Vandermonde Argument:

The matrix side of the original system is given by

$$\begin{bmatrix} b^0 & -e_0 & 0 & \cdots & 0 & 0 \\ b^1 & e_1 & -e_1 & \cdots & 0 & 0 \\ b^2 & 0 & e_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b^{m-1} & 0 & 0 & \cdots & e_{m-1} & -e_{m-1} \\ b^m & 0 & 0 & \cdots & 0 & e_m \end{bmatrix}^T \begin{bmatrix} \vec{f}^0(\lambda) \\ \vec{f}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vdots \\ \vec{f}^{m-1}(\lambda) \\ \vec{f}^m(\lambda) \end{bmatrix}, \quad \text{where} \quad \vec{f}^r(\lambda) = \begin{bmatrix} f_0^r(\lambda) \\ f_1^r(\lambda) \\ f_2^r(\lambda) \\ \vdots \\ f_{n-2}^r(\lambda) \\ f_{n-1}^r(\lambda) \end{bmatrix},$$

and

$$b^r = \begin{bmatrix} b_{00}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & b_{0(n-1)}^r \frac{1}{(i\lambda)^{n-1}} \\ \vdots & & \vdots \\ b_{n-1\,0}^r & \cdots & b_{(n-1)(n-1)}^r \end{bmatrix},$$

$$e_r = \begin{bmatrix} E_r(\lambda) & \cdots & E_r(\alpha^{n-1}\lambda)\,\alpha^{(n-1)(n-1)} \\ \vdots & \ddots & \vdots \\ E_r(\lambda) & \cdots & E_r(\alpha^{n-1}\lambda) \end{bmatrix}$$

Now, define B^r as follows:

$$B^r = \begin{bmatrix} \frac{1}{n} E_r(-\lambda) \sum_{j=0}^{n-1} b_{j0}^r \frac{1}{(i\lambda)^{n-1-j}} & \cdots & \frac{1}{n} E_r(-\lambda) \sum_{j=0}^{n-1} b_{j(n-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} E_r(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \alpha^{(n-1)(j+1)} b_{j0}^r \frac{1}{(i\lambda)^{n-1-j}} & \cdots & \frac{1}{n} E_r(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \alpha^{(n-1)(j+1)} b_{j(n-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \end{bmatrix}$$

Note that $e_r B^r = b^r$. This follows by examining the product $e_r B^r$: (t,s)-th entry of $e_r B^r$ is given by t-th row of e_r & s-th column of B^r

$$\frac{1}{n}\left[E_r(\lambda)\; E_r(\alpha\lambda)\alpha^{n-t}\; E_r(\alpha^2\lambda)\alpha^{2(n-t)} \cdots E_r(\alpha^{n-2}\lambda)\alpha^{(n-t)(n-2)}\; E_r(\alpha^{n-1}\lambda)\alpha^{(n-t)(n-1)}\right]$$

$$\begin{bmatrix} E_r(-\lambda) \sum_{j=0}^{n-1} b_{j(s-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \\ E_r(-\alpha\lambda) \sum_{j=0}^{n-1} b_{j(s-1)}^r \frac{\alpha^{j+1}}{(i\lambda)^{n-1-j}} \\ E_r(-\alpha^2\lambda) \sum_{j=0}^{n-1} b_{j(s-1)}^r \frac{\alpha^{2(j+1)}}{(i\lambda)^{n-1-j}} \\ \vdots \\ E_r(-\alpha^{n-2}\lambda) \sum_{j=0}^{n-1} b_{j(s-1)}^r \frac{\alpha^{(n-2)(j+1)}}{(i\lambda)^{n-1-j}} \\ E_r(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} b_{j(s-1)}^r \frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix} = \frac{1}{n} \sum_{j=0}^{n-1} b_{j(s-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \left[1 + \alpha^{n-t+j+1} + \alpha^{2(n-t+j+1)} + \cdots + \alpha^{(n-2)(n-t+j+1)} + \alpha^{(n-1)(n-t+j+1)}\right]$$

(2) Observe if $j = t-1$, then

$$1 + \alpha^{n-t+j+1} + \alpha^{2(n-t+j+1)} + \dots + \alpha^{(n-2)(n-t+j+1)} + \alpha^{(n-1)(n-t+j+1)} =$$

$$1 + \alpha^{n} + \alpha^{2n} + \dots + \alpha^{(n-2)n} + \alpha^{(n-1)n} = n.$$

If $j \neq t-1$, then

$$1 + \alpha^{n-t+j+1} + \dots + \alpha^{(n-1)(n-t+j+1)} = \sum_{k=0}^{n-1} \alpha^{(n-t+j+1)k} = \frac{\alpha^{(n-t+j+1)n} - 1}{\alpha^{n-t+j+1} - 1} = \frac{1-1}{\dots} = 0,$$

which follows by the geometric progression formula. This means that

$$\frac{1}{n} \sum_{j=0}^{n-1} b^{r}_{j(s-1)} \frac{1}{(i\lambda)^{n-1-j}} \left[1 + \alpha^{n-t+j+1} + \dots + \alpha^{(n-1)(n-t+j+1)} \right] =$$

$$\frac{1}{n} b^{r}_{(t-1)(s-1)} \frac{1}{(i\lambda)^{n-t}} n + \frac{1}{n} \sum_{\substack{j=0,\\ j \neq t-1}}^{n-1} b^{r}_{j(s-1)} \frac{1}{(i\lambda)^{n-1-j}} \cdot 0 = b^{r}_{(t-1)(s-1)} \frac{1}{(i\lambda)^{n-t}},$$

which is exactly the (t,s)-th entry of b^r. Thus, $e_r B^r = b^r$

$\Rightarrow (b^r)^T = (e_r B^r)^T$

$\Rightarrow (b^r)^T = (B^r)^T e_r^T$.

Also, note $(e_r I)^T = I^T e_r^T$

Now, we rewrite the system:

$$\begin{bmatrix} b^0 & -e_0 & 0 & \dots & 0 & 0 \\ b^1 & e_1 & -e_1 & \dots & 0 & 0 \\ b^2 & 0 & e_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b^{m-1} & 0 & 0 & \dots & e_{m-1} & -e_{m-1} \\ b^m & 0 & 0 & \dots & 0 & e_m \end{bmatrix}^T \begin{bmatrix} \vec{f}^{\,0}(\lambda) \\ \vec{f}^{\,1}(\lambda) \\ \vec{f}^{\,2}(\lambda) \\ \vdots \\ \vec{f}^{\,m-1}(\lambda) \\ \vec{f}^{\,m}(\lambda) \end{bmatrix} =$$

$$= \begin{bmatrix} (b^0)^T & (b^1)^T & (b^2)^T & \dots & (b^{m-1})^T & (b^m)^T \\ -e_0^T & e_1^T & 0 & \dots & 0 & 0 \\ 0 & -e_1^T & e_2^T & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e_{m-1}^T & 0 \\ 0 & 0 & 0 & \dots & -e_{m-1}^T & e_m^T \end{bmatrix} \begin{bmatrix} \vec{f}^{\,0}(\lambda) \\ \vec{f}^{\,1}(\lambda) \\ \vec{f}^{\,2}(\lambda) \\ \vdots \\ \vec{f}^{\,m-1}(\lambda) \\ \vec{f}^{\,m}(\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} (B^0)^T e_0^T & (B^1)^T e_1^T & (B^2)^T e_2^T & \dots & (B^{m-1})^T e_{m-1}^T & (B^m)^T e_m^T \\ -(e_0 I)^T & (e_0 I)^T & 0 & \dots & 0 & 0 \\ 0 & -(e_1 I)^T & (e_2 I)^T & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (e_{m-1} I)^T & 0 \\ 0 & 0 & 0 & \dots & -(e_{m-1} I)^T & (e_m I)^T \end{bmatrix} \begin{bmatrix} \vec{f}^{\,0}(\lambda) \\ \vec{f}^{\,1}(\lambda) \\ \vec{f}^{\,2}(\lambda) \\ \vdots \\ \vec{f}^{\,m-1}(\lambda) \\ \vec{f}^{\,m}(\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} (B^0)^T & (B^1)^T & (B^2)^T & \cdots & (B^{n-1})^T & (B^m)^T \\ -I^T & I^T & 0 & \cdots & 0 & 0 \\ 0 & -I^T & I^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & I^T & 0 \\ 0 & 0 & 0 & \cdots & -I^T & I^T \end{bmatrix} \begin{bmatrix} e_0^T & 0 & 0 & \cdots & 0 & 0 \\ 0 & e_1^T & 0 & \cdots & 0 & 0 \\ 0 & 0 & e_2^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_{m-1}^T & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_m^T \end{bmatrix} \begin{bmatrix} \vec{f}^0(\lambda) \\ \vec{f}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vdots \\ \vec{f}^{m-1}(\lambda) \\ \vec{f}^{n-1}(\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} B^0 & -I & 0 & \cdots & 0 & 0 \\ B^1 & I & -I & \cdots & 0 & 0 \\ B^2 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B^{m-1} & 0 & 0 & \cdots & I & -I \\ B^m & 0 & 0 & \cdots & 0 & I \end{bmatrix}^T \begin{bmatrix} e_0^T \vec{f}^0(\lambda) \\ e_1^T \vec{f}^1(\lambda) \\ e_2^T \vec{f}^2(\lambda) \\ \vdots \\ e_{m-1}^T \vec{f}^{n-1}(\lambda) \\ e_m^T \vec{f}^m(\lambda) \end{bmatrix}.$$

Finally, we have

$$e_r^T \vec{f}^r(\lambda) = \begin{bmatrix} E_r(\lambda) & E_r(\lambda) & \cdots & E_r(\lambda) \\ E_r(\alpha\lambda)\,\alpha^{n-1} & E_r(\alpha\lambda)\,\alpha^{n-2} & \cdots & E_r(\alpha\lambda) \\ \vdots & \vdots & \ddots & \\ E_r(\alpha^{n-1}\lambda)\,\alpha^{(n-1)(n-1)} & E_r(\alpha^{n-1}\lambda)\,\alpha^{(n-1)(n-2)} & \cdots & E_r(\alpha^{n-1}\lambda) \end{bmatrix} \begin{bmatrix} f_0^r(\lambda) \\ f_1^r(\lambda) \\ \vdots \\ f_{n-1}^r(\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} E_r(\lambda) f_0^r(\lambda) + E_r(\lambda) f_1^r(\lambda) + \cdots + E_r(\lambda) f_{n-1}^r(\lambda) \\ E_r(\alpha\lambda) f_0^r(\lambda)\alpha^{n-1} + E_r(\alpha\lambda) f_1^r(\lambda)\,\alpha^{n-2} + \cdots + E_r(\alpha\lambda) f_{n-1}^r(\lambda) \\ \vdots \\ E_r(\alpha^{n-1}\lambda) f_0^r(\lambda)\,\alpha^{(n-1)(n-1)} + E_r(\alpha^{n-1}\lambda) f_1^r(\lambda)\,\alpha^{(n-1)(n-2)} + \cdots + E_r(\alpha^{n-1}\lambda) f_{n-1}^r(\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{n-1} E_r(\lambda) f_k^r(\lambda) \\ \sum_{k=0}^{n-1} E_r(\alpha\lambda)\,\alpha^{n-1-k} f_k^r(\lambda) \\ \vdots \\ \sum_{k=0}^{n-1} E_r(\alpha^{n-1}\lambda)\,\alpha^{(n-1)(n-1-k)} f_k^r(\lambda) \end{bmatrix},$$

so that

$$\begin{bmatrix} e_0^T \vec{f}^0(\lambda) \\ e_1^T \vec{f}^1(\lambda) \\ \vdots \\ e_m^T \vec{f}^m(\lambda) \end{bmatrix} = \underbrace{\begin{bmatrix} E_r(\lambda) \sum_{k=0}^{n-1} f_k^r \\ E_r(\alpha\lambda) \sum_{k=0}^{n-1} \alpha^{n-1-k} f_k^r(\lambda) \\ E_r(\alpha^{n-1}\lambda) \sum_{k=0}^{n-1} \alpha^{(n-1)(n-1-k)} f_k^r(\lambda) \end{bmatrix}}_{r = 0, 1, \ldots, m.}$$

This completes the Vandermonde argument.