## IMVP Transform Pair

Let $x_0 = a < x_1 < \ldots < x_k = b$ be the partition of $[a, b]$, and consider the following problem:

$$(\partial_t + a(-i\partial_x)^n)q(x,t) = 0 \tag{PDE}$$
$$q(x,0) = f(x) \tag{IC}$$
$$Uq = \sum_{l=1}^{k} R_l \vec{\mathbf{q}}_l(x_{l-1}) + N_l \vec{\mathbf{q}}_l(x_l) = \vec{0}, \tag{MBC}$$

where $\vec{\mathbf{q}}_j(x_i) = \begin{bmatrix} q_j(x_i) \\ q_j'(x_i) \\ \vdots \\ q_j^{(n-1)}(x_i) \end{bmatrix}$, and $R_l, N_l$ are $n \times n$, full-rank matrices. Note that by the document IMVP Construction, there exists a sequence $\{P_l, Q_l\}_{l=1}^k$ where $P_l, Q_l$ are full-rank, $n \times n$ matrices, such that

$$U^+q = \sum_{l=1}^{k} P_l \vec{\mathbf{q}}_l(x_{l-1}) + Q_l \vec{\mathbf{q}}_l(x_l) = \vec{0}$$

is the adjoint multipoint BC. Now, let $\alpha = e^{2\pi i/n}$ and fix $s \in \{1, 2, \ldots, k\}$. Define the entries of the matrices $(M_s)^{\pm}(\lambda)$ entrywise by

$$(M_s)^+_{kj}(\lambda) = \sum_{r=0}^{n-1} (-i\alpha^{k-1}\alpha)^r (P_s)_{jr} \qquad (M_s)^-_{kj}(\lambda) = \sum_{r=0}^{n-1} (-i\alpha^{k-1}\alpha)^r (Q_s)_{jr}.$$

Then, define the matrix $(M_s)(\lambda)$ entrywise by

$$(M_s)_{kj}(\lambda) = (M_s)^+_{kj}(\lambda) + (M_s)^-_{kj}(\lambda) e^{-i\alpha^{k-1}\lambda},$$

and let $\Delta_s(\lambda) = \det[(M_s)(\lambda)]$, and define $X_s^{lj}$ as the $(n-1)\times(n-1)$ submatrix of $M_s$ with $(1,1)$ entry the $(l+1, j+1)$ entry of $M_s$. Thus, we have a sequence of matrices $\{M_s\}_{s=1}^k$ along with the associated sequences of determinants $\{M_s\}_{s=1}^k$ and matrices $\{X_s^{lj}\}_{s=1}^k$. Then, the transform pair is given by

$$f(x) \mapsto F(\lambda): \qquad F_\lambda(f) = \begin{cases} (F_1)^+_\lambda(f) & \text{if } \lambda \in (\Gamma_1)^+_0 \cup (\Gamma_1)^+_a \\ (F_1)^-_\lambda(f) & \text{if } \lambda \in (\Gamma_1)^-_0 \cup (\Gamma_1)^-_a \\ (F_2)^+_\lambda(f) & \text{if } \lambda \in (\Gamma_2)^+_0 \cup (\Gamma_2)^+_a \\ (F_2)^-_\lambda(f) & \text{if } \lambda \in (\Gamma_2)^-_0 \cup (\Gamma_2)^-_a \\ \ldots\ldots \\ (F_k)^+_\lambda(f) & \text{if } \lambda \in (\Gamma_k)^+_0 \cup (\Gamma_k)^+_a \\ (F_k)^-_\lambda(f) & \text{if } \lambda \in (\Gamma_k)^-_0 \cup (\Gamma_k)^-_a \end{cases}$$

$$F(\lambda) \mapsto f(x): \qquad f_x(F) = \sum_{s=1}^{k} \int_{\Gamma_s} e^{i\lambda x} F(\lambda)\, \mathrm{d}\lambda,$$

where, for $\lambda \in \mathbb{C}$ such that $\Delta_s(\lambda) \neq 0$,

$$(F_s)^+_\lambda = \frac{1}{2\pi\Delta_s(\lambda)} \sum_{l=1}^{n} \sum_{j=1}^{n} (-1)^{(n-1)(l+j)} \det[X_s^{lj}(\lambda)](M_s)^+_{1j}(\lambda) \int_{x_{s-1}}^{x_s} e^{-i\alpha^{l-1}\lambda x} f(x)\, \mathrm{d}x$$

$$(F_s)^-_\lambda = \frac{-e^{-i\lambda}}{2\pi\Delta_s(\lambda)} \sum_{l=1}^{n} \sum_{j=1}^{n} (-1)^{(n-1)(l+j)} \det[X_s^{lj}(\lambda)](M_s)^-_{1j}(\lambda) \int_{x_{s-1}}^{x_s} e^{-i\alpha^{l-1}\lambda x} f(x)\, \mathrm{d}x,$$

and various contours are defined by

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\begin{aligned}
\Gamma_s &= (\Gamma_s)_0 \cup (\Gamma_s)_a \\
\Gamma_s)_0 &= (\Gamma_s)_0^+ \cup (\Gamma_s)_0^- \\
(\Gamma_s)_0^+ &= \bigcup_{\sigma \in \mathbb{C}^+ :\ \Delta_s(\sigma)=0} C(\sigma, \epsilon) \\
(\Gamma_s)_0^- &= \bigcup_{\sigma \in \mathbb{C}^- :\ \Delta_s(\sigma)=0} C(\sigma, \epsilon) \\
(\Gamma_s)_a &= (\Gamma_s)_a^+ \cup (\Gamma_s)_a^- \\
(\Gamma_s)_a^\pm &\text{ is the boundary of the domain } \{\lambda \in \mathbb{C}^\pm : \Re(a\lambda^n) > 0\} \setminus \bigcup_{\sigma \in \mathbb{C} :\ \Delta_s(\sigma)=0} D(\sigma, 2\epsilon).
\end{aligned}
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Finally, let's prove that the transform pair, as defined above, actually works. In the formulation of the theorem, we borrow wording from Proposition 2.3 of Fokas and Smith's paper. We will also refer to this paper as the paper for convenience.

**Proposition 1.** *Let $S$ be a type I or type II operator. Then, for all $f \in \Phi$, and for all $x \in (0, 1)$,*

$$f_x(F_\lambda(f)) = f(x).$$

*Proof.* First, following almost the same argument as in the paper, it can be shown that for any $s = 1, \dots, k$, we have

$$\forall f \in C, \forall S, \qquad (F_s)_\lambda^+ - (F_s)_\lambda^- = \frac{1}{2\pi} \hat{f}_s(\lambda),$$

where $f_s$ refers to the restriction of $f$ to the interval $[x_{s-1}, x_s]$. Now, **we shall assume without proof** that for all $s = 1, \dots, k, (F_s)_\lambda^\pm = \mathcal{O}(\lambda^{-1})$ as $\lambda \to \infty$. In the paper, this assumption is justified by the well-posedness of the problem, but in our case, we have yet to make a similar connection. From the assumption, it follows that by Jordan's lemma and contour deformation

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\begin{aligned}
f_x(F_\lambda(f)) &= \sum_{s=1}^k \int_{\Gamma_s} e^{i\lambda x} F(\lambda) \, \mathrm{d}\lambda \\
&= \sum_{s=1}^k \left\{ \int_{(\Gamma_s)_0^+} + \int_{(\Gamma_s)_a^+} \right\} e^{i\lambda x} (F_s)_\lambda^+(f) \, \mathrm{d}\lambda + \left\{ \int_{(\Gamma_s)_0^-} + \int_{(\Gamma_s)_a^-} \right\} e^{i\lambda x} (F_s)_\lambda^-(f) \, \mathrm{d}\lambda \\
&= \sum_{s=1}^k \sum_{\sigma \in \mathbb{C} :\ \Im(\sigma) > \epsilon,\ \Delta_s(\sigma)=0} \left\{ \int_{C(\sigma,\epsilon)} - \int_{C(\sigma,2\epsilon)} \right\} e^{i\lambda x} (F_s)_\lambda^+(f) \, \mathrm{d}\lambda \\
&+ \sum_{\sigma \in \mathbb{C} :\ \Im(\sigma) < \epsilon,\ \Delta_s(\sigma)=0} \left\{ \int_{C(\sigma,\epsilon)} - \int_{C(\sigma,2\epsilon)} \right\} e^{i\lambda x} (F_s)_\lambda^-(f) \, \mathrm{d}\lambda \\
&+ \int_{\gamma_k} e^{i\lambda x} ((F_s)_\lambda^+ - (F_s)_\lambda^-) \, \mathrm{d}\lambda \\
&= \sum_{s=1}^k f_s(x) = f(x).
\end{aligned}
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□