Fundamentals of Operational Research Assignment 2

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(a)
Define:
x_t: the number of medical teams allocated in country t
$q_t(x_t)$: the additional person-years of life for country t with x_t medical teams
Decision Variable:

$$\max_{x_t, 0 \le x_t \le 5} \sum_{t=1}^{3} q_t(x_t)$$

Constraints:

$$\sum_{t=1}^{3} x_t \le 5$$
$$x_t \ge 0$$

(b)
$F(t, \beta)$ is the maximum additional person-years of life allocation for the stage that t countries with β available medical teams
The problem is to find F(3,5)
For $t > 1$:

$$F(t, \beta) = \max_{x_t, 0 \le x_t \le \bar{x}_t} \{q_t(x_t) + F(t-1, \beta - x_t)\}$$
$$s.t. x_t \le \beta$$

For $t = 1$:

$$F(1, \beta) = \max_{x_1, 0 \le x_1 \le \bar{x}_1} q_1(x_1)$$
$$s.t. x_1 \le \beta$$

(c)

$F(t,\beta) = \max_{x_t, 0 \le x_t \le \bar{x}_t}\{q_t(x_t) + F(t-1, \beta - x_t)\}$	Poss	Best
$F(1,0) = 0$	0	0
$F(1,1) = 45$	0-1	$\underline{1}$
$F(1,2) = 70$	0-2	2
$F(1,3) = 90$	0-3	3
$F(1,4) = 105$	0-4	4
$F(1,5) = 120$	0-5	5
$F(2,0) = 0 + 0 = 0$	0	0
$F(2,1) = \max\{q_2(0) + F(1,1), q_2(1) + F(1,0)\} = \max\{\underline{0+45}, 20+0\}$	0-1	0
$F(2,2) = \max\{q_2(0) + F(1,2), q_2(1) + F(1,1), q_2(2) + F(1,0)\} = \max\{\underline{0+70}, 20+45, 45+0\}$	0-2	0
$F(2,3) = \max\{q_2(0) + F(1,3), q_2(1) + F(1,2), q_2(2) + F(1,1), q_2(3) + F(1,0)\}$		
$= \max\{\underline{0+90}, \underline{20+70}, \underline{45+45}, 75+0\}$	0-3	0,1,2
$F(2,4) = \max\{q_2(0) + F(1,4), q_2(1) + F(1,3), q_2(2) + F(1,2), q_2(3) + F(1,1), q_2(4) + F(1,0)\}$		
$= \max\{0+105, 20+90, 45+70, \underline{75+45}, 110+0\}$	0-4	$\underline{3}$
$F(2,5) = \max\{q_2(0) + F(1,5), q_2(1) + F(1,4), q_2(2) + F(1,3), q_2(3) + F(1,2), q_2(4) + F(1,1), q_2(5) + F(1,0)\}$		
$= \max\{0+120, 20+105, 45+90, 75+70, \underline{110+45}, 150+0\}$	0-5	4
$F(3,5) = \max\{q_3(0) + F(2,5), q_3(1) + F(2,4), q_3(2) + F(2,3), q_3(3) + F(2,2), q_3(4) + F(2,1), q_3(5) + F(2,0)\}$		
$= \max\{0+155, \underline{50+120}, 70+90, 80+70, 100+45, 130+0\}$	0-5	$\underline{2}$

Solution: arrange 1 team for country 1, 3 teams for country 2, 1 team for country 3
Total maximum additional person-years of life for 3 countries: 170

(d)
The Lagrangian relaxation is:

$$\begin{aligned} &\max_{x_t:0\leq x_t\leq 5} \sum_{t=1}^{3} q_t(x_t) + \lambda(5-\sum_{t=1}^{3} x_t) \\ = &\max_{x_t:0\leq x_t\leq 5} 5\lambda + \sum_{t=1}^{3} \bar{q}_t(x_t), \bar{q}_t(x) = q_t(x_t) - \lambda x_t \end{aligned}$$

The function $\bar{q}_t(x_t)$ can be tabulated for $\lambda = 30$ below:

x	0	1	2	3	4	5
$\bar{q}_1(x)$	0	$\underline{15}$	10	0	-15	-30
$\bar{q}_2(x_t)$	$\underline{0}$	-10	-15	-15	-10	$\underline{0}$
$\bar{q}_3(x_t)$	0	$\underline{20}$	10	-10	-20	-20

After 'read-off', the solution under Lagrangian relaxation is: $x_1 = 1,\ x_2 = 0$ or $5,\ x_3 = 1$
And the objective function value becomes: $5\lambda + \sum_t \bar{q}_t(x_t) = 150 + 15 + 0 + 20 = 185$
As for the feasibility, $(x_1, x_2, x_3) = (1, 5, 1)$ is not feasible due to the constraint $\sum_{t=1}^{3} x_t \leq 5$, but $(x_1, x_2, x_3) = (1, 5, 1)$ is feasible with original objective function value $45 + 0 + 50 = 95$

(e)
From the infeasible solution $(x_1, x_2, x_3) = (1, 5, 1)$, the upper bound is 185
From the feasible solution $(x_1, x_2, x_3) = (1, 0, 1)$, the lower bound is 95
So the actual objective function value bound is: $95 \leq F(3, 5) \leq 185$