Construction of Adjoint Problem

Consider a closed interval $[a,b]$. Fix $n \in \mathbb{N}$, and let the differential operator be defined as

$$L := \sum_{k=0}^{n} a_k(t) \left(\frac{d}{dt}\right)^k, \text{ where } a_k(t) \in C^\infty[a,b] \text{ and } a_n(t) \neq 0 \ \forall t \in [a,b].$$

Fix $k \in \mathbb{N}$, and let $\pi = \{a = x_0 < x_1 < \ldots < x_k = b\}$ be a partition of $[a,b]$. Let the domain of L be given by the function space

$$\begin{aligned} C_\pi^{n-1}[a,b] = \Big\{ &f : [a,b] \to \mathbb{C} \text{ s.t. } \forall l \in \{1,2,\ldots,k\}, \\ &f_l := f\big|_{(\eta_{l-1},\eta_l)} \text{ admits an extension } g_l(t) \text{ to } [\eta_{l-1},\eta_l] \text{ s.t. } g_l \in C^{n-1}[\eta_{l-1},\eta_l] \Big\}. \end{aligned} \tag{1}$$

Consider a homogeneous multipoint BVP of rank m

$$\pi_m : Lq = 0, \qquad Uq = \vec{0},$$

where $U = (U_1,\ldots,U_m)$ is a multipoint boundary form with

$$U_i(q) = \sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l)], \qquad i \in \{1,\ldots,m\},$$

where $\alpha_{ijl}, \beta_{ijl} \in \mathbb{R}, q \in C_\pi^{n-1}[a,b]$. Our goal is to construct the adjoint multipoint BVP to π_m

$$\pi^+_{2nk-m} : L^+ q = 0, \qquad U^+ q = \vec{0},$$

with

$$L^+ := \sum_{k=0}^{n} (-1)^k \overline{a_k}(t) \left(\frac{d}{dt}\right)^k, \text{ where } \overline{a_k}(t) \text{ is the complex conjugate of } a_k(t),\ k = 0,\ldots,n,$$

and U^+ is an appropriate multipoint boundary form.

Green's Formula

For any $f, g \in C_\pi^{n-1}[a,b]$, application of Green's formula yields

$$\langle Lf, g\rangle - \langle f, L^+ g\rangle = \sum_{l=1}^{k} \sum_{p,q=0}^{n-1} [F_{p\,q}(x_l) f_l^{(p)}(x_l) g_l^{(q)}(x_l) - F_{p\,q}(x_{l-1}) f_l^{(p)}(x_{l-1}) g_l^{(q)}(x_{l-1})],$$

where $F(t)$ denotes an $n \times n$ boundary matrix at the point $t \in [a,b]$. From [1, p. 1286], the entries of $F(t)$ are given by

$$\begin{aligned} F_{p\,q}(t) &= \sum_{k=j}^{n-p-1} (-1)^k \binom{k}{j} \left(\frac{d}{dt}\right)^{k-j} a_{p+k+1}(t), && p+q < n-1 \\ F_{p\,q}(t) &= (-1)^q a_n(t), && p+q = n-1 \\ F_{p\,q}(t) &= 0, && p+q > n-1. \end{aligned}$$

Observe that since $\det(F(t)) = (a_0(t))^n \neq 0$, the matrix $F(t)$ is non-singular.

Our goal is to rewrite the Green's formula as a *semibilinear* form $\mathcal{S}$. First, let $\vec{f}_l := (f_l, \dots, f_l^{(n-1)})$, and observe that

$$\begin{aligned}
[fg]_l(t) := \sum_{p,q=0}^{n-1} F_{p\,q}(t) f_l^{(p)}(t) g_l^{(q)}(t) &= \sum_{p,q=0}^{n-1} \left[F_{p\,q} f_l^{(p)} g_l^{(q)}\right](t) \\
&= \sum_{q=0}^{n-1} \left[\left(\sum_{p=0}^{n-1} F_{p\,q} f_l^{(p)}\right) g_l^{(q)}\right](t) \\
&= F(t)\vec{f}_l(t) \cdot \vec{g}_l(t),
\end{aligned}$$

where $\cdot$ refers to dot product. The Green's formula can then be rewritten as

$$\langle Lf, g\rangle - \langle f, L^+ g\rangle = \sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k} F(x_l)\vec{f}_l(x_l) \cdot \vec{g}_l(x_l) - F(x_{l-1})\vec{f}_l(x_{l-1}) \cdot \vec{g}_l(x_{l-1}). \tag{2}$$

Note that

$$F(x_l)\vec{f}_l(x_l) \cdot \vec{g}_l(x_l) - F(x_{l-1})\vec{f}_l(x_{l-1}) \cdot \vec{g}_l(x_{l-1}) = \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_l(x_{l-1}) \\ \vec{g}_l(x_l) \end{bmatrix},$$

so that we obtain

$$\langle Lf, g\rangle - \langle f, L^+ g\rangle = \sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k} \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_l(x_{l-1}) \\ \vec{g}_l(x_l) \end{bmatrix}.$$

Now, expansion of the sum yields

$$\begin{aligned}
&\sum_{l=1}^{k} \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_l(x_{l-1}) \\ \vec{g}_l(x_l) \end{bmatrix}, \\
&= \begin{bmatrix} -F(x_0) & 0_{n\times n} \\ 0_{n\times n} & F(x_1) \end{bmatrix} \begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \end{bmatrix} + \dots + \begin{bmatrix} -F(x_{k-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_k) \end{bmatrix} \begin{bmatrix} \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} -F(x_0) & 0 & \dots & 0 & 0 \\ 0 & F(x_1) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -F(x_{k-1}) & 0 \\ 0 & 0 & \dots & 0 & F(x_k) \end{bmatrix}}_{2nk \times 2nk} \begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vec{f}_2(x_1) \\ \vec{f}_2(x_2) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vec{g}_2(x_1) \\ \vec{g}_2(x_2) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix} \\
&=: S \begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix} = \mathcal{S}\left(\begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix} \right),
\end{aligned} \tag{3}$$

where the matrix S is associated with the semibilinear form $\mathcal{S}$ and S is a block matrix where each block is $n \times n$. Further, note that the form $\mathcal{S}$ is the action of applying matrix S to the first argument and taking dot product of this result and the second argument. Thus, we managed to express the Green's Formula as a semibilinear form $\mathcal{S}$.

Boundary-Form Formula

We turn to characterising an adjoint multipoint boundary condition using an extension of boundary form formula that Linda derived in her work. First, recall that the multipoint boundary conditions are of the form

$$Uq = \begin{bmatrix} U_1(q) \\ \vdots \\ U_m(q) \end{bmatrix} = \vec{0},$$

with

$$U_i(q) = \sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l)], \qquad i \in \{1, \dots, m\},\ \alpha_{ijl}, \beta_{ijl} \in \mathbb{R}.$$

Note that $U_1, \dots, U_m$ are linearly independent when $\sum_{i=1}^{m} c_i U_i q = 0$ if and only if $c_i = 0$. When $U_1, \dots, U_m$ are linearly independent, we say that U has full rank m. For now, suppose that U has full rank, and define

$$\vec{q_l} = \begin{bmatrix} q_l \\ q_l' \\ \vdots \\ q_l^{(n-1)} \end{bmatrix}, M_l = \begin{bmatrix} \alpha_{1\ 0\ l} & \alpha_{1\ 1\ l} & \dots & \alpha_{1\ (n-1)\ l} \\ \alpha_{2\ 0\ l} & \alpha_{2\ 1\ l} & \dots & \alpha_{2\ (n-1)\ l} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \alpha_{m\ 1\ l} & \dots & \alpha_{m\ (n-1)\ l} \end{bmatrix}, N_l = \begin{bmatrix} \beta_{1\ 0\ l} & \beta_{1\ 1\ l} & \dots & \beta_{1\ (n-1)\ l} \\ \beta_{2\ 0\ l} & \beta_{2\ 1\ l} & \dots & \beta_{2\ (n-1)\ l} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m\ 0\ l} & \beta_{m\ 1\ l} & \dots & \beta_{m\ (n-1)\ l} \end{bmatrix}$$

Then,

$$\begin{aligned} Uq &= \begin{bmatrix} U_1(q) \\ \vdots \\ U_m(q) \end{bmatrix} \\ &= \sum_{l=1}^{k} \sum_{j=0}^{n-1} \begin{bmatrix} \alpha_{1\ j\ l} \\ \vdots \\ \alpha_{m\ j\ l} \end{bmatrix} q_l^{(j)}(x_{l-1}) + \begin{bmatrix} \beta_{1\ j\ l} \\ \vdots \\ \beta_{m\ j\ l} \end{bmatrix} q_l^{(j)}(x_l) \\ &= \sum_{l=1}^{k} \begin{bmatrix} \alpha_{1\ 0\ l} & \dots & \alpha_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \dots & \alpha_{m\ (n-1)\ l} \end{bmatrix} \begin{bmatrix} q_l(x_{l-1}) \\ \vdots \\ q_l^{(n-1)}(x_{l-1}) \end{bmatrix} + \begin{bmatrix} \beta_{1\ 0\ l} & \dots & \beta_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots \\ \beta_{m\ 0\ l} & \dots & \beta_{m\ (n-1)\ l} \end{bmatrix} \begin{bmatrix} q_l(x_l) \\ \vdots \\ q_l^{(n-1)}(x_l) \end{bmatrix} \\ &= \sum_{l=1}^{k} M_l \vec{q_l}(x_{l-1}) + N_l \vec{q_l}(x_l), \end{aligned} \tag{$\dagger$}$$

where M_l, N_l are $m \times n$ matrices. In addition, letting

$$\begin{bmatrix} M_l : N_l \end{bmatrix} = \begin{bmatrix} \alpha_{1\ 0\ l} & \dots & \alpha_{1\ (n-1)\ l} & \beta_{1\ 0\ l} & \dots & \beta_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \dots & \alpha_{m\ (n-1)\ l} & \beta_{m\ 0\ l} & \dots & \beta_{m\ (n-1)\ l} \end{bmatrix},$$

Finally, we can write

$$Uq = \sum_{l=1}^{k} \begin{bmatrix} M_l : N_l \end{bmatrix} \begin{bmatrix} \vec{q_l}(x_{l-1}) \\ \vec{q_l}(x_l) \end{bmatrix} = \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \end{bmatrix} \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}. \tag{$\star$}$$

Thus we have found two compact ways to write the multipoint boundary forms, namely ($\dagger$) and ($\star$). Next, we extend the notion of a complementary boundary form.

Definition 1. If $U = (U_1, \ldots, U_m)$ is any multipoint boundary form with $\text{rank}(U) = m$, and $U_c = (U_{m+1}, \ldots, U_{2nk})$ is a multipoint boundary form with $\text{rank}(U_c) = 2nk - m$ such that $\text{rank}(U_1, \ldots, U_{2nk}) = 2nk$, then U and U_c are **complementary multipoint boundary forms**.

Note that extending $U_1, \ldots, U_m$ to $U_1, \ldots, U_{2nk}$ is equivalent to embedding the matrices M_l, N_l in a $2nk \times 2nk$ non-singular matrix, i.e. we can write

$$\begin{aligned}
\begin{bmatrix} Uq \\ U_c q \end{bmatrix} &= \sum_{l=1}^{k} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{q_l}(x_{l-1}) \\ \vec{q_l}(x_l) \end{bmatrix} \\
&= \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \end{bmatrix} + \begin{bmatrix} M_2 & N_2 \\ \overline{M}_2 & \overline{N}_2 \end{bmatrix} \begin{bmatrix} \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \end{bmatrix} + \ldots + \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \begin{bmatrix} \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} M_1 & N_1 & M_2 & N_2 & \ldots & M_k & N_k \\ \overline{M}_1 & \overline{N}_1 & \overline{M}_2 & \overline{N}_2 & \ldots & \overline{M}_k & \overline{N}_k \end{bmatrix}}_{2nk \times 2nk} \underbrace{\begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}}_{2nk \times 1} \\
&=: H \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}.
\end{aligned} \tag{4}$$

where $\text{rank}(H) = 2nk$ and $\overline{M}_l, \overline{N}_l$ are $(2nk - m) \times n$ matrices. Just like the boundary form formula proven by Linda, the multipoint boundary form formula is motivated by the desire to express Green's formula as a combination of boundary forms U and U_c. Namely, we have:

Theorem 2 (Multipoint Boundary Form Formula)**.** *Given any boundary form U of rank m, and any complementary form U_c, there exist unique boundary forms U_c^+, U^+ of rank m and $2nk - m$, respectively, such that*

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g. \tag{5}$$

We will use the following proposition from Linda's capstone [3] in the proof of Theorem 2:

Proposition 1 (Prop. 2.12 in Linda's capstone)**.** *Let $\mathcal{S}$ be the semibilinear form associated with a nonsingular matrix S. Suppose $\vec{f} := Ff$ where F is a nonsingular matrix. Then, there exists a unique nonsingular matrix G such that if $\vec{g} = Gg$, then $\mathcal{S}(f, g) = \vec{f} \cdot \vec{g}$ for all f, g.*

Proof of Theorem 2. First, we have

$$\begin{bmatrix} Uf \\ U_c f \end{bmatrix} = H \begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix}.$$

From equation (3), we can write

$$\sum_{l=1}^{k}[fg]_l(x_l)-[fg]_l(x_{l-1})=\mathcal{S}\left(\begin{bmatrix}\vec{f_1}(x_0)\\ \vec{f_1}(x_1)\\ \vdots\\ \vec{f_k}(x_{k-1})\\ \vec{f_k}(x_k)\end{bmatrix},\begin{bmatrix}\vec{g_1}(x_0)\\ \vec{g_1}(x_1)\\ \vdots\\ \vec{g_k}(x_{k-1})\\ \vec{g_k}(x_k)\end{bmatrix}\right).$$

By Proposition 1, there exists a unique $2nk\times 2nk$ nonsingular matrix J such that

$$\mathcal{S}\left(\begin{bmatrix}\vec{f_1}(x_0)\\ \vec{f_1}(x_1)\\ \vdots\\ \vec{f_k}(x_{k-1})\\ \vec{f_k}(x_k)\end{bmatrix},\begin{bmatrix}\vec{g_1}(x_0)\\ \vec{g_1}(x_1)\\ \vdots\\ \vec{g_k}(x_{k-1})\\ \vec{g_k}(x_k)\end{bmatrix}\right)=H\begin{bmatrix}\vec{f_1}(x_0)\\ \vec{f_1}(x_1)\\ \vdots\\ \vec{f_k}(x_{k-1})\\ \vec{f_k}(x_k)\end{bmatrix}\cdot J\begin{bmatrix}\vec{g_1}(x_0)\\ \vec{g_1}(x_1)\\ \vdots\\ \vec{g_k}(x_{k-1})\\ \vec{g_k}(x_k)\end{bmatrix}.$$

Note that if S is the matrix associated with $\mathcal{S}$, then by Proposition 1, $J=(SH^{-1})^*$, where A^* refers to the conjugate transpose of matrix A.

Let U^+, U_c^+ be such that

$$\begin{bmatrix}U_c^+g\\ U^+g\end{bmatrix}=J\begin{bmatrix}\vec{g_1}(x_0)\\ \vec{g_1}(x_1)\\ \vdots\\ \vec{g_k}(x_{k-1})\\ \vec{g_k}(x_k)\end{bmatrix}.$$

Now, we obtain

$$\begin{aligned}\sum_{l=1}^{k}[fg]_l(x_l)-[fg]_l(x_{l-1})=\mathcal{S}\left(\begin{bmatrix}\vec{f_1}(x_0)\\ \vec{f_1}(x_1)\\ \vdots\\ \vec{f_k}(x_{k-1})\\ \vec{f_k}(x_k)\end{bmatrix},\begin{bmatrix}\vec{g_1}(x_0)\\ \vec{g_1}(x_1)\\ \vdots\\ \vec{g_k}(x_{k-1})\\ \vec{g_k}(x_k)\end{bmatrix}\right)&=H\begin{bmatrix}\vec{f_1}(x_0)\\ \vec{f_1}(x_1)\\ \vdots\\ \vec{f_k}(x_{k-1})\\ \vec{f_k}(x_k)\end{bmatrix}\cdot J\begin{bmatrix}\vec{g_1}(x_0)\\ \vec{g_1}(x_1)\\ \vdots\\ \vec{g_k}(x_{k-1})\\ \vec{g_k}(x_k)\end{bmatrix}\\ &=\begin{bmatrix}Uf\\ U_cf\end{bmatrix}\cdot\begin{bmatrix}U_c^+g\\ U^+g\end{bmatrix}\\ &=Uf\cdot U_c^+g+U_cf\cdot U^+g,\end{aligned}$$

which completes the proof. □

Theorem 2 allows us to define an adjoint multipoint boundary form. Namely,

Definition 3. Suppose $U=(U_1,\ldots,U_m)$ is a multipoint boundary form with $\text{rank}(U)=m$, along with the condition that $Uq=\vec{0}$ for functions $q\in C_\pi^{n-1}[a,b]$. If U^+ is any boundary form with $\text{rank}(U^+)=2nk-m$, determined as in Theorem 2, then the equation

$$U^+q=\vec{0}$$

is an **adjoint multipoint boundary form** to $Uq=\vec{0}$.

In turn, the above lets us define the adjoint multipoint problem:

Definition 4. Suppose $U = (U_1, \ldots, U_m)$ is a multipoint boundary form with $\text{rank}(U) = m$. Then, the problem of solving

$$\pi_m : Lq = 0, \qquad Uq = \vec{0},$$

is called a homogeneous multipoint boundary value problem of rank m. The problem of solving

$$\pi^+_{2nk-m} : L^+q = 0, \qquad U^+q = \vec{0},$$

is an **adjoint multipoint boundary value problem** to π_m.

The preceding construction allows us to state the following:

Proposition 2. *Let $f, g \in C^{n-1}_\pi[a, b]$ with $Uf = \vec{0}$ and $U^+g = \vec{0}$. Then, $\langle Lf, g\rangle = \langle f, L^+g\rangle$.*

Proof. We apply Green's formula and multipoint boundary form formula:

$$\langle Lf, g\rangle - \langle f, L^+g\rangle = \sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g = \vec{0} \cdot U_c^+ g + U_c f \cdot \vec{0} = 0. \qquad \square$$

Checking Adjointness

Finally, we extend Theorem 2.19 on Linda's Capstone [3].

Theorem 5. *The boundary condition $U^+g = \vec{0}$ is adjoint to $Uf = \vec{0}$ if and only if*

$$\sum_{l=1}^{k} M_l F^{-1}(x_{l-1}) P_l = \sum_{l=1}^{k} N_l F^{-1}(x_l) Q_l,$$

where $F(t)$ is the $n \times n$ matrix as given in Green's Formula subsection.

Recall that just how U is associated with a collection of $m \times n$ matrices M_l, N_l, such that

$$Uf = \sum_{l=1}^{k} M_l \vec{f_l}(x_{l-1}) + N_l \vec{f_l}(x_l), \qquad \text{rank}\left[M_1 : N_1 : \ \ldots \ : M_k : N_k\right] = m, \tag{6}$$

so is U^+ associated with $n \times (2nk - m)$ matrices P_l, Q_l, for $l = 1, \ldots, k$, such that

$$U^+g = \sum_{l=1}^{k} P_l^* \vec{g}_l(x_{l-1}) + Q_l^* \vec{g}_l(x_l), \qquad \text{rank}\left[P_1^* : Q_1^* : \ \ldots \ : P_k^* : Q_k^*\right] = 2nk - m. \tag{7}$$

Proof of Theorem 5. Suppose that $U^+f = \vec{0}$ is adjoint to $Uf = \vec{0}$. By definition of adjoint multipoint boundary condition, U^+ is determined as in Theorem 2. Thus, in determining U^+, there exist multipoint boundary forms U_c, U_c^+ of rank $2nk - m$ and m respectively, such that the multipoint boundary form formula (5) holds. As such, let matrices $\overline{M}_l, \overline{N}_l, \overline{P}_l, \overline{Q}_l$ be such that

$$U_c f = \sum_{l=1}^{k} \overline{M}_l \vec{f_l}(x_{l-1}) + \overline{N}_l \vec{f_l}(x_l), \qquad \text{rank}\left[\overline{M}_1 : \overline{N}_1 : \ \ldots \ : \overline{M}_k : \overline{N}_k\right] = 2nk - m \tag{8}$$

$$U_c^+ g = \sum_{l=1}^{k} \overline{P}_l^* \vec{g}_l(x_{l-1}) + \overline{Q}_l^* \vec{g}_l(x_l), \qquad \text{rank}\left[\overline{P}_1^* : \overline{Q}_1^* : \ \ldots \ : \overline{P}_k^* : \overline{Q}_k^*\right] = m \tag{9}$$

First, note that in the context of semibilinear form, we have $\mathcal{S}(f,g) = Sf \cdot g = f \cdot S^* g$, as given in Proposition 2.11 of Linda's capstone [3, p.18]. We use this to rewrite the multipoint boundary form formula (5) as follows:

$$\begin{aligned}
\sum_{l=1}^{k}[fg]_l(x_l) - [fg]_l(x_{l-1}) &= Uf \cdot U_c^+ g + U_c f \cdot U^+ g \\
&= \left(\sum_{l=1}^{k} M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l)\right) \cdot \left(\sum_{i=1}^{k} (\overline{P}_i)^* \vec{g}_i(x_{i-1}) + (\overline{Q}_i)^* \vec{g}_i(x_i)\right) \\
&+ \left(\sum_{l=1}^{k} \overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l)\right) \cdot \left(\sum_{i=1}^{k} P_i^* \vec{g}_i(x_{i-1}) + Q_i^* \vec{g}_i(x_i)\right) \qquad \text{(by equations (6), (7), (8), (9))} \\
&= \sum_{l=1}^{k}\sum_{i=1}^{k}\Bigg(\Big(M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l)\Big) \cdot \Big(\overline{P}_i^* \vec{g}_i(x_{i-1}) + \overline{Q}_i^* \vec{g}_i(x_i)\Big) \\
&+ \Big(\overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l)\Big) \cdot \Big(P_i^* \vec{g}_i(x_{i-1}) + Q_l^* \vec{g}_i(x_i)\Big)\Bigg),
\end{aligned}$$

where taking out the sum upfront follows due to distributivity and associativity of inner product. Moreover, using additivity of inner product and that $Sf \cdot g = f \cdot S^* g$, we write the above as

$$\begin{aligned}
\sum_{l=1}^{k}\sum_{i=1}^{k} &(\overline{Q}_i N_l + Q_i \overline{N}_l)\vec{f}_l(x_l) \cdot \vec{g}_i(x_i) + (\overline{P}_i N_l + P_i \overline{N}_l)\vec{f}_l(x_l) \cdot \vec{g}_i(x_{i-1}) \\
&+ (\overline{Q}_i M_l + Q_i \overline{M}_l)\vec{f}_l(x_{l-1}) \cdot \vec{g}_i(x_i) + (\overline{P}_i M_l + P_i \overline{M}_l)\vec{f}_l(x_{l-1}) \cdot \vec{g}_i(x_{i-1}).
\end{aligned} \tag{10}$$

From Green's formula (2), we have

$$\sum_{l=1}^{k}[fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k} F(x_l)\vec{f}_l(x_l) \cdot \vec{g}_l(x_l) - F(x_{l-1})\vec{f}_l(x_{l-1}) \cdot \vec{g}_l(x_{l-1}) \tag{11}$$

Note that equations (17) and (11) must be equal, and so, comparison of coefficients of inner product reveals that

$$\overline{Q}_i N_l + Q_i \overline{N}_l = \begin{cases} F(x_l) & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}; \qquad \overline{P}_i M_l + P_i \overline{M}_l = \begin{cases} -F(x_{l-1}) & \text{if } i = l \\ 0 & \text{otherwise} \end{cases};$$

$$\overline{P}_i N_l + P_i \overline{N}_l = 0 \quad \forall i; \qquad \overline{Q}_i M_l + Q_i \overline{M}_l = 0 \quad \forall i.$$

Thus, we have

$$\begin{bmatrix}
-F(x_0) & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & F(x_1) & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & -F(x_1) & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & F(x_{k-1}) & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & -F(x_{k-1}) & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & F(x_k)
\end{bmatrix}$$

$$= \begin{bmatrix}
\overline{P}_1 M_1 + P_1 \overline{M}_1 & 0 & \dots & 0 & 0 \\
0 & \overline{Q}_1 N_1 + Q_1 \overline{N}_1 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & \overline{P}_k M_k + P_k \overline{M}_k & 0 \\
0 & 0 & \dots & 0 & \overline{Q}_k N_k + Q_k \overline{N}_k
\end{bmatrix}.$$

Since the boundary matrix F is nonsingular on $[a, b]$, F is invertible, and so the block diagonal matrix on LHS must also be invertible. Premultiplying on both sides by the inverse of LHS block diagonal matrix yields

$$\begin{aligned}
E_{2nk\times 2nk} &= \begin{bmatrix} -F^{-1}(x_0)(\overline{P}_1 M_1 + P_1\overline{M}_1) & 0 & \dots & 0 \\ 0 & F^{-1}(x_1)(\overline{Q}_1 N_1 + Q_1\overline{N}_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F^{-1}(x_k)(\overline{Q}_k N_k + Q_k\overline{N}_k) \end{bmatrix} \\
&= \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 M_1 - F^{-1}(x_0)P_1\overline{M}_1 & 0 & \dots & 0 \\ 0 & F^{-1}(x_1)\overline{Q}_1 N_1 + F^{-1}(x_1)Q_1\overline{N}_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F^{-1}(x_k)\overline{Q}_k N_k + F^{-1}(x_k)Q_k\overline{N}_k \end{bmatrix} \\
&= \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix} \begin{bmatrix} M_1 & N_1 & \dots & M_k & N_k \\ \overline{M}_1 & \overline{N}_1 & \dots & \overline{M}_k & \overline{N}_k \end{bmatrix}, \qquad (*)
\end{aligned}$$

where $E_{j\times j}$ is the identity matrix of dimension j. Since the two matrices in $(*)$ are full rank, they are inverse to each other, and so we have

$$\begin{bmatrix} E_{m\times m} & 0_{m\times(2nk-m)} \\ 0_{(2nk-m)\times m} & E_{(2nk-m)\times(2nk-m)} \end{bmatrix} = \begin{bmatrix} M_1 & N_1 & \dots & M_k & N_k \\ \overline{M}_1 & \overline{N}_1 & \dots & \overline{M}_k & \overline{N}_k \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix},$$

which implies that

$$\begin{aligned}
-M_1F^{-1}(x_0)P_1 + N_1F^{-1}(x_1)Q_1 + \ldots - M_kF^{-1}(x_{k-1})P_k + N_kF^{-1}(x_k)Q_k &= 0_{m\times(2nk-m)} \\
\implies \sum_{l=1}^{k} M_l F^{-1}(x_{l-1})P_l = \sum_{l=1}^{k} N_l F^{-1}(x_l)Q_l.&
\end{aligned}$$

Now, we prove the "if" direction. Let U_1^+ be a multipoint boundary form of rank $2nk - m$ such that

$$U_1^+ g = \sum_{l=1}^{k} \mathcal{P}_l^* \vec{g}_l(x_{l-1}) + \mathcal{Q}_l^* \vec{g}_l(x_l),$$

for an appropriate collection of matrices $\mathcal{P}_l^*, \mathcal{Q}_l^*$, with

$$\operatorname{rank}\left[\mathcal{P}_1^* : \mathcal{Q}_1^* : \ \dots \ : \mathcal{P}_k^* : \mathcal{Q}_k^*\right] = 2nk - m$$

Suppose that

$$\sum_{l=1}^{k} M_l F^{-1}(x_{l-1})\mathcal{P}_l = \sum_{l=1}^{k} N_l F^{-1}(x_l)\mathcal{Q}_l$$

holds. Now, let $\mathbf{u}$ be a $2nk \times 1$ vector. Then, there exist $2nk - m$ linearly independent solutions of the system

$$\left[M_1 : N_1 : \ \dots \ : M_k : N_k\right]_{m\times 2nk} \mathbf{u} = \vec{0}.$$

By assumption, we have

$$\sum_{l=1}^{k} -M_l F(x_{l-1})^{-1}\mathcal{P}_l + N_l F(x_l)^{-1}\mathcal{Q}_l = 0_{m\times(2nk-m)},$$

so that

$$\begin{bmatrix} M_1 : N_1 : & \ldots & : M_k : N_k \end{bmatrix}_{m\times 2nk} \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix}_{2nk\times(2nk-m)} = 0_{m\times(2nk-m)}. \tag{12}$$

This means that the $2nk-m$ columns of the matrix

$$H_1 := \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix}$$

form the solution space of the system (12). Since rank $\begin{bmatrix} \mathcal{P}_1^* : \mathcal{Q}_1^* : & \ldots & : \mathcal{P}_k^* : \mathcal{Q}_k^* \end{bmatrix} = 2nk-m$,

$$\operatorname{rank} \begin{bmatrix} \mathcal{P}_1 \\ \mathcal{Q}_1 \\ \vdots \\ \mathcal{P}_k \\ \mathcal{Q}_k \end{bmatrix} = 2nk-m.$$

Since $F(x_{l-1}), F(x_l)$ are non-singular, $\operatorname{rank}(H_1) = 2nk-m$.

Now, if $U^+g = \sum_{l=1}^k P_l^* \vec{g}_l(x_{l-1}) + Q_l^* \vec{g}_l(x_l) = \vec{0}$ is a multipoint boundary condition adjoint to $Uf = \vec{0}$, then by multipoint boundary form formula we have that

$$\begin{aligned} \begin{bmatrix} Uf \\ U_c f \end{bmatrix} \cdot \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} &= \sum_{l=1}^k \sum_{i=1}^k \begin{bmatrix} M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l) \\ \overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \overline{P}_i^* \vec{g}_i(x_{i-1}) + \overline{Q}_i^* \vec{g}_i(x_i) \\ P_i^* \vec{g}_i(x_{i-1}) + Q_i^* \vec{g}_i(x_i) \end{bmatrix} \\ &= \sum_{l=1}^k \sum_{i=1}^k \left(\begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \right) \cdot \left(\begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix}^* \begin{bmatrix} \vec{g}_i(x_{i-1}) \\ \vec{g}_i(x_i) \end{bmatrix} \right) \\ &= \sum_{l=1}^k \sum_{i=1}^k \begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_i(x_{i-1}) \\ \vec{g}_i(x_i) \end{bmatrix}. \end{aligned} \tag{13}$$

In addition, by Green's formula (2), we have

$$\begin{bmatrix} Uf \\ U_c f \end{bmatrix} \cdot \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} = \sum_{l=1}^k \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_l(x_{l-1}) \\ \vec{g}_l(x_l) \end{bmatrix}. \tag{14}$$

Since equations (13) and (14) are equal, comparison of coefficients shows that we have

$$\begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} = \begin{cases} \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} & \text{if } i = l, \\ 0_{2n\times 2n} & \text{otherwise.} \end{cases}$$

Using the above relation, we obtain the equality

$$\begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}. \tag{15}$$

Since the matrix on LHS of (15) is invertible, we can premultiply both sides by this inverse to obtain

$$E_{2nk\times 2nk} = \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix} \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}$$

By Lemma 6:

$$= \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix}^{-1} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix}^{-1} \end{bmatrix} \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix}^{-1} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k)^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}}_{\Xi}. \tag{16}$$

Note that the two matrices in (16) are square, and that the matrix Ξ is full-rank. So, the matrix Λ must be the inverse of Ξ. In other words, the following holds:

$$\begin{bmatrix} E_{m\times m} & 0_{m\times(2nk-m)} \\ 0_{(2nk-m)\times m} & E_{(2nk-m)\times(2nk-m)} \end{bmatrix} = \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix}.$$

Thus, we have

$$\begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix} = 0_{m\times(2nk-m)}.$$

This means that

$$H := \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix}_{2nk\times(2nk-m)}$$

has rank $2nk-m$. Thus, columns H also form the solution space of the system (12), just like H_1 does. But this suggests that H_1 and H are the same up to a linear transformation, i.e. there exists a non-singular matrix A of size $(2nk-m)\times(2nk-m)$ such that

$$H_1 = \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix} = HA = \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix} A = \begin{bmatrix} -F^{-1}(x_0)P_1A \\ F^{-1}(x_1)Q_1A \\ \vdots \\ -F^{-1}(x_{k-1})P_kA \\ F^{-1}(x_k)Q_kA \end{bmatrix},$$

and so $P_lA = \mathcal{P}_l$ and $Q_lA = \mathcal{Q}_l$ for all $l = 1,\ldots,k$. Therefore,

$$U_1^+g = \sum_{l=1}^{k} \mathcal{P}_l^*\vec{g}_l(x_{l-1}) + \mathcal{Q}_l^*\vec{g}_l(x_l) = \sum_{l=1}^{k} A^*P_l^*\vec{g}_l(x_{l-1}) + A^*Q_l^*\vec{g}_l(x_l) = A^*U^+g.$$

Observe that $U^+g=\vec{0}$ implies $U_1^+g=\vec{0}$. Since A^* is nonsingular, it follows that $U^+g=\vec{0}$ if and only if $U_1^+g=\vec{0}$. Since $U^+g=\vec{0}$ is adjoint to $Uf=\vec{0}, U_1^+g=\vec{0}$ is adjoint to $Uf=\vec{0}$. This completes the proof. $\square$

Lemma 6. *For the relevant matrices $P_l, Q_l, \overline{P}_l, \overline{Q}_l, M_l, N_l, \overline{M}_l, \overline{N}_l$, we have*

$$\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix}\begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix}\begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk\times 2nk} = \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}_{2nk\times 2nk} \begin{bmatrix} M_1 : N_1 : & \ldots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \ldots & : \overline{M}_k : \overline{N}_k \end{bmatrix}_{2nk\times 2nk}.$$

Proof. First, observe that we can write

$$\begin{aligned} &\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix}\begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix}\begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix}_{2nk\times 2nk^2} \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk^2\times 2nk}. \end{aligned} \tag{17}$$

Now, let $\mathcal{V}$ and $\mathcal{W}$ be matrices given by:

$$\mathcal{V}_{2nk^2\times 2nk} = \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1}; \tag{18}$$

$$\mathcal{W}_{2nk\times 2nk^2} = \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix}. \tag{19}$$

Observe that

$$\mathcal{W}\mathcal{V} = \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1} \tag{20}$$

Substitute (17):

$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1} \tag{21}$$

Recall (15):

$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1} \tag{22}$$

Recall (3), (4), and the explicit definition for J as given in the proof of Theorem 2:

$$= (J^*)^{-1} S H^{-1} = (J^*)^{-1} J^* = E_{2nk\times 2nk}. \tag{23}$$

Thus, (17) can be rewritten as:

$$\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix}_{2nk\times 2nk^2} \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk^2\times 2nk}$$

$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} \mathcal{W}_{2nk\times 2nk^2} \mathcal{V}_{2nk^2\times 2nk} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}$$

$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} E_{2nk\times 2nk} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix} = \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix},$$

which completes the proof. $\square$

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