec{f}_l(x_l) \cdot \vec{g}_l(x_l) - F(x_{l-1}) \vec{f}_l(x_{l-1}) \cdot \vec{g}_l(x_{l-1}). \tag{10}$$

Note that equations (9) and (10) must be equal, and so, comparison of coefficients of inner product reveals that

$$\overline{Q}_i N_l + Q_i \overline{N}_l = \begin{cases} F(x_l) & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}; \qquad \overline{P}_i M_l + P_i \overline{M}_l = \begin{cases} -F(x_{l-1}) & \text{if } i = l \\ 0 & \text{otherwise} \end{cases};$$

$$\overline{P}_i N_l + P_i \overline{N}_l = 0 \quad \forall i; \qquad \overline{Q}_i M_l + Q_i \overline{M}_l = 0 \quad \forall i.$$

Thus, we have

$$\begin{bmatrix}
-F(x_0) & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & F(x_1) & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & -F(x_1) & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & F(x_{k-1}) & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & -F(x_{k-1}) & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & F(x_k)
\end{bmatrix}$$

$$= \begin{bmatrix}
\overline{P}_1 M_1 + P_1 \overline{M}_1 & 0 & \dots & 0 & 0 \\
0 & \overline{Q}_1 N_1 + Q_1 \overline{N}_1 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & \overline{P}_k M_k + P_k \overline{M}_k & 0 \\
0 & 0 & \dots & 0 & \overline{Q}_k N_k + Q_k \overline{N}_k
\end{bmatrix}.$$

Since the boundary matrix F is nonsingular on $[a, b]$, F is invertible, and so the block diagonal matrix on LHS must also be invertible. Premultiplying on both sides by the inverse of LHS block diagonal matrix yields

$$E_{2nk \times 2nk} = \begin{bmatrix}
-F^{-1}(x_0)(\overline{P}_1 M_1 + P_1 \overline{M}_1) & 0 & \dots & 0 \\
0 & F^{-1}(x_1)(\overline{Q}_1 N_1 + Q_1 \overline{N}_1) & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & F^{-1}(x_k)(\overline{Q}_k N_k + Q_k \overline{N}_k)
\end{bmatrix}$$

$$= \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1M_1 - F^{-1}(x_0)P_1\overline{M}_1 & 0 & \dots & 0 \\ 0 & F^{-1}(x_1)\overline{Q}_1N_1 + F^{-1}(x_1)Q_1\overline{N}_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F^{-1}(x_k)\overline{Q}_kN_k + F^{-1}(x_k)Q_k\overline{N}_k \end{bmatrix}$$

$$= \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix} \begin{bmatrix} M_1 & N_1 & \dots & M_k & N_k \\ \overline{M_1} & \overline{N_1} & \dots & \overline{M_k} & \overline{N_k} \end{bmatrix}, \qquad (*)$$

where $E_{j\times j}$ is the identity matrix of dimension j. Since the two matrices in $(*)$ are full rank, they are inverse to each other, and so we have

$$\begin{bmatrix} E_{m\times m} & 0_{m\times(2nk-m)} \\ 0_{(2nk-m)\times m} & E_{(2nk-m)\times(2nk-m)} \end{bmatrix} = \begin{bmatrix} M_1 & N_1 & \dots & M_k & N_k \\ \overline{M_1} & \overline{N_1} & \dots & \overline{M_k} & \overline{N_k} \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix},$$

which implies that

$$-M_1F^{-1}(x_0)P_1 + N_1F^{-1}(x_1)Q_1 + \dots - M_kF^{-1}(x_{k-1})P_k + N_kF^{-1}(x_k)Q_k = 0_{m\times(2nk-m)}$$
$$\implies \sum_{l=1}^{k} M_lF^{-1}(x_{l-1})P_l = \sum_{l=1}^{k} N_lF^{-1}(x_l)Q_l.$$

Now, we prove the "if" direction. Let $\mathcal{U}^+$ be a multipoint boundary form of rank $2nk-m$ such that

$$\mathcal{U}^+g = \sum_{l=1}^{k} \mathcal{P}_l^*\vec{g}_l(x_{l-1}) + \mathcal{Q}_l^*\vec{g}_l(x_l),$$

for an appropriate collection of matrices $\mathcal{P}_l^*, \mathcal{Q}_l^*$, with

$$\operatorname{rank}\left[\mathcal{P}_1^* : \mathcal{Q}_1^* : \ \dots \ : \mathcal{P}_k^* : \mathcal{Q}_k^*\right] = 2nk - m$$

Suppose that

$$\sum_{l=1}^{k} M_lF^{-1}(x_{l-1})\mathcal{P}_l = \sum_{l=1}^{k} N_lF^{-1}(x_l)\mathcal{Q}_l$$

holds. Now, let $\mathbf{u}$ be a $2nk\times 1$ vector. Then, there exist $2nk-m$ linearly independent solutions of the system

$$\left[M_1 : N_1 : \ \dots \ : M_k : N_k\right]_{m\times 2nk}\mathbf{u} = \vec{0}.$$

By assumption, we have

$$\sum_{l=1}^{k} -M_lF(x_{l-1})^{-1}\mathcal{P}_l + N_lF(x_l)^{-1}\mathcal{Q}_l = 0_{m\times(2nk-m)},$$

so that

$$\left[M_1 : N_1 : \ \dots \ : M_k : N_k\right]_{m\times 2nk} \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix}_{2nk\times(2nk-m)} = 0_{m\times(2nk-m)}. \qquad (11)$$

This means that the $2nk - m$ columns of the matrix

$$\mathcal{H} := \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix}$$

form the solution space of the system (11). Since rank $\left[\mathcal{P}_1^* : \mathcal{Q}_1^* : \ \dots \ : \mathcal{P}_k^* : \mathcal{Q}_k^*\right] = 2nk - m$,

$$\operatorname{rank} \begin{bmatrix} \mathcal{P}_1 \\ \mathcal{Q}_1 \\ \vdots \\ \mathcal{P}_k \\ \mathcal{Q}_k \end{bmatrix} = 2nk - m.$$

Since $F(x_{l-1}), F(x_l)$ are non-singular, $\operatorname{rank}(\mathcal{H}) = 2nk - m$.

Now, if $U^+ g = \sum_{l=1}^k P_l^* \vec{g}_l(x_{l-1}) + Q_l^* \vec{g}_l(x_l) = \vec{0}$ is a multipoint boundary condition adjoint to $Uf = \vec{0}$, then by multipoint boundary form formula we have that

$$\begin{aligned} \begin{bmatrix} Uf \\ U_c f \end{bmatrix} \cdot \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} &= \sum_{l=1}^k \sum_{i=1}^k \begin{bmatrix} M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l) \\ \overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \overline{P}_i^* \vec{g}_i(x_{i-1}) + \overline{Q}_i^* \vec{g}_i(x_i) \\ P_i^* \vec{g}_i(x_{i-1}) + Q_i^* \vec{g}_i(x_i) \end{bmatrix} \\ &= \sum_{l=1}^k \sum_{i=1}^k \left(\begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \right) \cdot \left(\begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix}^* \begin{bmatrix} \vec{g}_i(x_{i-1}) \\ \vec{g}_i(x_i) \end{bmatrix} \right) \\ &= \sum_{l=1}^k \sum_{i=1}^k \begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_i(x_{i-1}) \\ \vec{g}_i(x_i) \end{bmatrix}. \end{aligned} \tag{12}$$

In addition, by Green's formula (1), we have

$$\begin{bmatrix} Uf \\ U_c f \end{bmatrix} \cdot \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} = \sum_{l=1}^k \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_l(x_{l-1}) \\ \vec{g}_l(x_l) \end{bmatrix}. \tag{13}$$

Since equations (12) and (13) are equal, comparison of coefficients shows that we have

$$\begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} = \begin{cases} \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} & \text{if } i = l, \\ 0_{2n\times 2n} & \text{otherwise.} \end{cases}$$

Using the above relation, we obtain the equality

$$\begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}. \tag{14}$$

Since the matrix on LHS of (14) is invertible, we can premultiply both sides by this inverse to obtain

$$E_{2nk\times 2nk} = \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix} \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}$$

By Lemma 8:

$$
\begin{aligned}
&= \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix}^{-1} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix}^{-1} \end{bmatrix} \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix}^{-1} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k)^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix}}_{\Xi}. \qquad (15)
\end{aligned}
$$

Note that the two matrices in (15) are square, and that the matrix Ξ is full-rank. So, the matrix Λ must be the inverse of Ξ. In other words, the following holds:

$$
\begin{bmatrix} E_{m\times m} & 0_{m\times(2nk-m)} \\ 0_{(2nk-m)\times m} & E_{(2nk-m)\times(2nk-m)} \end{bmatrix} = \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix}.
$$

Thus, we have

$$
\begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix} = 0_{m\times(2nk-m)}.
$$

Now, observe that

$$
H := \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix}_{2nk\times(2nk-m)}
$$

has rank $2nk-m$. Thus, columns H also form the solution space of the system (11), just like $\mathcal{H}$ does. But this suggests that $\mathcal{H}$ and H are the same up to a linear transformation, i.e. there exists a non-singular matrix A of size $(2nk-m)\times(2nk-m)$ such that

$$
\mathcal{H} = \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix} = HA = \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix} A = \begin{bmatrix} -F^{-1}(x_0)P_1A \\ F^{-1}(x_1)Q_1A \\ \vdots \\ -F^{-1}(x_{k-1})P_kA \\ F^{-1}(x_k)Q_kA \end{bmatrix},
$$

and so $P_l A = \mathcal{P}_l$ and $Q_l A = \mathcal{Q}_l$ for all $l = 1, \dots, k$. Therefore,

$$\mathcal{U}^+ g = \sum_{l=1}^{k} \mathcal{P}_l^* \vec{g}_l(x_{l-1}) + \mathcal{Q}_l^* \vec{g}_l(x_l) = \sum_{l=1}^{k} A^* P_l^* \vec{g}_l(x_{l-1}) + A^* Q_l^* \vec{g}_l(x_l) = A^* U^+ g.$$

Observe that $U^+ g = \vec{0}$ implies $\mathcal{U}^+ g = \vec{0}$. Since A^* is nonsingular, it follows that $U^+ g = \vec{0}$ if and only if $\mathcal{U}^+ g = \vec{0}$. Since $U^+ g = \vec{0}$ is adjoint to $Uf = \vec{0}$, $\mathcal{U}^+ g = \vec{0}$ is adjoint to $Uf = \vec{0}$. This completes the proof. □

Lemma 8. *For the relevant matrices $P_l, Q_l, \overline{P}_l, \overline{Q}_l, M_l, N_l, \overline{M}_l, \overline{N}_l$, we have*

$$\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk \times 2nk}$$

$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}_{2nk \times 2nk} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}_{2nk \times 2nk}.$$

Kanıt. First, observe that we can write

$$\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix}_{2nk \times 2nk^2} \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk^2 \times 2nk}. \tag{16}$$

Now, let $\mathcal{V}$ and $\mathcal{W}$ be matrices given by:

$$\mathcal{V}_{2nk^2 \times 2nk} = \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1}; \tag{17}$$

$$\mathcal{W}_{2nk \times 2nk^2} = \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix}. \tag{18}$$

Observe that

$$\mathcal{W}\mathcal{V} = \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1} \tag{19}$$

Substitute (**??**):

$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1} \tag{20}$$

Recall (14):

$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1} \tag{21}$$

Recall (2), (3), and the explicit definition for J as given in the proof of Theorem 2:

$$= (J^*)^{-1} S H^{-1} = (J^*)^{-1} J^* = E_{2nk \times 2nk}. \tag{22}$$

Thus, (16) can be rewritten as:

$$\begin{aligned} &\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix}_{2nk \times 2nk^2} \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk^2 \times 2nk} \\ &= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} \mathcal{W}_{2nk \times 2nk^2} \mathcal{V}_{2nk^2 \times 2nk} \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix} \\ &= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} E_{2nk \times 2nk} \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix} = \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : \ \dots \ : \overline{M}_k : \overline{N}_k \end{bmatrix}, \end{aligned}$$

which completes the proof. □

2 Extension of the work by Pelloni & Smith

In this section, we extend the system that Pelloni & Smith derived in [2].

2.1 Formulation of the problem

Let $m, n \in \mathbb{N}$ be independent, let

$$\pi = \{0 = \eta_0 < \eta_1 < \ldots < \eta_m = 1\}$$

be a partition of a closed interval $[0, 1]$, and let

$$\varsigma^r_{kj}, \mathfrak{d}^r_{kj} \in \mathbb{C}, \quad \text{for } j \in \{0, 1, \ldots, mn-1\}, \quad k \in \{0, 1, \ldots, n-1\}, \quad r \in 0, \ldots, m.$$

Further, let

$$C^{n-1}_\pi[a, b] = \Big\{ f : [a, b] \to \mathbb{C} \text{ s.t. } \forall r \in \{1, 2, \ldots, m\}, \\ f_r := f\big|_{(\eta_{r-1},\eta_r)} \text{ admits an extension } g_r \text{ to } [\eta_{r-1}, \eta_r] \text{ s.t. } g_r \in C^{n-1}[\eta_{r-1}, \eta_r] \Big\}.$$

be the relevant function space. Consider the following *initial-multipoint value problem*:

$$[\partial_t + a(-i\partial_x)^n]q(x, t) = 0 \qquad (x, t) \in \mathbb{R} \times (0, T), \tag{23a}$$

$$q(x, 0) = q_0(x) \qquad x \in \mathbb{R}, \tag{23b}$$

$$\sum_{r=1}^{m}\sum_{k=0}^{n-1} \varsigma^r_{kj}\partial_x^{(k)} q(\eta_{r-1}, t) + \mathfrak{d}^r_{kj}\partial_x^{(k)} q(\eta_r, t) = v_j(t) \qquad t \in [0, T], \quad j \in \{0, 1, \ldots, mn-1\}, \tag{23c}$$

where $q(x, \cdot) \in C^{n-1}_\pi[0, 1], q_0 \in C^n[0, 1], v_j \in C^\infty[0, T]$, with $T > 0$ a fixed constant.

2.2 Global relations

First, we derive the relevant global relation. Fix $r \in \{1, 2, \ldots, m\}$, let ϕ_r denote the restriction of function ϕ to interval $[\eta_{r-1}, \eta_r]$, and observe the action of Fourier transform on the derivative operator:

$$\begin{aligned}\widehat{(-i\partial_x)^n\phi_r}(\lambda) &= \int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x}(-i\partial_x)^n\phi_r \,\mathrm{d}x \\ &= e^{-i\lambda x}\sum_{j=1}^{n-1}(-i)^{n+1-j}\lambda^{j-1}\phi_r^{(n-j)}(x)\big|^{x=\eta_r}_{x=\eta_{r-1}} + \lambda^n\int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x}\phi_r \,\mathrm{d}x \qquad (24) \\ &= e^{-i\lambda x}\sum_{k=0}^{n-1}(-i)^{k+1}\lambda^{n-k-1}\phi_r^{(k)}(x)\big|^{x=\eta_r}_{x=\eta_{r-1}} + \lambda^n\widehat{\phi_r}(\lambda), \qquad (25)\end{aligned}$$

where we performed integration by parts in (24) and relabeled indices as $k = n - j$ in (25). Applying the Fourier transform on the PDE (23a) on $[\eta_{r-1}, \eta_r]$ yields

$$\begin{aligned}0 &= \widehat{[\partial_t + a(-i\partial_x)^n]q_r}(\lambda, t) \\ &= [\partial_t + \lambda^n]\widehat{q_r}(\lambda, t) + a\sum_{k=0}^{n-1} e^{-i\lambda x}(-i)^{k+1}\lambda^{n-k-1}\partial_x^{(k)} q_r(x, t)\big|^{x=\eta_r}_{x=\eta_{r-1}}. \qquad (26)\end{aligned}$$

Multiplying (26) by $e^{a\lambda^n t}$ and integrating the result in time, we obtain

$$0 = e^{a\lambda^n t}\widehat{q_r}(\lambda; t) - \widehat{q_r}(\lambda; 0) + \sum_{k=0}^{n-1} a(-i)^{k+1}\lambda^{n-k-1}e^{-i\lambda x}\int_0^t e^{a\lambda^n s}\partial_x^k q_r(x, s)\big|^{x=\eta_r}_{x=\eta_{r-1}} \,\mathrm{d}s,$$

so that the we obtain the expression for the *global relation*

$$\begin{aligned}&\widehat{q_r}(\lambda;0) - e^{a\lambda^n t}\widehat{q_r}(\lambda;t)\\ &\qquad = \sum_{k=0}^{n-1} a(-i)^{k+1}\lambda^{n-k-1}\left(e^{-i\lambda\eta_r}\int_0^t e^{a\lambda^n s}\partial_x^k q_r(\eta_r,s)\,\mathrm{d}s - e^{-i\lambda\eta_{r-1}}\int_0^t e^{a\lambda^n s}\partial_x^k q_r(\eta_{r-1},s)\,\mathrm{d}s\right),\end{aligned} \tag{27}$$

valid for $t\in[0,T], \lambda\in\mathbb{C}, x\in[\eta_{r-1},\eta_r]$. Evaluating (27) at $\tau\in[0,T]$, we obtain the global relation at $\tau:$

$$\begin{aligned}&\widehat{q_r}(\lambda;0) - e^{a\lambda^n \tau}\widehat{q_r}(\lambda;\tau)\\ &\qquad = \sum_{k=0}^{n-1} a(-i)^{k+1}\lambda^{n-k-1}\left(e^{-i\lambda\eta_r}\int_0^\tau e^{a\lambda^n s}\partial_x^k q_r(\eta_r,s)\,\mathrm{d}s - e^{-i\lambda\eta_{r-1}}\int_0^\tau e^{a\lambda^n s}\partial_x^k q_r(\eta_{r-1},s)\,\mathrm{d}s\right).\end{aligned} \tag{28}$$

Now, we adopt the following notation: for $\lambda\in\mathbb{C}$ and $k\in\{0,\dots,n-1\}$, denote a primitive n^{th} root of unity

$$\alpha = e^{2\pi i/n},$$

an exponential function

$$E_r(\lambda) = e^{-i\lambda\eta_r},$$

coefficients

$$c_k(\lambda) = ia\lambda^{n-k-1}(-i)^k,$$

a time transform of the value $\partial_x^k q$ at $x=\eta_r$ as

$$g_k^r(\lambda) = g_k^r(\lambda,\tau) = c_k(\lambda)\int_0^\tau e^{a\lambda^n s}\partial_x^k q_r(\eta_r,s)\,\mathrm{d}s, \qquad r\in\{1,\dots,m\},$$

a time transform of the value $\partial_x^k q$ at $x=\eta_{r-1}$ as

$$f_k^r(\lambda) = f_k^r(\lambda,\tau) = c_k(\lambda)\int_0^\tau e^{a\lambda^n s}\partial_x^k q_r(\eta_{r-1},s)\,\mathrm{d}s, \qquad r\in\{1,\dots,m\}$$

the Fourier transform of the initial datum, restricted to (η_{r-1},η_r) as

$$\widehat{q_0^r}(\lambda) = \int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x}q_0(x)\,\mathrm{d}x,$$

and the Fourier transform of the solution at time τ, restricted to (η_{r-1},η_r) as

$$\widehat{q_\tau^r}(\lambda) = \int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x}q(x,\tau)\,\mathrm{d}x.$$

Using this notation, we can simplify the global relation (28) to

$$\widehat{q_0^r}(\lambda) - e^{a\lambda^n\tau}\widehat{q_\tau^r}(\lambda) = \sum_{k=0}^{n-1}\left[E_{r-1}(\lambda)f_k^r(\lambda) - E_r(\lambda)g_k^r(\lambda)\right]. \tag{29}$$

Now, we create the system of global relations. Consider the global relation in each of the rectangles $(x,t)\in[\eta_{r-1},\eta_r]\times[0,\tau], r\in\{1,\dots,m\}$, and $\lambda\in\mathbb{C}$. This yields a set of m global relations. Evaluating each relation at $\alpha, \alpha\lambda, \dots, \alpha^{n-1}\lambda$, and using the fact that $f_k^r(\alpha^p\lambda) = \alpha^{(n-1-k)p} f_k^r(\lambda), g_k^r(\alpha^p\lambda) = \alpha^{(n-1-k)p} g_k^r(\lambda)$ we obtain the following system of mn equations

$$\sum_{k=0}^{n-1} \alpha^{(n-1-k)p} \left[E_{r-1}(\alpha^p\lambda) f_k^r(\lambda) - E_r(\alpha^p\lambda) g_k^r(\lambda)\right] = \widehat{q_0^r}(\alpha^p\lambda) - e^{a\lambda^n\tau}\widehat{q_\tau^r}(\alpha^p\lambda), \qquad p \in \{0,1,\dots,n-1\}. \tag{30}$$

Now, we would like to write equations in (30) in a system form. Note

$$\begin{aligned}
&\sum_{k=0}^{n-1} \alpha^{(n-1-k)p} \left[E_{r-1}(\alpha^p\lambda) f_k^r(\lambda) - E_r(\alpha^p\lambda) g_k^r(\lambda)\right] \\
&\quad = \sum_{k=0}^{n-1} \alpha^{(n-1-k)p} E_{r-1}(\alpha^p\lambda) f_k^r(\lambda) - \sum_{k=0}^{n-1} \alpha^{(n-1-k)p} E_r(\alpha^p\lambda) g_k^r(\lambda) \\
&\quad = \left[\alpha^{(n-1)p} E_{r-1}(\alpha^p\lambda) f_0^r(\lambda) + \alpha^{(n-2)p} E_{r-1}(\alpha^p\lambda) f_1^r(\lambda) + \dots + \alpha^0 E_{r-1}(\alpha^p\lambda) f_{n-1}^r(\lambda)\right] \\
&\quad - \left[\alpha^{(n-1)p} E_r(\alpha^p\lambda) g_0^r(\lambda) + \alpha^{(n-2)p} E_r(\alpha^p\lambda) g_1^r(\lambda) + \dots + \alpha^0 E_r(\alpha^p\lambda) g_{n-1}^r(\lambda)\right] \\
&\quad = \begin{bmatrix} \alpha^{(n-1)p} E_{r-1}(\alpha^p\lambda) & \dots & \alpha^0 E_{r-1}(\alpha^p\lambda) & \alpha^{(n-1)p} E_r(\alpha^p\lambda) & \dots & \alpha^0 E_r(\alpha^p\lambda) \end{bmatrix} \begin{bmatrix} f_0^r(\lambda) \\ \vdots \\ f_{n-1}^r(\lambda) \\ g_0^r(\lambda) \\ \vdots \\ g_{n-1}^r(\lambda) \end{bmatrix}.
\end{aligned} \tag{31}$$

Evaluating (31) at $p = 0, 1, \dots, n-1$ yields the following system:

$$\begin{aligned}
&\begin{bmatrix}
\alpha^{(n-1)0} E_{r-1}(\alpha^0\lambda) & \dots & \alpha^0 E_{r-1}(\alpha^0\lambda) & \alpha^{(n-1)0} E_r(\alpha^p\lambda) & \dots & \alpha^0 E_r(\alpha^0\lambda) \\
\alpha^{(n-1)} E_{r-1}(\alpha\lambda) & \dots & \alpha^0 E_{r-1}(\alpha\lambda) & \alpha^{(n-1)} E_r(\alpha\lambda) & \dots & \alpha^0 E_r(\alpha\lambda) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha^{(n-1)(n-1)} E_{r-1}(\alpha^{(n-1)}\lambda) & \dots & \alpha^0 E_{r-1}(\alpha^{(n-1)}\lambda) & \alpha^{(n-1)(n-1)} E_r(\alpha^p\lambda) & \dots & \alpha^0 E_r(\alpha^{(n-1)}\lambda)
\end{bmatrix}
\begin{bmatrix} f_0^r(\lambda) \\ \vdots \\ f_{n-1}^r(\lambda) \\ g_0^r(\lambda) \\ \vdots \\ g_{n-1}^r(\lambda) \end{bmatrix} \\
&\qquad = \begin{bmatrix} \widehat{q_0^r}(\alpha^0\lambda) \\ \vdots \\ \widehat{q_0^r}(\alpha^{(n-1)}\lambda) \end{bmatrix} - e^{a\lambda^n\tau} \begin{bmatrix} \widehat{q_\tau^r}(\alpha^0\lambda) \\ \vdots \\ \widehat{q_\tau^r}(\alpha^{(n-1)}\lambda) \end{bmatrix}.
\end{aligned} \tag{32}$$

For notational convenience, let

$$\vec{f^r}(\lambda) = \begin{bmatrix} f_0^r(\lambda) \\ \vdots \\ f_{n-1}^r(\lambda) \end{bmatrix}, \quad \vec{g^r}(\lambda) = \begin{bmatrix} g_0^r(\lambda) \\ \vdots \\ g_{n-1}^r(\lambda) \end{bmatrix} \quad \vec{\widehat{q_0^r}}(\lambda) = \begin{bmatrix} \widehat{q_0^r}(\alpha^0\lambda) \\ \vdots \\ \widehat{q_0^r}(\alpha^{(n-1)}\lambda) \end{bmatrix}, \quad \vec{\widehat{q_\tau^r}}(\lambda) = \begin{bmatrix} \widehat{q_\tau^r}(\alpha^0\lambda) \\ \vdots \\ \widehat{q_\tau^r}(\alpha^{(n-1)}\lambda) \end{bmatrix},$$

$$e_r = \begin{bmatrix}
E_r(\lambda) & E_r(\alpha\lambda)\alpha^{(n-1)} & \dots & E_r(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-1)} \\
E_r(\lambda) & E_r(\alpha\lambda)\alpha^{(n-2)} & \dots & E_r(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-2)} \\
\vdots & \vdots & \ddots & \vdots \\
E_r(\lambda) & E_r(\alpha\lambda) & \dots & E_r(\alpha^{(n-1)}\lambda)
\end{bmatrix}$$

for $r \in \{0, 1, \dots, m\}$, and e_r are $n \times n$ matrices. Then, we can write (32) in a more compact form:

$$\begin{bmatrix} e_{r-1}^T : -e_r^T \end{bmatrix} \begin{bmatrix} \vec{f^r}(\lambda) \\ \vec{g^r}(\lambda) \end{bmatrix} = \vec{\widehat{q_0^r}}(\lambda) - e^{(a\lambda^n\tau)} \vec{\widehat{q_\tau^r}}(\lambda), \qquad r \in \{1, \dots, m\}, \tag{33}$$

where

$$\left[e_{r-1}^T : -e_r^T\right] = \begin{bmatrix} (e_{r-1}^T)_{1\ 1} & \dots & (e_{r-1}^T)_{1\ n} & -(e_r^T)_{1\ 1} & \dots & -(e_r^T)_{1\ n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ (e_{r-1}^T)_{n\ 1} & \dots & (e_{r-1}^T)_{n\ n} & -(e_r^T)_{n\ 1} & \dots & -(e_r^T)_{n\ n} \end{bmatrix}.$$

2.3 Multipoint conditions

We would also like to rewrite the multipoint conditions

$$\sum_{r=1}^{m}\sum_{k=0}^{n-1} \varsigma_{kj}^r \partial_x^k q(\eta_{r-1},t) + ꝺ_{kj}^r \partial_x^k q(\eta_r,t) = v_j(t),$$

where $t \in [0,T]$, $j \in \{0,1,\dots,mn-1\}$. Multiplying by c_k and $e^{a\lambda^n t}$, and applying the time transform at time $\tau \in [0,T]$ yields

$$\begin{aligned}\sum_{r=1}^{m}\sum_{k=0}^{n-1} \varsigma_{kj}^r \frac{(-a)}{i^n c_k(\lambda)} c_k(\lambda) \int_0^\tau e^{a\lambda^n s} \partial_x^k q(\eta_{r-1},s)\,\mathrm{d}s \\ + ꝺ_{kj}^r \frac{(-a)}{i^n c_k(\lambda)} c_k(\lambda) \int_0^\tau e^{a\lambda^n s} \partial_x^k q(\eta_r,s)\,\mathrm{d}s = \frac{(-a)}{i^n}\int_0^\tau e^{a\lambda^n s} v_j(s)\,\mathrm{d}s := h_j(\lambda),\end{aligned}$$

so that the conditions become

$$\sum_{r=1}^{m}\sum_{k=0}^{n-1} \varsigma_{kj}^r \frac{(-a)}{i^n c_k(\lambda)} f_k^r(\lambda) + ꝺ_{kj}^r \frac{(-a)}{i^n c_k(\lambda)} g_k^r(\lambda) = h_j(\lambda). \tag{34}$$

Now, expand the sum in (34) over k :

$$\begin{aligned} h_j(\lambda) &= \sum_{r=1}^{m}\sum_{k=0}^{n-1} \varsigma_{kj}^r \frac{(-a)}{i^n c_k(\lambda)} f_k^r(\lambda) + ꝺ_{kj}^r \frac{(-a)}{i^n c_k(\lambda)} g_k^r(\lambda) \\ &= \sum_{r=1}^{m} \varsigma_{0j}^r \frac{(-a)}{i^n c_0(\lambda)} f_0^r(\lambda) + \varsigma_{1j}^r \frac{(-a)}{i^n c_1(\lambda)} f_1^r(\lambda) + \dots + \varsigma_{(n-1)j}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} f_{n-1}^r(\lambda) \\ &\quad + ꝺ_{0j}^r \frac{(-a)}{i^n c_0(\lambda)} g_0^r(\lambda) + ꝺ_{1j}^r \frac{(-a)}{i^n c_1(\lambda)} g_1^r(\lambda) + \dots + ꝺ_{(n-1)j}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} g_{n-1}^r(\lambda) \\ &= \sum_{r=1}^{m} \begin{bmatrix} \varsigma_{0j}^r \frac{(-a)}{i^n c_0(\lambda)} & \dots & \varsigma_{(n-1)j}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} & ꝺ_{0j}^r \frac{(-a)}{i^n c_0(\lambda)} & \dots & ꝺ_{(n-1)j}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} \end{bmatrix} \begin{bmatrix} \vec{f^r}(\lambda) \\ \vec{g^r}(\lambda) \end{bmatrix}. \end{aligned} \tag{35}$$

Evaluating (35) at $j = 0,1,\dots,mn-1$ and combining the resultant equations, we obtain:

$$\sum_{r=1}^{m} \begin{bmatrix} \varsigma_{00}^r \frac{(-a)}{i^n c_0(\lambda)} & \dots & \varsigma_{(n-1)0}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} & ꝺ_{00}^r \frac{(-a)}{i^n c_0(\lambda)} & \dots & ꝺ_{(n-1)0}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} \\ \varsigma_{01}^r \frac{(-a)}{i^n c_0(\lambda)} & \dots & \varsigma_{(n-1)1}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} & ꝺ_{01}^r \frac{(-a)}{i^n c_0(\lambda)} & \dots & ꝺ_{(n-1)1}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \varsigma_{0(mn-1)}^r \frac{(-a)}{i^n c_0(\lambda)} & \dots & \varsigma_{(n-1)(mn-1)}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} & ꝺ_{0(mn-1)}^r \frac{(-a)}{i^n c_0(\lambda)} & \dots & ꝺ_{(n-1)(mn-1)}^r \frac{(-a)}{i^n c_{n-1}(\lambda)} \end{bmatrix} \begin{bmatrix} \vec{f^r}(\lambda) \\ \vec{g^r}(\lambda) \end{bmatrix} = \begin{bmatrix} h_0(\lambda) \\ h_1(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \end{bmatrix}. \tag{36}$$

For $k = 0,1,\dots,m-1$, define the following matrices

$$Ç_k^r = \begin{bmatrix} \varsigma_{0kn}^r \frac{1}{(i\lambda)^{n-1}} & \varsigma_{1kn}^r \frac{1}{(i\lambda)^{n-2}} & \dots & \varsigma_{(n-1)kn}^r \\ \varsigma_{0(kn+1)}^r \frac{1}{(i\lambda)^{n-1}} & \varsigma_{1(kn+1)}^r \frac{1}{(i\lambda)^{n-2}} & \dots & \varsigma_{(n-1)(kn+1)}^r \\ \vdots & \vdots & \ddots & \vdots \\ \varsigma_{0((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1}} & \varsigma_{1((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-2}} & \dots & \varsigma_{(n-1)((k+1)n-1)}^r \end{bmatrix}^T, \qquad n \times n \text{ block;}$$

$$\underset{\sim}{\mathrm{D}}_k^r = \begin{bmatrix} \underset{\sim}{d}^r_{0kn}\frac{1}{(i\lambda)^{n-1}} & \underset{\sim}{d}^r_{1kn}\frac{1}{(i\lambda)^{n-2}} & \cdots & \underset{\sim}{d}^r_{(n-1)kn} \\ \underset{\sim}{d}^r_{0(kn+1)}\frac{1}{(i\lambda)^{n-1}} & \underset{\sim}{d}^r_{1(kn+1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \underset{\sim}{d}^r_{(n-1)(kn+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \underset{\sim}{d}^r_{0((k+1)n-1)}\frac{1}{(i\lambda)^{n-1}} & \underset{\sim}{d}^r_{1((k+1)n-1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \underset{\sim}{d}^r_{(n-1)((k+1)n-1)} \end{bmatrix}^T, \qquad n \times n \text{ block.}$$

Then, we can rewrite the system in (36) as

$$\begin{aligned}
\underbrace{\begin{bmatrix} h_0(\lambda) \\ h_1(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \end{bmatrix}}_{mn \times 1} &= \sum_{r=1}^{m} \begin{bmatrix} (\underset{\sim}{\mathrm{C}}_0^r)^T & : & (\underset{\sim}{\mathrm{D}}_0^r)^T \\ (\underset{\sim}{\mathrm{C}}_1^r)^T & : & (\underset{\sim}{\mathrm{D}}_1^r)^T \\ & \vdots & \\ (\underset{\sim}{\mathrm{C}}_{m-1}^r)^T & : & (\underset{\sim}{\mathrm{C}}_{m-1}^r)^T \end{bmatrix} \begin{bmatrix} \vec{f^r}(\lambda) \\ \vec{g^r}(\lambda) \end{bmatrix} \\
&= \begin{bmatrix} (\underset{\sim}{\mathrm{C}}_0^1)^T & : & (\underset{\sim}{\mathrm{D}}_0^1)^T \\ (\underset{\sim}{\mathrm{C}}_1^1)^T & : & (\underset{\sim}{\mathrm{D}}_1^1)^T \\ & \vdots & \\ (\underset{\sim}{\mathrm{C}}_{m-1}^1)^T & : & (\underset{\sim}{\mathrm{C}}_{m-1}^1)^T \end{bmatrix} \begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \end{bmatrix} + \begin{bmatrix} (\underset{\sim}{\mathrm{C}}_0^2)^T & : & (\underset{\sim}{\mathrm{D}}_0^2)^T \\ (\underset{\sim}{\mathrm{C}}_1^2)^T & : & (\underset{\sim}{\mathrm{D}}_1^2)^T \\ & \vdots & \\ (\underset{\sim}{\mathrm{C}}_{m-1}^2)^T & : & (\underset{\sim}{\mathrm{C}}_{m-1}^2)^T \end{bmatrix} \begin{bmatrix} \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \end{bmatrix} \\
&\quad + \ \dots \ + \begin{bmatrix} (\underset{\sim}{\mathrm{C}}_0^m)^T & : & (\underset{\sim}{\mathrm{D}}_0^m)^T \\ (\underset{\sim}{\mathrm{C}}_1^m)^T & : & (\underset{\sim}{\mathrm{D}}_1^m)^T \\ & \vdots & \\ (\underset{\sim}{\mathrm{C}}_{m-1}^m)^T & : & (\underset{\sim}{\mathrm{C}}_{m-1}^m)^T \end{bmatrix} \begin{bmatrix} \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} (\underset{\sim}{\mathrm{C}}_0^1)^T & : & (\underset{\sim}{\mathrm{D}}_0^1)^T & : & (\underset{\sim}{\mathrm{C}}_0^2)^T & : & (\underset{\sim}{\mathrm{D}}_0^2)^T & : \dots : & (\underset{\sim}{\mathrm{C}}_0^m)^T & : & (\underset{\sim}{\mathrm{D}}_0^m)^T \\ (\underset{\sim}{\mathrm{C}}_1^1)^T & : & (\underset{\sim}{\mathrm{D}}_1^1)^T & : & (\underset{\sim}{\mathrm{C}}_1^2)^T & : & (\underset{\sim}{\mathrm{D}}_1^2)^T & : \dots : & (\underset{\sim}{\mathrm{C}}_1^m)^T & : & (\underset{\sim}{\mathrm{D}}_1^m)^T \\ & & & & & & \vdots & & & & \\ (\underset{\sim}{\mathrm{C}}_{m-1}^1)^T & : & (\underset{\sim}{\mathrm{D}}_{m-1}^1)^T & : & (\underset{\sim}{\mathrm{C}}_{m-1}^2)^T & : & (\underset{\sim}{\mathrm{D}}_{m-1}^2)^T & : \dots : & (\underset{\sim}{\mathrm{C}}_{m-1}^m)^T & : & (\underset{\sim}{\mathrm{D}}_{m-1}^m)^T \end{bmatrix}}_{mn \times 2mn} \underbrace{\begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \\ \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \\ \vdots \\ \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix}}_{2mn \times 1}.
\end{aligned} \tag{37}$$

The equation (37) gives a convenient way to express the multipoint conditions.

2.4 The Dirichlet-to-Neumann map in $\mathcal{B}$ form

We use the results in the previous two subsections to define a system, whose solution will aid us in finding the Dirichlet-to-Neumann map. First, recall from the Global Relations subsection, we have the equation (33), reproduced below:

$$\begin{bmatrix} e_{r-1}^T : -e_r^T \end{bmatrix} \begin{bmatrix} \vec{f^r}(\lambda) \\ \vec{g^r}(\lambda) \end{bmatrix} = \widehat{\vec{q}_0^{\,r}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q}_\tau^{\,r}}(\lambda), \qquad r \in \{1, \dots, m\}. \tag{38}$$

Evaluating the equation in (38) at $r = 1, \ldots, m$ yields

$$
\begin{aligned}
\begin{bmatrix} e_0^T : -e_1^T \end{bmatrix} \begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \end{bmatrix} &= \widehat{\vec{q}_0^{\,1}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q}_\tau^{\,1}}(\lambda) \\
\begin{bmatrix} e_1^T : -e_2^T \end{bmatrix} \begin{bmatrix} \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \end{bmatrix} &= \widehat{\vec{q}_0^{\,2}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q}_\tau^{\,2}}(\lambda) \\
&\vdots \\
\begin{bmatrix} e_{m-2}^T : -e_{m-1}^T \end{bmatrix} \begin{bmatrix} \vec{f}^{m-1}(\lambda) \\ \vec{g}^{m-1}(\lambda) \end{bmatrix} &= \widehat{\vec{q}_0^{\,m-1}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q}_\tau^{\,m-1}}(\lambda) \\
\begin{bmatrix} e_{m-1}^T : -e_m^T \end{bmatrix} \begin{bmatrix} \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda) \end{bmatrix} &= \widehat{\vec{q}_0^{\,m}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q}_\tau^{\,m}}(\lambda).
\end{aligned}
\tag{39}
$$

Combining the global relations in (39), we obtain the following system:

$$
\underbrace{\begin{bmatrix}
e_0^T : -e_1^T & 0 & \ldots & 0 & 0 \\
0 & e_1^T : -e_2^T & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & e_{m-2}^T : -e_{m-1}^T & 0 \\
0 & 0 & \ldots & 0 & e_{m-1}^T : -e_m^T
\end{bmatrix}}_{mn \times 2mn}
\underbrace{\begin{bmatrix}
\vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^{m-1}(\lambda) \\ \vec{g}^{m-1}(\lambda) \\ \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda)
\end{bmatrix}}_{2mn \times 1}
=
\underbrace{\begin{bmatrix}
\widehat{\vec{q}_0^{\,1}}(\lambda) \\ \widehat{\vec{q}_0^{\,2}}(\lambda) \\ \vdots \\ \widehat{\vec{q}_0^{\,m-1}}(\lambda) \\ \widehat{\vec{q}_0^{\,m}}(\lambda)
\end{bmatrix}}_{mn \times 1}
- e^{a\lambda^n \tau}
\underbrace{\begin{bmatrix}
\widehat{\vec{q}_\tau^{\,1}}(\lambda) \\ \widehat{\vec{q}_\tau^{\,2}}(\lambda) \\ \vdots \\ \widehat{\vec{q}_\tau^{\,m-1}}(\lambda) \\ \widehat{\vec{q}_\tau^{\,m}}(\lambda)
\end{bmatrix}}_{mn \times 1}.
\tag{40}
$$

Combining (40) and (37), we arrive at the following system

$$
\mathcal{B}
\underbrace{\begin{bmatrix}
\vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^{m-1}(\lambda) \\ \vec{g}^{m-1}(\lambda) \\ \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda)
\end{bmatrix}}_{2mn \times 1}
=
\underbrace{\begin{bmatrix}
h_0(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \\ \widehat{\vec{q}_0^{\,1}}(\lambda) \\ \widehat{\vec{q}_0^{\,2}}(\lambda) \\ \vdots \\ \widehat{\vec{q}_0^{\,m-1}}(\lambda) \\ \widehat{\vec{q}_0^{\,m}}(\lambda)
\end{bmatrix}}_{2mn \times 1}
- e^{a\lambda^n \tau}
\underbrace{\begin{bmatrix}
0 \\ \vdots \\ 0 \\ \widehat{\vec{q}_\tau^{\,1}}(\lambda) \\ \widehat{\vec{q}_\tau^{\,2}}(\lambda) \\ \vdots \\ \widehat{\vec{q}_\tau^{\,m-1}}(\lambda) \\ \widehat{\vec{q}_\tau^{\,m}}(\lambda)
\end{bmatrix}}_{2mn \times 1},
\tag{41}
$$

where

$$
\mathcal{B} =
\underbrace{\begin{bmatrix}
(\text{Ç}_0^1)^T & (\text{Ḑ}_0^1)^T & (\text{Ç}_0^2)^T & (\text{Ḑ}_0^2)^T & \ldots & (\text{Ç}_0^{m-1})^T & (\text{Ḑ}_0^{m-1})^T & (\text{Ç}_0^m)^T & (\text{Ḑ}_0^m)^T \\
(\text{Ç}_1^1)^T & (\text{Ḑ}_1^1)^T & (\text{Ç}_1^2)^T & (\text{Ḑ}_1^2)^T & \ldots & (\text{Ç}_1^{m-1})^T & (\text{Ḑ}_1^{m-1})^T & (\text{Ç}_1^m)^T & (\text{Ḑ}_1^m)^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
(\text{Ç}_{m-1}^1)^T & (\text{Ḑ}_{m-1}^1)^T & (\text{Ç}_{m-1}^2)^T & (\text{Ḑ}_{m-1}^2)^T & \ldots & (\text{Ç}_{m-1}^{m-1})^T & (\text{Ḑ}_{m-1}^{m-1})^T & (\text{Ç}_{m-1}^m)^T & (\text{Ḑ}_{m-1}^m)^T \\
e_0^T & -e_1^T & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & e_1^T & -e_2^T & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & e_{m-2}^T & -e_{m-1}^T & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & e_{m-1}^T & -e_m^T
\end{bmatrix}}_{2mn \times 2mn}
$$

is a block matrix where each block is $n \times n$. Solving this system will help obtain the Dirichlet-to-Neumann map. We shall refer to this system as the *D-to-N map in $\mathcal{B}$ form.*

2.5 The Dirichlet-to-Neumann map in $\mathcal{A}$ form

We seek to simplify the system (41). First, recall the matrices

$$\text{\c{C}}^r_k = \begin{bmatrix} \text{\c{c}}^r_{0kn}\frac{1}{(i\lambda)^{n-1}} & \text{\c{c}}^r_{1kn}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{c}}^r_{(n-1)kn} \\ \text{\c{c}}^r_{0(kn+1)}\frac{1}{(i\lambda)^{n-1}} & \text{\c{c}}^r_{1(kn+1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{c}}^r_{(n-1)(kn+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \text{\c{c}}^r_{0((k+1)n-1)}\frac{1}{(i\lambda)^{n-1}} & \text{\c{c}}^r_{1((k+1)n-1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{c}}^r_{(n-1)((k+1)n-1)} \end{bmatrix}^T, \qquad r = 1, \dots, m,$$

$$\text{\c{D}}^r_k = \begin{bmatrix} \text{\c{d}}^r_{0kn}\frac{1}{(i\lambda)^{n-1}} & \text{\c{d}}^r_{1kn}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{d}}^r_{(n-1)kn} \\ \text{\c{d}}^r_{0(kn+1)}\frac{1}{(i\lambda)^{n-1}} & \text{\c{d}}^r_{1(kn+1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{d}}^r_{(n-1)(kn+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \text{\c{d}}^r_{0((k+1)n-1)}\frac{1}{(i\lambda)^{n-1}} & \text{\c{d}}^r_{1((k+1)n-1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{d}}^r_{(n-1)((k+1)n-1)} \end{bmatrix}^T, \qquad r = 1, \dots, m,$$

$$e_r = \begin{bmatrix} E_r(\lambda) & E_r(\alpha\lambda)\alpha^{(n-1)} & \dots & E_r(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-1)} \\ E_r(\lambda) & E_r(\alpha\lambda)\alpha^{(n-2)} & \dots & E_r(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ E_r(\lambda) & E_r(\alpha\lambda) & \dots & E_r(\alpha^{(n-1)}\lambda) \end{bmatrix} \qquad r = 0, 1, \dots, m,$$

where both matrices are $n \times n$, and $k = 0, \dots, m-1$. Now, define the matrices

$$\text{\c{S}}^r_k = \frac{1}{n}\begin{bmatrix} E_{r-1}(-\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ kn}\frac{1}{(i\lambda)^{n-1-j}} & \dots & E_{r-1}(-\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ ((k+1)n-1)}\frac{1}{(i\lambda)^{n-1-j}} \\ E_{r-1}(-\alpha\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ kn}\frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} & \dots & E_{r-1}(-\alpha\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ ((k+1)n-1)}\frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} \\ \vdots & \ddots & \vdots \\ E_{r-1}(-\alpha^{n-1}\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ kn}\frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} & \dots & E_{r-1}(-\alpha^{n-1}\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ ((k+1)n-1)}\frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix};$$

$$\text{\c{T}}^r_k = \frac{1}{n}\begin{bmatrix} E_{r}(-\lambda)\sum_{j=0}^{n-1}\text{\c{d}}^r_{j\ kn}\frac{1}{(i\lambda)^{n-1-j}} & \dots & E_{r}(-\lambda)\sum_{j=0}^{n-1}\text{\c{d}}^r_{j\ ((k+1)n-1)}\frac{1}{(i\lambda)^{n-1-j}} \\ E_{r}(-\alpha\lambda)\sum_{j=0}^{n-1}\text{\c{d}}^r_{j\ kn}\frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} & \dots & E_{r}(-\alpha\lambda)\sum_{j=0}^{n-1}\text{\c{d}}^r_{j\ ((k+1)n-1)}\frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} \\ \vdots & \ddots & \vdots \\ E_{r}(-\alpha^{n-1}\lambda)\sum_{j=0}^{n-1}\text{\c{d}}^r_{j\ kn}\frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} & \dots & E_{r}(-\alpha^{n-1}\lambda)\sum_{j=0}^{n-1}\text{\c{d}}^r_{j\ ((k+1)n-1)}\frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix}.$$

The matrices $\text{\c{S}}^r_k, \text{\c{T}}^r_k$ have the following convenient property:

Lemma 9. *For the relevant matrices $e_r, \text{\c{C}}^r_k, \text{\c{D}}^r_k, \text{\c{S}}^r_k, \text{\c{T}}^r_k$, we have*

$$e_{r-1}\text{\c{S}}^r_k = \text{\c{C}}^r_k, \qquad e_r\text{\c{T}}^r_k = \text{\c{D}}^r_k,$$

where $r = 1, \dots, m$ and $k = 0, \dots, m-1$.

Kanıt. Fix r and k, and consider the product $e_{r-1}\text{\c{S}}^r_k$. Observe that the (t,s)-th entry of the product $e_{r-1}\text{\c{S}}^r_k$ is given by the t-th row of e_{r-1} times s-th column of $\text{\c{S}}^r_k$. Thus, we have:

$$\begin{aligned}(e_{r-1}\text{\c{S}}^r_k)_{(t,s)} &= \frac{1}{n}\begin{bmatrix} E_{r-1}(\lambda) & E_{r-1}(\alpha\lambda)\alpha^{n-t} & \dots & E_{r-1}(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-t)} \end{bmatrix} \\ &\qquad \begin{bmatrix} E_{r-1}(-\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ (s-1)}\frac{1}{(i\lambda)^{n-1-j}} \\ E_{r-1}(-\alpha\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ (s-1)}\frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} \\ \vdots \\ E_{r-1}(-\alpha^{n-1}\lambda)\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ (s-1)}\frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix} \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\text{\c{c}}^r_{j\ (s-1)}\frac{1}{(i\lambda)^{n-1-j}}\Big[E_{r-1}(\lambda)E_{r-1}(-\lambda) + E_{r-1}(\alpha\lambda)E_{r-1}(-\alpha\lambda)\alpha^{n-t}\alpha^{j+1}\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \ldots + E_{r-1}(\alpha^{(n-1)}\lambda)E_{r-1}(-\alpha^{n-1}\lambda)\alpha^{(n-1)(n-t)}\alpha^{(n-1)(j+1)}\Big] \\
&= \frac{1}{n}\sum_{j=0}^{n-1} \varsigma^r_{j\ (s-1)} \frac{1}{(i\lambda)^{n-1-j}} \Big[1 + \alpha^{n-t}\alpha^{j+1} + \alpha^{2(n-t)}\alpha^{2(j+1)} \\
&\qquad\qquad + \ldots + \alpha^{(n-2)(n-t)}\alpha^{(n-2)(j+1)} + \alpha^{(n-1)(n-t)}\alpha^{(n-1)(j+1)}\Big] \\
&= \frac{1}{n}\sum_{j=0}^{n-1} \varsigma^r_{j\ (s-1)} \frac{1}{(i\lambda)^{n-1-j}} \Big[1 + \alpha^{n-t+j+1} + \alpha^{2(n-t+j+1)} \\
&\qquad\qquad + \ldots + \alpha^{(n-2)(n-t+j+1)} + \alpha^{(n-1)(n-t+j+1)}\Big].
\end{aligned} \tag{42}
$$

Consider the inner sum in (42):

Case 1: $j = t-1$. If $j = t-1$, then

$$
\begin{aligned}
1 + \alpha^{n-t+j+1} &+ \alpha^{2(n-t+j+1)} + \ldots + \alpha^{(n-2)(n-t+j+1)} + \alpha^{(n-1)(n-t+j+1)} \\
&= 1 + \alpha^{n-t+t-1+1} + \alpha^{2(n-t+t-1+1)} + \ldots + \alpha^{(n-2)(n-t+t-1+1)} + \alpha^{(n-1)(n-t+t-1+1)} \\
&= 1 + \alpha^n + \alpha^{2n} + \ldots + \alpha^{(n-2)n} + \alpha^{(n-1)n} \\
&= 1 + 1 + 1 + \ldots + 1 + 1 \\
&= n,
\end{aligned}
$$

where the second last equality follows since α's are primitive roots of unity.

Case 2: $j \neq t-1$. If $j \neq t-1$, then $\alpha^{n-t+j+1} \neq 1$, and so we treat the term $1 + \ldots + \alpha^{(n-1)(n-t+j+1)}$ as a geometric progression with a common ratio $\alpha^{n-t+j+1}$. Thus, by geometric progression formula,

$$
\begin{aligned}
1 + \alpha^{n-t+j+1} &+ \alpha^{2(n-t+j+1)} + \ldots + \alpha^{(n-2)(n-t+j+1)} + \alpha^{(n-1)(n-t+j+1)} \\
&= \sum_{k=0}^{n-1} \alpha^{(n-t+j+1)k} \\
&= \frac{\alpha^{(n-t+j+1)n} - 1}{\alpha^{n-t+j+1} - 1} \\
&= 0,
\end{aligned}
$$

where the last equality follows since since $\alpha^n = 1$.

Thus, by the above analysis, we have

$$
\begin{aligned}
(e_{r-1}\S^r_k)_{(t,s)} &= \frac{1}{n}\sum_{j=0}^{n-1} \varsigma^r_{j\ (s-1)} \frac{1}{(i\lambda)^{n-1-j}} \left[1 + \alpha^{n-t+j+1} + \ldots + \alpha^{(n-1)(n-t+j+1)}\right] \\
&= \varsigma^r_{(t-1)\ (s-1)} \frac{1}{(i\lambda)^{n-t}} + \frac{1}{n}\sum_{\substack{j=0 \\ j\neq t-1}}^{n-1} \varsigma^r_{j\ (s-1)} \frac{1}{(i\lambda)^{n-1-j}} \underbrace{\left[1 + \alpha^{n-t+j+1} + \ldots + \alpha^{(n-1)(n-t+j+1)}\right]}_{=0} \\
&= \varsigma^r_{(t-1)\ (s-1)} \frac{1}{(i\lambda)^{n-t}}.
\end{aligned}
$$

But $\varsigma^r_{(t-1)\ (s-1)} \dfrac{1}{(i\lambda)^{n-t}}$ is exactly the (t,s)-th entry of $Ç^r_k$, and so we have $e_{r-1}\S^r_k = Ç^r_k$. The proof that $e_r Ţ^r_k = Ḑ^r_k$ is analogous. The proof is complete. □

By lemma 9, we have $(Ç^r_k)^T = (\S^r_k)^T e^T_{r-1}$ and $(Ḑ^r_k)^T = (Ţ^r_k)^T e^T_r$. This allows to rewrite the system

(41) as follows:

$$
\begin{aligned}
&\underbrace{\begin{bmatrix}
(\text{\c{C}}_0^1)^T & (\text{\c{D}}_0^1)^T & (\text{\c{C}}_0^2)^T & (\text{\c{D}}_0^2)^T & \cdots & (\text{\c{C}}_0^m)^T & (\text{\c{D}}_0^m)^T \\
(\text{\c{C}}_1^1)^T & (\text{\c{D}}_1^1)^T & (\text{\c{C}}_1^2)^T & (\text{\c{D}}_1^2)^T & \cdots & (\text{\c{C}}_1^m)^T & (\text{\c{D}}_1^m)^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(\text{\c{C}}_{m-1}^1)^T & (\text{\c{D}}_{m-1}^1)^T & (\text{\c{C}}_{m-1}^2)^T & (\text{\c{D}}_{m-1}^2)^T & \cdots & (\text{\c{C}}_{m-1}^m)^T & (\text{\c{D}}_{m-1}^m)^T \\
e_0^T & -e_1^T & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & e_1^T & -e_2^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & e_{m-1}^T & -e_m^T
\end{bmatrix}}_{2mn\times 2mn}
\underbrace{\begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda) \end{bmatrix}}_{2mn\times 1} \\
&= \begin{bmatrix}
(\text{\c{S}}_0^1)^T e_0^T & (\text{\c{T}}_0^1)^T e_1^T & (\text{\c{S}}_0^2)^T e_1^T & (\text{\c{T}}_0^2)^T e_2^T & \cdots & (\text{\c{S}}_0^m)^T e_{m-1}^T & (\text{\c{T}}_0^m)^T e_m^T \\
(\text{\c{S}}_1^1)^T e_0^T & (\text{\c{T}}_1^1)^T e_1^T & (\text{\c{S}}_1^2)^T e_1^T & (\text{\c{T}}_1^2)^T e_2^T & \cdots & (\text{\c{S}}_1^m)^T e_{m-1}^T & (\text{\c{T}}_1^m)^T e_m^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(\text{\c{S}}_{m-1}^1)^T e_0^T & (\text{\c{T}}_{m-1}^1)^T e_1^T & (\text{\c{S}}_{m-1}^2)^T e_1^T & (\text{\c{T}}_{m-1}^2)^T e_2^T & \cdots & (\text{\c{S}}_{m-1}^m)^T e_{m-1}^T & (\text{\c{T}}_{m-1}^m)^T e_m^T \\
I^T e_0^T & -I^T e_1^T & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I^T e_1^T & -I^T e_2^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & I^T e_{m-1}^T & -I^T e_m^T
\end{bmatrix}
\begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix}
(\text{\c{S}}_0^1)^T & (\text{\c{T}}_0^1)^T & (\text{\c{S}}_0^2)^T & (\text{\c{T}}_0^2)^T & \cdots & (\text{\c{S}}_0^m)^T & (\text{\c{T}}_0^m)^T \\
(\text{\c{S}}_1^1)^T & (\text{\c{T}}_1^1)^T & (\text{\c{S}}_1^2)^T & (\text{\c{T}}_1^2)^T & \cdots & (\text{\c{S}}_1^m)^T & (\text{\c{T}}_1^m)^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(\text{\c{S}}_{m-1}^1)^T & (\text{\c{T}}_{m-1}^1)^T & (\text{\c{S}}_{m-1}^2)^T & (\text{\c{T}}_{m-1}^2)^T & \cdots & (\text{\c{S}}_{m-1}^m)^T & (\text{\c{T}}_{m-1}^m)^T \\
I^T & -I^T & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I^T & -I^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & I^T & -I^T
\end{bmatrix}}_{2mn\times 2mn} \\
&\qquad \underbrace{\begin{bmatrix}
e_0^T & & & & & & & 0 \\
& e_1^T & & & & & & \\
& & e_1^T & & & & & \\
& & & e_2^T & & & & \\
& & & & \ddots & & & \\
& & & & & e_{m-1}^T & & \\
& & & & & & e_{m-1}^T & \\
0 & & & & & & & e_m^T
\end{bmatrix}}_{2mn\times 2mn}
\underbrace{\begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda) \end{bmatrix}}_{2mn\times 1} \\
&= \underbrace{\begin{bmatrix}
(\text{\c{S}}_0^1)^T & (\text{\c{T}}_0^1)^T & (\text{\c{S}}_0^2)^T & (\text{\c{T}}_0^2)^T & \cdots & (\text{\c{S}}_0^m)^T & (\text{\c{T}}_0^m)^T \\
(\text{\c{S}}_1^1)^T & (\text{\c{T}}_1^1)^T & (\text{\c{S}}_1^2)^T & (\text{\c{T}}_1^2)^T & \cdots & (\text{\c{S}}_1^m)^T & (\text{\c{T}}_1^m)^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(\text{\c{S}}_{m-1}^1)^T & (\text{\c{T}}_{m-1}^1)^T & (\text{\c{S}}_{m-1}^2)^T & (\text{\c{T}}_{m-1}^2)^T & \cdots & (\text{\c{S}}_{m-1}^m)^T & (\text{\c{T}}_{m-1}^m)^T \\
I^T & -I^T & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I^T & -I^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & I^T & -I^T
\end{bmatrix}}_{2mn\times 2mn}
\underbrace{\begin{bmatrix} e_0^T \vec{f}^1(\lambda) \\ e_1^T \vec{g}^1(\lambda) \\ e_1^T \vec{f}^2(\lambda) \\ e_2^T \vec{g}^2(\lambda) \\ \vdots \\ e_{m-1}^T \vec{f}^m(\lambda) \\ e_m^T \vec{g}^m(\lambda) \end{bmatrix}}_{2mn\times 1}, \qquad (43)
\end{aligned}
$$

where I is the $n \times n$ identity matrix. We finally rewrite the system (41) as follows:

$$
\mathcal{A} \underbrace{\begin{bmatrix} e_0^T \vec{f}^1(\lambda) \\ e_1^T \vec{g}^1(\lambda) \\ e_1^T \vec{f}^2(\lambda) \\ e_2^T \vec{g}^2(\lambda) \\ \vdots \\ e_{m-1}^T \vec{f}^m(\lambda) \\ e_m^T \vec{g}^m(\lambda) \end{bmatrix}}_{2mn \times 1} = \underbrace{\begin{bmatrix} h_0(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \\ \widehat{\vec{q}_0^1}(\lambda) \\ \widehat{\vec{q}_0^2}(\lambda) \\ \vdots \\ \widehat{\vec{q}_0^{m-1}}(\lambda) \\ \widehat{\vec{q}_0^m}(\lambda) \end{bmatrix}}_{2mn \times 1} - e^{a\lambda^n t} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \widehat{\vec{q}_t^1}(\lambda) \\ \widehat{\vec{q}_t^2}(\lambda) \\ \vdots \\ \widehat{\vec{q}_t^{m-1}}(\lambda) \\ \widehat{\vec{q}_t^m}(\lambda) \end{bmatrix}}_{2mn \times 1}, \tag{44}
$$

where

$$
\mathcal{A} = \underbrace{\begin{bmatrix} (\underset{\sim}{\mathrm{S}}_0^1) & (\underset{\sim}{\mathrm{S}}_1^1) & \dots & (\underset{\sim}{\mathrm{S}}_{m-1}^1) & I & 0 & \dots & 0 & 0 \\ (\underset{\sim}{\mathrm{T}}_0^1) & (\underset{\sim}{\mathrm{T}}_1^1) & \dots & (\underset{\sim}{\mathrm{T}}_{m-1}^1) & -I & 0 & \dots & 0 & 0 \\ (\underset{\sim}{\mathrm{S}}_0^2) & (\underset{\sim}{\mathrm{S}}_1^2) & \dots & (\underset{\sim}{\mathrm{S}}_{m-1}^2) & 0 & I & \dots & 0 & 0 \\ (\underset{\sim}{\mathrm{T}}_0^2) & (\underset{\sim}{\mathrm{T}}_1^2) & \dots & (\underset{\sim}{\mathrm{T}}_{m-1}^2) & 0 & -I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (\underset{\sim}{\mathrm{S}}_0^m) & (\underset{\sim}{\mathrm{S}}_1^m) & \dots & (\underset{\sim}{\mathrm{S}}_{m-1}^m) & 0 & 0 & \dots & 0 & I \\ (\underset{\sim}{\mathrm{T}}_0^m) & (\underset{\sim}{\mathrm{T}}_1^m) & \dots & (\underset{\sim}{\mathrm{T}}_{m-1}^m) & 0 & 0 & \dots & 0 & -I \end{bmatrix}^T}_{2mn \times 2mn}
$$

is a block matrix where each block is $n \times n$. The system (44) has the advantage that the main matrix is easier to compute. We refer to the system (44) as the *D-to-N map in $\mathcal{A}$ form*.

Kaynaklar

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