Fundamentals of Optimization Homework 3

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(4.1)
$\bar{x} \in R^n$ is the unique optimal solution of $(P) \to c^T\bar{d} > 0$ for all feasible directions $\bar{d} \in R^n$ at $\bar{x}$ such that $\bar{d} \neq 0$:
$\bar{x}$ is the unique potimal solution, and it is a vertex. And we have the optimal value $\bar{z} = c^T\bar{x}$. As there is no any other optimal solution, we assume $x^* \neq \bar{x}$ as a feasible solution, and so that $z^* = c^T x^*$. z^* is just a normal feasible value but not optimal, so $z^* = c^T x^* > \bar{z} = c^T\bar{x}$. Let $\lambda\bar{d} = \lambda(x^* - \bar{x})$ as the feasible direction at $\bar{x}$ (in which $\lambda > 0$). In this case, $\lambda c^T\bar{d} = \lambda(c^T x^* - c^T\bar{x}) = \lambda(z^* - \bar{z}) > 0$ for $\bar{d} = x^* - \bar{x} \neq 0$.

$c^T\bar{d} > 0$ for all feasible directions $\bar{d} \in R^n$ at $\bar{x}$ such that $\bar{d} \neq 0 \to \bar{x} \in R^n$ is the unique optimal solution of (P):
We consider an arbitrary feasible solution of (P), as x^*. Let the feasible direction at $\bar{x}$ as $\lambda\bar{d} = \lambda(x^* - \bar{x}) \neq 0$ (in which $\lambda > 0$), which also implies that $x^* \neq \bar{x}$. Under the condition that for all feasible directions, $c^T\bar{d} > 0$, we find $\lambda c^T\bar{d} = \lambda(c^T x^* - c^T\bar{x}) > 0$. Let the z^* as the corresponding feasible objective function value for feasible solution x^*, we have $z^* - \bar{z} > 0$ for z^* as arbitrary feasible solution x^*'s value. In this case, we prove $\bar{z}$ is the optimal value and $\bar{x}$ is the only optimal feasible solution.

(4.2)
From 15.3.2, we know that, if $\bar{d} \in R^n$ is a feasible diretion at $\hat{x}$, then $A\bar{d} = A_B\bar{d}_B + A_N\bar{d}_N = 0$, $\hat{x}_B + \lambda^*\bar{d}_B \geq 0$, and $\bar{d}_N \geq 0$. When considering $-\bar{d}$, its $\bar{d}_N \leq 0$. In a contradiction, if $-\bar{d}$ is a feasible direction, its $\bar{d}_N$ should also satisfy $\bar{d}_N \geq 0$. Since $\bar{d}_N \geq 0$ as well as $\bar{d}_N \leq 0$, we obtain $\bar{d}_N = 0$ if $-\bar{d}$ is a feasible direction. Futhermore, $A\bar{d} = A_B\bar{d}_B + A_N\bar{d}_N = A_B\bar{d}_B = 0$, we can obtain $\bar{d}_B = -(A_B)^{-1}A_N\bar{d}_N = 0$ due to $\bar{d} = 0$. As $\bar{d}_B = 0$ and $\bar{d}_N = 0$, $\bar{d} = 0$, which is conflict with $\bar{d} \neq 0$, so from this contradiction, we prove that $-\bar{d}$ could not be a feasible direction.

(5.1)
Following the similar method of Proposition 17.1, under the condition that there exists $\bar{c}_{j^*} = c_{j^*} - c_B^T(A_B)^{-1}A^j = 0$ for $j \in N$, we construct a feasible direction $d \in R^n$ at the nondegenerate optimal vertex x^* as follows: Set $d_{j^*} = 1$ and $d_j = 0$ for each $j \in N \backslash j^*$. $d_N \geq 0$. $d_B = -(A_B)^{-1}A_N d_N \in R^m$ and $Ad = A_B d_B + A_N d_N = 0$. As d is a feasible direction at x^*, as long as there exists some real number $\lambda^* > 0$ such that $x^* + \lambda^* d \in P$, we can say for $\lambda \in [0, \lambda^*]$ there are infinite optimal solutions.
It it clear that $A(x^* + \lambda d) = Ax^* + \lambda Ad = b + 0 = b$ for any $\lambda \in R$, and $x_N^* + \lambda d_N = \lambda d_N \geq 0$ for any $\lambda \geq 0$. Now consider $x_B^* + \lambda d_B$:
Case 1: if d_B has at least one negative component, then define $\lambda^* = \min_{j \in B: d_j < 0} \frac{-x_j^*}{d_j}$. Since x^* is nondegenerate, we obtain $x_j^* > 0$ for each $j \in B$. Therefore, $\lambda^* > 0$ and it is finite. As $x_B^* + \lambda d_B \geq 0$ for each $\lambda \in [0, \lambda^*]$. In this case, $x^* + \lambda d \in P$ for each $\lambda \in [0, \lambda^*]$. For $\lambda \in [0, \lambda^*]$, we have $c^T(x^* + \lambda d) = c^T x^* + \lambda \sum_{j \in N} \bar{c}_j d_j$. Here, for $j \in N \backslash j^*$, $d_j = 0$, while for $j = j^*$, $\bar{c}_{j^*} = 0$. So $c^T x^* + \lambda \sum_{j \in N} \bar{c}_j d_j = c^T x^*$, that these feasible solutions have the same optimal value, and the LP problem has infinite optimal solutions.
Case 2: if $d_B \geq 0$, $x_B^* + \lambda d_B \geq 0$ for any $\lambda \geq 0$. Therefore, $x^* + \lambda d \in P$ for all $\lambda \geq 0$. $c^T(x^* + \lambda d) = c^T x^*$ (the same process like above, only λ's value range changes). So in this case, we also prove that there would be infinite optimal solutions.
Overall, we prove that if there exist $\bar{c}_j = 0, j \in N$ for nondegenerate optimal vertex x^*, then (P) has an infinite number of optimal solutions.

(5.2)
x^* is the unique optimal solution of $(P) \to$ if the reduced costs of all nonbasic variables are strictly positive:
As x^* is the unique optimal solution, $\bar{c}_j \geq 0$, $j \in N$, or if there exists $\bar{c}_j < 0$, there would be better solution. As for the cases that $\bar{c}_j = 0$, $j \in N$, from (5.1), we know that there would be infinite optimal solution and x^* is no longer unique, so this condition is also not suitable. In this case, we prove that if x^* is the unique optimal solution, then for every $j \in N$, $\bar{c}_j > 0$.

if the reduced costs of all nonbasic variables are strictly positive *to* x^* is the unique optimal solution of (P):
For contradiction, we set $\bar{x} \neq x^*$ and it is also an optimal solution, considering $d = \bar{x} - x^*$. By (5.1) of Exercise3, there exists index $k \in N$ that $\bar{x}_k > 0$, whereas $x_k^* = 0$. $d_k = \bar{x}_k = \bar{x}_k - x_k^* = \bar{x}_k > 0$. By (5.1) in this homework above, $c^T d = \sum_{n \in N} \bar{c}_n d_n$, in which $\bar{c}_k d_k > 0$, $0 = c^T > 0$, it is a contradiction.