Unified Transform Method for Multipoint Problems

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August 2, 2019

1 Adjoint of an ordinary differential operator

In this section, we extend the construction of an adjoint problem, mostly following a similar argument as given by Linda in [3].

1.1 Formulation of the problem

Consider a closed interval $[a, b]$. Fix $n \in \mathbb{N}$, and let the differential operator be defined as

$$L := \sum_{k=0}^{n} a_k(t) \left(\frac{d}{dt}\right)^k, \text{ where } a_k(t) \in C^\infty[a, b] \text{ and } a_n(t) \neq 0 \ \forall t \in [a, b].$$

Fix $k \in \mathbb{N}$, and let $\{a = x_0 < x_1 < \ldots < x_k = b\}$ be a partition of $[a, b]$. Let the domain of L be given by the function space

$$\begin{aligned} C_\pi^{n-1}[a, b] = \Big\{ & f : [a, b] \to \mathbb{C} \text{ s.t. } \forall l \in \{1, 2, \ldots, k\}, \\ & f_l := f\big|_{(\eta_{l-1}, \eta_l)} \text{ admits an extension } g_l \text{ to } [\eta_{l-1}, \eta_l] \text{ s.t. } g_l \in C^{n-1}[\eta_{l-1}, \eta_l] \Big\}. \end{aligned}$$

Consider a homogeneous multipoint BVP of rank m

$$\pi : Lq = 0, \qquad Uq = \vec{0},$$

where $U = (U_1, \ldots, U_m)$ is a vector multipoint form with

$$U_i(q) = \sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l)], \qquad i \in \{1, \ldots, m\},$$

where $\alpha_{ijl}, \beta_{ijl} \in \mathbb{R}, q \in C_\pi^{n-1}[a, b]$. Our goal is to construct an adjoint multipoint value problem (MVP) to π

$$\pi^+ : L^+ q = 0, \qquad U^+ q = \vec{0},$$

with

$$L^+ := \sum_{k=0}^{n} (-1)^k \left(\overline{a_k}(t) \frac{d}{dt}\right)^k, \text{ where } \overline{a_k}(t) \text{ is the complex conjugate of } a_k(t),\ k = 0, \ldots, n,$$

and U^+ is an appropriate vector multipoint form.

1.2 Green's formula

For any $f, g \in C_\pi^{n-1}[a,b]$, application of Green's formula yields

$$\langle Lf, g\rangle - \langle f, L^+g\rangle = \sum_{l=1}^{k}\sum_{p,q=0}^{n-1}[F_{p\,q}(x_l)f_l^{(p)}(x_l)g_l^{(q)}(x_l) - F_{p\,q}(x_{l-1})f_l^{(p)}(x_{l-1})g_l^{(q)}(x_{l-1})],$$

where $F(t)$ denotes an $n\times n$ boundary matrix at the point $t\in[a,b]$. From [1, p. 1286], the entries of $F(t)$ are given by

$$\begin{aligned}
F_{p\,q}(t) &= \sum_{k=j}^{n-p-1}(-1)^k\binom{k}{j}\left(\frac{d}{dt}\right)^{k-j}a_{p+k+1}(t), && p+q<n-1\\
F_{p\,q}(t) &= (-1)^q a_n(t), && p+q=n-1\\
F_{p\,q}(t) &= 0, && p+q>n-1.
\end{aligned}$$

Observe that since $\det(F(t)) = (a_0(t))^n \neq 0$, the matrix $F(t)$ is non-singular.

Our goal is to rewrite the Green's formula as a *semibilinear* form $\mathcal{S}$. First, let $\vec{f_l} := (f_l,\dots,f_l^{(n-1)})$, and observe that

$$\begin{aligned}
[fg]_l(t) := \sum_{p,q=0}^{n-1}F_{p\,q}(t)f_l^{(p)}(t)g_l^{(q)}(t) &= \sum_{p,q=0}^{n-1}\left[F_{p\,q}f_l^{(p)}g_l^{(q)}\right](t)\\
&= \sum_{q=0}^{n-1}\left[\left(\sum_{p=0}^{n-1}F_{p\,q}f_l^{(p)}\right)g_l^{(q)}\right](t)\\
&= F(t)\vec{f_l}(t)\cdot\vec{g_l}(t),
\end{aligned}$$

where $\cdot$ refers to dot product. The Green's formula can then be rewritten as

$$\langle Lf, g\rangle - \langle f, L^+g\rangle = \sum_{l=1}^{k}[fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k}F(x_l)\vec{f_l}(x_l)\cdot\vec{g_l}(x_l) - F(x_{l-1})\vec{f_l}(x_{l-1})\cdot\vec{g_l}(x_{l-1}). \quad (1)$$

Note that

$$F(x_l)\vec{f_l}(x_l)\cdot\vec{g_l}(x_l) - F(x_{l-1})\vec{f_l}(x_{l-1})\cdot\vec{g_l}(x_{l-1}) = \begin{bmatrix}-F(x_{l-1}) & 0_{n\times n}\\ 0_{n\times n} & F(x_l)\end{bmatrix}\begin{bmatrix}\vec{f_l}(x_{l-1})\\ \vec{f_l}(x_l)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_l}(x_{l-1})\\ \vec{g_l}(x_l)\end{bmatrix},$$

so that we obtain

$$\langle Lf, g\rangle - \langle f, L^+g\rangle = \sum_{l=1}^{k}[fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k}\begin{bmatrix}-F(x_{l-1}) & 0_{n\times n}\\ 0_{n\times n} & F(x_l)\end{bmatrix}\begin{bmatrix}\vec{f_l}(x_{l-1})\\ \vec{f_l}(x_l)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_l}(x_{l-1})\\ \vec{g_l}(x_l)\end{bmatrix}.$$

Now, expansion of the sum yields

$$\begin{aligned}
&\sum_{l=1}^{k}\begin{bmatrix}-F(x_{l-1}) & 0_{n\times n}\\ 0_{n\times n} & F(x_l)\end{bmatrix}\begin{bmatrix}\vec{f_l}(x_{l-1})\\ \vec{f_l}(x_l)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_l}(x_{l-1})\\ \vec{g_l}(x_l)\end{bmatrix}\\
&= \begin{bmatrix}-F(x_0) & 0_{n\times n}\\ 0_{n\times n} & F(x_1)\end{bmatrix}\begin{bmatrix}\vec{f_1}(x_0)\\ \vec{f_1}(x_1)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_1}(x_0)\\ \vec{g_1}(x_1)\end{bmatrix} + \dots + \begin{bmatrix}-F(x_{k-1}) & 0_{n\times n}\\ 0_{n\times n} & F(x_k)\end{bmatrix}\begin{bmatrix}\vec{f_k}(x_{k-1})\\ \vec{f_k}(x_k)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_k}(x_{k-1})\\ \vec{g_k}(x_k)\end{bmatrix}\\
&= \underbrace{\begin{bmatrix}-F(x_0) & 0 & \dots & 0 & 0\\ 0 & F(x_1) & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & -F(x_{k-1}) & 0\\ 0 & 0 & \dots & 0 & F(x_k)\end{bmatrix}}_{2nk\times 2nk}\begin{bmatrix}\vec{f_1}(x_0)\\ \vec{f_1}(x_1)\\ \vec{f_2}(x_1)\\ \vec{f_2}(x_2)\\ \vdots\\ \vec{f_k}(x_{k-1})\\ \vec{f_k}(x_k)\end{bmatrix}\cdot\begin{bmatrix}\vec{g_1}(x_0)\\ \vec{g_1}(x_1)\\ \vec{g_2}(x_1)\\ \vec{g_2}(x_2)\\ \vdots\\ \vec{g_k}(x_{k-1})\\ \vec{g_k}(x_k)\end{bmatrix}
\end{aligned}$$

$$=: S \begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix} \cdot \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix} = \mathcal{S}\left(\begin{bmatrix} \vec{f_1}(x_0) \\ \vec{f_1}(x_1) \\ \vdots \\ \vec{f_k}(x_{k-1}) \\ \vec{f_k}(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g_1}(x_0) \\ \vec{g_1}(x_1) \\ \vdots \\ \vec{g_k}(x_{k-1}) \\ \vec{g_k}(x_k) \end{bmatrix} \right), \tag{2}$$

where the matrix S is associated with the semibilinear form $\mathcal{S}$ and S is a block matrix where each block is $n \times n$. Further, note that the form $\mathcal{S}$ is the action of applying matrix S to the first argument and taking dot product of this result and the second argument. Thus, we managed to express the Green's Formula as a semibilinear form $\mathcal{S}$.

1.3 Boundary Form formula

We turn to characterising an adjoint multipoint condition using an extension of boundary form formula that Linda derived in her work. First, recall that the multipoint condition

$$Uq = \begin{bmatrix} U_1(q) \\ \vdots \\ U_m(q) \end{bmatrix} = \vec{0},$$

with

$$U_i(q) = \sum_{l=1}^{k} \sum_{j=0}^{n-1} [\alpha_{ijl} q_l^{(j)}(x_{l-1}) + \beta_{ijl} q_l^{(j)}(x_l)], \qquad i \in \{1, \dots, m\},\ \alpha_{ijl}, \beta_{ijl} \in \mathbb{R}.$$

Note that $U_1, \dots, U_m$ are linearly independent when $\sum_{i=1}^m c_i U_i q = 0$ if and only if $c_i = 0$. When $U_1, \dots, U_m$ are linearly independent, we say that U has full rank m. For now, suppose that U has full rank, and define

$$\vec{q_l} = \begin{bmatrix} q_l \\ q_l' \\ \vdots \\ q_l^{(n-1)} \end{bmatrix}, M_l = \begin{bmatrix} \alpha_{1\ 0\ l} & \alpha_{1\ 1\ l} & \dots & \alpha_{1\ (n-1)\ l} \\ \alpha_{2\ 0\ l} & \alpha_{2\ 1\ l} & \dots & \alpha_{2\ (n-1)\ l} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \alpha_{m\ 1\ l} & \dots & \alpha_{m\ (n-1)\ l} \end{bmatrix}, N_l = \begin{bmatrix} \beta_{1\ 0\ l} & \beta_{1\ 1\ l} & \dots & \beta_{1\ (n-1)\ l} \\ \beta_{2\ 0\ l} & \beta_{2\ 1\ l} & \dots & \beta_{2\ (n-1)\ l} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m\ 0\ l} & \beta_{m\ 1\ l} & \dots & \beta_{m\ (n-1)\ l} \end{bmatrix}$$

Then,

$$\begin{aligned} Uq &= \begin{bmatrix} U_1(q) \\ \vdots \\ U_m(q) \end{bmatrix} \\ &= \sum_{l=1}^{k} \sum_{j=0}^{n-1} \begin{bmatrix} \alpha_{1\ j\ l} \\ \vdots \\ \alpha_{m\ j\ l} \end{bmatrix} q_l^{(j)}(x_{l-1}) + \begin{bmatrix} \beta_{1\ j\ l} \\ \vdots \\ \beta_{m\ j\ l} \end{bmatrix} q_l^{(j)}(x_l) \\ &= \sum_{l=1}^{k} \begin{bmatrix} \alpha_{1\ 0\ l} & \dots & \alpha_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \dots & \alpha_{m\ (n-1)\ l} \end{bmatrix} \begin{bmatrix} q_l(x_{l-1}) \\ \vdots \\ q_l^{(n-1)}(x_{l-1}) \end{bmatrix} + \begin{bmatrix} \beta_{1\ 0\ l} & \dots & \beta_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots \\ \beta_{m\ 0\ l} & \dots & \beta_{m\ (n-1)\ l} \end{bmatrix} \begin{bmatrix} q_l(x_l) \\ \vdots \\ q_l^{(n-1)}(x_l) \end{bmatrix} \\ &= \sum_{l=1}^{k} M_l \vec{q_l}(x_{l-1}) + N_l \vec{q_l}(x_l), \end{aligned} \tag{†}$$

where M_l, N_l are $m \times n$ matrices. In addition, letting

$$[M_l : N_l] = \begin{bmatrix} \alpha_{1\ 0\ l} & \dots & \alpha_{1\ (n-1)\ l} & \beta_{1\ 0\ l} & \dots & \beta_{1\ (n-1)\ l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m\ 0\ l} & \dots & \alpha_{m\ (n-1)\ l} & \beta_{m\ 0\ l} & \dots & \beta_{m\ (n-1)\ l} \end{bmatrix},$$

we can write

$$Uq = \sum_{l=1}^{k} \begin{bmatrix} M_l : N_l \end{bmatrix} \begin{bmatrix} \vec{q_l}(x_{l-1}) \\ \vec{q_l}(x_l) \end{bmatrix} = \begin{bmatrix} M_1 : N_1 : \ \dots \ : M_k : N_k \end{bmatrix} \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}. \tag{$\star$}$$

Thus we have found two compact ways to write the vector multipoint form, namely (†) and ($\star$). Next, we extend the notion of a complementary boundary form.

Definition 1. If $U = (U_1, \dots, U_m)$ is any vector multipoint form with $\operatorname{rank}(U) = m$, and $U_c = (U_{m+1}, \dots, U_{2nk})$ is a vector multipoint form with $\operatorname{rank}(U_c) = 2nk - m$ such that $\operatorname{rank}(U_1, \dots, U_{2nk}) = 2nk$, then U and U_c are **complementary vector multipoint forms**.

Note that extending $U_1, \dots, U_m$ to $U_1, \dots, U_{2nk}$ is equivalent to embedding the matrices M_l, N_l in a $2nk \times 2nk$ non-singular matrix, i.e. we can write

$$\begin{aligned} \begin{bmatrix} Uq \\ U_c q \end{bmatrix} &= \sum_{l=1}^{k} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{q_l}(x_{l-1}) \\ \vec{q_l}(x_l) \end{bmatrix} \\ &= \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \end{bmatrix} + \begin{bmatrix} M_2 & N_2 \\ \overline{M}_2 & \overline{N}_2 \end{bmatrix} \begin{bmatrix} \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \end{bmatrix} + \dots + \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \begin{bmatrix} \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} M_1 & N_1 & M_2 & N_2 & \dots & M_k & N_k \\ \overline{M}_1 & \overline{N}_1 & \overline{M}_2 & \overline{N}_2 & \dots & \overline{M}_k & \overline{N}_k \end{bmatrix}}_{2nk \times 2nk} \underbrace{\begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}}_{2nk \times 1} \\ &=: H \begin{bmatrix} \vec{q_1}(x_0) \\ \vec{q_1}(x_1) \\ \vec{q_2}(x_1) \\ \vec{q_2}(x_2) \\ \vdots \\ \vec{q_k}(x_{k-1}) \\ \vec{q_k}(x_k) \end{bmatrix}. \end{aligned} \tag{3}$$

where $\operatorname{rank}(H) = 2nk$ and $\overline{M}_l, \overline{N}_l$ are $(2nk - m) \times n$ matrices. Just like the boundary form formula proven by Linda, the multipoint form formula is motivated by the desire to express Green's formula as a combination of vector boundary forms U and U_c. Namely, we have:

Theorem 2 (Multipoint Form Formula)**.** *Given any vector multipoint form U of rank m, and any complementary vector form U_c, there exist unique vector multipoint forms U_c^+, U^+ of rank m and $2nk-m$, respectively, such that*

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g. \tag{4}$$

We will use the following proposition from Linda's capstone [3] in the proof of Theorem 2:

Proposition 3 (Prop. 2.12 in Linda's capstone)**.** *Let $\mathcal{S}$ be the semibilinear form associated with a nonsingular matrix S. Suppose $\vec{f} := Ff$ where F is a nonsingular matrix. Then, there exists a unique nonsingular matrix G such that if $\vec{g} = Gg$, then $\mathcal{S}(f, g) = \vec{f} \cdot \vec{g}$ for all f, g.*

Proof of Theorem 2. First, we have

$$\begin{bmatrix} Uf \\ U_c f \end{bmatrix} = H \begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix}.$$

From equation (2), we can write

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = \mathcal{S}\left(\begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix} \right).$$

By Proposition 3, there exists a unique $2nk \times 2nk$ nonsingular matrix J such that

$$\mathcal{S}\left(\begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix} \right) = H \begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix} \cdot J \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix}.$$

Note that if S is the matrix associated with $\mathcal{S}$, then by Proposition 3, $J = (SH^{-1})^*$, where A^* refers to the conjugate transpose of matrix A.

Let U^+, U_c^+ be such that

$$\begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} = J \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix}.$$

Now, we obtain

$$\begin{aligned} \sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = \mathcal{S}\left(\begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix}, \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix} \right) &= H \begin{bmatrix} \vec{f}_1(x_0) \\ \vec{f}_1(x_1) \\ \vdots \\ \vec{f}_k(x_{k-1}) \\ \vec{f}_k(x_k) \end{bmatrix} \cdot J \begin{bmatrix} \vec{g}_1(x_0) \\ \vec{g}_1(x_1) \\ \vdots \\ \vec{g}_k(x_{k-1}) \\ \vec{g}_k(x_k) \end{bmatrix} \\ &= \begin{bmatrix} Uf \\ U_c f \end{bmatrix} \cdot \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} \\ &= Uf \cdot U_c^+ g + U_c f \cdot U^+ g, \end{aligned}$$

which completes the proof. □

Theorem 2 allows us to define an adjoint multipoint condition. Namely,

Definition 4. Suppose $U = (U_1, \ldots, U_m)$ is a vector multipoint form with $\operatorname{rank}(U) = m$, along with the condition that $Uq = \vec{0}$ for functions $q \in C_\pi^{n-1}[a, b]$. If U^+ is any vector multipoint form with $\operatorname{rank}(U^+) = 2nk - m$, determined as in Theorem 2, then the equation

$$U^+ q = \vec{0}$$

is an **adjoint multipoint condition** to $Uq = \vec{0}$.

In turn, the above lets us define the adjoint multipoint problem:

Definition 5. Suppose $U = (U_1, \ldots, U_m)$ is a vector multipoint form with $\text{rank}(U) = m$. Then, the problem of solving

$$\pi : Lq = 0, \qquad Uq = \vec{0},$$

is called a homogeneous multipoint value problem of rank m. The problem of solving

$$\pi^+ : L^+q = 0, \qquad U^+q = \vec{0},$$

is an **adjoint multipoint value problem** to π.

The preceding construction allows us to state the following:

Proposition 6. *Let $f, g \in C_\pi^{n-1}[a, b]$ with $Uf = \vec{0}$ and $U^+g = \vec{0}$. Then, $\langle Lf, g\rangle = \langle f, L^+g\rangle$.*

Proof. We apply the multipoint form formula (4)

$$\langle Lf, g\rangle - \langle f, L^+g\rangle = Uf \cdot U_c^+ g + U_c f \cdot U^+ g = \vec{0} \cdot U_c^+ g + U_c f \cdot \vec{0} = 0,$$

which completes the proof. □

1.4 Checking adjointness

Finally, we extend Theorem 2.19 on Linda's Capstone [3].

Theorem 7. *The multipoint condition $U^+g = \vec{0}$ is adjoint to $Uf = \vec{0}$ if and only if*

$$\sum_{l=1}^{k} M_l F^{-1}(x_{l-1}) P_l = \sum_{l=1}^{k} N_l F^{-1}(x_l) Q_l,$$

where $F(t)$ is the $n \times n$ matrix as given in Green's formula subsection.

Recall that just how U is associated with a collection of $m \times n$ matrices M_l, N_l, such that

$$Uf = \sum_{l=1}^{k} M_l \vec{f_l}(x_{l-1}) + N_l \vec{f_l}(x_l), \qquad \text{rank}\left[M_1 : N_1 : \ \ldots \ : M_k : N_k\right] = m, \tag{5}$$

so is U^+ associated with $n \times (2nk - m)$ matrices P_l, Q_l, for $l = 1, \ldots, k$, such that

$$U^+g = \sum_{l=1}^{k} P_l^* \vec{g}_l(x_{l-1}) + Q_l^* \vec{g}_l(x_l), \qquad \text{rank}\left[P_1^* : Q_1^* : \ \ldots \ : P_k^* : Q_k^*\right] = 2nk - m. \tag{6}$$

Proof of Theorem 7. Suppose that $U^+f = \vec{0}$ is adjoint to $Uf = \vec{0}$. By definition of adjoint multipoint condition, U^+ is determined as in Theorem 2. Thus, in determining U^+, there exist vector multipoint forms U_c, U_c^+ of rank $2nk - m$ and m respectively, such that the multipoint form formula (4) holds. As such, let matrices $\overline{M}_l, \overline{N}_l, \overline{P}_l, \overline{Q}_l$ be such that

$$U_c f = \sum_{l=1}^{k} \overline{M}_l \vec{f_l}(x_{l-1}) + \overline{N}_l \vec{f_l}(x_l), \qquad \text{rank}\left[\overline{M}_1 : \overline{N}_1 : \ \ldots \ : \overline{M}_k : \overline{N}_k\right] = 2nk - m \tag{7}$$

$$U_c^+ g = \sum_{l=1}^{k} \overline{P}_l^* \vec{g}_l(x_{l-1}) + \overline{Q}_l^* \vec{g}_l(x_l), \qquad \text{rank}\left[\overline{P}_1^* : \overline{Q}_1^* : \ \ldots \ : \overline{P}_k^* : \overline{Q}_k^*\right] = m \tag{8}$$

First, note that in the context of semibilinear form, we have $\mathcal{S}(f, g) = Sf \cdot g = f \cdot S^*g$, as given in Proposition 2.11 of Linda's capstone [3, p.18]. We use this to rewrite the multipoint form formula (4) as follows:

$$\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = Uf \cdot U_c^+ g + U_c f \cdot U^+ g$$

$$
\begin{aligned}
&= \left(\sum_{l=1}^{k} M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l)\right) \cdot \left(\sum_{i=1}^{k} (\overline{P}_i)^* \vec{g}_i(x_{i-1}) + (\overline{Q}_i)^* \vec{g}_i(x_i)\right) \\
&+ \left(\sum_{l=1}^{k} \overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l)\right) \cdot \left(\sum_{i=1}^{k} P_i^* \vec{g}_i(x_{i-1}) + Q_i^* \vec{g}_i(x_i)\right) \qquad \text{(by equations (5), (6), (7), (8))} \\
&= \sum_{l=1}^{k}\sum_{i=1}^{k} \Bigg(\left(M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l)\right) \cdot \left(\overline{P}_i^* \vec{g}_i(x_{i-1}) + \overline{Q}_i^* \vec{g}_i(x_i)\right) \\
&+ \left(\overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l)\right) \cdot \left(P_i^* \vec{g}_i(x_{i-1}) + Q_l^* \vec{g}_i(x_i)\right)\Bigg),
\end{aligned}
$$

where taking out the sum upfront follows due to distributivity and associativity of inner product. Moreover, using additivity of inner product and that $Sf \cdot g = f \cdot S^* g$, we write the above as

$$
\begin{aligned}
\sum_{l=1}^{k}\sum_{i=1}^{k} &(\overline{Q}_i N_l + Q_i \overline{N}_l) \vec{f}_l(x_l) \cdot \vec{g}_i(x_i) + (\overline{P}_i N_l + P_i \overline{N}_l) \vec{f}_l(x_l) \cdot \vec{g}_i(x_{i-1}) \\
&+ (\overline{Q}_i M_l + Q_i \overline{M}_l) \vec{f}_l(x_{l-1}) \cdot \vec{g}_i(x_i) + (\overline{P}_i M_l + P_i \overline{M}_l) \vec{f}_l(x_{l-1}) \cdot \vec{g}_i(x_{i-1}).
\end{aligned} \tag{9}
$$

From Green's formula (1), we have

$$
\sum_{l=1}^{k} [fg]_l(x_l) - [fg]_l(x_{l-1}) = \sum_{l=1}^{k} F(x_l) \vec{f}_l(x_l) \cdot \vec{g}_l(x_l) - F(x_{l-1}) \vec{f}_l(x_{l-1}) \cdot \vec{g}_l(x_{l-1}). \tag{10}
$$

Note that equations (9) and (10) must be equal, and so, comparison of coefficients of inner product reveals that

$$
\begin{aligned}
\overline{Q}_i N_l + Q_i \overline{N}_l &= \begin{cases} F(x_l) & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}; \qquad & \overline{P}_i M_l + P_i \overline{M}_l &= \begin{cases} -F(x_{l-1}) & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}; \\
\overline{P}_i N_l + P_i \overline{N}_l &= 0 \qquad \forall i; & \overline{Q}_i M_l + Q_i \overline{M}_l &= 0 \qquad \forall i.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\begin{bmatrix}
-F(x_0) & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & F(x_1) & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & -F(x_1) & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & F(x_{k-1}) & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & -F(x_{k-1}) & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & F(x_k)
\end{bmatrix} \\
&= \begin{bmatrix}
\overline{P}_1 M_1 + P_1 \overline{M}_1 & 0 & \dots & 0 & 0 \\
0 & \overline{Q}_1 N_1 + Q_1 \overline{N}_1 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & \overline{P}_k M_k + P_k \overline{M}_k & 0 \\
0 & 0 & \dots & 0 & \overline{Q}_k N_k + Q_k \overline{N}_k
\end{bmatrix}.
\end{aligned} \tag{11}
$$

Since the boundary matrix F is nonsingular on $[a, b]$, F is invertible, and so the block diagonal matrix on LHS of (11) must also be invertible. Premultiplying on both sides by the inverse of LHS of block diagonal matrix yields

$$
E_{2nk \times 2nk} = \begin{bmatrix}
-F^{-1}(x_0)(\overline{P}_1 M_1 + P_1 \overline{M}_1) & 0 & \dots & 0 \\
0 & F^{-1}(x_1)(\overline{Q}_1 N_1 + Q_1 \overline{N}_1) & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & F^{-1}(x_k)(\overline{Q}_k N_k + Q_k \overline{N}_k)
\end{bmatrix}
$$

$$= \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 M_1 - F^{-1}(x_0)P_1\overline{M}_1 & 0 & \dots & 0 \\ 0 & F^{-1}(x_1)\overline{Q}_1 N_1 + F^{-1}(x_1)Q_1\overline{N}_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F^{-1}(x_k)\overline{Q}_k N_k + F^{-1}(x_k)Q_k\overline{N}_k \end{bmatrix}$$

$$= \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix} \begin{bmatrix} M_1 & N_1 & \dots & M_k & N_k \\ \overline{M_1} & \overline{N_1} & \dots & \overline{M_k} & \overline{N_k} \end{bmatrix}, \tag{$*$}$$

where $E_{j\times j}$ is the identity matrix of dimension j. Since the two matrices in $(*)$ are full rank, they are inverse to each other, and so we have

$$\begin{bmatrix} E_{m\times m} & 0_{m\times(2nk-m)} \\ 0_{(2nk-m)\times m} & E_{(2nk-m)\times(2nk-m)} \end{bmatrix} = \begin{bmatrix} M_1 & N_1 & \dots & M_k & N_k \\ \overline{M_1} & \overline{N_1} & \dots & \overline{M_k} & \overline{N_k} \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix},$$

which implies that

$$-M_1F^{-1}(x_0)P_1 + N_1F^{-1}(x_1)Q_1 + \ldots - M_kF^{-1}(x_{k-1})P_k + N_kF^{-1}(x_k)Q_k = 0_{m\times(2nk-m)}$$
$$\implies \sum_{l=1}^{k} M_lF^{-1}(x_{l-1})P_l = \sum_{l=1}^{k} N_lF^{-1}(x_l)Q_l.$$

Now, we prove the "if" direction. Let $\mathcal{U}^+$ be a multipoint form of rank $2nk-m$ such that

$$\mathcal{U}^+ g = \sum_{l=1}^{k} \mathcal{P}_l^* \vec{g}_l(x_{l-1}) + \mathcal{Q}_l^* \vec{g}_l(x_l),$$

for an appropriate collection of matrices $\mathcal{P}_l^*, \mathcal{Q}_l^*$, with

$$\operatorname{rank}\left[\mathcal{P}_1^* : \mathcal{Q}_1^* : \ \dots \ : \mathcal{P}_k^* : \mathcal{Q}_k^*\right] = 2nk-m$$

Suppose that

$$\sum_{l=1}^{k} M_lF^{-1}(x_{l-1})\mathcal{P}_l = \sum_{l=1}^{k} N_lF^{-1}(x_l)\mathcal{Q}_l$$

holds. Now, let $\mathbf{u}$ be a $2nk\times 1$ vector. Then, there exist $2nk-m$ linearly independent solutions of the system

$$\left[M_1 : N_1 : \ \dots \ : M_k : N_k\right]_{m\times 2nk} \mathbf{u} = \vec{0}.$$

By assumption, we have

$$\sum_{l=1}^{k} -M_lF(x_{l-1})^{-1}\mathcal{P}_l + N_lF(x_l)^{-1}\mathcal{Q}_l = 0_{m\times(2nk-m)},$$

so that

$$\left[M_1 : N_1 : \ \dots \ : M_k : N_k\right]_{m\times 2nk} \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix}_{2nk\times(2nk-m)} = 0_{m\times(2nk-m)}. \tag{12}$$

This means that the $2nk - m$ columns of the matrix

$$\mathcal{H} := \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix}$$

form the solution space of the system (12). Since rank $\left[\mathcal{P}_1^* : \mathcal{Q}_1^* : \ \dots \ : \mathcal{P}_k^* : \mathcal{Q}_k^*\right] = 2nk - m$,

$$\operatorname{rank} \begin{bmatrix} \mathcal{P}_1 \\ \mathcal{Q}_1 \\ \vdots \\ \mathcal{P}_k \\ \mathcal{Q}_k \end{bmatrix} = 2nk - m.$$

Since $F(x_{l-1}), F(x_l)$ are non-singular, $\operatorname{rank}(\mathcal{H}) = 2nk - m$.

Now, if $U^+ g = \sum_{l=1}^k P_l^* \vec{g}_l(x_{l-1}) + Q_l^* \vec{g}_l(x_l) = \vec{0}$ is a multipoint condition adjoint to $Uf = \vec{0}$, then by multipoint form formula we have that

$$\begin{aligned}
\begin{bmatrix} Uf \\ U_c f \end{bmatrix} \cdot \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} &= \sum_{l=1}^k \sum_{i=1}^k \begin{bmatrix} M_l \vec{f}_l(x_{l-1}) + N_l \vec{f}_l(x_l) \\ \overline{M}_l \vec{f}_l(x_{l-1}) + \overline{N}_l \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \overline{P}_i^* \vec{g}_i(x_{i-1}) + \overline{Q}_i^* \vec{g}_i(x_i) \\ P_i^* \vec{g}_i(x_{i-1}) + Q_i^* \vec{g}_i(x_i) \end{bmatrix} \\
&= \sum_{l=1}^k \sum_{i=1}^k \left(\begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \right) \cdot \left(\begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix}^* \begin{bmatrix} \vec{g}_i(x_{i-1}) \\ \vec{g}_i(x_i) \end{bmatrix} \right) \\
&= \sum_{l=1}^k \sum_{i=1}^k \begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_i(x_{i-1}) \\ \vec{g}_i(x_i) \end{bmatrix}.
\end{aligned} \tag{13}$$

In addition, by Green's formula (1), we have

$$\begin{bmatrix} Uf \\ U_c f \end{bmatrix} \cdot \begin{bmatrix} U_c^+ g \\ U^+ g \end{bmatrix} = \sum_{l=1}^k \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} \begin{bmatrix} \vec{f}_l(x_{l-1}) \\ \vec{f}_l(x_l) \end{bmatrix} \cdot \begin{bmatrix} \vec{g}_l(x_{l-1}) \\ \vec{g}_l(x_l) \end{bmatrix}. \tag{14}$$

Since equations (13) and (14) are equal, comparison of coefficients shows that we have

$$\begin{bmatrix} \overline{P}_i & P_i \\ \overline{Q}_i & Q_i \end{bmatrix} \begin{bmatrix} M_l & N_l \\ \overline{M}_l & \overline{N}_l \end{bmatrix} = \begin{cases} \begin{bmatrix} -F(x_{l-1}) & 0_{n\times n} \\ 0_{n\times n} & F(x_l) \end{bmatrix} & \text{if } i = l, \\ 0_{2n\times 2n} & \text{otherwise.} \end{cases}$$

Using the above relation, we obtain the equality

$$\begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}. \tag{15}$$

Since the matrix on LHS of (15) is invertible, we can premultiply both sides by this inverse to obtain

$$E_{2nk\times 2nk} = \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix} \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}$$

By Lemma 8:

$$
\begin{aligned}
&= \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix}^{-1} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k) \end{bmatrix}^{-1} \end{bmatrix} \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} -F(x_0) & 0 \\ 0 & F(x_1) \end{bmatrix}^{-1} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} -F(x_{k-1}) & 0 \\ 0 & F(x_k)^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}}_{\Xi}. \qquad (16)
\end{aligned}
$$

Note that the two matrices in (16) are square, and that the matrix Ξ is full-rank. So, the matrix Λ must be the inverse of Ξ. In other words, the following holds:

$$
\begin{bmatrix} E_{m\times m} & 0_{m\times(2nk-m)} \\ 0_{(2nk-m)\times m} & E_{(2nk-m)\times(2nk-m)} \end{bmatrix} = \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)\overline{P}_1 & -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)\overline{Q}_1 & F^{-1}(x_1)Q_1 \\ \vdots & \vdots \\ -F^{-1}(x_{k-1})\overline{P}_k & -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)\overline{Q}_k & F^{-1}(x_k)Q_k \end{bmatrix}.
$$

Thus, we have

$$
\begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \end{bmatrix} \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix} = 0_{m\times(2nk-m)}.
$$

Now, observe that

$$
H := \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix}_{2nk\times(2nk-m)}
$$

has rank $2nk-m$. Thus, columns H also form the solution space of the system (12), just like $\mathcal{H}$ does. But this suggests that $\mathcal{H}$ and H are the same up to a linear transformation, i.e. there exists a non-singular matrix A of size $(2nk-m)\times(2nk-m)$ such that

$$
\mathcal{H} = \begin{bmatrix} -F(x_0)^{-1}\mathcal{P}_1 \\ F(x_1)^{-1}\mathcal{Q}_1 \\ \vdots \\ -F(x_{k-1})^{-1}\mathcal{P}_k \\ F(x_k)^{-1}\mathcal{Q}_k \end{bmatrix} = HA = \begin{bmatrix} -F^{-1}(x_0)P_1 \\ F^{-1}(x_1)Q_1 \\ \vdots \\ -F^{-1}(x_{k-1})P_k \\ F^{-1}(x_k)Q_k \end{bmatrix} A = \begin{bmatrix} -F^{-1}(x_0)P_1A \\ F^{-1}(x_1)Q_1A \\ \vdots \\ -F^{-1}(x_{k-1})P_kA \\ F^{-1}(x_k)Q_kA \end{bmatrix},
$$

and so $P_l A = \mathcal{P}_l$ and $Q_l A = \mathcal{Q}_l$ for all $l = 1, \dots, k$. Therefore,

$$\mathcal{U}^+ g = \sum_{l=1}^{k} \mathcal{P}_l^* \vec{g}_l(x_{l-1}) + \mathcal{Q}_l^* \vec{g}_l(x_l) = \sum_{l=1}^{k} A^* P_l^* \vec{g}_l(x_{l-1}) + A^* Q_l^* \vec{g}_l(x_l) = A^* U^+ g.$$

Observe that $U^+ g = \vec{0}$ implies $\mathcal{U}^+ g = \vec{0}$. Since A^* is nonsingular, it follows that $U^+ g = \vec{0}$ if and only if $\mathcal{U}^+ g = \vec{0}$. Since $U^+ g = \vec{0}$ is adjoint to $Uf = \vec{0}$, $\mathcal{U}^+ g = \vec{0}$ is adjoint to $Uf = \vec{0}$. This completes the proof. □

Lemma 8. *For the relevant matrices $P_l, Q_l, \overline{P}_l, \overline{Q}_l, M_l, N_l, \overline{M}_l, \overline{N}_l$, we have*

$$\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk\times 2nk}$$
$$= \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}_{2nk\times 2nk} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}_{2nk\times 2nk}.$$

Proof. First, observe that we can write

$$\begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix}_{2nk\times 2nk^2} \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix}_{2nk^2\times 2nk}. \quad (17)$$

Now, let $\mathcal{V}$ and $\mathcal{W}$ be matrices given by:

$$\mathcal{V}_{2nk^2\times 2nk} = \begin{bmatrix} \begin{bmatrix} M_1 & N_1 \\ \overline{M}_1 & \overline{N}_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} M_k & N_k \\ \overline{M}_k & \overline{N}_k \end{bmatrix} \end{bmatrix} \begin{bmatrix} M_1 : N_1 : & \dots & : M_k : N_k \\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k \end{bmatrix}^{-1}; \quad (18)$$

$$\mathcal{W}_{2nk\times 2nk^2} = \begin{bmatrix} \overline{P}_1 : P_1 \\ \overline{Q}_1 : Q_1 \\ \vdots \\ \overline{P}_k : P_k \\ \overline{Q}_k : Q_k \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} \overline{P}_1 & P_1 \\ \overline{Q}_1 & Q_1 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} \overline{P}_k & P_k \\ \overline{Q}_k & Q_k \end{bmatrix} \end{bmatrix}. \quad (19)$$

Observe that

$$\mathcal{W}\mathcal{V}=\begin{bmatrix}\overline{P}_1 : P_1\\ \overline{Q}_1 : Q_1\\ \vdots\\ \overline{P}_k : P_k\\ \overline{Q}_k : Q_k\end{bmatrix}^{-1}\begin{bmatrix}\begin{bmatrix}\overline{P}_1 & P_1\\ \overline{Q}_1 & Q_1\end{bmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{bmatrix}\overline{P}_k & P_k\\ \overline{Q}_k & Q_k\end{bmatrix}\end{bmatrix}\begin{bmatrix}\begin{bmatrix}M_1 & N_1\\ \overline{M}_1 & \overline{N}_1\end{bmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{bmatrix}M_k & N_k\\ \overline{M}_k & \overline{N}_k\end{bmatrix}\end{bmatrix}\begin{bmatrix}M_1 : N_1 : & \dots & : M_k : N_k\\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k\end{bmatrix}^{-1}$$

Substitute (17):

$$=\begin{bmatrix}\overline{P}_1 : P_1\\ \overline{Q}_1 : Q_1\\ \vdots\\ \overline{P}_k : P_k\\ \overline{Q}_k : Q_k\end{bmatrix}^{-1}\begin{bmatrix}\begin{bmatrix}\overline{P}_1 & P_1\\ \overline{Q}_1 & Q_1\end{bmatrix}\begin{bmatrix}M_1 & N_1\\ \overline{M}_1 & \overline{N}_1\end{bmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{bmatrix}\overline{P}_k & P_k\\ \overline{Q}_k & Q_k\end{bmatrix}\begin{bmatrix}M_k & N_k\\ \overline{M}_k & \overline{N}_k\end{bmatrix}\end{bmatrix}\begin{bmatrix}M_1 : N_1 : & \dots & : M_k : N_k\\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k\end{bmatrix}^{-1}$$

Recall (15):

$$=\begin{bmatrix}\overline{P}_1 : P_1\\ \overline{Q}_1 : Q_1\\ \vdots\\ \overline{P}_k : P_k\\ \overline{Q}_k : Q_k\end{bmatrix}^{-1}\begin{bmatrix}\begin{bmatrix}-F(x_0) & 0\\ 0 & F(x_1)\end{bmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{bmatrix}-F(x_{k-1}) & 0\\ 0 & F(x_k)\end{bmatrix}\end{bmatrix}\begin{bmatrix}M_1 : N_1 : & \dots & : M_k : N_k\\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k\end{bmatrix}^{-1}$$

Recall (2), (3), and the explicit definition for J as given in the proof of Theorem 2:

$$=(J^*)^{-1}SH^{-1}=(J^*)^{-1}J^*=E_{2nk\times 2nk}.$$

Thus, (17) can be rewritten as:

$$\begin{bmatrix}\begin{bmatrix}\overline{P}_1 & P_1\\ \overline{Q}_1 & Q_1\end{bmatrix}\begin{bmatrix}M_1 & N_1\\ \overline{M}_1 & \overline{N}_1\end{bmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{bmatrix}\overline{P}_k & P_k\\ \overline{Q}_k & Q_k\end{bmatrix}\begin{bmatrix}M_k & N_k\\ \overline{M}_k & \overline{N}_k\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}\begin{bmatrix}\overline{P}_1 & P_1\\ \overline{Q}_1 & Q_1\end{bmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{bmatrix}\overline{P}_k & P_k\\ \overline{Q}_k & Q_k\end{bmatrix}\end{bmatrix}_{2nk\times 2nk^2}\begin{bmatrix}\begin{bmatrix}M_1 & N_1\\ \overline{M}_1 & \overline{N}_1\end{bmatrix} & & 0\\ & \ddots & \\ 0 & & \begin{bmatrix}M_k & N_k\\ \overline{M}_k & \overline{N}_k\end{bmatrix}\end{bmatrix}_{2nk^2\times 2nk}$$

$$=\begin{bmatrix}\overline{P}_1 : P_1\\ \overline{Q}_1 : Q_1\\ \vdots\\ \overline{P}_k : P_k\\ \overline{Q}_k : Q_k\end{bmatrix}\mathcal{W}_{2nk\times 2nk^2}\mathcal{V}_{2nk^2\times 2nk}\begin{bmatrix}M_1 : N_1 : & \dots & : M_k : N_k\\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k\end{bmatrix}$$

$$=\begin{bmatrix}\overline{P}_1 : P_1\\ \overline{Q}_1 : Q_1\\ \vdots\\ \overline{P}_k : P_k\\ \overline{Q}_k : Q_k\end{bmatrix}E_{2nk\times 2nk}\begin{bmatrix}M_1 : N_1 : & \dots & : M_k : N_k\\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k\end{bmatrix}=\begin{bmatrix}\overline{P}_1 : P_1\\ \overline{Q}_1 : Q_1\\ \vdots\\ \overline{P}_k : P_k\\ \overline{Q}_k : Q_k\end{bmatrix}\begin{bmatrix}M_1 : N_1 : & \dots & : M_k : N_k\\ \overline{M}_1 : \overline{N}_1 : & \dots & : \overline{M}_k : \overline{N}_k\end{bmatrix},$$

which completes the proof. □

2 Extension of the work by Pelloni & Smith

In this section, we extend the system that Pelloni & Smith derived in [2].

2.1 Formulation of the problem

Let $m, n \in \mathbb{N}$ be independent, let

$$\pi = \{0 = \eta_0 < \eta_1 < \ldots < \eta_m = 1\}$$

be a partition of a closed interval $[0,1]$, and let

$$\varsigma^r_{kj}, \text{ḑ}^r_{kj} \in \mathbb{C}, \quad \text{for } j \in \{0,1,\ldots,mn-1\}, \quad k \in \{0,1,\ldots,n-1\}, \quad r \in 0,\ldots,m.$$

Further, let

$$C^{n-1}_\pi[a,b] = \Big\{ f : [a,b] \to \mathbb{C} \text{ s.t. } \forall r \in \{1,2,\ldots,m\},$$
$$f_r := f\big|_{(\eta_{r-1},\eta_r)} \text{ admits an extension } g_r \text{ to } [\eta_{r-1},\eta_r] \text{ s.t. } g_r \in C^{n-1}[\eta_{r-1},\eta_r] \Big\}.$$

be the relevant function space. Consider the following *initial-multipoint value problem*:

$$[\partial_t + a(-i\partial_x)^n]q(x,t) = 0 \qquad (x,t) \in \mathbb{R} \times (0,T), \tag{20a}$$
$$q(x,0) = q_0(x) \qquad x \in \mathbb{R}, \tag{20b}$$
$$\sum_{r=1}^{m}\sum_{k=0}^{n-1} \varsigma^r_{kj}\partial_x^{(k)}q(\eta_{r-1},t) + \text{ḑ}^r_{kj}\partial_x^{(k)}q(\eta_r,t) = v_j(t) \qquad t \in [0,T], \quad j \in \{0,1,\ldots,mn-1\}, \tag{20c}$$

where $q(x,\cdot) \in C^{n-1}_\pi[0,1], q_0 \in C^n[0,1], v_j \in C^\infty[0,T]$, with $T > 0$ a fixed constant.

2.2 Global relations

First, we derive the relevant global relation. Fix $r \in \{1,2,\ldots,m\}$, let ϕ_r denote the restriction of function ϕ to interval $[\eta_{r-1},\eta_r]$, and observe the action of Fourier transform on the derivative operator:

$$\begin{aligned}
\widehat{(-i\partial_x)^n\phi_r}(\lambda) &= \int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x}(-i\partial_x)^n\phi_r \,\mathrm{d}x \\
&= e^{-i\lambda x}\sum_{j=1}^{n-1}(-i)^{n+1-j}\lambda^{j-1}\phi_r^{(n-j)}(x)\big|_{x=\eta_{r-1}}^{x=\eta_r} + \lambda^n\int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x}\phi_r\,\mathrm{d}x \qquad (21)\\
&= e^{-i\lambda x}\sum_{k=0}^{n-1}(-i)^{k+1}\lambda^{n-k-1}\phi_r^{(k)}(x)\big|_{x=\eta_{r-1}}^{x=\eta_r} + \lambda^n\widehat{\phi_r}(\lambda), \qquad (22)
\end{aligned}$$

where we performed integration by parts in (22) and relabeled indices as $k = n-j$ in (23). Applying the Fourier transform on the PDE (21a) on $[\eta_{r-1},\eta_r]$ yields

$$\begin{aligned}
0 &= \widehat{[\partial_t + a(-i\partial_x)^n]q_r}(\lambda,t) \\
&= [\partial_t + \lambda^n]\widehat{q_r}(\lambda,t) + a\sum_{k=0}^{n-1} e^{-i\lambda x}(-i)^{k+1}\lambda^{n-k-1}\partial_x^{(k)}q_r(x,t)\big|_{x=\eta_{r-1}}^{x=\eta_r}. \qquad (23)
\end{aligned}$$

Multiplying (24) by $e^{a\lambda^n t}$ and integrating the result in time, we obtain

$$0 = e^{a\lambda^n t}\widehat{q_r}(\lambda;t) - \widehat{q_r}(\lambda;0) + \sum_{k=0}^{n-1} a(-i)^{k+1}\lambda^{n-k-1}e^{-i\lambda x}\int_0^t e^{a\lambda^n s}\partial_x^k q_r(x,s)\big|_{x=\eta_{r-1}}^{x=\eta_r}\,\mathrm{d}s,$$

so that the we obtain the expression for the *global relation*

$$\widehat{q_r}(\lambda;0) - e^{a\lambda^n t}\widehat{q_r}(\lambda;t)$$

$$= \sum_{k=0}^{n-1} a(-i)^{k+1}\lambda^{n-k-1}\left(e^{-i\lambda\eta_r}\int_0^t e^{a\lambda^n s}\partial_x^k q_r(\eta_r,s)\,\mathrm{d}s - e^{-i\lambda\eta_{r-1}}\int_0^t e^{a\lambda^n s}\partial_x^k q_r(\eta_{r-1},s)\,\mathrm{d}s\right), \tag{24}$$

valid for $t\in[0,T],\lambda\in\mathbb{C},x\in[\eta_{r-1},\eta_r]$. Evaluating (25) at $\tau\in[0,T]$, we obtain the global relation at τ :

$$\begin{aligned}&\widehat{q_r}(\lambda;0) - e^{a\lambda^n\tau}\widehat{q_r}(\lambda;\tau)\\ &\quad= \sum_{k=0}^{n-1} a(-i)^{k+1}\lambda^{n-k-1}\left(e^{-i\lambda\eta_r}\int_0^\tau e^{a\lambda^n s}\partial_x^k q_r(\eta_r,s)\,\mathrm{d}s - e^{-i\lambda\eta_{r-1}}\int_0^\tau e^{a\lambda^n s}\partial_x^k q_r(\eta_{r-1},s)\,\mathrm{d}s\right).\end{aligned} \tag{25}$$

Now, we adopt the following notation: for $\lambda\in\mathbb{C}$ and $k\in\{0,\dots,n-1\}$, denote a primitive n^{th} root of unity

$$\alpha = e^{2\pi i/n},$$

an exponential function

$$E_r(\lambda) = e^{-i\lambda\eta_r},$$

coefficients

$$c_k(\lambda) = ia\lambda^{n-k-1}(-i)^k,$$

a time transform of the value $\partial_x^k q$ at $x=\eta_r$ as

$$g_k^r(\lambda) = g_k^r(\lambda,\tau) = c_k(\lambda)\int_0^\tau e^{a\lambda^n s}\partial_x^k q_r(\eta_r,s)\,\mathrm{d}s, \qquad r\in\{1,\dots,m\},$$

a time transform of the value $\partial_x^k q$ at $x=\eta_{r-1}$ as

$$f_k^r(\lambda) = f_k^r(\lambda,\tau) = c_k(\lambda)\int_0^\tau e^{a\lambda^n s}\partial_x^k q_r(\eta_{r-1},s)\,\mathrm{d}s, \qquad r\in\{1,\dots,m\}$$

the Fourier transform of the initial datum, restricted to (η_{r-1},η_r) as

$$\widehat{q_0^r}(\lambda) = \int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x}q_0(x)\,\mathrm{d}x,$$

and the Fourier transform of the solution at time τ, restricted to (η_{r-1},η_r) as

$$\widehat{q_\tau^r}(\lambda) = \int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x}q(x,\tau)\,\mathrm{d}x.$$

Using this notation, we can simplify the global relation (26) to

$$\widehat{q_0^r}(\lambda) - e^{a\lambda^n\tau}\widehat{q_\tau^r}(\lambda) = \sum_{k=0}^{n-1}\left[E_{r-1}(\lambda)f_k^r(\lambda) - E_r(\lambda)g_k^r(\lambda)\right]. \tag{26}$$

Now, we create the system of global relations. Consider the global relation in each of the rectangles $(x,t)\in[\eta_{r-1},\eta_r]\times[0,\tau], r\in\{1,\dots,m\}$, and $\lambda\in\mathbb{C}$. This yields a set of m global relations. Evaluating each relation at $\alpha,\alpha\lambda,\dots,\alpha^{n-1}\lambda$, and using the fact that $f_k^r(\alpha^p\lambda) = \alpha^{(n-1-k)p}f_k^r(\lambda), g_k^r(\alpha^p\lambda) = \alpha^{(n-1-k)p}g_k^r(\lambda)$ we obtain the following system of mn equations

$$\sum_{k=0}^{n-1}\alpha^{(n-1-k)p}\left[E_{r-1}(\alpha^p\lambda)f_k^r(\lambda) - E_r(\alpha^p\lambda)g_k^r(\lambda)\right] = \widehat{q_0^r}(\alpha^p\lambda) - e^{a\lambda^n\tau}\widehat{q_\tau^r}(\alpha^p\lambda), \qquad p\in\{0,1,\dots,n-1\}. \tag{27}$$

Now, we would like to write equations in (28) in a system form. Note

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\alpha^{(n-1-k)p}\left[E_{r-1}(\alpha^p\lambda)f_k^r(\lambda)-E_r(\alpha^p\lambda)g_k^r(\lambda)\right]\\
&\quad=\sum_{k=0}^{n-1}\alpha^{(n-1-k)p}E_{r-1}(\alpha^p\lambda)f_k^r(\lambda)-\sum_{k=0}^{n-1}\alpha^{(n-1-k)p}E_r(\alpha^p\lambda)g_k^r(\lambda)\\
&\quad=\left[\alpha^{(n-1)p}E_{r-1}(\alpha^p\lambda)f_0^r(\lambda)+\alpha^{(n-2)p}E_{r-1}(\alpha^p\lambda)f_1^r(\lambda)+\ldots+\alpha^0E_{r-1}(\alpha^p\lambda)f_{n-1}^r(\lambda)\right]\\
&\quad-\left[\alpha^{(n-1)p}E_r(\alpha^p\lambda)g_0^r(\lambda)+\alpha^{(n-2)p}E_r(\alpha^p\lambda)g_1^r(\lambda)+\ldots+\alpha^0E_r(\alpha^p\lambda)g_{n-1}^r(\lambda)\right]\\
&\quad=\begin{bmatrix}\alpha^{(n-1)p}E_{r-1}(\alpha^p\lambda) & \ldots & \alpha^0E_{r-1}(\alpha^p\lambda) & \alpha^{(n-1)p}E_r(\alpha^p\lambda) & \ldots & \alpha^0E_r(\alpha^p\lambda)\end{bmatrix}\begin{bmatrix}f_0^r(\lambda)\\ \vdots\\ f_{n-1}^r(\lambda)\\ g_0^r(\lambda)\\ \vdots\\ g_{n-1}^r(\lambda)\end{bmatrix}. \qquad (28)
\end{aligned}
$$

Evaluating (29) at $p=0,1,\ldots,n-1$ yields the following system:

$$
\begin{bmatrix}
\alpha^{(n-1)0}E_{r-1}(\alpha^0\lambda) & \ldots & \alpha^0E_{r-1}(\alpha^0\lambda) & \alpha^{(n-1)0}E_r(\alpha^p\lambda) & \ldots & \alpha^0E_r(\alpha^0\lambda)\\
\alpha^{(n-1)}E_{r-1}(\alpha\lambda) & \ldots & \alpha^0E_{r-1}(\alpha\lambda) & \alpha^{(n-1)}E_r(\alpha\lambda) & \ldots & \alpha^0E_r(\alpha\lambda)\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
\alpha^{(n-1)(n-1)}E_{r-1}(\alpha^{(n-1)}\lambda) & \ldots & \alpha^0E_{r-1}(\alpha^{(n-1)}\lambda) & \alpha^{(n-1)(n-1)}E_r(\alpha^p\lambda) & \ldots & \alpha^0E_r(\alpha^{(n-1)}\lambda)
\end{bmatrix}
\begin{bmatrix}f_0^r(\lambda)\\ \vdots\\ f_{n-1}^r(\lambda)\\ g_0^r(\lambda)\\ \vdots\\ g_{n-1}^r(\lambda)\end{bmatrix}
$$

$$
=\begin{bmatrix}\widehat{q_0^r}(\alpha^0\lambda)\\ \vdots\\ \widehat{q_0^r}(\alpha^{(n-1)}\lambda)\end{bmatrix}-e^{a\lambda^n\tau}\begin{bmatrix}\widehat{q_\tau^r}(\alpha^0\lambda)\\ \vdots\\ \widehat{q_\tau^r}(\alpha^{(n-1)}\lambda)\end{bmatrix}. \qquad (29)
$$

For notational convenience, let

$$
\vec{f^r}(\lambda)=\begin{bmatrix}f_0^r(\lambda)\\ \vdots\\ f_{n-1}^r(\lambda)\end{bmatrix},\quad \vec{g^r}(\lambda)=\begin{bmatrix}g_0^r(\lambda)\\ \vdots\\ g_{n-1}^r(\lambda)\end{bmatrix}\quad \vec{\widehat{q_0^r}}(\lambda)=\begin{bmatrix}\widehat{q_0^r}(\alpha^0\lambda)\\ \vdots\\ \widehat{q_0^r}(\alpha^{(n-1)}\lambda)\end{bmatrix},\quad \vec{\widehat{q_\tau^r}}(\lambda)=\begin{bmatrix}\widehat{q_\tau^r}(\alpha^0\lambda)\\ \vdots\\ \widehat{q_\tau^r}(\alpha^{(n-1)}\lambda)\end{bmatrix},
$$

$$
e_r=\begin{bmatrix}
E_r(\lambda) & E_r(\alpha\lambda)\alpha^{(n-1)} & \ldots & E_r(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-1)}\\
E_r(\lambda) & E_r(\alpha\lambda)\alpha^{(n-2)} & \ldots & E_r(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-2)}\\
\vdots & \vdots & \ddots & \vdots\\
E_r(\lambda) & E_r(\alpha\lambda) & \ldots & E_r(\alpha^{(n-1)}\lambda)
\end{bmatrix}
$$

for $r\in\{0,1,\ldots,m\}$, and e_r are $n\times n$ matrices. Then, we can write (30) in a more compact form:

$$
\left[e_{r-1}^T : -e_r^T\right]\begin{bmatrix}\vec{f^r}(\lambda)\\ \vec{g^r}(\lambda)\end{bmatrix}=\vec{\widehat{q_0^r}}(\lambda)-e^{(a\lambda^n\tau)}\vec{\widehat{q_\tau^r}}(\lambda), \qquad r\in\{1,\ldots,m\}, \qquad (30)
$$

where

$$
\left[e_{r-1}^T : -e_r^T\right]=\begin{bmatrix}
(e_{r-1}^T)_{1\ 1} & \ldots & (e_{r-1}^T)_{1\ n} & -(e_r^T)_{1\ 1} & \ldots & -(e_r^T)_{1\ n}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
(e_{r-1}^T)_{n\ 1} & \ldots & (e_{r-1}^T)_{n\ n} & -(e_r^T)_{n\ 1} & \ldots & -(e_r^T)_{n\ n}
\end{bmatrix}.
$$

2.3 Multipoint conditions

We would also like to rewrite the multipoint conditions

$$\sum_{r=1}^{m}\sum_{k=0}^{n-1} \text{ç}^r_{kj}\partial_x^k q(\eta_{r-1},t) + \text{ḑ}^r_{kj}\partial_x^k q(\eta_r,t) = v_j(t),$$

where $t \in [0,T], j \in \{0,1,\dots,mn-1\}$. Multiplying by c_k and $e^{a\lambda^n t}$, and applying the time transform at time $\tau \in [0,T]$ yields

$$\begin{aligned}\sum_{r=1}^{m}\sum_{k=0}^{n-1} \text{ç}^r_{kj}\frac{(-a)}{i^n c_k(\lambda)}c_k(\lambda)\int_0^\tau e^{a\lambda^n s}\partial_x^k q(\eta_{r-1},s)\,\mathrm{d}s \\ + \text{ḑ}^r_{kj}\frac{(-a)}{i^n c_k(\lambda)}c_k(\lambda)\int_0^\tau e^{a\lambda^n s}\partial_x^k q(\eta_r,s)\,\mathrm{d}s = \frac{(-a)}{i^n}\int_0^\tau e^{a\lambda^n s}v_j(s)\,\mathrm{d}s := h_j(\lambda),\end{aligned}$$

so that the conditions become

$$\sum_{r=1}^{m}\sum_{k=0}^{n-1} \text{ç}^r_{kj}\frac{(-a)}{i^n c_k(\lambda)}f^r_k(\lambda) + \text{ḑ}^r_{kj}\frac{(-a)}{i^n c_k(\lambda)}g^r_k(\lambda) = h_j(\lambda). \tag{31}$$

Now, expand the sum in (32) over k :

$$\begin{aligned} h_j(\lambda) &= \sum_{r=1}^{m}\sum_{k=0}^{n-1}\ \text{ç}^r_{kj}\frac{(-a)}{i^n c_k(\lambda)}f^r_k(\lambda) + \text{ḑ}^r_{kj}\frac{(-a)}{i^n c_k(\lambda)}g^r_k(\lambda) \\ &= \sum_{r=1}^{m}\ \text{ç}^r_{0j}\frac{(-a)}{i^n c_0(\lambda)}f^r_0(\lambda) + \text{ç}^r_{1j}\frac{(-a)}{i^n c_1(\lambda)}f^r_1(\lambda) + \dots + \text{ç}^r_{(n-1)j}\frac{(-a)}{i^n c_{n-1}(\lambda)}f^r_{n-1}(\lambda) \\ &\qquad + \text{ḑ}^r_{0j}\frac{(-a)}{i^n c_0(\lambda)}g^r_0(\lambda) + \text{ḑ}^r_{1j}\frac{(-a)}{i^n c_1(\lambda)}g^r_1(\lambda) + \dots + \text{ḑ}^r_{(n-1)j}\frac{(-a)}{i^n c_{n-1}(\lambda)}g^r_{n-1}(\lambda) \\ &= \sum_{r=1}^{m}\ \begin{bmatrix} \text{ç}^r_{0j}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & \text{ç}^r_{(n-1)j}\frac{(-a)}{i^n c_{n-1}(\lambda)} & \text{ḑ}^r_{0j}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & \text{ḑ}^r_{(n-1)j}\frac{(-a)}{i^n c_{n-1}(\lambda)}\end{bmatrix}\begin{bmatrix}\vec{f^r}(\lambda)\\ \vec{g^r}(\lambda)\end{bmatrix}. \end{aligned} \tag{32}$$

Evaluating (33) at $j = 0,1,\dots,mn-1$ and combining the resultant equations, we obtain:

$$\begin{aligned}\sum_{r=1}^{m}\begin{bmatrix} \text{ç}^r_{00}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & \text{ç}^r_{(n-1)0}\frac{(-a)}{i^n c_{n-1}(\lambda)} & \text{ḑ}^r_{00}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & \text{ḑ}^r_{(n-1)0}\frac{(-a)}{i^n c_{n-1}(\lambda)} \\ \text{ç}^r_{01}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & \text{ç}^r_{(n-1)1}\frac{(-a)}{i^n c_{n-1}(\lambda)} & \text{ḑ}^r_{01}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & \text{ḑ}^r_{(n-1)1}\frac{(-a)}{i^n c_{n-1}(\lambda)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \text{ç}^r_{0(mn-1)}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & \text{ç}^r_{(n-1)(mn-1)}\frac{(-a)}{i^n c_{n-1}(\lambda)} & \text{ḑ}^r_{0(mn-1)}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & \text{ḑ}^r_{(n-1)(mn-1)}\frac{(-a)}{i^n c_{n-1}(\lambda)}\end{bmatrix}\begin{bmatrix}\vec{f^r}(\lambda)\\ \vec{g^r}(\lambda)\end{bmatrix} \\ = \begin{bmatrix} h_0(\lambda)\\ h_1(\lambda)\\ \vdots \\ h_{mn-1}(\lambda)\end{bmatrix}.\end{aligned} \tag{33}$$

For $k = 0,1,\dots,m-1$, define the following matrices

$$\text{Ç}^r_k = \begin{bmatrix} \text{ç}^r_{0kn}\frac{1}{(i\lambda)^{n-1}} & \text{ç}^r_{1kn}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{ç}^r_{(n-1)kn} \\ \text{ç}^r_{0(kn+1)}\frac{1}{(i\lambda)^{n-1}} & \text{ç}^r_{1(kn+1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{ç}^r_{(n-1)(kn+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \text{ç}^r_{0((k+1)n-1)}\frac{1}{(i\lambda)^{n-1}} & \text{ç}^r_{1((k+1)n-1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{ç}^r_{(n-1)((k+1)n-1)}\end{bmatrix}^T, \qquad n\times n \text{ block;}$$

$$\text{Ḑ}^r_k = \begin{bmatrix} \text{ḑ}^r_{0kn}\frac{1}{(i\lambda)^{n-1}} & \text{ḑ}^r_{1kn}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{ḑ}^r_{(n-1)kn} \\ \text{ḑ}^r_{0(kn+1)}\frac{1}{(i\lambda)^{n-1}} & \text{ḑ}^r_{1(kn+1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{ḑ}^r_{(n-1)(kn+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \text{ḑ}^r_{0((k+1)n-1)}\frac{1}{(i\lambda)^{n-1}} & \text{ḑ}^r_{1((k+1)n-1)}\frac{1}{(i\lambda)^{n-2}} & \cdots & \text{ḑ}^r_{(n-1)((k+1)n-1)}\end{bmatrix}^T, \qquad n\times n \text{ block.}$$

Then, we can rewrite the system in (34) as

$$
\begin{aligned}
\underbrace{\begin{bmatrix} h_0(\lambda) \\ h_1(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \end{bmatrix}}_{mn \times 1} &= \sum_{r=1}^{m} \begin{bmatrix} (\text{Ç}_0^r)^T & : & (\text{Ḑ}_0^r)^T \\ (\text{Ç}_1^r)^T & : & (\text{Ḑ}_1^r)^T \\ & \vdots & \\ (\text{Ç}_{m-1}^r)^T & : & (\text{Ç}_{m-1}^r)^T \end{bmatrix} \begin{bmatrix} \vec{f^r}(\lambda) \\ \vec{g^r}(\lambda) \end{bmatrix} \\
&= \begin{bmatrix} (\text{Ç}_0^1)^T & : & (\text{Ḑ}_0^1)^T \\ (\text{Ç}_1^1)^T & : & (\text{Ḑ}_1^1)^T \\ & \vdots & \\ (\text{Ç}_{m-1}^1)^T & : & (\text{Ç}_{m-1}^1)^T \end{bmatrix} \begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \end{bmatrix} + \begin{bmatrix} (\text{Ç}_0^2)^T & : & (\text{Ḑ}_0^2)^T \\ (\text{Ç}_1^2)^T & : & (\text{Ḑ}_1^2)^T \\ & \vdots & \\ (\text{Ç}_{m-1}^2)^T & : & (\text{Ç}_{m-1}^2)^T \end{bmatrix} \begin{bmatrix} \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \end{bmatrix} \\
&\quad + \ \dots \ + \begin{bmatrix} (\text{Ç}_0^m)^T & : & (\text{Ḑ}_0^m)^T \\ (\text{Ç}_1^m)^T & : & (\text{Ḑ}_1^m)^T \\ & \vdots & \\ (\text{Ç}_{m-1}^m)^T & : & (\text{Ç}_{m-1}^m)^T \end{bmatrix} \begin{bmatrix} \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} (\text{Ç}_0^1)^T & : & (\text{Ḑ}_0^1)^T & : & (\text{Ç}_0^2)^T & : & (\text{Ḑ}_0^2)^T & : & \dots & : & (\text{Ç}_0^m)^T & : & (\text{Ḑ}_0^m)^T \\ (\text{Ç}_1^1)^T & : & (\text{Ḑ}_1^1)^T & : & (\text{Ç}_1^2)^T & : & (\text{Ḑ}_1^2)^T & : & \dots & : & (\text{Ç}_1^m)^T & : & (\text{Ḑ}_1^m)^T \\ & & & & & & \vdots & & & & & & \\ (\text{Ç}_{m-1}^1)^T & : & (\text{Ḑ}_{m-1}^1)^T & : & (\text{Ç}_{m-1}^2)^T & : & (\text{Ḑ}_{m-1}^2)^T & : & \dots & : & (\text{Ç}_{m-1}^m)^T & : & (\text{Ḑ}_{m-1}^m)^T \end{bmatrix}}_{mn \times 2mn} \underbrace{\begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \\ \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \\ \vdots \\ \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix}}_{2mn \times 1}.
\end{aligned} \tag{34}
$$

The equation (35) gives a convenient way to express the multipoint conditions.

2.4 The Dirichlet-to-Neumann map in $\mathcal{B}$ form

We use the results in the previous two subsections to define a system, whose solution will aid us in finding the Dirichlet-to-Neumann map. First, recall from the Global Relations subsection, we have the equation (31), reproduced below:

$$
\begin{bmatrix} e_{r-1}^T : -e_r^T \end{bmatrix} \begin{bmatrix} \vec{f^r}(\lambda) \\ \vec{g^r}(\lambda) \end{bmatrix} = \widehat{\vec{q_0^r}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q_\tau^r}}(\lambda), \qquad r \in \{1, \dots, m\}. \tag{35}
$$

Evaluating the equation in (36) at $r = 1, \dots, m$ yields

$$
\begin{aligned}
\begin{bmatrix} e_0^T : -e_1^T \end{bmatrix} \begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \end{bmatrix} &= \widehat{\vec{q_0^1}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q_\tau^1}}(\lambda) \\
\begin{bmatrix} e_1^T : -e_2^T \end{bmatrix} \begin{bmatrix} \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \end{bmatrix} &= \widehat{\vec{q_0^2}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q_\tau^2}}(\lambda) \\
&\vdots \\
\begin{bmatrix} e_{m-2}^T : -e_{m-1}^T \end{bmatrix} \begin{bmatrix} \vec{f^{m-1}}(\lambda) \\ \vec{g^{m-1}}(\lambda) \end{bmatrix} &= \widehat{\vec{q_0^{m-1}}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q_\tau^{m-1}}}(\lambda) \\
\begin{bmatrix} e_{m-1}^T : -e_m^T \end{bmatrix} \begin{bmatrix} \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix} &= \widehat{\vec{q_0^m}}(\lambda) - e^{a\lambda^n \tau} \widehat{\vec{q_\tau^m}}(\lambda).
\end{aligned} \tag{36}
$$

Combining the global relations in (37), we obtain the following system:

$$\underbrace{\begin{bmatrix} e_0^T : -e_1^T & 0 & \dots & 0 & 0 \\ 0 & e_1^T : -e_2^T & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e_{m-2}^T : -e_{m-1}^T & 0 \\ 0 & 0 & \dots & 0 & e_{m-1}^T : -e_m^T \end{bmatrix}}_{mn \times 2mn} \underbrace{\begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \\ \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \\ \vdots \\ \vec{f^{m-1}}(\lambda) \\ \vec{g^{m-1}}(\lambda) \\ \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix}}_{2mn \times 1} = \underbrace{\begin{bmatrix} \widehat{\vec{q_0^1}}(\lambda) \\ \widehat{\vec{q_0^2}}(\lambda) \\ \vdots \\ \widehat{\vec{q_0^{m-1}}}(\lambda) \\ \widehat{\vec{q_0^m}}(\lambda) \end{bmatrix}}_{mn \times 1} - e^{a\lambda^n \tau} \underbrace{\begin{bmatrix} \widehat{\vec{q_\tau^1}}(\lambda) \\ \widehat{\vec{q_\tau^2}}(\lambda) \\ \vdots \\ \widehat{\vec{q_\tau^{m-1}}}(\lambda) \\ \widehat{\vec{q_\tau^m}}(\lambda) \end{bmatrix}}_{mn \times 1}. \tag{37}$$

Combining (38) and (35), we arrive at the following system

$$\mathcal{B} \underbrace{\begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \\ \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \\ \vdots \\ \vec{f^{m-1}}(\lambda) \\ \vec{g^{m-1}}(\lambda) \\ \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix}}_{2mn \times 1} = \underbrace{\begin{bmatrix} h_0(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \\ \widehat{\vec{q_0^1}}(\lambda) \\ \widehat{\vec{q_0^2}}(\lambda) \\ \vdots \\ \widehat{\vec{q_0^{m-1}}}(\lambda) \\ \widehat{\vec{q_0^m}}(\lambda) \end{bmatrix}}_{2mn \times 1} - e^{a\lambda^n \tau} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \widehat{\vec{q_\tau^1}}(\lambda) \\ \widehat{\vec{q_\tau^2}}(\lambda) \\ \vdots \\ \widehat{\vec{q_\tau^{m-1}}}(\lambda) \\ \widehat{\vec{q_\tau^m}}(\lambda) \end{bmatrix}}_{2mn \times 1}, \tag{38}$$

where

$$\mathcal{B} = \underbrace{\begin{bmatrix} (\text{Ç}_0^1)^T & (\text{Ḑ}_0^1)^T & (\text{Ç}_0^2)^T & (\text{Ḑ}_0^2)^T & \dots & (\text{Ç}_0^{m-1})^T & (\text{Ḑ}_0^{m-1})^T & (\text{Ç}_0^m)^T & (\text{Ḑ}_0^m)^T \\ (\text{Ç}_1^1)^T & (\text{Ḑ}_1^1)^T & (\text{Ç}_1^2)^T & (\text{Ḑ}_1^2)^T & \dots & (\text{Ç}_1^{m-1})^T & (\text{Ḑ}_1^{m-1})^T & (\text{Ç}_1^m)^T & (\text{Ḑ}_1^m)^T \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ (\text{Ç}_{m-1}^1)^T & (\text{Ḑ}_{m-1}^1)^T & (\text{Ç}_{m-1}^2)^T & (\text{Ḑ}_{m-1}^2)^T & \dots & (\text{Ç}_{m-1}^{m-1})^T & (\text{Ḑ}_{m-1}^{m-1})^T & (\text{Ç}_{m-1}^m)^T & (\text{Ḑ}_{m-1}^m)^T \\ e_0^T & -e_1^T & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & e_1^T & -e_2^T & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & e_{m-2}^T & -e_{m-1}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & e_{m-1}^T & -e_m^T \end{bmatrix}}_{2mn \times 2mn}$$

is a block matrix where each block is $n \times n$. Solving this system will help obtain the Dirichlet-to-Neumann map. We shall refer to this system as the *D-to-N map in $\mathcal{B}$ form*.

2.5 The Dirichlet-to-Neumann map in $\mathcal{A}$ form

We seek to simplify the system (39). First, recall the matrices

$$\text{Ç}_k^r = \begin{bmatrix} \text{ç}_{0kn}^r \frac{1}{(i\lambda)^{n-1}} & \text{ç}_{1kn}^r \frac{1}{(i\lambda)^{n-2}} & \dots & \text{ç}_{(n-1)kn}^r \\ \text{ç}_{0(kn+1)}^r \frac{1}{(i\lambda)^{n-1}} & \text{ç}_{1(kn+1)}^r \frac{1}{(i\lambda)^{n-2}} & \dots & \text{ç}_{(n-1)(kn+1)}^r \\ \vdots & \vdots & \ddots & \vdots \\ \text{ç}_{0((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1}} & \text{ç}_{1((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-2}} & \dots & \text{ç}_{(n-1)((k+1)n-1)}^r \end{bmatrix}^T, \qquad r = 1, \dots, m,$$

$$\text{\c{D}}_k^r = \begin{bmatrix} \text{\c{d}}_{0kn}^r \frac{1}{(i\lambda)^{n-1}} & \text{\c{d}}_{1kn}^r \frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{d}}_{(n-1)kn}^r \\ \text{\c{d}}_{0(kn+1)}^r \frac{1}{(i\lambda)^{n-1}} & \text{\c{d}}_{1(kn+1)}^r \frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{d}}_{(n-1)(kn+1)}^r \\ \vdots & \vdots & \ddots & \vdots \\ \text{\c{d}}_{0((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1}} & \text{\c{d}}_{1((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-2}} & \cdots & \text{\c{d}}_{(n-1)((k+1)n-1)}^r \end{bmatrix}^T, \qquad r = 1, \dots, m,$$

$$e_r = \begin{bmatrix} E_r(\lambda) & E_r(\alpha\lambda)\alpha^{(n-1)} & \dots & E_r(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-1)} \\ E_r(\lambda) & E_r(\alpha\lambda)\alpha^{(n-2)} & \dots & E_r(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ E_r(\lambda) & E_r(\alpha\lambda) & \dots & E_r(\alpha^{(n-1)}\lambda) \end{bmatrix} \qquad r = 0, 1, \dots, m,$$

where both matrices are $n \times n$, and $k = 0, \dots, m-1$. Now, define the matrices

$$\text{\c{S}}_k^r = \frac{1}{n} \begin{bmatrix} E_{r-1}(-\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ kn}^r \frac{1}{(i\lambda)^{n-1-j}} & \dots & E_{r-1}(-\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ ((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \\ E_{r-1}(-\alpha\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ kn}^r \frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} & \dots & E_{r-1}(-\alpha\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ ((k+1)n-1)}^r \frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} \\ \vdots & \ddots & \vdots \\ E_{r-1}(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ kn}^r \frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} & \dots & E_{r-1}(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ ((k+1)n-1)}^r \frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix};$$

$$\text{\c{T}}_k^r = \frac{1}{n} \begin{bmatrix} E_r(-\lambda) \sum_{j=0}^{n-1} \text{\c{d}}_{j\ kn}^r \frac{1}{(i\lambda)^{n-1-j}} & \dots & E_r(-\lambda) \sum_{j=0}^{n-1} \text{\c{d}}_{j\ ((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \\ E_r(-\alpha\lambda) \sum_{j=0}^{n-1} \text{\c{d}}_{j\ kn}^r \frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} & \dots & E_r(-\alpha\lambda) \sum_{j=0}^{n-1} \text{\c{d}}_{j\ ((k+1)n-1)}^r \frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} \\ \vdots & \ddots & \vdots \\ E_r(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \text{\c{d}}_{j\ kn}^r \frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} & \dots & E_r(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \text{\c{d}}_{j\ ((k+1)n-1)}^r \frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix}.$$

The matrices $\text{\c{S}}_k^r, \text{\c{T}}_k^r$ have the following convenient property:

Lemma 9. *For the relevant matrices $e_r, \text{\c{C}}_k^r, \text{\c{D}}_k^r, \text{\c{S}}_k^r, \text{\c{T}}_k^r$, we have*

$$e_{r-1}\text{\c{S}}_k^r = \text{\c{C}}_k^r, \qquad e_r \text{\c{T}}_k^r = \text{\c{D}}_k^r,$$

where $r = 1, \dots, m$ and $k = 0, \dots, m-1$.

Proof. Fix r and k, and consider the product $e_{r-1}\text{\c{S}}_k^r$. Observe that the (t, s)-th entry of the product $e_{r-1}\text{\c{S}}_k^r$ is given by the t-th row of e_{r-1} times s-th column of $\text{\c{S}}_k^r$. Thus, we have:

$$\begin{aligned}
(e_{r-1}\text{\c{S}}_k^r)_{(t,s)} &= \frac{1}{n} \begin{bmatrix} E_{r-1}(\lambda) & E_{r-1}(\alpha\lambda)\alpha^{n-t} & \dots & E_{r-1}(\alpha^{(n-1)}\lambda)\alpha^{(n-1)(n-t)} \end{bmatrix} \\
&\qquad \begin{bmatrix} E_{r-1}(-\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ (s-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \\ E_{r-1}(-\alpha\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ (s-1)}^r \frac{\alpha^{(j+1)}}{(i\lambda)^{n-1-j}} \\ \vdots \\ E_{r-1}(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \text{\c{c}}_{j\ (s-1)}^r \frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix} \\
&= \frac{1}{n} \sum_{j=0}^{n-1} \text{\c{c}}_{j\ (s-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \Big[E_{r-1}(\lambda)E_{r-1}(-\lambda) + E_{r-1}(\alpha\lambda)E_{r-1}(-\alpha\lambda)\alpha^{n-t}\alpha^{j+1} \\
&\qquad + \dots + E_{r-1}(\alpha^{(n-1)}\lambda)E_{r-1}(-\alpha^{n-1}\lambda)\alpha^{(n-1)(n-t)}\alpha^{(n-1)(j+1)} \Big] \\
&= \frac{1}{n} \sum_{j=0}^{n-1} \text{\c{c}}_{j\ (s-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \Big[1 + \alpha^{n-t}\alpha^{j+1} + \alpha^{2(n-t)}\alpha^{2(j+1)} \\
&\qquad + \dots + \alpha^{(n-2)(n-t)}\alpha^{(n-2)(j+1)} + \alpha^{(n-1)(n-t)}\alpha^{(n-1)(j+1)} \Big] \\
&= \frac{1}{n} \sum_{j=0}^{n-1} \text{\c{c}}_{j\ (s-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \Big[1 + \alpha^{n-t+j+1} + \alpha^{2(n-t+j+1)} \\
&\qquad + \dots + \alpha^{(n-2)(n-t+j+1)} + \alpha^{(n-1)(n-t+j+1)} \Big].
\end{aligned} \tag{39}$$

Consider the inner sum in (40):

Case 1: $j = t-1$. If $j = t-1$, then

$$\begin{aligned}
1 + \alpha^{n-t+j+1} &+ \alpha^{2(n-t+j+1)} + \ldots + \alpha^{(n-2)(n-t+j+1)} + \alpha^{(n-1)(n-t+j+1)} \\
&= 1 + \alpha^{n-t+t-1+1} + \alpha^{2(n-t+t-1+1)} + \ldots + \alpha^{(n-2)(n-t+t-1+1)} + \alpha^{(n-1)(n-t+t-1+1)} \\
&= 1 + \alpha^{n} + \alpha^{2n} + \ldots + \alpha^{(n-2)n} + \alpha^{(n-1)n} \\
&= 1 + 1 + 1 + \ldots + 1 + 1 \\
&= n,
\end{aligned}$$

where the second last equality follows since α's are primitive roots of unity.

Case 2: $j \neq t-1$. If $j \neq t-1$, then $\alpha^{n-t+j+1} \neq 1$, and so we treat the term $1 + \ldots + \alpha^{(n-1)(n-t+j+1)}$ as a geometric progression with a common ratio $\alpha^{n-t+j+1}$. Thus, by geometric progression formula,

$$\begin{aligned}
1 + \alpha^{n-t+j+1} &+ \alpha^{2(n-t+j+1)} + \ldots + \alpha^{(n-2)(n-t+j+1)} + \alpha^{(n-1)(n-t+j+1)} \\
&= \sum_{k=0}^{n-1} \alpha^{(n-t+j+1)k} \\
&= \frac{\alpha^{(n-t+j+1)n} - 1}{\alpha^{n-t+j+1} - 1} \\
&= 0,
\end{aligned}$$

where the last equality follows since since $\alpha^n = 1$.

Thus, by the above analysis, we have

$$\begin{aligned}
(e_{r-1}\c{S}_k^r)_{(t,s)} &= \frac{1}{n}\sum_{j=0}^{n-1} \c{c}^r_{j\ (s-1)} \frac{1}{(i\lambda)^{n-1-j}} \left[1 + \alpha^{n-t+j+1} + \ldots + \alpha^{(n-1)(n-t+j+1)}\right] \\
&= \c{c}^r_{(t-1)\ (s-1)} \frac{1}{(i\lambda)^{n-t}} + \frac{1}{n}\sum_{\substack{j=0 \\ j\neq t-1}}^{n-1} \c{c}^r_{j\ (s-1)} \frac{1}{(i\lambda)^{n-1-j}} \underbrace{\left[1 + \alpha^{n-t+j+1} + \ldots + \alpha^{(n-1)(n-t+j+1)}\right]}_{=0} \\
&= \c{c}^r_{(t-1)\ (s-1)} \frac{1}{(i\lambda)^{n-t}}.
\end{aligned}$$

But $\c{c}^r_{(t-1)\ (s-1)} \dfrac{1}{(i\lambda)^{n-t}}$ is exactly the (t,s)-th entry of $\c{C}^r_k$, and so we have $e_{r-1}\c{S}^r_k = \c{C}^r_k$. The proof that $e_r\c{T}^r_k = \c{D}^r_k$ is analogous. The proof is complete. □

By lemma 9, we have $(\c{C}^r_k)^T = (\c{S}^r_k)^T e^T_{r-1}$ and $(\c{D}^r_k)^T = (\c{T}^r_k)^T e^T_r$. This allows to rewrite the system (39) as follows:

$$\underbrace{\begin{bmatrix}
(\c{C}^1_0)^T & (\c{D}^1_0)^T & (\c{C}^2_0)^T & (\c{D}^2_0)^T & \ldots & (\c{C}^m_0)^T & (\c{D}^m_0)^T \\
(\c{C}^1_1)^T & (\c{D}^1_1)^T & (\c{C}^2_1)^T & (\c{D}^2_1)^T & \ldots & (\c{C}^m_1)^T & (\c{D}^m_1)^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(\c{C}^1_{m-1})^T & (\c{D}^1_{m-1})^T & (\c{C}^2_{m-1})^T & (\c{D}^2_{m-1})^T & \ldots & (\c{C}^m_{m-1})^T & (\c{D}^m_{m-1})^T \\
e^T_0 & -e^T_1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & e^T_1 & -e^T_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & e^T_{m-1} & -e^T_m
\end{bmatrix}}_{2mn \times 2mn}
\underbrace{\begin{bmatrix}
\vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \\ \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \\ \vdots \\ \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda)
\end{bmatrix}}_{2mn \times 1}$$

$$
= \begin{bmatrix}
(\underset{\sim}{\mathrm{S}}{}^{1}_{0})^T e_0^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{0})^T e_1^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{0})^T e_1^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{0})^T e_2^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{0})^T e_{m-1}^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{0})^T e_m^T \\
(\underset{\sim}{\mathrm{S}}{}^{1}_{1})^T e_0^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{1})^T e_1^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{1})^T e_1^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{1})^T e_2^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{1})^T e_{m-1}^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{1})^T e_m^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(\underset{\sim}{\mathrm{S}}{}^{1}_{m-1})^T e_0^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{m-1})^T e_1^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{m-1})^T e_1^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{m-1})^T e_2^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{m-1})^T e_{m-1}^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{m-1})^T e_m^T \\
I^T e_0^T & -I^T e_1^T & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I^T e_1^T & -I^T e_2^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & I^T e_{m-1}^T & -I^T e_m^T
\end{bmatrix}
\begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \\ \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \\ \vdots \\ \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix}
$$

$$
= \underbrace{\begin{bmatrix}
(\underset{\sim}{\mathrm{S}}{}^{1}_{0})^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{0})^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{0})^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{0})^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{0})^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{0})^T \\
(\underset{\sim}{\mathrm{S}}{}^{1}_{1})^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{1})^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{1})^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{1})^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{1})^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{1})^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(\underset{\sim}{\mathrm{S}}{}^{1}_{m-1})^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{m-1})^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{m-1})^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{m-1})^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{m-1})^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{m-1})^T \\
I^T & -I^T & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I^T & -I^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & I^T & -I^T
\end{bmatrix}}_{2mn \times 2mn}
$$

$$
\underbrace{\begin{bmatrix}
e_0^T & & & & & & & 0 \\
& e_1^T & & & & & & \\
& & e_1^T & & & & & \\
& & & e_2^T & & & & \\
& & & & \ddots & & & \\
& & & & & e_{m-1}^T & & \\
& & & & & & e_{m-1}^T & \\
0 & & & & & & & e_m^T
\end{bmatrix}}_{2mn \times 2mn}
\underbrace{\begin{bmatrix} \vec{f^1}(\lambda) \\ \vec{g^1}(\lambda) \\ \vec{f^2}(\lambda) \\ \vec{g^2}(\lambda) \\ \vdots \\ \vec{f^m}(\lambda) \\ \vec{g^m}(\lambda) \end{bmatrix}}_{2mn \times 1}
$$

$$
= \underbrace{\begin{bmatrix}
(\underset{\sim}{\mathrm{S}}{}^{1}_{0})^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{0})^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{0})^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{0})^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{0})^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{0})^T \\
(\underset{\sim}{\mathrm{S}}{}^{1}_{1})^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{1})^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{1})^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{1})^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{1})^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{1})^T \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(\underset{\sim}{\mathrm{S}}{}^{1}_{m-1})^T & (\underset{\sim}{\mathrm{T}}{}^{1}_{m-1})^T & (\underset{\sim}{\mathrm{S}}{}^{2}_{m-1})^T & (\underset{\sim}{\mathrm{T}}{}^{2}_{m-1})^T & \cdots & (\underset{\sim}{\mathrm{S}}{}^{m}_{m-1})^T & (\underset{\sim}{\mathrm{T}}{}^{m}_{m-1})^T \\
I^T & -I^T & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I^T & -I^T & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & I^T & -I^T
\end{bmatrix}}_{2mn \times 2mn}
\underbrace{\begin{bmatrix} e_0^T \vec{f^1}(\lambda) \\ e_1^T \vec{g^1}(\lambda) \\ e_1^T \vec{f^2}(\lambda) \\ e_2^T \vec{g^2}(\lambda) \\ \vdots \\ e_{m-1}^T \vec{f^m}(\lambda) \\ e_m^T \vec{g^m}(\lambda) \end{bmatrix}}_{2mn \times 1}, \tag{40}
$$

where I is the $n \times n$ identity matrix. We finally rewrite the system (39) as follows:

$$
\mathcal{A} \underbrace{\begin{bmatrix} e_0^T \vec{f^1}(\lambda) \\ e_1^T \vec{g^1}(\lambda) \\ e_1^T \vec{f^2}(\lambda) \\ e_2^T \vec{g^2}(\lambda) \\ \vdots \\ e_{m-1}^T \vec{f^m}(\lambda) \\ e_m^T \vec{g^m}(\lambda) \end{bmatrix}}_{2mn \times 1}
= \underbrace{\begin{bmatrix} h_0(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \\ \widehat{\vec{q_0^1}}(\lambda) \\ \widehat{\vec{q_0^2}}(\lambda) \\ \vdots \\ \widehat{\vec{q_0^{m-1}}}(\lambda) \\ \widehat{\vec{q_0^m}}(\lambda) \end{bmatrix}}_{2mn \times 1}
- e^{a\lambda^n t} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \widehat{\vec{q_t^1}}(\lambda) \\ \widehat{\vec{q_t^2}}(\lambda) \\ \vdots \\ \widehat{\vec{q_t^{m-1}}}(\lambda) \\ \widehat{\vec{q_t^m}}(\lambda) \end{bmatrix}}_{2mn \times 1}, \tag{41}
$$

where

$$\mathcal{A} = \underbrace{\begin{bmatrix} (\underset{\sim}{\mathrm{S}}^1_0) & (\underset{\sim}{\mathrm{S}}^1_1) & \dots & (\underset{\sim}{\mathrm{S}}^1_{m-1}) & I & 0 & \dots & 0 & 0 \\ (\underset{\sim}{\mathrm{T}}^1_0) & (\underset{\sim}{\mathrm{T}}^1_1) & \dots & (\underset{\sim}{\mathrm{T}}^1_{m-1}) & -I & 0 & \dots & 0 & 0 \\ (\underset{\sim}{\mathrm{S}}^2_0) & (\underset{\sim}{\mathrm{S}}^2_1) & \dots & (\underset{\sim}{\mathrm{S}}^2_{m-1}) & 0 & I & \dots & 0 & 0 \\ (\underset{\sim}{\mathrm{T}}^2_0) & (\underset{\sim}{\mathrm{T}}^2_1) & \dots & (\underset{\sim}{\mathrm{T}}^2_{m-1}) & 0 & -I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (\underset{\sim}{\mathrm{S}}^m_0) & (\underset{\sim}{\mathrm{S}}^m_1) & \dots & (\underset{\sim}{\mathrm{S}}^m_{m-1}) & 0 & 0 & \dots & 0 & I \\ (\underset{\sim}{\mathrm{T}}^m_0) & (\underset{\sim}{\mathrm{T}}^m_1) & \dots & (\underset{\sim}{\mathrm{T}}^m_{m-1}) & 0 & 0 & \dots & 0 & -I \end{bmatrix}^T}_{2mn \times 2mn}$$

is a block matrix where each block is $n \times n$. The system (42) has the advantage that the main matrix is easier to compute. We refer to the system (42) as the *D-to-N map in $\mathcal{A}$ form.*

References

[1] Nelson Dunford and Jacob T. Schwartz, *Linear Operators II*, Interscience, 1963.

[2] Beatrice Pelloni and David A. Smith, *Nonlocal and multipoint boundary value problems for linear evolution equations*, • (2018).

[3] Linfan Xiao, *Algorithmic solution of high order partial differential equations in julia via the fokas transform method*, 2018.