

## Summary of Locker's Self-Adjointness For Multipoint Differential Operators

The set-up is as follows: given a closed interval  $[a, b]$ , fix  $n \in \mathbb{N}$ , so that the differential operator is given by

$$\tau := \sum_{k=0}^n a_k(t) \left( \frac{d}{dt} \right)^k, \text{ where } a_k(t) \in C^\infty[a, b] \text{ and } a_n(t) \neq 0 \forall t \in [a, b].$$

Let  $S := L^2[a, b]$  with the standard inner product, and let  $H^n[a, b]$  be a subspace of  $S$  consisting of  $f \in C^{n-1}[a, b]$ , with  $f^{(n-1)}$  absolutely continuous (i.e. differentiable almost everywhere), and  $f^{(n)} \in S$ . Fix  $m \in \mathbb{N}$ , and let  $\pi = \{a = x_0 < x_1 < \dots < x_m = b\}$  be a partition of  $[a, b]$ . Finally, let  $H^n(\pi)$  be a collection of  $f \in S$  such that

1. On each subinterval  $[x_{l-1}, x_l]$ ,  $f(t)$  possesses right-hand and left-hand limits at the endpoints  $x_{l-1}$  and  $x_l$  respectively. Moreover, let  $f_l$  denote the function  $f$  on subinterval  $(x_{l-1}, x_l)$ , and let  $f_l(x_{l-1}) = f(x_{l-1}^+)$  and  $f_l(x_l) = f(x_l^-)$ . We call  $f_1, \dots, f_m$  the components of  $f$  and denote this by writing  $f = (f_1, \dots, f_m)$ .
2. For  $l = 1, \dots, m$ ,  $f_l \in H^n[x_{l-1}, x_l]$ .

We define a *multipoint boundary value* (MBV) to be a linear functional  $B$  on  $H^n(\pi)$  of the form

$$B(f) = \sum_{l=1}^m \sum_{j=0}^{n-1} [\alpha_{jl} f_l^{(j)}(x_{l-1}) + \beta_{jl} f_l^{(j)}(x_l)]$$

where  $f = (f_1, \dots, f_m)$ , and  $\alpha_{jl}, \beta_{jl} \in \mathbb{R}$ . Note that since every boundary value consists of a double sum, and for all  $l, j$  we expect that  $f_l^{(j)}(x_{l-1})$  and  $f_l^{(j)}(x_l)$  to be linearly independent, the space of all boundary values has dimension  $2mn$ .

Now, suppose we are given a set of  $k$  linearly independent MBVs

$$B_i(f) = \sum_{l=1}^m \sum_{j=0}^{n-1} [\alpha_{ijl} f_l^{(j)}(x_{l-1}) + \beta_{ijl} f_l^{(j)}(x_l)], \quad i \in \{1, \dots, k\}.$$

Let  $L$  be an operator given by  $Lf = \tau f$ , whose domain is

$$\mathcal{D}(L) = \{f \in H^n(\pi) \mid B_i(f) = 0, i = 1, \dots, k\}.$$

Then  $L$  is a *multipoint differential operator*. Observe that density of  $\mathcal{D}(L)$  in  $S$  ensures that  $L$  has a well-defined adjoint  $L^*$ . Our goal is to obtain an explicit formula for  $L^*$  and its domain. First, recall the Green's formula from Dunford & Schwartz: given  $f, g \in H^n(\pi)$ ,

$$\langle \tau f, g \rangle - \langle f, \tau^* g \rangle = \sum_{l=1}^m \sum_{p,q=0}^{n-1} [F_{x_l}^{pq}(\tau) f_l^{(p)}(x_l) g_l^{(q)}(x_l) - F_{x_{l-1}}^{pq}(\tau) f_l^{(p)}(x_{l-1}) g_l^{(q)}(x_{l-1})],$$

where  $F_t$  denotes an  $n \times n$  boundary matrix for  $\tau$  at the point  $t \in [a, b]$ . From Dunford & Schwartz, the entries of  $F_t$  are given by

$$\begin{aligned} F_t^{pq}(\tau) &= \sum_{k=j}^{n-p-1} (-1)^k \binom{k}{j} \left( \frac{d}{dt} \right)^{k-j} a_{p+k+1}(t), & p+q < n-1 \\ F_t^{pq}(\tau) &= (-1)^q a_n(t), & p+q = n-1 \\ F_t^{pq}(\tau) &= 0, & p+q > n-1. \end{aligned}$$

Consider a linear system of equations

$$\sum_{l=1}^m \sum_{j=0}^{n-1} [\alpha_{ijl} x_{jl} + \beta_{ijl} y_{jl}] = 0, \quad i = 1, \dots, k.$$

Since the list  $\{B_i\}_{i=1}^k$  is linearly independent, the above system has rank  $k$ , and so  $\dim \text{range}\{B_i\}_{i=1}^k = k$ . Now, since the system is homogeneous, the solution space must have the same dimension as the null space of the system. Since  $\dim\{B_i\}_{i=1}^k = 2mn$ , it follows by the fundamental theorem of linear maps that

$$\dim \text{null}\{B_i\}_{i=1}^k = \dim\{B_i\}_{i=1}^k - \dim \text{range}\{B_i\}_{i=1}^k = 2mn - k.$$

Thus, let  $[x_{ijl}, y_{ijl}], i = 1, \dots, 2mn - k$  be the set of solutions of the above system that also form a basis for the solution space. Define

$$\alpha_{ijl}^* = - \sum_{p=0}^{n-1} x_{ipl} F_{x_{l-1}}^{p, q}(\tau) \quad \text{and} \quad \beta_{ijl}^* = \sum_{p=0}^{n-1} y_{ipl} F_{x_l}^{p, q}(\tau),$$

where  $i = 1, \dots, 2mn - k, j = 0, \dots, n - 1, l = 1, \dots, m$ . Finally, let

$$B_i^*(f) = \sum_{l=1}^m \sum_{j=0}^{n-1} [\alpha_{ijl}^* f_l^{(j)}(x_{l-1}) + \beta_{ijl}^* f_l^{(j)}(x_l)], \quad i \in \{1, \dots, 2mn - k\}.$$

We refer to  $B_i^*$  as *adjoint* multipoint boundary values. In the following theorem, we prove that the above construction is valid.

**Theorem 1.** *The adjoint operator  $L^*$  is the multipoint differential operator defined by*

$$\mathcal{D}(L^*) = \{f \in H^n(\pi) \mid B_i^*(f) = 0, i = 1, \dots, 2mn - k\}, L^* f = \tau^* f.$$

*Proof.* First, we will define  $L_0$  to be what we think is the adjoint of  $L$ . That is, let  $L_0$  be the linear operator in  $S$  whose domain consists of all functions  $f \in H^n(\pi)$  satisfying  $B_i^*(f) = 0$  for  $i = 1, \dots, 2mn - k$  with  $L_0 f = \tau^* f$ . We want to show that  $L_0 = L^*$ . First, we show that  $\mathcal{D}(L_0) \subseteq \mathcal{D}(L^*)$ .

Let  $g \in \mathcal{D}(L_0)$  and set  $g^* = L_0 g = \tau^* g$ . Then, let  $f \in \mathcal{D}(L)$ . Now, we want to show that  $\langle Lf, g \rangle = \langle f, L_0 g \rangle = \langle f, g^* \rangle$ . Recall that the numbers

$$x_{jl} = f_l^{(j)}(x_{l-1}), \quad y_{jl} = f_l^{(j)}(x_l),$$

form the solutions to the system

$$\sum_{l=1}^m \sum_{j=0}^{n-1} [\alpha_{ijl} x_{jl} + \beta_{ijl} y_{jl}] = 0, \quad i = 1, \dots, k.$$

Also recall that as defined earlier,  $[x_{ijl}, y_{ijl}], i = 1, \dots, 2mn - k$  is the set of solutions of the above system which form a basis for the solution space. So, by definition of a basis, there exist constants  $c_1, \dots, c_{2mn-k}$  such that

$$f_l^{(j)}(x_{l-1}) = \sum_{i=1}^{2mn-k} c_i x_{ijl} \quad \text{and} \quad f_l^{(j)}(x_l) = \sum_{i=1}^{2mn-k} c_i y_{ijl}.$$

Now, we apply Green's formula to obtain

$$\begin{aligned}
\langle Lf, g \rangle - \langle f, g^* \rangle &= \langle \tau f, g \rangle - \langle f, \tau^* g \rangle \\
&= \sum_{l=1}^m \sum_{p,q=0}^{n-1} [F_{x_l}^{pq}(\tau) f_l^{(p)}(x_l) g_l^{(q)}(x_l) - F_{x_{l-1}}^{pq}(\tau) f_l^{(p)}(x_{l-1}) g_l^{(q)}(x_{l-1})] \\
\text{(Substitute for the } f_l^{(j)}) &= \sum_{l=1}^m \sum_{p,q=0}^{n-1} \sum_{i=1}^{2mn-k} c_i [F_{x_l}^{pq}(\tau) y_{ipl} g_l^{(q)}(x_l) - F_{x_{l-1}}^{pq}(\tau) x_{ipl} g_l^{(q)}(x_{l-1})] \\
\text{(Substitute in } \alpha^*, \beta^*) &= \sum_{i=1}^{2mn-k} c_i \sum_{l=1}^m \sum_{q=0}^{n-1} [\beta_{iql}^* g_l^{(q)}(x_l) + \alpha_{iql}^* g_l^{(q)}(x_{l-1})] \\
&= \sum_{i=1}^{2mn-k} c_i B_i^*(g) \\
\text{(By definition of } g) &= 0
\end{aligned}$$

Since  $f \in \mathcal{D}(L)$  is arbitrary, and  $\langle Lf, g \rangle = \langle f, L_0 g \rangle = \langle f, g^* \rangle$ , we can conclude that  $g \in \mathcal{D}(L^*)$ , which implies  $\mathcal{D}(L_0) \subseteq \mathcal{D}(L^*)$ .

To complete the proof, it remains to show that  $\mathcal{D}(L^*) \subseteq \mathcal{D}(L_0)$ . Let  $g \in \mathcal{D}(L^*)$ . Now, we want to show that  $g \in H^n(\pi)$  and that  $B_i^*(g) = 0$ , which would imply that  $g \in \mathcal{D}(L_0)$  by definition of  $L_0$ . Fix an integer  $l$  with  $1 \leq l \leq m$ , and let  $\bar{g}$  denote the restriction of  $g$  to the interval  $[x_{l-1}, x_l]$ . Let  $\bar{f}$  be any function in  $H^n[x_{l-1}, x_l]$  having its support in the open interval  $(x_{l-1}, x_l)$ . Then, we can extend  $\bar{f}$  to  $f$  defined on  $[a, b]$  by making it 0 outside of  $[x_{l-1}, x_l]$ . The extension of  $f$  belongs in  $\mathcal{D}(L^*)$  because  $f \in H^n(\pi)$ , and  $B_i(f) = 0$  (because it is 0 at all boundary points). Then,

$$0 = \langle Lf, g \rangle - \langle f, L^* g \rangle = \int_{x_{l-1}}^{x_l} (\tau \bar{f}) \bar{g} - \int_{x_{l-1}}^{x_l} \bar{f} (L^* g).$$

By Theorem 10 of [1, p. 1294], the above implies that  $\bar{g}$  is equal a.e to a function in  $H^n[x_{l-1}, x_l]$  and that  $L^* g = \tau^* \bar{g}$  a.e. on  $[x_{l-1}, x_l]$ . Since this holds for all  $l$ , we can conclude  $g \in H^n(\pi)$  and  $L^* g = \tau^* g$ .

Next, we want to show that  $B_i^*(g) = 0$ . Fix an integer  $i$  with  $1 \leq i \leq 2mn - k$  and choose a function  $\sigma = (\sigma_1, \dots, \sigma_m) \in H^n(\pi)$  such that  $\sigma_l^{(j)}(x_{l-1}) = x_{ijl}$  and  $\sigma_l^{(j)}(x_l) = y_{ijl}$ . That is, evaluating  $\sigma$  at each boundary point yields the set of solutions that form the basis for the solution space. Clearly,  $\sigma \in \mathcal{D}(L)$ , and from Green's formula

$$\begin{aligned}
0 &= \langle L\sigma, g \rangle - \langle \sigma, L^* g \rangle \\
&= \langle \tau \sigma, g \rangle - \langle \sigma, \tau^* g \rangle \\
&= \sum_{l=1}^m \sum_{p,q=0}^{n-1} [F_{x_l}^{pq}(\tau) \sigma_l^{(p)}(x_l) g_l^{(q)}(x_l) - F_{x_{l-1}}^{pq}(\tau) \sigma_l^{(p)}(x_{l-1}) g_l^{(q)}(x_{l-1})] \\
&= \sum_{l=1}^m \sum_{p,q=0}^{n-1} [F_{x_l}^{pq}(\tau) y_{ipl} g_l^{(q)}(x_l) - F_{x_{l-1}}^{pq}(\tau) x_{ipl} g_l^{(q)}(x_{l-1})] \\
&= \sum_{l=1}^m \sum_{q=0}^{n-1} [\beta_{iql}^* g_l^{(q)}(x_l) + \alpha_{iql}^* g_l^{(q)}(x_{l-1})] \\
&= B_i^*(g)
\end{aligned}$$

So, we have shown that if given  $g \in \mathcal{D}(L^*)$ ,  $B_i^*(g) = 0$ . That together with  $g \in H^n(\pi)$  proven earlier implies that  $g \in \mathcal{D}(L_0)$ . Thus,  $L_0 = L^*$ .  $\square$

### Explanation of Example 3

Consider the following. Let  $S = L^2[-1, 1]$ , let  $\tau f = f''$ , let  $\pi = \{-1, 0, 1\}$ , and let

$$B_1(f) = f(-1) = f_1(-1),$$

$$B_2(f) = f(1) = f_2(1),$$

$$B_3(f) = f(0^-) - f(0^+) = f_1(0) - f_2(0),$$

$$B_4(f) = f(0^-) - f'(0^-) + f'(0^+) = f_1(0) - f'_1(0) + f'_2(0)$$

We want all these  $B$  to be equal to 0. Now, all we need to do to find the adjoint multi-point boundary value problem for the problem we just described, is to find the  $X$  matrix and the  $F$  matrix. Before that, let's rewrite the above problem into a system of linear equations.

$$1x_{01} + 0x_{11} + 0y_{01} + 0y_{11} + 0x_{02} + 0x_{12} + 0y_{02} + 0y_{12} = 0$$

$$0x_{01} + 0x_{11} + 0y_{01} + 0y_{11} + 0x_{02} + 0x_{12} + 1y_{02} + 0y_{12} = 0$$

$$0x_{01} + 0x_{11} + 1y_{01} + 0y_{11} - 1x_{02} + 0x_{12} + 0y_{02} + 0y_{12} = 0$$

$$0x_{01} + 0x_{11} + 1y_{01} - 1y_{11} + 0x_{02} + 1x_{12} + 0y_{02} + 0y_{12} = 0$$

Now, the notational conversion from the  $f$  form into the  $x$  or  $y$  form isn't obvious, so I will explain the notation a bit more here.

First, note that we are considering  $\pi = \{-1, 0, 1\}$ , so we are concerned with 3 points, and the 2 intervals among those points. These two intervals can be described by two functions separately, namely  $f_1$  and  $f_2$ . That is why when we talk about the middle point, i.e. the point 0, we need to indicate whether we are talking about the left or right limit. Right limit means we are talking about  $f_2$ , and left limit means we are talking about  $f_1$ .

Now, let's talk about this  $a_{bc}$  notation.

- $a$  can be  $x$  or  $y$ .  $x$  means right limit, and  $y$  means left limit.
- $b$  is a number from 0 to  $n - 1$ , which tells us something about the  $b$ th derivative at that point.
- $c$  tells us which  $f$  to use. If  $c$  is 1, we are talking about  $f_1$ . If  $c$  is 2, we are talking about  $f_2$ , etc.

For example,  $x_{01}$  means that we are talking about the right limit, 0th derivative, and  $f_1$ . Since  $f_1$  describes the interval between points  $-1$  and  $0$ , and we are talking about the right limit, we can deduce that it means we are talking about  $f(-1)$ .

Next, the  $X$  given by Locker is

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Without bothering too much with the formal definition of  $X$ , Locker simply describes this matrix as "the general solution" of the linear system of equations written above, and that it contains a basis for the solution space. This  $X$  was not really obvious to me, so I created my own  $X'$  from my understanding of the problem.

$$X' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 2 & -1 & 0 & 0 \end{bmatrix}$$

You can test that this works. Just take any row of the  $X'$  matrix, and you'll notice that there are 8 columns, which correspond to the values of the  $x$  and the  $y$  to be substituted into the system of equations. For instance, the first row of  $X'$  tells us that setting  $x_{11} = 1$  and everything else to 0 is a solution for the system. Note that each row is linearly independent. However, at this point I noticed that you can get the  $X$  matrix from  $X'$  by doing some row operations. For example the second row of  $X$  is simply the fourth row of  $X'$  minus the third row of  $X'$ .

Then, to get the adjoint  $X^*$  or in my case  $X'^*$ , we just need to do multiplication with the matrix  $F$ . The big  $F$  matrix is actually defined by putting smaller  $F_t$  matrices together in the following way:

$$F = \begin{bmatrix} -F_{x_0} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & F_{x_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -F_{x_1} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & F_{x_2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -F_{x_{m-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{x_m} \end{bmatrix}$$

$m$  just tell us the number of points we are concerned with  $-1$ . So in our example,  $m$  is 2, because  $3 - 1 = 2$ . The small  $F_t$  matrices are defined in Dunford-Schwartz:

$$F_t^{pq}(\tau) = \sum_{k=q}^{n-p-1} (-1)^k \binom{k}{q} \left( \frac{d}{dt} \right)^{k-q} a_{p+k+1}(t), \quad p+q \leq n-1$$

$$F_t^{pq}(\tau) = 0, \quad p+q > n-1.$$

with  $0 \leq p, q \leq n-1$  and  $p$  describing the row, and  $q$  describing the columns. Also,  $n$  is the order of the differential operator. Luckily, the  $F_t$  for this problem is rather simple:

$$F_t = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ for all } t \in [-1, 1].$$

Now, we can write  $F$  out explicitly as

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Using the  $X$  given in Locker to find  $X^*$ , we get

$$X^* = XF = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

But if we were to use my  $X'$ , we get :

$$X'^* = X'F = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 2 & 0 & 0 \end{bmatrix}.$$

Again, you can get  $X^*$  from  $X'^*$  by doing row operations. So, we can now do the conversion like we did earlier and turn this into 4 new boundary values. We can use either  $X^*$  or  $X'^*$ , because they are equivalent, but the one Locker used  $X^*$  happens to give out the simplified version. Perhaps this is why he chose a more unintuitive  $X$  at the start. Anyway, the equations come out as:

$$\begin{aligned} B_1(f) &= -f(-1) = -f_1(-1), \\ B_2(f) &= f(0^-) - f'(0^-) + f'(0^+) = f_1(0) - f'_1(0) + f'_2(0) \\ B_3(f) &= f(0^-) - f(0^+) = f_1(0) - f_2(0), \\ B_4(f) &= f(1) = f_2(1), \end{aligned}$$

which is the same system as the one given as the start. So the operator is self-adjoint.

## References

- [1] N. Dunford and J. T. Schwartz. *Linear Operators II*. Interscience, New York, 1963.
- [2] Locker John. Self-Adjointness for multi-point differential operators. *Pacific Journal of Mathematics*, 45, 1973.