Algorithmic Construction of an Adjoint of an Ordinary Differential Operator in julia

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Abstract

Given an operator, its adjoint is a linear transformation that can be used to characterise a wide range of phenomena in mathematics and physics. In this project we seek to characterise an adjoint of an ordinary differential operator with multipoint homogeneous conditions. Extending the ideas of Linfan Xiao, we show that the operator and its adjoint satisfy the multipoint form formula, and provide an explicit checking test to determine whether a given set o1f multipoint conditions satisfies the adjoint equation. Finally, we implement a method of deriving the adjoint multipoint conditions in programming language julia, thus providing a fast way to compute the adjoint.

Formulation of the Problem

Consider a closed interval $[a, b]$. Fix $n \in \mathbb{N}$, and let the differential operator be defined as

$$L := \sum_{k=0}^{n} a_k(t) \left(\frac{d}{dt}\right)^k, \text{ where } a_k(t) \in C^\infty[a,b] \text{ and } a_n(t) \neq 0 \,\forall t \in [a,b]. \tag{1}$$

Fix $k \in \mathbb{N}$, and let $\pi = \{a = x_0 < x_1 < \ldots < x_k = b\}$ be a partition of $[a, b]$. Consider a homogeneous multipoint BVP of rank m

$$\pi_m : Lq = 0, \qquad Uq = \vec{0},$$

where $U = (U_1, \ldots, U_m)$ is a multipoint boundary form with

$$U_i(q) = \sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l)], \qquad i \in \{1, \ldots, m\}, \tag{2}$$

where $\alpha_{ijl}, \beta_{ijl} \in \mathbb{R}$, $q \in C_\pi^{n-1}[a, b]$. Our goal is to construct the adjoint multipoint value problem (MVP) to π_m

$$\pi^+_{2nk-m} : L^+ q = 0, \qquad U^+ q = \vec{0},$$

with

$$L^+ := \sum_{k=0}^{n} (-1)^k \overline{a_k}(t) \left(\frac{d}{dt}\right)^k,$$

where $\overline{a_k}(t)$ is the complex conjugate of $a_k(t)$, $k = 0, \ldots, n$, and U^+ is an appropriate multipoint boundary form. Observe that for π^+_{2nk-m} to be an adjoint problem to π_m, we must have

$$\langle Lq, q \rangle = \langle q, L^+ q \rangle.$$

Multipoint Boundary Form Formula

Definition 1 *If $U = (U_1, \ldots, U_m)$ is any multipoint boundary form with $\operatorname{rank}(U) = m$, and $U_c = (U_{m+1}, \ldots, U_{2nk})$ is a multipoint boundary form with $\operatorname{rank}(U_c) = 2nk - m$ such that $\operatorname{rank}(U_1, \ldots, U_{2nk}) = 2nk$, then U and U_c are* ***complementary multipoint boundary forms****.*

Theorem 2 (Multipoint Boundary Form Formula) *Given any boundary form U of rank m, and any complementary form U_c, there exist unique boundary forms U_c^+, U^+ of rank m and $2nk - m$, respectively, such that*

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g. \tag{3}$$

Definition of the Adjoint

Theorem 2 allows us to define an adjoint multipoint boundary form. Namely,

Definition 3 *Suppose $U = (U_1, \ldots, U_m)$ is a multipoint boundary form with $\operatorname{rank}(U) = m$, along with the condition that $Uq = \vec{0}$ for functions $q \in C_\pi^{n-1}[a, b]$. If U^+ is any boundary form with $\operatorname{rank}(U^+) = 2nk - m$, determined as in Theorem 2, then the equation*

$$U^+ q = \vec{0}$$

is an ***adjoint multipoint boundary form*** *to $Uq = \vec{0}$.*

In turn, the above lets us define the adjoint multipoint problem:

Definition 4 *Suppose $U = (U_1, \ldots, U_m)$ is a multipoint boundary form with $\operatorname{rank}(U) = m$. Then, the problem of solving*

$$\pi_m : Lq = 0, \qquad Uq = \vec{0},$$

is called a homogeneous multipoint boundary value problem of rank m. The problem of solving

$$\pi^+_{2nk-m} : L^+ q = 0, \qquad U^+ q = \vec{0},$$

is an ***adjoint multipoint value problem*** *to π_m.*

The preceding construction allows us to state the following:

Proposition 5 *Let $f, g \in C_\pi^{n-1}[a, b]$ with $Uf = \vec{0}$ and $U^+ g = \vec{0}$. Then, $\langle Lf, g \rangle = \langle f, L^+ g \rangle$.*

We apply multipoint boundary form formula:

$$\begin{aligned}\langle Lf, g \rangle - \langle f, L^+ g \rangle = \sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) &= Uf \cdot U_c^+ g + U_c f \cdot U^+ g \\ &= \vec{0} \cdot U_c^+ g + U_c f \cdot \vec{0} \\ &= 0.\end{aligned}$$

Checking Adjointness

Note that we can rewrite the multipoint conditions Uf, $U_c f$, $U_c^+ g$, $U^+ g$ as follows:

$$\begin{bmatrix} Uf \\ U_c f \end{bmatrix} = \sum_{l=1}^{k} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \text{ and } \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} = \sum_{l=1}^{k} \begin{bmatrix} \overline{P}_l & P_l \\ \overline{Q}_l & Q_l \end{bmatrix}^* \begin{bmatrix} \vec{g}_l(x_{l-1}) \\ \vec{g}_l(x_l) \end{bmatrix}. \tag{4}$$

Then, we have the following result:

Lemma 6 *For the relevant matrices $P_l, Q_l, \overline{P}_l, \overline{Q}_l, M_l, N_l, \overline{M}_l, \overline{N}_l$, we have*

$$\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk \times 2nk}$$

$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}_{2nk \times 2nk} \begin{bmatrix} M_1 : N_1 : & \ldots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \ldots & : \overline{M}_k : \overline{N}_k \end{bmatrix}_{2nk \times 2nk}.$$

Using lemma (6), the following theorem is proven.

Theorem 7 *The boundary condition $U^+ g = \vec{0}$ is adjoint to $Uf = \vec{0}$ if and only if*

$$\sum_{l=1}^{k} M_l F^{-1}(x_{l-1}) P_l = \sum_{l=1}^{k} N_l F^{-1}(x_l) Q_l,$$

where $F(t)$ is the $n \times n$ boundary matrix.

Implementation in julia

The proof of Theorem 2 provides an explicit way to construct the matrices P_l, Q_l in (4), which we use to define a function $\mathrm{get\_adjointU}$. Furthermore, we use Theorem 7 to define a function $\mathrm{check\_adjointU}$, to check whether the multipoint conditions obtained from $\mathrm{get\_adjointU}$ satisfy the adjoint equation.

input : A partitioned interval π, the list of functions $a_k(t)$ from (1), multipoint conditions (2)

output: Adjoint Multipoint Conditions

begin

$L \longleftarrow (\pi, \{a_k(t)\}_{k=0}^n)$;

$\mathrm{aDerivMatrix} \longleftarrow \begin{bmatrix} a_0 & \ldots & a_0^{n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \ldots & a_{n-1}^{n-1} \end{bmatrix}$;

$\mathrm{adjointU} \longleftarrow \mathrm{get\_adjointU}(L, U, \mathrm{aDerivMatrix})$;

if $\mathrm{check\_adjointU}(L, U, \mathrm{adjointU}) = \mathbf{true}$ **then**

return $\mathrm{adjointU}$;

else

return error``*Adjoint found is not valid*"

end

end

Future Work

As explained in [**DF**], the adjoint boundary conditions were used to define a transform pair, that in turn would be used to solve an initial boundary value problem with a linear evolution equation and two-point boundary conditions. Thus, one possible direction is an initial *multipoint* value problem with a linear evolution equation and multipoint boundary conditions, in which we could use adjoint multipoint conditions to define the relevant transform pair.

References

1 Nelson Dunford and Jacob T. Schwartz, *Linear Operators II*, Interscience, 1963.

2 David A. Smith and Athanasios S. Fokas, *Evolution PDEs and augmented eigenfunction. Finite interval*, 2016.

3 Beatrice Pelloni and David A. Smith, *Nonlocal and multipoint boundary value problems for linear evolution equations*, 2018.

4 Linfan Xiao, *Algorithmic solution of high order partial differential equations in julia via the fokas transform method*, 2018.

5 Sultan Aitzhan, *Construction of the adjoint multipoint problem*, 2019.

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