Nonlocal problems for linear evolution equations

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Papers

Peter Miller, S. *The diffusion equation with nonlocal data* J. Math. Anal. Appl. **466** 2. 1119–1143 (2018) arXiv:1708.00972

Beatrice Pelloni, S. *Nonlocal and multipoint boundary value problems for linear evolution equations* Stud. Appl. Math. **141** 1. 46–88 (2018) arXiv:1511.07244

Nonlocal problems: mathematical motivation

Integrable evolution equations in $2+1$ tend to be nonlocal equations.

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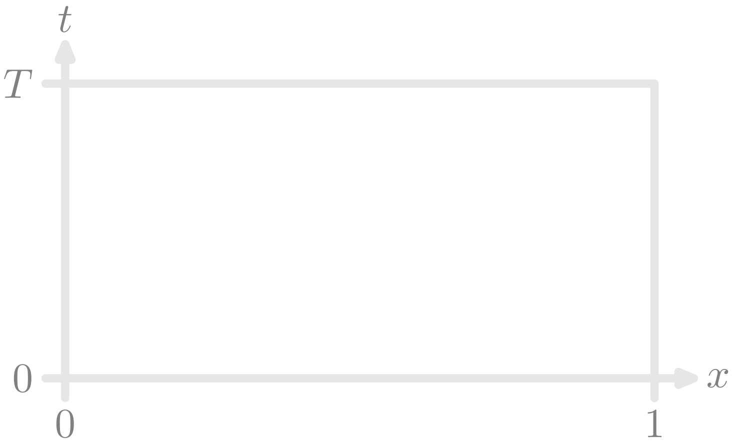
See also: Deconinck Vasan 2013, Fokas Pelloni 2005, Biondini et. al. 2019(?).

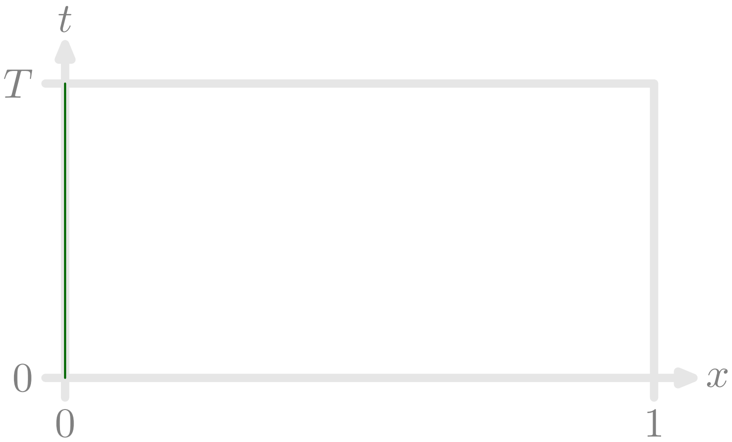
Boundary / Multipoint / Nonlocal conditions

Domain



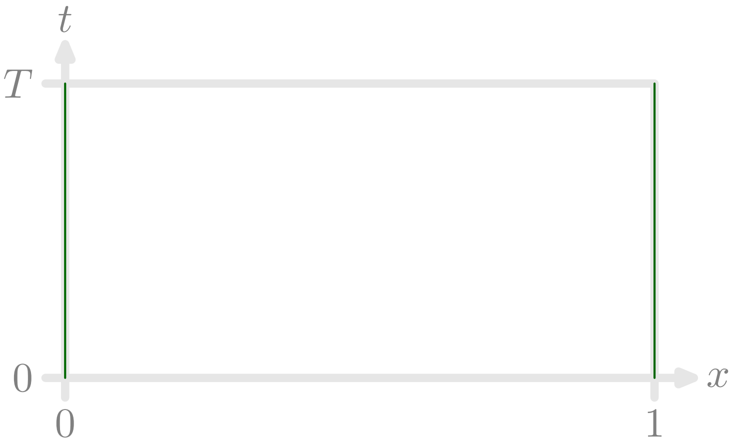
Boundary / Multipoint / Nonlocal conditions

Boundary condition



Boundary / Multipoint / Nonlocal conditions

Coupled boundary condition

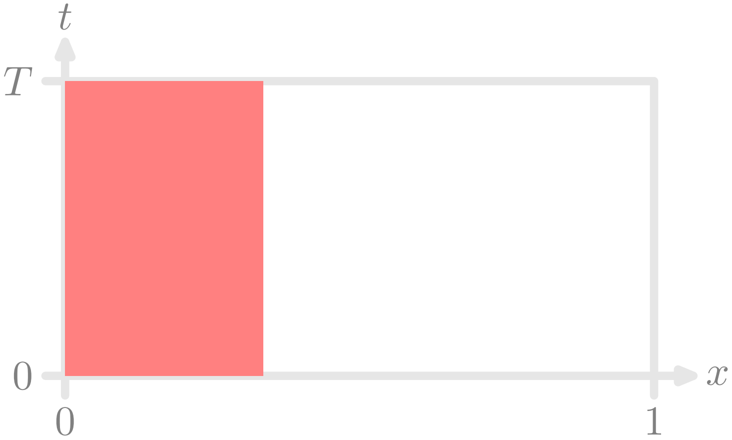


Boundary / Multipoint / Nonlocal conditions

Multipoint condition

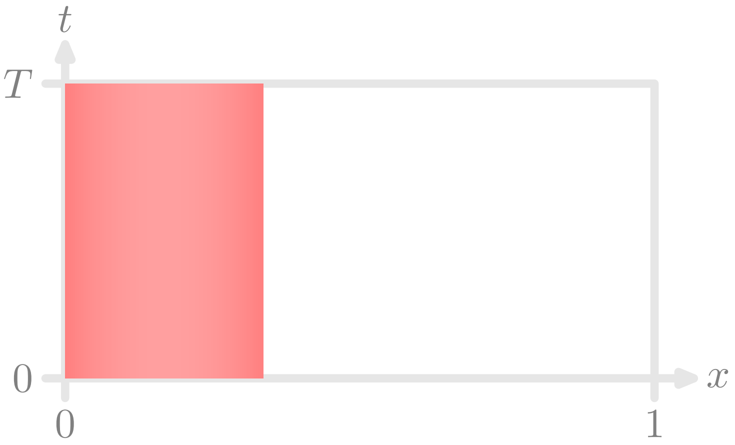
Boundary / Multipoint / Nonlocal conditions

Nonlocal condition



Boundary / Multipoint / Nonlocal conditions

Multipoint condition arises as limit of nonlocal conditions



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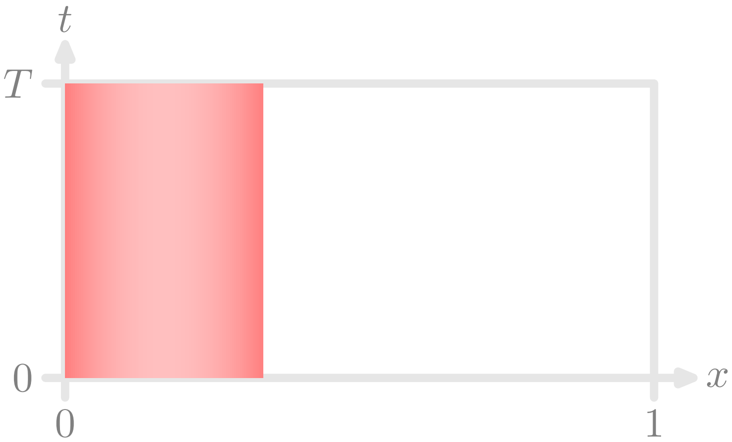

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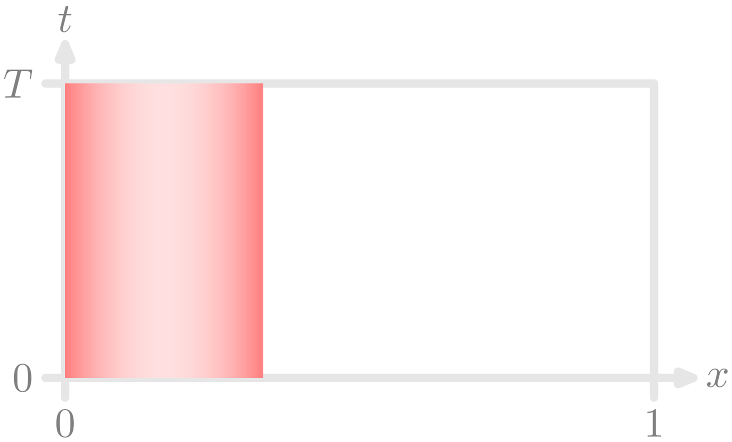

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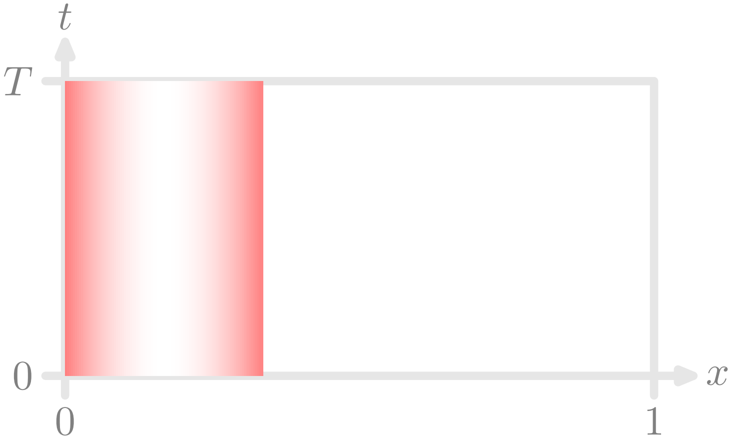

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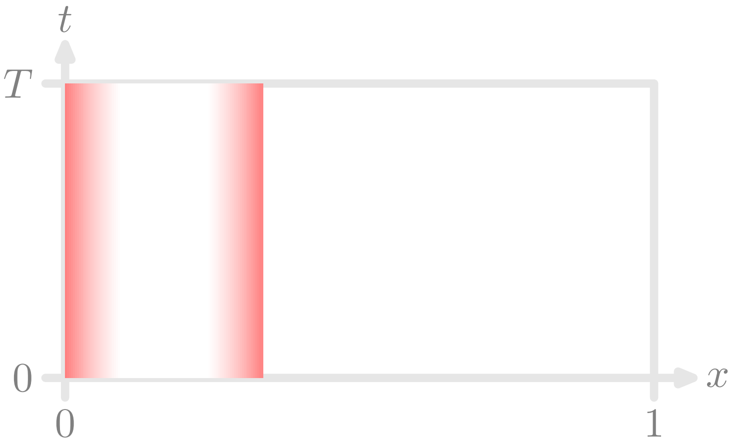

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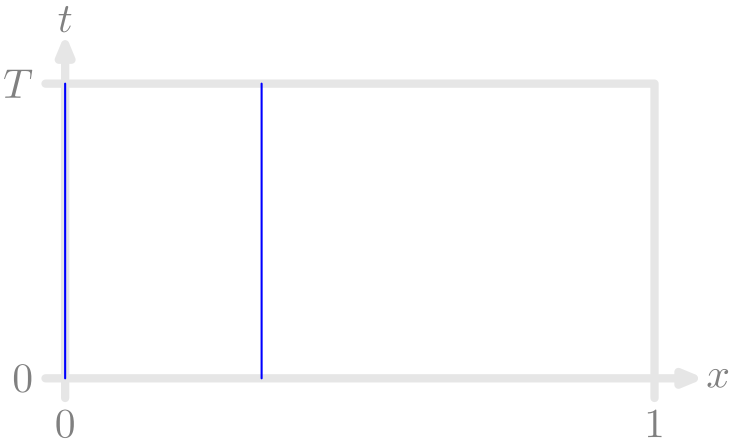

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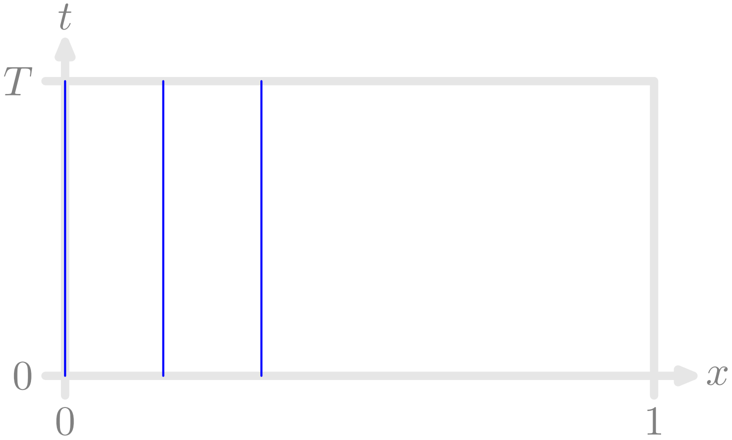

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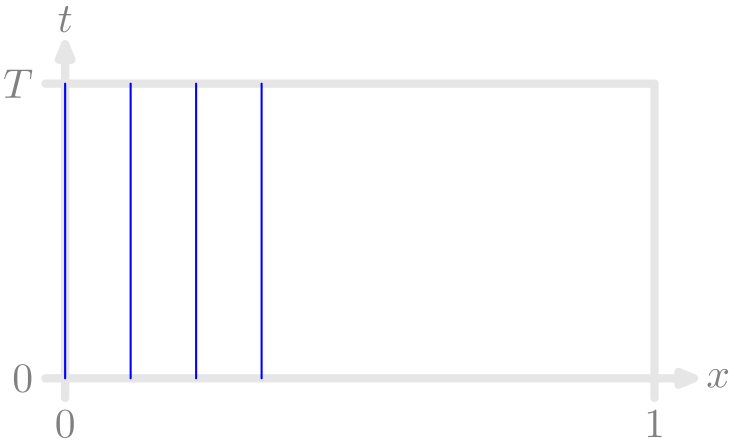

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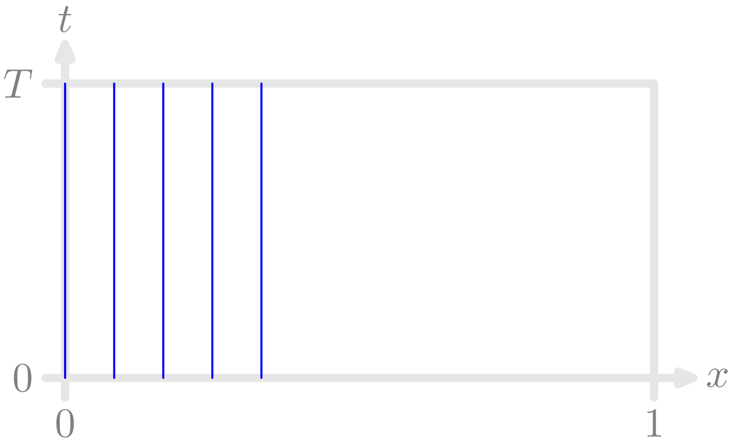

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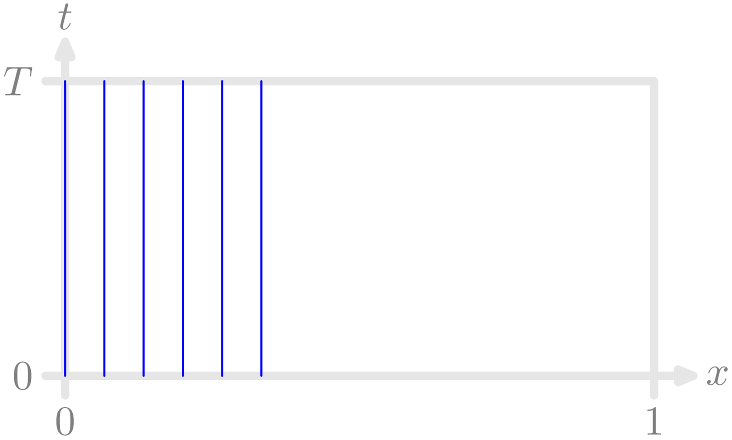

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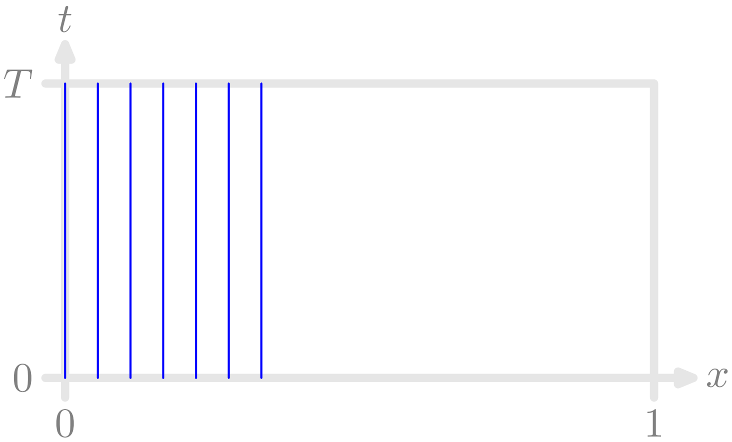

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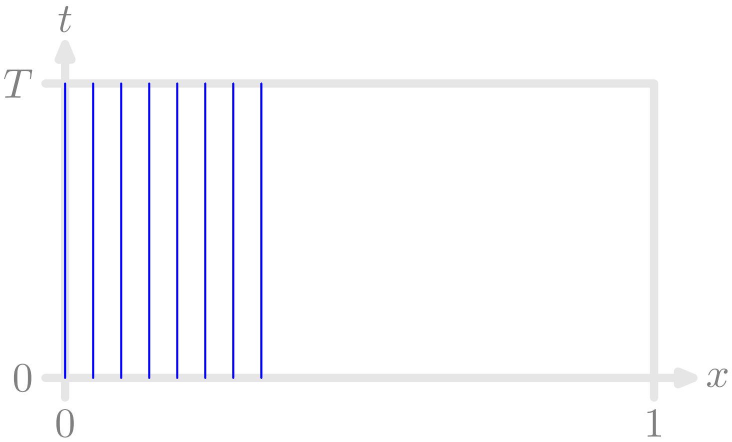

Piecewise linear K

Suppose $K = 1/a$ on $(0, a)$, and 0 elsewhere. Study

$$[\partial_t - \partial_x^2]q(x,t) = 0, \qquad q(x,0) = q_0(x), \qquad q_x(1,t) = g_1(t),$$
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Differentiate nonlocal condition in t, and apply PDE:

$$\frac{1}{a}\int_0^a \partial_x^2 q(x,t)dx = g_0'(t).$$

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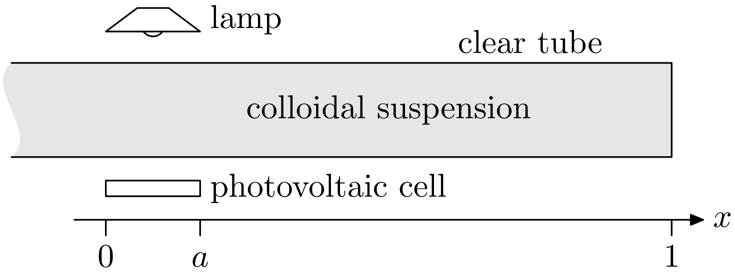
Evaluate integral:

$$q_x(a,t) - q_x(0,t) = a g_0'(t).$$

Nonlocal problems: physical motivation



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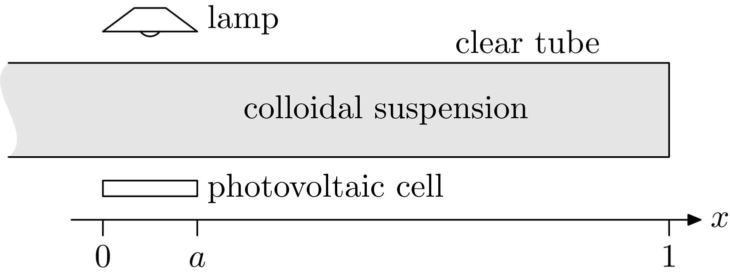

For q, concentration of dispersed substance,
initial-*boundary* value problem:

$$
\begin{aligned}
[\partial_t - \partial_x^2]q(x,t) &= 0 && (x,t) \in (0,1) \times (0,T), \\
q(x,0) &= q_0(x) && x \in [0,1], \\
q_x(1,t) &= 0 && t \in [0,T], \\
q(0,t) &= \gamma(t) && t \in [0,T],
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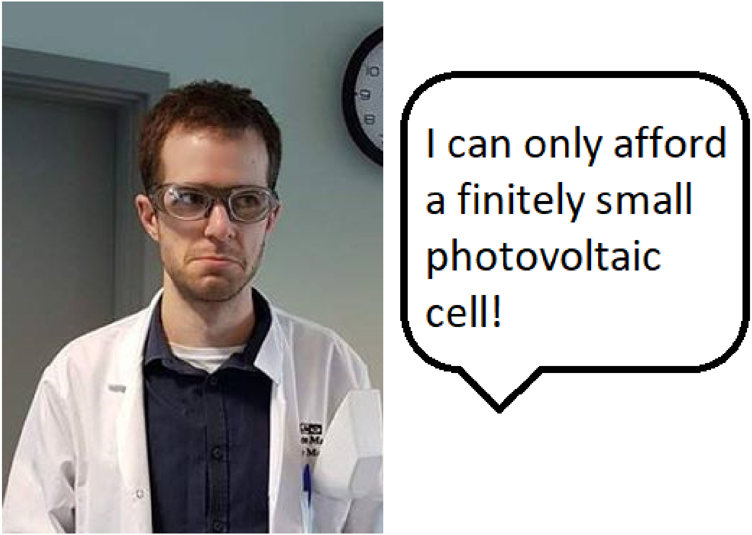

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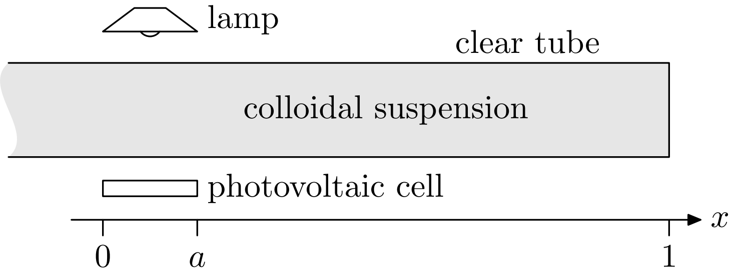

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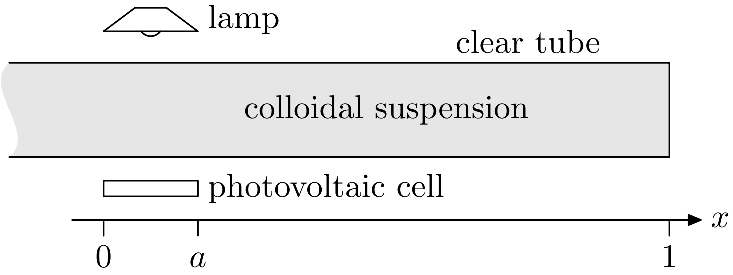
\end{aligned}$$

Nonlocal problems: physical motivation



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\int_0^1 K(x)q(x,t)dx &= g_0(t) && t \in [0,T],
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Overview of Fokas transform method

- Stage 1: assuming existence of a solution, obtain implicit integral representation & "global relation".
- Stage 2: continuing from stage 1, obtain explicit integral representation of "solution" by implementing **D**ata-**to**-u**N**known map.
  Have now established uniqueness of solution.
- Stage 3: show that the "solution" obtained in stage 2 truly satisfies the problem.
  Have now established existence of solution.

Overview of Fokas transform method: Stage 1

1a Apply Fourier transform $\phi \mapsto \hat{\phi}$, defined by $\int_{-\infty}^{\infty} e^{-i\lambda x}\phi(x)dx$, to PDE:

$$\left[\frac{d}{dt} + \lambda^2\right] \hat{q}(\lambda; t) = 0$$

Overview of Fokas transform method: Stage 1

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$$\left[\frac{d}{dt} + \lambda^2\right]\hat{q}(\lambda;t) = -e^{-i\lambda y}\big(q_x(y,t)+i\lambda q(y,t)\big)+e^{-i\lambda z}\big(q_x(z,t)+i\lambda q(z,t)\big)$$

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Solve the ODE, pretending right side is data, for the *Global relation*:

$$\begin{aligned}&\hat{q}_0(\lambda;y,z) - e^{\lambda^2 t}\hat{q}(\lambda;y,z,t)\\ &\qquad = e^{-i\lambda y}\left[i\lambda f_0(\lambda;y,t) + f_1(\lambda;y,t)\right] - e^{-i\lambda z}\left[i\lambda f_0(\lambda;z,t) + f_1(\lambda;z,t)\right]\end{aligned}$$

where

$$\hat{q}_0(\lambda;y,z) = \int_y^z e^{-i\lambda x}q_0(x)dx$$

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1b Evaluate global relation at $y = 0$, $z = 1$. Use inverse Fourier transform
$$2\pi q(x, t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{q}_0(\lambda; 0, 1)\, d\lambda$$
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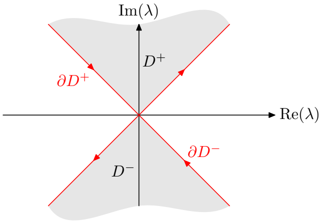
Define $D^\pm = \{\lambda \in \mathbb{C}^\pm : \mathrm{Re}(\lambda^2) < 0\}$. Deform contours using Jordan's lemma.



Overview of Fokas transform method: Stage 2 (BVP)

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$\hat{q}_0(\lambda; 0, 1) - e^{\lambda^2 t}\hat{q}(\lambda; 0, 1, t) =$
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For $m \in \mathbb{N}$ consider partition $0 = \eta_0 < \eta_1 < \eta_2 < \ldots < \eta_m = 1$ and two general $(m+1)$-point conditions $k = 0, 1$:

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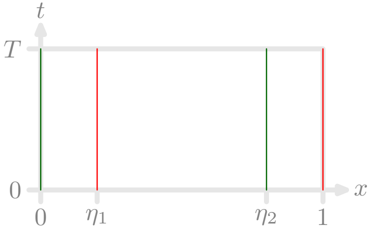

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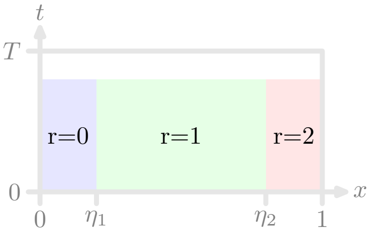

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$f_j(\lambda; \eta_r, t)$, $\qquad r = 0, 1, \ldots, m$, as if $\hat{q}(\lambda; \eta_r, \eta_{r+1}, t)$ is data.

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Stage 2c (MVP): more details

Linear system for Multipoint value problem:

$X'\mathcal{B} = \mathcal{Y} + Y$ where

$$X' = \Big(\overbrace{f_0(\lambda;\eta_r), \dots, f_{n-1}(\lambda;\eta_r)}^{r=0,1,\dots,m} \Big),$$

$$Y = \Big(\tilde{g}_0(\lambda), \dots, \tilde{g}_{n-1}(\lambda), \overbrace{-\hat{q}_0(\lambda;\eta_{r-1},\eta_r), -\hat{q}_0(\alpha\lambda;\eta_{r-1},\eta_r), \dots, -\hat{q}_0(\alpha^{n-1}\lambda;\eta_{r-1},\eta_r)}^{r=1,2,\dots,m} \Big),$$

$$\mathcal{Y} = e^{\lambda^2 t}\Big(0, \dots, 0, \overbrace{\hat{q}(\lambda;\eta_{r-1},\eta_r,t), \hat{q}(\alpha\lambda;\eta_{r-1},\eta_r,t), \dots, \hat{q}(\alpha^{n-1}\lambda;\eta_{r-1},\eta_r,t)}^{r=1,2,\dots,m} \Big),$$

$$\mathcal{B} = \begin{pmatrix} \mathfrak{b}^0 & -e_0 & 0 & \cdots & 0 & 0 \\ \mathfrak{b}^1 & e_1 & -e_1 & \cdots & 0 & 0 \\ \mathfrak{b}^2 & 0 & e_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathfrak{b}^{m-1} & 0 & 0 & \cdots & e_{m-1} & -e_{m-1} \\ \mathfrak{b}^m & 0 & 0 & \cdots & 0 & e_m \end{pmatrix},$$

in which

$\mathfrak{b}^r$ is a full $n \times n$ matrix of monomials in λ^{-1} encoding the multipoint conditions,

e_r is a full $n \times n$ matrix of exponentials.

Stage 2c (MVP): more details

Rewrite the MVP system as

$X\mathcal{A} = \mathcal{Y} + Y$ where

entries in X are linear combinations of entries in X',

$$\mathcal{A} = \begin{pmatrix} \beta^0 & -I & 0 & \cdots & 0 & 0 \\ \beta^1 & I & -I & \cdots & 0 & 0 \\ \beta^2 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta^{m-1} & 0 & 0 & \cdots & I & -I \\ \beta^m & 0 & 0 & \cdots & 0 & I \end{pmatrix},$$

in which

β^r is a full $n \times n$ matrix of polynomials in λ^{-1} encoding the multipoint conditions,

I is the $n \times n$ identity matrix.

This system can be solved by hand for general m.

Stage 2c (MVP): more details

Solving by hand, we obtain a Riemann Sum.

Under continuum limit $m \to \infty$, obtain an integral.

So we have solved the full Nonlocal Problem too ...

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We can still implement stage 3 to establish existence & validity of solution representation, but how do we show uniqueness?

Deckert & Maple 1963 show uniqueness in the case that K is constant.

Try to generalise Deckert & Maple 1963. Difficult.

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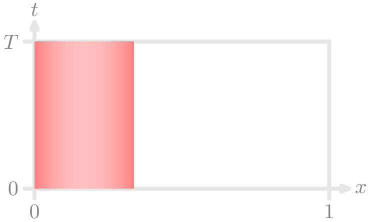
Consider chemistry problem:

$$\int_0^1 K(x)q(x,t)dx = g_0(t), \qquad q_x(1,t) = g_1(t).$$



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Evaluate $z = 1$, integrate against $e^{i\lambda y}K(y)$, use $\lambda \mapsto -\lambda$; obtain 2 more global relations:

$$\int_0^1 e^{i\lambda y}K(y)\hat{q}_0(\lambda; y, 1)dy - e^{\lambda^2 t}\int_0^1 e^{i\lambda y}K(y)\hat{q}(\lambda; y, 1, t)dy = i\lambda\int_0^1 K(y)f_0(\lambda; y, t)dy + \int_0^1 K(y)f_1(\lambda; y, t)dy - e^{-i\lambda}\left[i\lambda f_0(\lambda; 1, t)\int_0^1 e^{i\lambda y}K(y)dy + f_1(\lambda; 1, t)\int_0^1 e^{i\lambda y}K(y)dy\right]$$

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$i\lambda \int_0^1 K(y)f_0(\lambda;y,t)dy + \int_0^1 K(y)f_1(\lambda;y,t)dy$
$-e^{-i\lambda}\left[i\lambda f_0(\lambda;1,t)\int_0^1 e^{i\lambda y}K(y)dy + f_1(\lambda;1,t)\int_0^1 e^{i\lambda y}K(y)dy\right]$

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Analysis of stages 2e & 3

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$\dfrac{\zeta^\pm(\lambda)}{\Delta(\lambda)} =$

	BVP/MVP	NVP
heat	$\mathcal{O}(\lvert\lambda\rvert^{-1})$	$\mathcal{O}(\lvert\lambda\rvert^{-1})$
Schrödinger	$\mathcal{O}(\lvert\lambda\rvert^{-1})$	$\mathcal{O}(1)$

as $\lambda \to \infty$ from $\overline{D^\pm}$.

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(i) Δ has (at most) finitely many zeros in $\overline{D^\pm}$.
Asymptotic / geometric argument based on R. Langer 1931.

(ii) $\dfrac{\zeta^\pm(\lambda)}{\Delta(\lambda)} \to 0$ as $\lambda \to \infty$ from within $\overline{D^\pm}$.
Careful asymptotic analysis.

$\dfrac{\zeta^\pm(\lambda)}{\Delta(\lambda)} =$

	BVP/MVP	NVP
heat	$\mathcal{O}(\lvert\lambda\rvert^{-1})$	$\mathcal{O}(\lvert\lambda\rvert^{-1})$
Schrödinger	$\mathcal{O}(\lvert\lambda\rvert^{-1})$	$\mathcal{O}(1)$

as $\lambda \to \infty$ from $\overline{D^\pm}$.

Nonlocal problems for linear Scrödinger must have K a δ function at $x = 0, 1$ in order to apply the Fokas method.

Similar for evolution equations of odd order.

Contrasting methods for stage 2 (MVP)

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Adapting approach taken for nonlocal problems to multipoint problems gives D-to-N map as linear system of rank $n(n+1)$, independent of m.

Thanks

Thank you

More on Fokas method / unified transform method:
http://unifiedmethod.azurewebsites.net/