Summary of PS2018a

Introduction & Formulation of the Problem

In this paper, the authors use the Fokas transform to solve multipoint boundary value problems for linear PDEs of arbitrary order of the form

$$q_t + a(-i\partial_x)^n q = 0, \qquad x \in (0,1), \qquad t > 0.$$

More specifically, let $m, n \in \mathbb{N}, 0 = \eta_0 < \eta_1 < \ldots < \eta_m = 1$, and $b^r_{kj} \in \mathbb{C}$, where $k, j \in \{0, \ldots, n-1\}, r \in \{0, \ldots, m\}$. Consider the following problem

$$q_t + a(-i\partial_x)^n q = 0, \qquad (x,t) \in (0,1) \times (0,T) \tag{1}$$

$$q(x,0) = q_0(x) \qquad x \in [0,1] \tag{2}$$

$$\sum_{k=0}^{n-1}\sum_{r=0}^{m} b^r_{kj}\partial_x^k q(\eta_r, t) = g_j(t) \qquad t \in [0,T],\ j \in \{0,1,\ldots,n-1\}, \tag{3}$$

where (3) refers to the multipoint conditions. The following theorem is the main result of the paper.

Theorem 1. *Suppose $q(x,t)$ is the solution of initial-$(m+1)$ point value of problem of order n. Then, $q(x,t)$ admits the integral representation*

$$2\pi q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - a\lambda^n t} q_0(\lambda)\mathrm{d}\lambda - \int_{\partial D_R^+} e^{i\lambda x - a\lambda^n t}\sum_{k=0}^{n-1} f_k^0(\lambda;\tau)\mathrm{d}\lambda - \int_{\partial D_R^-} e^{i\lambda x - a\lambda^n t}\sum_{k=0}^{n-1} f_k^m(\lambda;\tau)\mathrm{d}\lambda, \tag{4}$$

where q_0, D_R, f_k^j are appropriately defined as in the paper.

Nonlocal Boundary Conditions

The authors show that the multipoint conditions (3) are in fact equivalent to a mix of multipoint and nonlocal conditions

$$\sum_{k=0}^{n-1}\sum_{r=0}^{m} b^r_{kj}\partial_x^k q(\eta_r, t) = g_j(t) \qquad t \in [0,T],\ j \in \{0,1,\ldots,J\},$$

$$\sum_{k=0}^{n-1}\sum_{r=0}^{m} \int_{\eta_{r-1}}^{\eta_r} b^r_{kj} x^k q(x,t)\mathrm{d}x = g_j(t) \qquad t \in [0,T],\ j \in \{J+1,\ldots,n-1\}. \tag{BC}$$

This means that we can use the Fokas method to solve BVPs not only with multipoint conditions but also with certain nonlocal conditions, namely the nonlocal data when variable coefficients are polynomials x^k. The backward implication is as follows: differentiate (BC) with respect to t, and apply the PDE to obtain:

$$\sum_{k=0}^{n-1}\sum_{r=0}^{m} \int_{\eta_{r-1}}^{\eta_r} b^r_{kj} x^k \partial_x^n q(x,t)\mathrm{d}x = \frac{i^n}{a}\frac{\mathrm{d}}{\mathrm{d}t} g_j(t).$$

Integrate the above by parts $k+1$ times, so that

$$\begin{aligned}\sum_{k=0}^{n-1}\sum_{r=0}^{m}\sum_{p=0}^{k}(-1)^p b^r_{kj}[\eta_r^{k-p}\partial_x^{n-p-1}q(\eta_r,t) - \eta_{r-1}^{k-p}\partial_x^{n-p-1}q(\eta_{r-1},t)] &= \sum_{k=0}^{n-1}\sum_{r=0}^{m}\sum_{p=0}^{k}(-1)^p \tilde{b}^r_{kj}\partial_x^{n-p-1}q(\eta_r,t)] \\ &= \sum_{k=0}^{n-1}\sum_{r=0}^{m}\hat{b}^r_{kj}\partial_x^{n-p-1}q(\eta_r,t)],\end{aligned}$$

where both equalities follow by appropriate renaming as on page 4. Thus, for $j \in \{J+1, \ldots, n-1\}$, we have a corresponding multipoint condition

$$\sum_{k=0}^{n-1} \sum_{r=0}^{m} \hat{b}_{kj}^{r} \partial_x^{n-p-1} q(\eta_r, t)]x = \frac{i^n}{a} \frac{\mathrm{d}}{\mathrm{d}t} g_j(t).$$

The forward implication is proved by reversing the argument.

Fokas Transform and its Implementation for IMVP

Recalling (3.1) on page 5, we have the *global relation* on $[\eta_{r-1}, \eta_r]$

$$\hat{q}_0^r(\lambda) - e^{a\lambda^n t} \hat{q}_\tau^r(\lambda) = e^{-i\lambda\eta_{r-1}} \sum_{k=0}^{n-1} c_k(\lambda) \int_0^t e^{a\lambda^n s} \partial_x^k q(\eta_{r-1}, s) \mathrm{d}s - e^{-i\lambda\eta_r} \sum_{k=0}^{n-1} c_k(\lambda) \int_0^t e^{a\lambda^n - i\lambda s} \partial_x^k q(\eta_r, s) \mathrm{d}s.$$

Defining $\alpha, E_r(\lambda), \hat{q}_0^r(\lambda), \hat{q}_\tau^r(\lambda)$, and $f_k^r(\lambda)$ as on page 7, the global relation becomes

$$\hat{q}_0^r(\lambda) - e^{a\lambda^n t} \hat{q}_\tau^r(\lambda) = E_{r-1}(\lambda) \sum_{k=0}^{n-1} f_k^{r-1}(\lambda) - E_r(\lambda) \sum_{k=0}^{n-1} f_k^r(\lambda). \tag{5}$$

Evaluating (5) at $\lambda, \alpha\lambda, \ldots, \alpha^{n-1}\lambda$ and using that $f_k^r(\alpha\lambda) = \alpha^{n-1-k} f_k^r(\lambda)$, we arrive at a system of mn equations

$$\hat{q}_0^r(\alpha^p\lambda) - e^{a\lambda^n t} \hat{q}_\tau^r(\alpha^p\lambda) = \sum_{k=0}^{n-1} \alpha^{(n-1-k)p} [E_{r-1}(\alpha^p\lambda) f_k^{r-1}(\lambda) - E_r(\alpha^p\lambda) f_k^r(\lambda)],$$
$$r \in \{1, \ldots, m\}, p \in \{0, 1, \ldots, n-1\},$$

and $(m+1)n$ unknown functions $f_k^r, k \in \{0, 1, \ldots, n-1\}, r \in \{0, \ldots, m\}$. Applying time transform on multipoint conditions (3), we obtain one more equation where we need to solve for $\{f_k^r\}$. Thus, we arrive at the system of size $(m+1)n$, of the form

$$\mathcal{B}(\underbrace{f_0^r(\lambda), \ldots, f_{n-1}^r(\lambda)}_{r=0,1,\ldots,m}) = (h_0(\lambda), \ldots, h_{n-1}(\lambda), \underbrace{-\hat{q}_0^r(\lambda), \ldots, -\hat{q}_0^r(\alpha^{n-1}\lambda)}_{r=1,2,\ldots,m})$$
$$+ e^{a\lambda^n \tau}(0, \ldots, 0, \underbrace{-\hat{q}_\tau^r(\lambda), \ldots, -\hat{q}_\tau^r(\alpha^{n-1}\lambda)}_{r=1,2,\ldots,m}),$$

where the matrix $\mathcal{B}$ is defined on page 9. The task of solving for f_k^r can be simplified with some modifications as on page 9 of the paper, and then we can solve explicitly for $\{f_k^r\}$.

Finally, note that any expression for $\{f_k^r\}$ contains unknown functions $\hat{q}_\tau^r(\lambda)$. However, it can be shown that these functions evaluate to 0 when integrated over the contour, so that the solution is indeed given as in (4).

References

[1] Pelloni, B. and Smith, D.A. 2018. *Nonlocal and Multipoint Boundary Value Problems for Linear Evolution Equations.*