Summary of "Nonlocal and multipoint boundary value problems for linear evolution equations" paper

First, the paper describes the problems that we are dealing with in section 2: Let $m, n \in \mathbb{N}$ be independent,

$$0 = \eta_0 < \eta_1 < \cdots < \eta_m = 1$$

and

$$b_{k\,j} \in C \text{ for } k, j \in \{0, \ldots, n-1\}, r \in \{0, \ldots, m\}$$

$$\begin{aligned} [\partial_t + a(-i\partial_x)^n]q(x,t) &= 0 && (x,t) \in (0,1) \times (0,T), \\ q(x,0) &= q_0(x) && x \in [0,1], \\ \sum_{k=0}^{n-1} \sum_{r=0}^{m} b_{k\,j} \partial_x^k q(\eta_r, t) &= g_j(t) && t \in [0,T], j \in \{0,1,\ldots,n-1\} \end{aligned}$$

Because there are so many variables in play here, I will describe them in words:

- a is the coefficient in front of the ∂_x and a has to fulfill some conditions for the PDE to be well-posed.
- n is the order of the PDE
- r counts the multipoints from 0 up to m (so there are m+1 points)
- k counts the order from 0 to $n-1$. (because our multipoint conditions may contain derivatives up to 1 less than the order of the PDE)
- j counts the equations which describe the multipoint conditions (and there should be n of these equations)
- g is some known function of t
- q is the function of x and t that we want to find.

Then, in 2.1 paper also describes how the problem described above also generalizes to other non-local conditions, where instead of the multipoint boundary conditions, we are given something in the form,

$$\begin{aligned} \sum_{k=0}^{n-1} \sum_{r=0}^{m} b_{k\,j} \partial_x^k q(\eta_r, t) &= g_j(t) && t \in [0,T], j \in \{0,1,\ldots,J\} \\ \sum_{k=0}^{n-1} \sum_{r=0}^{m} \int_{\eta_{r-1}}^{\eta_r} b_{k\,j} \partial_x^k q(x,t) &= g_j(t) && t \in [0,T], j \in \{J+1,\ldots,n-1\} \end{aligned}$$

with $J \in \{-1, \ldots, n-1\}$, given that the initial datum q_0 is compatible with the conditions.

Next, the paper derives the formal representation of the solution using the Fokas transform. The method used here is different from what we did in class, where we manually applied the Fourier transform and integrating to get the global relation. Instead, the paper invents some new notations and uses Green's theorem to simplify the process. Ultimately, this is the representation

presented in (3.10),

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - a\lambda^n t}\hat{q}_0(\lambda)d\lambda - \int_{\partial D_R^+} e^{i\lambda x - a\lambda^n t}\int_0^\tau e^{a\lambda^n s}\sum_{k=0}^{n-1} c_k(\lambda)\partial_x^k q(0,s)dsd\lambda - \int_{\partial D_R^-} e^{i\lambda(x-1) - a\lambda^n t}\int_0^\tau e^{a\lambda^n s}\sum_{k=0}^{n-1} c_k(\lambda)\partial_x^k q(1,s)dsd\lambda.$$

Then, the paper defines some new notations to make the previous equation more concise and to gear it more towards multipoint conditions. The most notable new notation is the time transform of the value of $\partial_x^k q$ at $x = \eta_r$:

$$f_k^r(\lambda) = c_k(\lambda)\int_0^\tau e^{a\lambda^n s}\partial_x^k q(\eta_r, s)ds$$

with $r \in \{0,\dots,m\}$, $\tau \in [0,T]$. So, we can rewrite the solution representation as:

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - a\lambda^n t}\hat{q}_0(\lambda)d\lambda - \int_{\partial D_R^+} e^{i\lambda x - a\lambda^n t}\sum_{k=0}^{n-1} f_k^0(\lambda,\tau)d\lambda - \int_{\partial D_R^-} e^{i\lambda(x-1) - a\lambda^n t}\sum_{k=0}^{n-1} f_k^m(\lambda,\tau)d\lambda.$$

Note that now all we need to do is find $\sum_{k=0}^{n-1} f_k^0(\lambda,\tau)$ and $\sum_{k=0}^{n-1} f_k^m(\lambda,\tau)$ and substitute them into the above equation. To do so, we need to use the global relations in each of the intervals $[\eta_{r-1},\eta_r]$, so we have a set of m global relations (there are $m+1$ points so m intervals). Also, each relation can be evaluated at $\lambda, \alpha\lambda, \dots, \alpha^{n-1}\lambda$, with α being the nth root of 1, so for each relation we can get n equations. The global relations can be written as follows

$$\sum_{k=0}^{n-1}\alpha^{(n-1-k)p}[E_{r-1}(a^p\lambda)f_k^{r-1}(\lambda) - E_r(a^p\lambda)f_k^r(\lambda)] = \hat{q}_0^r(a^p\lambda) - e^{a\lambda^n\tau}\hat{q}_\tau^r(a^p\lambda)$$

with exponential function E; and with $\hat{q}$ being the Fourier transform of q restricted to the interval $[\eta_{r-1},\eta_r]$; and $p \in \{0,\dots,n-1\}$.

Now, we have mn equations, but there are $(m+1)n$ unknowns, which are the $f_k^r(\lambda)$, because there are n possible ks and $m+1$ possible rs. The n additional equations can be obtained using the n multipoint conditions and taking the time transform of the known g_j, and we will call the result of that h_j. Ultimately, we can write the problem in terms of matrix multiplication:

$$FB = (h_0(\lambda),\dots,h_{n-1}(\lambda),\overbrace{-\hat{q}_0^r(\lambda),-\hat{q}_0^r(\alpha\lambda),\cdots-\hat{q}_0^r(\alpha^{n-1}\lambda)}^{\text{r=1,2,...,m}}) + e^{a\lambda^n\tau}(0,\dots,0,\overbrace{\hat{q}_\tau^r(\lambda),\hat{q}_\tau^r(\alpha\lambda),\dots\hat{q}_\tau^r(\alpha^{n-1}\lambda)}^{\text{r=1,2,...,m}})$$

with F being the matrix of the unknowns f_k^r and B being a very complicated matrix that describes the equations we define earlier. Note that the right side of this equation are all known, except the $\hat{q}_\tau^r$ terms, which will disappear later due to Jordan's lemma, so we simply pretend that we know them for now. Then, after some rearranging of the equation done in the paper, we use Cramer's rule to solve for the f_k^r we need in the form $\sum_{k=0}^{n-1} f_k^0(\lambda,\tau)$ and $\sum_{k=0}^{n-1} f_k^m(\lambda,\tau)$. Then, substitute them back into the solution representation and we are done. There are also examples of how this method is applied to PDEs of 2nd and 3rd order.