Order $n = 3$ problems

Problem 1

Let $x_0 = 0 < x_1 = 0.25 < x_2 = 0.7 < x_3 = 1$ be the partition of $[0, 1]$, and consider the following problem:

$$\partial_t q - \partial_{xxx} q = 0 \tag{PDE}$$

$$q(x, 0) = f(x) \tag{IC}$$

$$Uq = \sum_{l=1}^{3} R_l \begin{bmatrix} q_l^{(0)}(x_{l-1}) \\ q_l^{(1)}(x_{l-1}) \\ q_l^{(2)}(x_{l-1}) \end{bmatrix} + N_l \begin{bmatrix} q_l^{(0)}(x_l) \\ q_l^{(1)}(x_l) \\ q_l^{(2)}(x_l) \end{bmatrix} = \vec{0}, \tag{BC}$$

where

$$R_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & -3 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -5 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix};$$

$$N_1 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 3 & 1 & -2 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$$

Adjoint conditions are

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0.5 & -0.5 & -3 \\ 0 & -1 & -2/3 \\ -1 & 0 & 5 \\ -1 & 0 & 1/3 \\ -1 & 1 & -7/3 \\ 3 & -3 & 2/3 \\ -3 & 3 & -8/3 \\ -2 & 1 & -1/3 \\ 0 & 0 & -16/3 \end{bmatrix};$$

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & -0.5 & -5.5 \\ 5/3 & -1 & -7 \\ 0 & 0 & 10 \\ -1/3 & 0 & 2 \\ -8/3 & 1 & 9 \\ 19/3 & -3 & -22 \\ -22/3 & 4 & 23 \\ -11/3 & 2 & 13 \\ -4/3 & 1 & -5 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} -5.75 & -0.5 & 3 \\ -7 & 0 & 13/3 \\ 10.5 & 0 & -5 \\ 1 & 0 & -5/3 \\ 11 & 0 & -19/3 \\ -25.5 & -1 & 47/3 \\ 27 & 0 & -50/3 \\ 14.5 & 0 & -28/3 \\ -5 & 0 & 8/3 \end{bmatrix}.$$

Problem 2

Let $x_0 = 0 < x_1 = 0.2 < x_2 = 1$ be the partition of $[0, 1]$, and consider the following problem:

$$\partial_t q - \partial_{xxx} q = 0 \qquad \text{(PDE)}$$

$$q(x, 0) = f(x) \qquad \text{(IC)}$$

$$Uq = \sum_{l=1}^{2} R_l \begin{bmatrix} q_l^{(0)}(x_{l-1}) \\ q_l^{(1)}(x_{l-1}) \\ q_l^{(2)}(x_{l-1}) \end{bmatrix} + N_l \begin{bmatrix} q_l^{(0)}(x_l) \\ q_l^{(1)}(x_l) \\ q_l^{(2)}(x_l) \end{bmatrix} = \vec{0}, \qquad \text{(BC)}$$

where

$$R_1 = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix};$$

$$N_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad N_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Order $n = 2$ problems

Problem 3

Let $x_0 = 0 < x_1 = 0.5 < x_2 = 1$ be the partition of $[0, 1]$, and consider the following problem:

$$\partial_t q - i\partial_{xx} q = 0 \qquad \text{(PDE)}$$
$$q(x,0) = f(x) \qquad \text{(IC)}$$
$$Uq = \sum_{l=1}^{2} R_l \begin{bmatrix} q_l^{(0)}(x_{l-1}) \\ q_l^{(1)}(x_{l-1}) \end{bmatrix} + N_l \begin{bmatrix} q_l^{(0)}(x_l) \\ q_l^{(1)}(x_l) \end{bmatrix} = \vec{0}, \qquad \text{(BC)}$$

where

$$R_1 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & -2 \end{bmatrix} \quad N_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \quad N_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 4 \end{bmatrix}.$$

Problem 4

Let $x_0 = 0 < x_1 = 0.25 < x_2 = 0.5 < x_3 = 1$ be the partition of $[0, 1]$, and consider the following problem:

$$\partial_t q - i\partial_{xx} q = 0 \qquad \text{(PDE)}$$
$$q(x,0) = f(x) \qquad \text{(IC)}$$
$$Uq = \sum_{l=1}^{3} R_l \begin{bmatrix} q_l^{(0)}(x_{l-1}) \\ q_l^{(1)}(x_{l-1}) \end{bmatrix} + N_l \begin{bmatrix} q_l^{(0)}(x_l) \\ q_l^{(1)}(x_l) \end{bmatrix} = \vec{0}, \qquad \text{(BC)}$$

where

$$R_1 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & -2 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix};$$
$$N_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \\ 1 & -2 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 4 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ 1 & -1 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Problem 5

Let $x_0 = 0 < x_1 = 0.25 < x_2 = 0.5 < x_3 = 1$ be the partition of $[0, 1]$, and consider the following problem:

$$\begin{aligned} \partial_t q - i\partial_{xx} q &= 0 && \text{(PDE)} \\ q(x, 0) &= f(x) && \text{(IC)} \\ Uq = \sum_{l=1}^{3} R_l \begin{bmatrix} q_l^{(0)}(x_{l-1}) \\ q_l^{(1)}(x_{l-1}) \end{bmatrix} + N_l \begin{bmatrix} q_l^{(0)}(x_l) \\ q_l^{(1)}(x_l) \end{bmatrix} &= \vec{0}, && \text{(BC)} \end{aligned}$$

where

$$R_1 = \begin{bmatrix} 3 & -2 \\ 0 & 3 \\ 0 & 0 \\ -1 & 1 \\ -1 & -1 \\ 3 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 4 & 0 \\ 0 & -1 \\ 0 & -1 \\ 0 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 0 & 1 \\ -2 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix};$$

$$N_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 2 & 2 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \\ 1 & -2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}.$$