Locker Theorem 1

Theorem 1. *The adjoint operator L^* is the multipoint differential operator defined by*

$$\mathcal{D}(L^*) = \{f \in H^n(\pi) | B_i^*(f) = 0, i = 1, \ldots, 2mn - k\}, L^* f = \tau^* f$$

Proof. First, we will define L_0 to be what we think is the adjoint of L. That is, let L_0 be the linear operator in S whose domain consists of all functions $f \in H^n(\pi)$ satisfying $B_i^*(f) = 0$ for $i = 1, \ldots, 2mn - k$ with $L_0 = \tau^* f$. We want to show that $L_0 = L^*$. First, we show that $\mathcal{D}(L_0) \subsetneqq \mathcal{D}(L^*)$.

Let $g \in \mathcal{D}(L_0)$ and set $g^* = L_0 g = \tau^* g$. Then, let $f \in \mathcal{D}(L)$. Now, we want to show that $\langle Lf, g\rangle = \langle f, L_0 g\rangle = \langle f, g^*\rangle$. Recall that the numbers

$$x_{jl} = f_l^{(j)}(x_{l-1}), \quad y_{jl} = f_l^{(j)}(x_l),$$

form the solutions to the system

$$\sum_{l=1}^{m} \sum_{j=0}^{n-1} [\alpha_{ijl} x_{jl} + \beta_{ijl} y_{jl}] = 0, \quad i = 1, \ldots, k.$$

Also recall that as defined earlier, $[x_{ijl}, y_{ijl}], i = 1, \ldots, 2mn - k$ is the set of solutions of the above system which form a basis for the solution space. So, by definition of a basis, there exist constants $c_1, \ldots, c_{2mn-k}$ such that

$$f_l^{(j)}(x_{l-1}) = \sum_{i=1}^{2mn-k} c_i x_{ijl} \text{ and } f_l^{(j)}(x_l) = \sum_{i=1}^{2mn-k} c_i y_{ijl}.$$

Now, we can applying Green's formula and get

$$\begin{aligned}
\langle Lf, g\rangle - \langle f, g^*\rangle &= \langle \tau f, g\rangle - \langle f, \tau^* g\rangle \\
&= \sum_{l=1}^{m} \sum_{p,q=0}^{n-1} [F_{x_l}^{pq}(\tau) f_l^{(p)}(x_l) g_l^{(q)}(x_l) - F_{x_{l-1}}^{pq}(\tau) f_l^{(p)}(x_{l-1}) g_l^{(q)}(x_{l-1})] \\
\text{(Substitute for the } f_l^{(j)}) \quad &= \sum_{l=1}^{m} \sum_{p,q=0}^{n-1} \sum_{i=1}^{2mn-k} c_i [F_{x_l}^{pq}(\tau) y_{ipl} g_l^{(q)}(x_l) - F_{x_{l-1}}^{pq}(\tau) x_{ipl} g_l^{(q)}(x_{l-1})] \\
\text{(Substitute in } \alpha^*, \beta^*) \quad &= \sum_{i=1}^{2mn-k} c_i \sum_{l=1}^{m} \sum_{q=0}^{n-1} [\beta_{iql}^* g_l^{(q)}(x_l) + \alpha_{iql}^* g_l^{(q)}(x_{l-1})] \\
&= \sum_{i=1}^{2mn-k} c_i B_i^*(g) \\
&= 0 \quad \text{(by definition of } g)
\end{aligned}$$

Since $f \in \mathcal{D}(L)$ is arbitrary, and $\langle Lf, g\rangle = \langle f, L_0 g\rangle = \langle f, g^*\rangle$, we can conclude that $g \in \mathcal{D}(L^*)$, which implies $\mathcal{D}(L_0) \subsetneqq \mathcal{D}(L^*)$.

To complete the proof, it is sufficient to show that $\mathcal{D}(L^*) \subsetneqq \mathcal{D}(L_0)$. Let $g \in \mathcal{D}(L^*)$. Now, we want to show that $g \in H^n(\pi)$ and that $B_i^*(g) = 0$, which would imply that $g \in \mathcal{D}(L_0)$ by definition of L_0. Fix an integer l with $1 \leq l \leq m$, and let $\bar{g}$ denote the restriction of g to the interval $[x_{l-1}, x_l]$. Let $\bar{f}$ be any function in $H^n[x_{l-1}, x_l]$ having its its support in the open interval (x_{l-1}, x_l). Then, we can extend $\bar{f}$ to f defined on $[a, b]$ by making it 0 outside of

$[x_{l-1}, x_l]$. The extension of f belongs in $\mathcal{D}(L^*)$ because $f \in H^n(\pi)$, and $B_i(f) = 0$ (because it is 0 at all the boundaries). Then,

$$0 = \langle Lf, g\rangle - \langle f, L^* g\rangle = \int_{x_{l-1}}^{x_l} (\tau \bar{f})\bar{g} - \int_{x_{l-1}}^{x_l} \bar{f}(L^* g).$$

By Theorem 10 of [2, p. 1294], the above implies that $\bar{g}$ is equal a.e to a function in $H^n[x_{l-1}, x_l]$ and that $L^* = \tau^* \bar{g}$ a.e. on $[x_{l-1}, x_l]$. Since this holds for all l, we can conclude $g \in H^n(\pi)$ and $L^* g = \tau^* g$.

Next, we want to show that $B_i^*(g) = 0$. Fix in integer i with $1 \le i \le 2mn - k$ and choose a function $\sigma = (\sigma_1, \ldots, \sigma_m) \in H^n(\pi)$ such that $\sigma_l^{(j)}(x_{l-1}) = x_{ijl}$ and $\sigma_l^{(j)}(x_l) = y_{ijl}$. That is, evaluating σ at each boundary point yields the set of solutions that form the basis for the solution space. Clearly, $\sigma \in \mathcal{D}(L)$, and from Green's formula

$$\begin{aligned}
0 &= \langle L\sigma, g\rangle - \langle \sigma, L^* g\rangle \\
&= \langle \tau\sigma, g\rangle - \langle \sigma, \tau^* g\rangle \\
&= \sum_{l=1}^{m} \sum_{p,q=0}^{n-1} [F_{x_l}^{p\,q}(\tau)\sigma_l^{(p)}(x_l) g_l^{(q)}(x_l) - F_{x_{l-1}}^{p\,q}(\tau)\sigma_l^{(p)}(x_{l-1}) g_l^{(q)}(x_{l-1})] \\
&= \sum_{l=1}^{m} \sum_{p,q=0}^{n-1} [F_{x_l}^{p\,q}(\tau) y_{ipl} g_l^{(q)}(x_l) - F_{x_{l-1}}^{p\,q}(\tau) x_{ipl} g_l^{(q)}(x_{l-1})] \\
&= \sum_{l=1}^{m} \sum_{q=0}^{n-1} [\beta_{iql}^* g_l^{(q)}(x_l) + \alpha_{iql}^* g_l^{(q)}(x_{l-1})] \\
&= B_i^*(g)
\end{aligned}$$

So, we have shown that if given $g \in \mathcal{D}(L^*)$, we know $B_i^*(g) = 0$. That together with $g \in H^n(\pi)$ proven earlier implies that $g \in \mathcal{D}(L_0)$. So $L_0 = L^*$. $\square$