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Algorithmic Construction of an Adjoint of an Ordinary Differential Operator in Julia

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Abstract

Given an operator, its adjoint is a linear transformation that can be used to characterise a wide range of phenomena in mathematics and physics. In this project we seek to characterise an adjoint of an ordinary differential operator with multipoint homogeneous conditions. Extending the ideas of Linfan Xiao, we show that the operator and its adjoint satisfy the multipoint form formula. In doing so, we provide an explicit construction of adjoint multipoint conditions, and provide an explicit checking test to determine whether a given set of multipoint conditions satisfies the adjoint equation. Finally, we implement a method of deriving the adjoint multipoint conditions in programming language julia, thus providing a fast way to compute the adjoint.

Multipoint Conditions

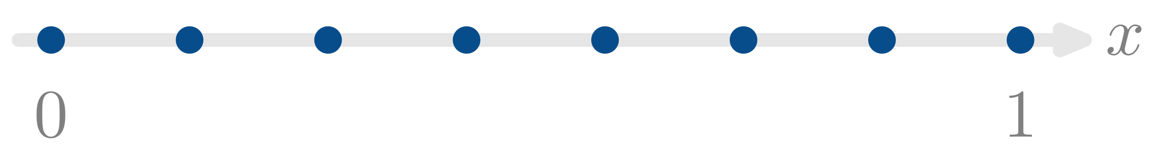
Consider an interval $[0,1]$, and suppose that we want specify the behaviour of $q(x)$ on this interval not just at the endpoints but also at finitely many points in the interior. Then, we can represent one such specification by an expression

$$\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l),$$

where x_{l-1}, x_l are two adjacent points, q_l is the function q but restricted to the interval (x_{l-1}, x_l), and $q_l^{(j)}$ is the j^{th} derivative of q_l. Combining all such specifications yields an expression of the form

$$\sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{jl} q_l^{(j)}(x_{l-1}) + \beta_{jl} q_l^{(j)}(x_l)]$$

which we call a **multipoint form**. Then, requiring that a multipoint form be equal to some number, and combining these equations yileds one vector equation, which we term as a **multipoint condition**. The image below provides an intuitive idea of the multipoint conditions.



Formulation of the Problem

Consider a closed interval $[a,b]$. Fix $n \in \mathbb{N}$, and let the differential operator be defined as

$$L := \sum_{k=0}^{n} a_k(x) \left(\frac{d}{dx}\right)^k, \text{ where } a_k(x) \in C^\infty[a,b] \text{ and } a_n(x) \neq 0 \, \forall x \in [a,b]. \tag{1}$$

Fix $k \in \mathbb{N}$, and let $\{a = x_0 < x_1 < \ldots < x_k = b\}$ be a partition of $[a,b]$. Consider a homogeneous multipoint value problem (MVP)

$$\pi : Lq = 0, \qquad Uq = \vec{0},$$

where $U = (U_1, \ldots, U_m)$ is a vector multipoint form with

$$U_i(q) = \sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l)], \qquad i \in \{1, \ldots, m\}, \tag{2}$$

where $\alpha_{ijl}, \beta_{ijl} \in \mathbb{R}$, and q is sufficiently smooth. Our goal is to construct the adjoint multipoint value problem to π

$$\pi^+ : L^+ q = 0, \qquad U^+ q = \vec{0},$$

with

$$L^+ := \sum_{k=0}^{n} (-1)^k \left(\overline{a_k}(t) \frac{d}{dt}\right)^k,$$

where $\overline{a_k}(t)$ is the complex conjugate of $a_k(t)$, $k = 0, \ldots, n$, and U^+ is an appropriate vector multipoint form. Observe that for π^+ to be an adjoint problem to π, we must have

$$\langle Lq, q\rangle = \langle q, L^+ q\rangle.$$

Multipoint Form Formula

Definition 1 *If $U = (U_1, \ldots, U_m)$ is any vector multipoint form with $\operatorname{rank}(U) = m$, and $U_c = (U_{m+1}, \ldots, U_{2nk})$ is a vector multipoint form with $\operatorname{rank}(U_c) = 2nk - m$ such that $\operatorname{rank}(U_1, \ldots, U_{2nk}) = 2nk$, then U and U_c are* ***complementary vector multipoint forms****.*

Theorem 2 (Multipoint Form Formula) *Given any vector multipoint form U of rank m, and any complementary vector form U_c, there exist unique vector multipoint forms U_c^+, U^+ of rank m and $2nk - m$, respectively, such that*

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g. \tag{3}$$

Definition of the Adjoint

Theorem 2 allows us to define an adjoint vector multipoint form. Namely,

Definition 3 *Suppose $U = (U_1, \ldots, U_m)$ is a vector multipoint form with $\operatorname{rank}(U) = m$, along with the condition that $Uq = \vec{0}$ for functions q sufficiently smooth. If U^+ is any vector multipoint form with $\operatorname{rank}(U^+) = 2nk - m$, determined as in Theorem 2, then the equation*

$$U^+ q = \vec{0}$$

is an ***adjoint multipoint condition*** *to $Uq = \vec{0}$.*

In turn, the above lets us define the adjoint multipoint problem:

Definition 4 *Suppose $U = (U_1, \ldots, U_m)$ is a vector multipoint form with $\operatorname{rank}(U) = m$. Then, the problem of solving*

$$\pi : Lq = 0, \qquad Uq = \vec{0},$$

is called a homogeneous multipoint value problem. The problem of solving

$$\pi^+ : L^+ q = 0, \qquad U^+ q = \vec{0},$$

is an ***adjoint multipoint value problem****.*

The preceding construction allows us to state the following:

Proposition 5 *For f, g sufficiently smooth, suppose $Uf = \vec{0}$ and $U^+ g = \vec{0}$. Then, $\langle Lf, g\rangle = \langle f, L^+ g\rangle$.*

Proof. We apply the multipoint form formula (3)

$$\langle Lf, g\rangle - \langle f, L^+ g\rangle = Uf \cdot U_c^+ g + U_c f \cdot U^+ g = \vec{0} \cdot U_c^+ g + U_c f \cdot \vec{0} = 0,$$

which completes the proof. ∎

Checking Adjointness

Note that for the appropriate matrices M_l, N_l, P_l, Q_l, we can rewrite the multipoint conditions $Uf, U^+ g$ as follows:

$$Uf = \sum_{l=1}^{k} \begin{bmatrix} M_l & N_l \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \text{ and } U^+ g = \sum_{l=1}^{k} \begin{bmatrix} P_l^* & Q_l^* \end{bmatrix} \begin{bmatrix} \vec{g}_l(x_{l-1}) \\ \vec{g}_l(x_l) \end{bmatrix}, \tag{4}$$

where $\vec{f}_l = (f_l^{(0)}, f_l, \ldots, f_l^{(n-1)})$ is an $n \times 1$ column vector of derivatives. The above notation allows to devise the following theorem.

Theorem 6 *The boundary condition $U^+ g = \vec{0}$ is adjoint to $Uf = \vec{0}$ if and only if*

$$\sum_{l=1}^{k} M_l F^{-1}(x_{l-1}) P_l = \sum_{l=1}^{k} N_l F^{-1}(x_l) Q_l,$$

where $F(t)$ is the $n \times n$ boundary matrix.

Implementation in Julia

The proof of Theorem 2 provides an explicit way to construct the matrices P_l, Q_l in (4), which we use in defining a function *get_adjointU*. Furthermore, we use Theorem 6 to define a function *check_adjointU*, to check whether the multipoint conditions obtained from *get_adjointU* satisfy the adjoint equation.

input : The partitioned interval, the list of functions $a_k(t)$ from (1), vector multipoint form U from (2)
output: Adjoint Multipoint Conditions

begin
$L \longleftarrow (\pi, \{a_k(t)\}_{k=0}^{n})$
aDerivMatrix $\longleftarrow \begin{bmatrix} a_0 & \ldots & a_0^{n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \ldots & a_{n-1}^{n-1} \end{bmatrix}$
adjointU $\longleftarrow$ *get_adjointU*$(L, U,$ *aDerivMatrix*$)$
if *check_adjointU*$(L, U,$ *adjointU*$) =$ ***true*** **then**
  return *adjointU*
else
  return *error: "Adjoint found is not valid"*
end
end

Future Work

As explained in [2], the adjoint boundary conditions may be used to define a transform pair, that in turn can be used to solve an initial boundary value problem for a linear evolution equation with two-point boundary conditions. Thus, one possible direction is solving an initial *multipoint* value problem for a linear evolution equation with multipoint boundary conditions, in which we could use adjoint multipoint conditions to define the relevant transform pair.

References

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