Derivation of the Global Relation:

Fix $r \in \{1, 2, \dots, m\}$. Then,

$$\widehat{(-i\partial_x)^n \varphi^r}(\lambda) = \int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x} (-i\partial_x)^n \varphi(x)\, dx$$

$$= e^{-i\lambda x} \sum_{j=1}^{n} (-i)^{n+1-j} \lambda^{j-1} \varphi^{(n-j)}(x) \Big|_{x=\eta_{r-1}}^{x=\eta_r} + \lambda^n \int_{\eta_{r-1}}^{\eta_r} e^{-i\lambda x} \varphi(x)\, dx$$

$$= e^{-i\lambda x} \sum_{k=0}^{n-1} (-i)^{k+1} \lambda^{n-k-1} \varphi^{(k)}(x) \Big|_{x=\eta_{r-1}}^{x=\eta_r} + \lambda^n \hat{\varphi}^r(\lambda).$$

Now, $\left[\partial_t + a(-i\partial_x)^n\right] q(x,t) = 0.$

$$\Rightarrow \widehat{\left[\partial_t + a(-i\partial_x)^n\right] q}(\lambda, t) = 0$$

$$\Rightarrow \sum_{k=0}^{n-1} (-i)^{k+1} \lambda^{n-k-1} a \Big(e^{-i\lambda\eta_r} \int_0^{\tau} e^{a\lambda^n s} \partial_x^k q(\eta_r, s)\, ds$$

$$- e^{-i\lambda\eta_{r-1}} \int_0^{\tau} e^{a\lambda^n s} \partial_x^k q(\eta_{r-1}, s)\, ds \Big)$$

$$= \hat{q}^r(\lambda; 0) - e^{a\lambda^n \tau} \hat{q}^r(\lambda; \tau),$$

at $\tau \in [0, T]$.

$\Rightarrow$ Let $E_r(\lambda) = e^{-i\lambda\eta_r}$,

$C_k(\lambda) = a\lambda^{n-k-1} i (-i)^k$,

$g_k^r(\lambda; t) = C_k(\lambda) \int_0^t e^{a\lambda^n s} q_r^{(k)}(\eta_r, s)\, ds$

$f_k^r(\lambda; t) = C_k(\lambda) \int_0^t e^{a\lambda^n s} q_r^{(k)}(\eta_{r-1}, s)\, ds.$

Then, $\hat{q}^r(\lambda, 0) - e^{a\lambda^n t} \hat{q}^r(\lambda, t)$

$$\sum_{k=0}^{n-1} \left[E_{r-1}(\lambda) f_k^r(\lambda; t) - E_r(\lambda) g_k^r(\lambda; t)\right]$$

Thus, we derived the Global Relation.

Now, we have m global relations. By evaluating each of the global relation at $\lambda, \alpha\lambda, \alpha^2\lambda, \ldots, \alpha^{n-1}\lambda$, & using the fact that

$$f_k^r(\alpha^p\lambda) = \alpha^{(n-1-k)p} f_k^r(\lambda)$$

$$g_k^r(\alpha^p\lambda) = \alpha^{(n-1-k)p} g_k^r(\lambda),$$

for $p = 0, 1, \ldots, n-1$, we obtain a system of mn equations

$$\sum_{k=0}^{n-1} \alpha^{(n-1-k)p} \left[E_{r-1}(\alpha^p\lambda) f_k^r(\lambda) - E_r(\alpha^p\lambda) g_k^r(\lambda) \right]$$

$$= \hat{q}_0^r(\alpha^p\lambda) - e^{a\lambda^n\tau} \hat{g}_\tau^r(\alpha^p\lambda)$$

⇒ ~~ex~~ write down the system

⇒ go back to multipoint....

⇒ ~~we~~ go through Vandermonde

⇒ meet on Monday.

Global Relations:

Note
$$\sum_{k=0}^{n-1} \alpha^{(n-1-k)p} E_{r-1}(\alpha^p\lambda) f_k^r(\lambda) - \alpha^{(n-1-k)p} E_r(\alpha^p\lambda) g_k^r(\lambda) =$$

$$\alpha^{(n-1)p} E_{r-1}(\alpha^p\lambda) f_0^r(\lambda) + \alpha^{(n-2)p} E_{r-1}(\alpha^p\lambda) f_1^r(\lambda) + \dots + \alpha^{0\cdot p} E_{r-1}(\alpha^p\lambda) f_{n-1}^r(\lambda)$$

$$- \left(\alpha^{(n-1)p} E_r(\alpha^p\lambda) g_0^r(\lambda) + \alpha^{(n-2)p} E_r(\alpha^p\lambda) g_1^r(\lambda) + \dots + \alpha^{0\cdot p} E_r(\alpha^p\lambda) g_{n-1}^r(\lambda)\right) =$$

$$= \left[\alpha^{(n-1)p} E_{r-1}(\alpha^p\lambda), \dots, \alpha^{0\cdot p} E_{r-1}(\alpha^p\lambda), -\alpha^{(n-1)p} E_r(\alpha^p\lambda), \dots, -\alpha^{0\cdot p} E_r(\alpha^p\lambda)\right] \begin{bmatrix} f_0^r(\lambda) \\ \vdots \\ f_{n-1}^r(\lambda) \\ g_0^r(\lambda) \\ \vdots \\ g_{n-1}^r(\lambda) \end{bmatrix}$$

$$= \hat{q}_0^r(\alpha^p\lambda) - e^{a\lambda^n t} \hat{q}_t^r(\alpha^p\lambda).$$

Evaluating the above at $p = 0, 1, \dots, n-1$, we have

$$\begin{bmatrix} \alpha^{(n-1)\cdot 0} E_{r-1}(\lambda) & \cdots & \alpha^{0\cdot p} E_{r-1}(\lambda) & -\alpha^{(n-1)\cdot 0} E_r(\lambda) & \cdots & -\alpha^{0\cdot p} E_r(\lambda) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha^{(n-1)(n-1)} E_{r-1}(\alpha^{n-1}\lambda) & \cdots & \alpha^{0(n-1)} E_{r-1}(\alpha^{n-1}\lambda) & -\alpha^{(n-1)(n-1)} E_r(\alpha^{n-1}\lambda) & \cdots & \alpha^{0\cdot n-1} E_r(\alpha^{n-1}\lambda) \end{bmatrix} \begin{bmatrix} f^r(\lambda) \\ \vdots \\ g_{n-1}^r(\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{q}_0^r(\alpha^0\lambda) \\ \vdots \\ \hat{q}_0^r(\alpha^{n-1}\lambda) \end{bmatrix} - e^{a\lambda^n t} \begin{bmatrix} \hat{q}_T^r(\alpha^0\lambda) \\ \vdots \\ \hat{q}_T^r(\alpha^{n-1}\lambda) \end{bmatrix}$$

Letting

$$e_r = \begin{bmatrix} \alpha^{(n-1)\cdot 0} E_r(\lambda) & \cdots & \alpha^{0\cdot p} E_r(\lambda) \\ \vdots & \ddots & \vdots \\ \alpha^{(n-1)(n-1)} E_r(\alpha^{n-1}\lambda) & \cdots & \alpha^{0(n-1)} E_r(\alpha^{n-1}\lambda) \end{bmatrix}^T$$

$$\vec{f}^r(\lambda) = \begin{bmatrix} f_0^r(\lambda) \\ \vdots \\ f_{n-1}^r(\lambda) \end{bmatrix}, \quad \vec{g}^r(\lambda) = \begin{bmatrix} g_0^r(\lambda) \\ \vdots \\ g_{n-1}^r(\lambda) \end{bmatrix}$$

$$\vec{\hat{q}}_0^r(\lambda) = \begin{bmatrix} \hat{q}_0^r(\alpha^0\lambda) \\ \vdots \\ \hat{q}_0^r(\alpha^{n-1}\lambda) \end{bmatrix}, \quad \vec{\hat{q}}_T^r(\lambda) = \begin{bmatrix} \hat{q}_T^r(\alpha^0\lambda) \\ \vdots \\ \hat{q}_T^r(\alpha^{n-1}\lambda) \end{bmatrix},$$

the system becomes $[(e_{r-1})^T, -(e_r)^T] \begin{bmatrix} \vec{f}^r(\lambda) \\ \vec{g}^r(\lambda) \end{bmatrix} = \vec{\hat{q}}_0^r(\lambda) - e^{a\lambda^n t} \vec{\hat{q}}_T^r(\lambda)$.

Boundary Conditions:

Our conditions are $\displaystyle\sum_{r=1}^{m}\sum_{k=0}^{n-1} c_{kj}^{r}\,\partial_x^{(k)} q(\eta_{r-1},t) + d_{kj}^{r}\,\partial_x^{(k)} q(\eta_r,t) = v_j(t),$

$j \in (0,\dots,mn-1).$

Applying the time transform, we obtain

$$\sum_{r=1}^{m}\sum_{k=0}^{n-1} c_{kj}^{r}\,\frac{(-a)}{i^n c_k(\lambda)}\,c_k(\lambda)\int_0^T e^{a\lambda^n s}\,\partial_x^{(k)} q(\eta_{r-1},s)\,ds \,+$$

$$d_{kj}^{r}\,\frac{(-a)}{i^n c_k(\lambda)}\,c_k(\lambda)\int_0^T e^{a\lambda^n s}\,\partial_x^{(k)} q(\eta_r,s)\,ds = \frac{(-a)}{i^n}\int_0^T e^{a\lambda^n s}\,v_j(s)\,ds$$

$$=: h_j(\lambda).$$

$$\Rightarrow \sum_{r=1}^{m}\sum_{k=0}^{n-1} c_{kj}^{r}\,\frac{(-a)}{i^n c_k(\lambda)}\,f_k^r(\lambda) + d_{kj}^{r}\,\frac{(-a)}{i^n c_k(\lambda)}\,g_k^r(\lambda) = h_j(\lambda)$$

Expand the sum over k:

$$\sum_{r=1}^{m} c_{0j}^{r}\,\frac{(-a)}{i^n c_0(\lambda)}\,f_0^r(\lambda) + c_{1j}^{r}\,\frac{(-a)}{i^n c_1(\lambda)}\,f_1^r(\lambda) + \dots + c_{(n-1)j}^{r}\,\frac{(-a)}{i^n c_{n-1}(\lambda)}\,f_{n-1}^r(\lambda)$$

$$+\, d_{0j}^{r}\,\frac{(-a)}{i^n c_0(\lambda)}\,g_0^r(\lambda) + d_{1j}^{r}\,\frac{(-a)}{i^n c_1(\lambda)}\,g_1^r(\lambda) + \dots + d_{(n-1)j}^{r}\,\frac{(-a)}{i^n c_{n-1}(\lambda)}\,g_{n-1}^r(\lambda) = h_j(\lambda)$$

$$\Rightarrow \sum_{r=1}^{m}\left[c_{0j}^{r}\,\frac{(-a)}{i^n c_0(\lambda)},\dots,c_{(n-1)j}^{r}\,\frac{(-a)}{i^n c_{n-1}(\lambda)},\; d_{0j}^{r}\,\frac{(-a)}{i^n c_0(\lambda)},\dots,d_{(n-1)j}^{r}\,\frac{(-a)}{i^n c_{n-1}(\lambda)}\right]\begin{bmatrix}\vec{f}^r(\lambda)\\ \vec{g}^r(\lambda)\end{bmatrix}$$

$$= h_j(\lambda).$$

Evaluate the above for $j = 0,\dots,mn-1$:

$$\sum_{r=1}^{m}\underbrace{\left[\begin{array}{ccc|ccc} c_{00}^{r}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & c_{(n-1)0}^{r}\frac{(-a)}{i^n c_{n-1}(\lambda)} & d_{00}^{r}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & d_{(n-1)0}^{r}\frac{(-a)}{i^n c_{n-1}(\lambda)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{0(mn-1)}^{r}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & c_{(n-1)(mn-1)}^{r}\frac{(-a)}{i^n c_{n-1}(\lambda)} & d_{0(mn-1)}^{r}\frac{(-a)}{i^n c_0(\lambda)} & \cdots & d_{(n-1)(mn-1)}^{r}\frac{(-a)}{i^n c_{n-1}(\lambda)} \end{array}\right]}_{mn \times 2n}\underbrace{\begin{bmatrix}\vec{f}^r(\lambda)\\ \vec{g}^r(\lambda)\end{bmatrix}}_{2n\times 1} = \underbrace{\begin{bmatrix}h_0(\lambda)\\ \vdots \\ h_{mn-1}(\lambda)\end{bmatrix}}_{mn\times 1}$$

For $k = 0, \dots, m-1$, define

$$\varphi_k^r = \begin{bmatrix} \varphi_{0\,kn}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & \varphi_{(n-1)\,kn}^r \\ \vdots & \ddots & \vdots \\ \varphi_{0\,((k+1)n-1)}^r \frac{1}{(i\lambda)^{(n-1)}} & \cdots & \varphi_{(n-1)\,((k+1)n-1)}^r \end{bmatrix}^T \quad n \times n$$

$$D_k^r = \begin{bmatrix} d_{0\,kn}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & d_{(n-1)\,kn}^r \\ \vdots & \ddots & \vdots \\ d_{0\,((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & d_{(n-1)\,((k+1)n-1)}^r \end{bmatrix}^T \quad n \times n$$

Then, we can rewrite the system on page 2 as

$$\sum_{r=1}^{m} \begin{bmatrix} (\varphi_1^r)^T & (D_1^r)^T \\ (\varphi_2^r)^T & (D_2^r)^T \\ \vdots & \vdots \\ (\varphi_{m-1}^r)^T & (D_{m-1}^r)^T \end{bmatrix} \begin{bmatrix} \vec{f}^r(\lambda) \\ \vec{g}^r(\lambda) \end{bmatrix} = \begin{bmatrix} h_0(\lambda) \\ \vdots \\ \vdots \\ h_{mn-1}(\lambda) \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} (\varphi_0^1)^T & (D_0^1)^T \\ \vdots & \vdots \\ (\varphi_{m-1}^1)^T & (D_{m-1}^1)^T \end{bmatrix} \begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \end{bmatrix} + \begin{bmatrix} (\varphi_0^2)^T & (D_0^2)^T \\ \vdots & \vdots \\ (\varphi_{m-1}^2)^T & (D_{m-1}^2)^T \end{bmatrix} \begin{bmatrix} \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \end{bmatrix} + \dots + \begin{bmatrix} (\varphi_0^m)^T & (D_0^m)^T \\ \vdots & \vdots \\ (\varphi_{m-1}^m)^T & (D_{m-1}^m)^T \end{bmatrix} \times \begin{bmatrix} \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda) \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} (\varphi_0^1)^T & (D_0^1)^T & (\varphi_0^2)^T & (D_0^2)^T & \cdots & (\varphi_0^m)^T & (D_0^m)^T \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (\varphi_{m-1}^1)^T & (D_{m-1}^1)^T & (\varphi_{m-1}^2)^T & (D_{m-1}^2)^T & \cdots & (\varphi_{m-1}^m)^T & (D_{m-1}^m)^T \end{bmatrix}}_{mn \times 2mn} \underbrace{\begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda) \end{bmatrix}}_{2mn \times 1} = \underbrace{\begin{bmatrix} h_0(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \end{bmatrix}}_{mn \times 1}$$

(5) Defining System #1.

From Global Relations, we have

$$[(e_0)^T; -(e_1)^T]\begin{bmatrix}\vec{f}^1(\lambda)\\ \vec{g}^1(\lambda)\end{bmatrix} = \hat{\vec{g}}_0^1(\lambda) - e^{a\lambda^n t}\hat{\vec{g}}_\sigma^1(\lambda) =: q_1$$

$$[(e_1)^T \;\; -(e_2)^T]\begin{bmatrix}\vec{f}^2(\lambda)\\ \vec{g}^2(\lambda)\end{bmatrix} = \hat{\vec{g}}_0^2(\lambda) - e^{a\lambda^n t}\hat{\vec{g}}_\sigma^2(\lambda) =: q_2$$

$$\vdots$$

$$[(e_{m-2})^T \;\; -(e_{m-1})^T]\begin{bmatrix}\vec{f}^{m-1}(\lambda)\\ \vec{g}^{m-1}(\lambda)\end{bmatrix} = \hat{\vec{g}}_0^{m-1}(\lambda) - e^{a\lambda^n t}\hat{\vec{g}}_\sigma^{m-1}(\lambda) =: q_{m-1}$$

$$[(e_{m-1})^T \;\; -(e_m)^T]\begin{bmatrix}\vec{f}^m(\lambda)\\ \vec{g}^m(\lambda)\end{bmatrix} = \hat{\vec{g}}_0^m(\lambda) - e^{a\lambda^n t}\hat{\vec{g}}_\sigma^m(\lambda) =: q_m$$

Combining the above, we obtain

$$\begin{bmatrix}
e_0^T & -e_1^T & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & e_1^T & -e_2^T & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & e_{m-2}^T & -e_{m-1}^T & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & e_{m-1}^T & -e_m^T
\end{bmatrix}
\begin{bmatrix}\vec{f}^1(\lambda)\\ \vec{g}^1(\lambda)\\ \vec{f}^2(\lambda)\\ \vec{g}^2(\lambda)\\ \vdots\\ \vec{f}^m(\lambda)\\ \vec{g}^m(\lambda)\end{bmatrix}
=
\begin{bmatrix}q_1\\ q_2\\ \vdots\\ q_{m-1}\\ q_m\end{bmatrix}$$

Finally, we combine the above system with the system of BC:

$$\underbrace{\begin{bmatrix}
(C_0^1)^T & (D_0^1)^T & (C_0^2)^T & (D_0^2)^T & \cdots & (C_0^{m-1})^T & (D_0^{m-1})^T & (C_0^m)^T & (D_0^m)^T\\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\
(C_{m-1}^1)^T & (D_{m-1}^1)^T & (C_{m-1}^2)^T & (D_{m-1}^2)^T & \cdots & (C_{m-1}^{m-1})^T & (D_{m-1}^{m-1})^T & (C_{m-1}^m)^T & (D_{m-1}^m)^T\\
e_0^T & -e_1^T & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & e_1^T & -e_2^T & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & e_{m-2}^T & -e_{m-1}^T & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & e_{m-1}^T & -e_m^T
\end{bmatrix}}_{2mn\times 2mn}
\underbrace{\begin{bmatrix}\vec{f}^1(\lambda)\\ \vec{g}^1(\lambda)\\ \vec{f}^2(\lambda)\\ \vec{g}^2(\lambda)\\ \vdots\\ \vec{f}^m(\lambda)\\ \vec{g}^m(\lambda)\end{bmatrix}}_{2mn\times 1}$$

$$= \underbrace{\begin{bmatrix}h_0(\lambda)\\ \vdots\\ h_{mn-1}(\lambda)\\ \hat{\vec{g}}_0^1(\lambda)\\ \vdots\\ \hat{\vec{g}}_0^m(\lambda)\end{bmatrix} - e^{a\lambda^n t}\begin{bmatrix}0\\ \vdots\\ 0\\ \hat{\vec{g}}_\sigma^1(\lambda)\\ \vdots\\ \hat{\vec{g}}_\sigma^m(\lambda)\end{bmatrix}}_{2mn\times 1}.$$

⑥. Finally, taking the transpose will yield the desired system:

$$\begin{bmatrix} F_0^1 & \cdots & F_{m-1}^1 & e_0 & 0 & \cdots & 0 & 0 \\ D_0^1 & \cdots & D_{m-1}^1 & -e_1 & 0 & \cdots & 0 & 0 \\ F_0^2 & \cdots & F_{m-1}^2 & 0 & e_1 & \cdots & 0 & 0 \\ D_0^2 & \cdots & D_{m-1}^2 & 0 & -e_2 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ F_0^{m-1} & \cdots & F_{m-1}^{m-1} & 0 & 0 & \cdots & e_{m-2} & 0 \\ D_0^{m-1} & \cdots & D_{m-1}^{m-1} & 0 & 0 & \cdots & -e_{m-1} & 0 \\ F_0^m & \cdots & F_{m-1}^m & 0 & 0 & \cdots & 0 & e_{m-1} \\ D_0^m & \cdots & D_{m-1}^m & 0 & 0 & \cdots & 0 & e_m \end{bmatrix}^T \begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^{m-1}(\lambda) \\ \vec{g}^{m-1}(\lambda) \\ \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda) \end{bmatrix} = \begin{bmatrix} h_0(\lambda) \\ \vdots \\ h_{mn-1}(\lambda) \\ \vec{\hat{g}}_0^1(\lambda) \\ \vdots \\ \vec{\hat{g}}_0^m(\lambda) \end{bmatrix} - e^{a\lambda' d} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vec{\hat{g}}_t^1(\lambda) \\ \vdots \\ \vec{\hat{g}}_t^m(\lambda) \end{bmatrix}$$

where

$$F_k^r = \begin{bmatrix} F_{0\,kn}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & F_{(n-1)\,kn}^r \\ \vdots & \ddots & \vdots \\ F_{0\,((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & F_{(n-1)\,((k+1)n-1)}^r \end{bmatrix}^T,$$

$$D_k^r = \begin{bmatrix} d_{0\,kn}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & d_{(n-1)\,kn}^r \\ \vdots & \ddots & \vdots \\ d_{0\,((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & d_{(n-1)\,((k+1)n-1)}^r \end{bmatrix}^T,$$

$$\vec{f}^r(\lambda) = \begin{bmatrix} f_0^r(\lambda) \\ \vdots \\ f_{n-1}^r(\lambda) \end{bmatrix}, \quad \vec{g}^r(\lambda) = \begin{bmatrix} g_0^r(\lambda) \\ g_{n-1}^r(\lambda) \end{bmatrix}, \quad \vec{\hat{q}}_0^r(\lambda) = \begin{bmatrix} \hat{q}_0^r(a^0\lambda) \\ \vdots \\ \hat{q}_0^r(a^{n-1}\lambda) \end{bmatrix}, \quad \vec{\hat{q}}_t^r(\lambda) = \begin{bmatrix} \hat{q}_t^r(a^0\lambda) \\ \vdots \\ \hat{q}_t^r(a^{n-1}\lambda) \end{bmatrix}$$