Summary of Locker's Self-Adjointness For Multipoint Differential Operators

The set-up is as follows: given a closed interval $[a, b]$, fix $n \in \mathbb{N}$, so that the differential operator is given by

$$\tau := \sum_{k=0}^{n} a_k(t) \left(\frac{d}{dt}\right)^k, \text{ where } a_k(t) \in C^\infty[a, b] \text{ and } a_n(t) \neq 0 \ \forall t \in [a, b].$$

Let $S := L^2[a, b]$ with the standard inner product, and let $H^n[a, b]$ be a subspace of S consisting of $f \in C^{n-1}[a, b]$, with $f^{(n-1)}$ absolutely continuous (i.e. differentiable almost everywhere), and $f^{(n)} \in S$. Fix $m \in \mathbb{N}$, and let $\pi = \{a = x_0 < x_1 < \ldots < x_m = b\}$ be a partition of $[a, b]$. Finally, let $H^n(\pi)$ be a collection of $f \in S$ such that

1. On each subinterval $[x_{l-1}, x_l], f(t)$ possesses right-hand and left-hand limits at the endpoints x_{l-1} and x_l respectively. Moreover, let f_l denote the function f on subinterval (x_{l-1}, x_l), and let $f_l(x_{l-1}) = f(x_{l-1}^+)$ and $f_l(x_l) = f(x_l^-)$. We call $f_1, \ldots, f_m$ the components of f and denote this by writing $f = (f_1, \ldots, f_m)$.

2. For $l = 1, \ldots, m, f_l \in H^n[x_{l-1}, x_l]$.

We define a *multipoint boundary value* (MBV) to be a linear functional B on $H^n(\pi)$ of the form

$$B(f) = \sum_{l=1}^{m} \sum_{j=0}^{n-1} [\alpha_{jl} f_l^{(j)}(x_{l-1}) + \beta jl f_l^{(j)}(x_l)]$$

where $f = (f_1, \ldots, f_m)$, and $\alpha_{jl}, \beta_{jl} \in \mathbb{R}$. Note that since every boundary value consists of a double sum, and for all l, j we expect that $f_l^{(j)}(x_{l-1})$ and $f_l^{(j)}(x_l)$ to be linearly independent, the space of all boundary values has dimension $2mn$.

Now, suppose we are given a set of k linearly independent MBVs

$$B_i(f) = \sum_{l=1}^{m} \sum_{j=0}^{n-1} [\alpha_{ijl} f_l^{(j)}(x_{l-1}) + \beta ijl f_l^{(j)}(x_l)], \qquad i \in \{1, \ldots, k\}.$$

Let L be an operator given by $Lf = \tau f$, whose domain is

$$\mathcal{D}(L) = \{f \in H^n(\pi) \mid B_i(f) = 0, i = 1, \ldots, k\}.$$

Then L is a *multipoint differential operator*. Observe that density of $\mathcal{D}(L)$ in S ensures that L has a well-defined adjoint L^*. Our goal is to obtain an explicit formula for L^* and its domain. First, recall the Green's formula from Dunford & Schwartz: given $f, g \in H^n(\pi)$,

$$\langle \tau f, g \rangle - \langle f, \tau^* g \rangle = \sum_{l=1}^{m} \sum_{p,q=0}^{n-1} [F_{x_l}^{p\,q}(\tau) f_l^{(p)}(x_l) g_l^{(q)}(x_l) - F_{x_{l-1}}^{p\,q}(\tau) f_l^{(p)}(x_{l-1}) g_l^{(q)}(x_{l-1})],$$

where F_t denotes an n×n boundary matrix for τ at the point $t \in [a, b]$. From Dunford & Schwartz, the entries of F_t are given by

$$\begin{aligned}
F_t^{p\,q}(\tau) &= \sum_{k=j}^{n-p-1} (-1)^k \binom{k}{j} \left(\frac{d}{dt}\right)^{k-j} a_{p+k+1}(t), && p + q < n - 1 \\
F_t^{p\,q}(\tau) &= (-1)^q a_n(t), && p + q = n - 1 \\
F_t^{p\,q}(\tau) &= 0, && p + q > n - 1.
\end{aligned}$$

Consider a linear system of equations

$$\sum_{l=1}^{m}\sum_{j=0}^{n-1}[\alpha_{ijl}x_{jl}+\beta_{ijl}y_{jl}]=0,\quad i=1,\dots,k.$$

Since the list $\{B_i\}_{i=1}^{k}$ is linearly independent, the above system has rank k, and so dim range$\{B_i\}_{i=1}^{k}=k$. Now, since the system is homogeneous, the solution space must have the same dimension as the null space of the system. Since $\dim\{B_i\}_{i=1}^{k}=2mn$, it follows by the fundamental theorem of linear maps that

$$\dim\text{ null}\{B_i\}_{i=1}^{k}=\dim\{B_i\}_{i=1}^{k}-\dim\text{ range}\{B_i\}_{i=1}^{k}=2mn-k.$$

Thus, let $[x_{ijl},y_{ijl}], i=1,\dots,2mn-k$ be the set of solutions of the above system that also form a basis for the solution space. Define

$$\alpha^*_{ijl}=-\sum_{p=0}^{n-1}x_{ipl}F^{p,\ q}_{x_{l-1}}(\tau)\qquad\text{and}\qquad\beta^*_{ijl}=\sum_{p=0}^{n-1}y_{ipl}F^{p,\ q}_{x_l}(\tau),$$

where $i=1,\dots,2mn-k, j=0,\dots,n-1, l=1,\dots,m$. Finally, let

$$B^*_i(f)=\sum_{l=1}^{m}\sum_{j=0}^{n-1}[\alpha^*_{ijl}f_l^{(j)}(x_{l-1})+\beta ijl^*f_l^{(j)}(x_l)],\qquad i\in\{1,\dots,2mn-k\}.$$

We refer to B^*_i as *adjoint* multipoint boundary values. In the following theorem, we prove that the above construction is valid.

Theorem 1. *The adjoint operator L^* is the multipoint differential operator defined by*

$$\mathcal{D}(L^*)=\{f\in H^n(\pi)\mid B^*_i(f)=0,i=1,\dots,2mn-k\},L^*f=\tau^*f.$$

Proof. First, we will define L_0 to be what we think is the adjoint of L. That is, let L_0 be the linear operator in S whose domain consists of all functions $f\in H^n(\pi)$ satisfying $B^*_i(f)=0$ for $i=1,\dots,2mn-k$ with $L_0f=\tau^*f$. We want to show that $L_0=L^*$. First, we show that $\mathcal{D}(L_0)\subseteqq\mathcal{D}(L^*)$.

Let $g\in\mathcal{D}(L_0)$ and set $g^*=L_0g=\tau^*g$. Then, let $f\in\mathcal{D}(L)$. Now, we want to show that $\langle Lf,g\rangle=\langle f,L_0g\rangle=\langle f,g^*\rangle$. Recall that the numbers

$$x_{jl}=f_l^{(j)}(x_{l-1}),\quad y_{jl}=f_l^{(j)}(x_l),$$

form the solutions to the system

$$\sum_{l=1}^{m}\sum_{j=0}^{n-1}[\alpha_{ijl}x_{jl}+\beta_{ijl}y_{jl}]=0,\quad i=1,\dots,k.$$

Also recall that as defined earlier, $[x_{ijl},y_{ijl}], i=1,\dots,2mn-k$ is the set of solutions of the above system which form a basis for the solution space. So, by definition of a basis, there exist constants $c_1,\dots,c_{2mn-k}$ such that

$$f_l^{(j)}(x_{l-1})=\sum_{i=1}^{2mn-k}c_ix_{ijl}\text{ and }f_l^{(j)}(x_l)=\sum_{i=1}^{2mn-k}c_iy_{ijl}.$$

Now, we apply Green's formula to obtain

$$\begin{aligned}
\langle Lf, g\rangle - \langle f, g^*\rangle &= \langle \tau f, g\rangle - \langle f, \tau^* g\rangle \\
&= \sum_{l=1}^{m}\sum_{p,q=0}^{n-1}[F^{p\,q}_{x_l}(\tau)f_l^{(p)}(x_l)g_l^{(q)}(x_l) - F^{p\,q}_{x_{l-1}}(\tau)f_l^{(p)}(x_{l-1})g_l^{(q)}(x_{l-1})] \\
\text{(Substitute for the } f_l^{(j)}\text{)} \quad &= \sum_{l=1}^{m}\sum_{p,q=0}^{n-1}\sum_{i=1}^{2mn-k} c_i[F^{p\,q}_{x_l}(\tau)y_{ipl}g_l^{(q)}(x_l) - F^{p\,q}_{x_{l-1}}(\tau)x_{ipl}g_l^{(q)}(x_{l-1})] \\
\text{(Substitute in } \alpha^*, \beta^*\text{)} \quad &= \sum_{i=1}^{2mn-k} c_i \sum_{l=1}^{m}\sum_{q=0}^{n-1}[\beta^*_{iql}g_l^{(q)}(x_l) + \alpha^*_{iql}g_l^{(q)}(x_{l-1})] \\
&= \sum_{i=1}^{2mn-k} c_i B_i^*(g) \\
\text{(By definition of } g\text{)} \quad &= 0
\end{aligned}$$

Since $f \in \mathcal{D}(L)$ is arbitrary, and $\langle Lf, g\rangle = \langle f, L_0 g\rangle = \langle f, g^*\rangle$, we can conclude that $g \in \mathcal{D}(L^*)$, which implies $\mathcal{D}(L_0) \subseteqq \mathcal{D}(L^*)$.

To complete the proof, it remains to show that $\mathcal{D}(L^*) \subseteqq \mathcal{D}(L_0)$. Let $g \in \mathcal{D}(L^*)$. Now, we want to show that $g \in H^n(\pi)$ and that $B_i^*(g) = 0$, which would imply that $g \in \mathcal{D}(L_0)$ by definition of L_0. Fix an integer l with $1 \leq l \leq m$, and let $\bar{g}$ denote the restriction of g to the interval $[x_{l-1}, x_l]$. Let $\bar{f}$ be any function in $H^n[x_{l-1}, x_l]$ having its support in the open interval (x_{l-1}, x_l). Then, we can extend $\bar{f}$ to f defined on $[a, b]$ by making it 0 outside of $[x_{l-1}, x_l]$. The extension of f belongs in $\mathcal{D}(L^*)$ because $f \in H^n(\pi)$, and $B_i(f) = 0$ (because it is 0 at all boundary points). Then,

$$0 = \langle Lf, g\rangle - \langle f, L^* g\rangle = \int_{x_{l-1}}^{x_l} (\tau\bar{f})\bar{g} - \int_{x_{l-1}}^{x_l} \bar{f}(L^* g).$$

By Theorem 10 of [1, p. 1294], the above implies that $\bar{g}$ is equal a.e to a function in $H^n[x_{l-1}, x_l]$ and that $L^* g = \tau^* \bar{g}$ a.e. on $[x_{l-1}, x_l]$. Since this holds for all l, we can conclude $g \in H^n(\pi)$ and $L^* g = \tau^* g$.

Next, we want to show that $B_i^*(g) = 0$. Fix an integer i with $1 \leq i \leq 2mn - k$ and choose a function $\sigma = (\sigma_1, \ldots, \sigma_m) \in H^n(\pi)$ such that $\sigma_l^{(j)}(x_{l-1}) = x_{ijl}$ and $\sigma_l^{(j)}(x_l) = y_{ijl}$. That is, evaluating σ at each boundary point yields the set of solutions that form the basis for the solution space. Clearly, $\sigma \in \mathcal{D}(L)$, and from Green's formula

$$\begin{aligned}
0 &= \langle L\sigma, g\rangle - \langle \sigma, L^* g\rangle \\
&= \langle \tau\sigma, g\rangle - \langle \sigma, \tau^* g\rangle \\
&= \sum_{l=1}^{m}\sum_{p,q=0}^{n-1}[F^{p\,q}_{x_l}(\tau)\sigma_l^{(p)}(x_l)g_l^{(q)}(x_l) - F^{p\,q}_{x_{l-1}}(\tau)\sigma_l^{(p)}(x_{l-1})g_l^{(q)}(x_{l-1})] \\
&= \sum_{l=1}^{m}\sum_{p,q=0}^{n-1}[F^{p\,q}_{x_l}(\tau)y_{ipl}g_l^{(q)}(x_l) - F^{p\,q}_{x_{l-1}}(\tau)x_{ipl}g_l^{(q)}(x_{l-1})] \\
&= \sum_{l=1}^{m}\sum_{q=0}^{n-1}[\beta^*_{iql}g_l^{(q)}(x_l) + \alpha^*_{iql}g_l^{(q)}(x_{l-1})] \\
&= B_i^*(g)
\end{aligned}$$

So, we have shown that if given $g \in \mathcal{D}(L^*)$, $B_i^*(g) = 0$. That together with $g \in H^n(\pi)$ proven earlier implies that $g \in \mathcal{D}(L_0)$. Thus, $L_0 = L^*$. $\square$

References

[1] N. Dunford and J. T. Schwartz. *Linear Operators II.* Interscience, New York, 1963.

[2] Locker John. Self-Adjointness for multi-point differential operators. *Pacific Journal of Mathematics*, 45, 1973.