Extension of the Vandermonde Argument

Recall the matrix side of the original system we're dealing with:

$$\underbrace{\begin{bmatrix} (\varphi_0^1)^T & (D_0^1)^T & (\varphi_0^2)^T & (D_0^2)^T & \cdots & (\varphi_0^{m-1})^T & (D_0^{m-1})^T & (\varphi_0^m)^T & (D_0^m)^T \\ \vdots & \vdots & \vdots & \vdots & & & & & \\ (\varphi_{m-1}^1)^T & (D_{m-1}^1)^T & (\varphi_{m-1}^2)^T & (D_{m-1}^2)^T & \cdots & (\varphi_{m-1}^{m-1})^T & (D_{m-1}^{m-1})^T & (\varphi_{m-1}^m)^T & (D_{m-1}^m)^T \\ e_0^T & -e_1^T & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & e_1^T & -e_2^T & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e_{m-2}^T & -e_{m-1}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & e_{m-1}^T & -e_m^T \end{bmatrix}}_{2mn \times 2mn,} \underbrace{\begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^m(\lambda) \\ \vec{g}^m(\lambda) \end{bmatrix}}_{2mn \times 1,}$$

where e_r, $\vec{f}^r(\lambda)$, $\vec{g}^r(\lambda)$ are defined as usual, and

$$\varphi_k^r = \begin{bmatrix} \varphi_{0\,kn}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & \varphi_{0\,((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1}} \\ \vdots & \ddots & \\ \varphi_{(n-1)\,kn}^r & \cdots & \varphi_{(n-1)\,((k+1)n-1)}^r \end{bmatrix}, \qquad \begin{aligned} r &= 1, \dots, m \\ k &= 0, \dots, m-1 \end{aligned}$$

$$D_k^m = \begin{bmatrix} d_{0\,kn}^r \frac{1}{(i\lambda)^{n-1}} & \cdots & d_{0\,((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1}} \\ \vdots & \ddots & \vdots \\ d_{(n-1)\,kn}^r & \cdots & d_{(n-1)\,((k+1)n-1)}^r \end{bmatrix}, \qquad \begin{aligned} n &= 1, \dots, m \\ k &= 0, \dots, m-1 \end{aligned}$$

Now, define

$$\phi_k^r = \begin{bmatrix} \frac{1}{n} E_{r-1}(-\lambda) \sum_{j=0}^{n-1} \varphi_{j\,kn}^r \frac{1}{(i\lambda)^{n-1-j}} & \cdots & \frac{1}{n} E_{r-1}(-\lambda) \sum_{j=0}^{n-1} \varphi_{j\,((k+1)n-1)}^r \frac{1}{(i\lambda)^{n-1-j}} \\ \vdots & & \\ \frac{1}{n} E_{r-1}(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \varphi_{j\,kn}^r \frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} & \cdots & \frac{1}{n} E_{r-1}(-\alpha^{n-1}\lambda) \sum_{j=0}^{n-1} \varphi_{j\,((k+1)n-1)}^r \frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix}$$

and consider the product $e_{r-1} \phi_k^r$.

Since the (t,s)-th entry of $e_{r-1}\,\xi_k^r$ is given by t-th row of e_{r-1} times s-th column of ξ_k^r, we have

$$\frac{1}{n}\left[E_{r-1}(\lambda)\quad E_{r-1}(\alpha\lambda)\,\alpha^{n-t}\quad E_{r-1}(\alpha^2\lambda)\,\alpha^{2(n-t)}\ \cdots\ E_{r-1}(\alpha^{n-1}\lambda)\,\alpha^{(n-1)(n-t)}\right]\begin{bmatrix} E_{r-1}(-\lambda)\sum_{j=0}^{n-1}\varphi_{j(s-1)}^r\frac{1}{(i\lambda)^{n-1-j}} \\ E_{r-1}(-\alpha\lambda)\sum_{j=0}^{n-1}\varphi_{j(s-1)}^r\frac{\alpha^{j+1}}{(i\lambda)^{n-1-j}} \\ \vdots \\ E_{r-1}(-\alpha^{n-1}\lambda)\sum_{j=0}^{n-1}\varphi_{j(s-1)}^r\frac{\alpha^{(j+1)(n-1)}}{(i\lambda)^{n-1-j}} \end{bmatrix}$$

$$= \frac{1}{n}\sum_{j=0}^{n-1}\varphi_{j(s-1)}^r\frac{1}{(i\lambda)^{n-1-j}}\Big[\overbrace{E_{r-1}(\lambda)E_{r-1}(-\lambda)}^{1} + \overbrace{E_{r-1}(\alpha\lambda)E_{r-1}(-\alpha\lambda)}^{1}\,\alpha^{n-t+j+1}+\cdots+$$

$$+\underbrace{E_{r-1}(\alpha^{n-1}\lambda)\,E_{r-1}(-\alpha^{n-1}\lambda)}_{1}\ \alpha^{(n-1)(j+1+n-t)}\Big]$$

$$= \frac{1}{n}\sum_{j=0}^{n-1}\varphi_{j(s-1)}^r\frac{1}{(i\lambda)^{n-1-j}}\underbrace{\left[1+\alpha^{n-t+j+1}+\cdots+\alpha^{(n-1)(j+1+n-t)}\right]}_{\substack{\text{if } j=t-1,\ \text{sum}=n \\ \text{if } j\neq t-1,\ \text{sum}=0.}} = \varphi_{(t-1)(s-1)}^r\frac{1}{(i\lambda)^{n-t}}$$

Thus, the (t,s)-th entry of $e_{r-1}\,\xi_k^r$ is $\varphi_{(t-1)(s-1)}^r\frac{1}{(i\lambda)^{n-t}}$, which is exactly the (t,s)-th entry of φ_k^r. Thus, $\underline{e_{r-1}\,\xi_k^r = \varphi_k^r}$, for $r=1,\dots,m$, & $k=0,\dots,m-1$.

Now, defining

$$\mathcal{D}_k^r = \begin{bmatrix} \frac{1}{n}E_r(-\lambda)\sum_{j=0}^{n-1}d_{j\,kn}^r\frac{1}{(i\lambda)^{n-1-j}} & \cdots & \frac{1}{n}E_r(-\lambda)\sum_{j=0}^{n-1}d_{j\,((k+1)n-1)}^r\frac{1}{(i\lambda)^{n-1-j}} \\ \vdots & \ddots & \vdots \\ \frac{1}{n}E_r(-\alpha^{n-1}\lambda)\sum_{j=0}^{n-1}d_{j\,kn}^r\frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} & \cdots & \frac{1}{n}E_r(-\alpha^{n-1}\lambda)\sum_{j=0}^{n-1}d_{j\,((k+1)n-1)}^r\frac{\alpha^{(n-1)(j+1)}}{(i\lambda)^{n-1-j}} \end{bmatrix},$$

& arguing in a similar way as above, we obtain that $\underline{e_r\,\mathcal{D}_k^r = \mathcal{D}_k^r}$, for $r=1,\dots,m$, & $k=0,\dots,m-1$.

Remark: Observe how similar ξ_k^r, $\mathcal{D}_k^r$ are to matrices $\underline{B^r}$ in the original system. We only had to take care of E_{r-1}, E_r to make the system work.

Finally, we simplify the system. First, note that $e_{r-1}\varsigma_k^r = \varsigma_k^r \Rightarrow (\varsigma_k^r)^T = (\varsigma_k^r)^T e_{r-1}^T$, and $e_r \mathcal{D}_k^r = \mathcal{D}_k^r \Rightarrow (\mathcal{D}_k^r)^T = (\mathcal{D}_k^r)^T e_r^T$. With this in mind, we obtain

$$\begin{bmatrix} (\varsigma_0^1)^T & (\mathcal{D}_0^1)^T & (\varsigma_0^2)^T & (\mathcal{D}_0^2)^T & \cdots & (\varsigma_0^{M-1})^T & (\mathcal{D}_0^{M-1})^T & (\varsigma_0^M)^T & (\mathcal{D}_0^M)^T \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ (\varsigma_{M-1}^1)^T & (\mathcal{D}_{M-1}^1)^T & (\varsigma_{M-1}^2)^T & (\mathcal{D}_{M-1}^2)^T & \cdots & (\varsigma_{M-1}^{M-1})^T & (\mathcal{D}_{M-1}^{M-1})^T & (\varsigma_{M-1}^M)^T & (\mathcal{D}_{M-1}^M)^T \\ e_0^T & -e_1^T & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & e_1^T & -e_2^T & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e_{M-2}^T & -e_{M-1}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & e_{M-1}^T & -e_M^T \end{bmatrix} \begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^M(\lambda) \\ \vec{g}^M(\lambda) \end{bmatrix} =$$

$$= \begin{bmatrix} (\varsigma_0^1)^T e_0^T & (\mathcal{D}_0^1)^T e_1^T & (\varsigma_0^2)^T e_1^T & (\mathcal{D}_0^2)^T e_2^T & \cdots & (\varsigma_0^M)^T e_{M-1}^T & (\mathcal{D}_0^M)^T e_M^T \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (\varsigma_{M-1}^1)^T e_0^T & (\mathcal{D}_{M-1}^1)^T e_1^T & (\varsigma_{M-1}^2)^T e_1^T & (\mathcal{D}_{M-1}^2)^T e_2^T & \cdots & (\varsigma_{M-1}^M)^T e_{M-1}^T & (\mathcal{D}_{M-1}^M)^T e_M^T \\ I^T e_0^T & -I^T e_1^T & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I^T e_1^T & -I^T e_2^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e_{M-1}^T & -e_M^T \end{bmatrix} \begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vec{f}^2(\lambda) \\ \vec{g}^2(\lambda) \\ \vdots \\ \vec{f}^M(\lambda) \\ \vec{g}^M(\lambda) \end{bmatrix} =$$

$$\begin{bmatrix} (\varsigma_0^1)^T & (\mathcal{D}_0^1)^T & (\varsigma_0^2)^T & (\mathcal{D}_0^2)^T & \cdots & (\varsigma_0^M)^T & (\mathcal{D}_0^M)^T \\ \vdots & & & & & & \\ (\varsigma_{M-1}^1)^T & (\mathcal{D}_{M-1}^1)^T & (\varsigma_{M-1}^2)^T & (\mathcal{D}_{M-1}^2)^T & \cdots & (\varsigma_{M-1}^M)^T & (\mathcal{D}_{M-1}^M)^T \\ I^T & -I^T & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I^T & -I^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & e_{M-1}^T & -e_M^T \end{bmatrix} \begin{bmatrix} e_0^T & 0 & & & & & \\ 0 & e_1^T & 0 & & & & \\ & 0 & e_1^T & 0 & & & \\ & & 0 & e_2^T & & & \\ & & & & \ddots & & \\ & & & & & e_{M-1}^T & 0 \\ & & & & 0 & e_{M-1}^T & 0 \\ & & & & & 0 & e_M^T \end{bmatrix} \begin{bmatrix} \vec{f}^1(\lambda) \\ \vec{g}^1(\lambda) \\ \vdots \\ \vec{f}^M(\lambda) \\ \vec{g}^M(\lambda) \end{bmatrix}$$

Applying the identity-like diagonal block matrix to the column vector, and taking the transpose out in the main matrix yields:

$\vdots$

$$\begin{bmatrix} \mathcal{S}^1_0 & \cdots & \mathcal{S}^1_{M-1} & I & 0 & \cdots & 0 & 0 \\ \mathcal{D}^1_0 & \cdots & \mathcal{D}^1_{M-1} & -I & 0 & \cdots & 0 & 0 \\ \mathcal{S}^2_0 & \cdots & \mathcal{S}^2_{M-1} & 0 & I & \cdots & 0 & 0 \\ \mathcal{D}^2_0 & \cdots & \mathcal{D}^2_{M-1} & 0 & -I & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathcal{S}^{M-1}_0 & \cdots & \mathcal{S}^{M-1}_{M-1} & 0 & 0 & \cdots & I & 0 \\ \mathcal{D}^{M-1}_0 & \cdots & \mathcal{D}^{M-1}_{M-1} & 0 & 0 & \cdots & -I & 0 \\ \mathcal{S}^M_0 & \cdots & \mathcal{S}^M_{M-1} & 0 & 0 & \cdots & 0 & I \\ \mathcal{D}^M_0 & \cdots & \mathcal{D}^M_{M-1} & 0 & 0 & \cdots & 0 & -I \end{bmatrix} \begin{bmatrix} e_0^T \vec{f}^1(\lambda) \\ e_1^T \vec{g}^1(\lambda) \\ e_1^T \vec{f}^2(\lambda) \\ e_2^T \vec{g}^2(\lambda) \\ \vdots \\ e_{M-2}^T \vec{f}^{M-1}(\lambda) \\ e_{M-1}^T \vec{g}^{M-1}(\lambda) \\ e_{M-1}^T \vec{f}^M(\lambda) \\ e_M^T \vec{g}^M(\lambda) \end{bmatrix}$$

Therefore we have arrived at the desired, simplified system.