Time Series: Assignment 2

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1. Linear filter

(a)

Assume the straight line trend to be $u_t = a + bt$. To have the relationship of $u'_t = u_t$ for $t = 1, \dots, n$, we could substitute the formula and obtain:

$$
\begin{aligned}
u'_t &= \sum_{j=\alpha}^{\beta} K_j u_{t+j} \\
&= \sum_{j=\alpha}^{\beta} K_j (a + b(t+j)) \\
&= (a + bt) \sum_{j=\alpha}^{\beta} K_j + b \sum_{j=\alpha}^{\beta} jK_j
\end{aligned}
$$

So the coefficients should satisfy:

$$\sum_{j=\alpha}^{\beta} K_j = 1$$

$$\sum_{j=\alpha}^{\beta} jK_j = 0$$

Now, given $K_{-2} = \frac{1}{9},\ K_{-1} = \frac{2}{9},\ K_0 = \frac{3}{9},\ K_1 = \frac{2}{9},\ K_2 = \frac{1}{9}$, we have:

$$\sum_{j=\alpha}^{\beta} K_j = \frac{1}{9} + \frac{2}{9} + \frac{3}{9} + \frac{2}{9} + \frac{1}{9} = 1$$

$$\sum_{j=\alpha}^{\beta} jK_j = -2 \times \frac{1}{9} - 1 \times \frac{2}{9} + 0 \times \frac{3}{9} + 1 \times \frac{2}{9} + 2 \times \frac{1}{9} = 0$$

(actually, since the filter is symmetric, the second condition is automatically satisfied.)

In this case, the filter does not distort linear trend, so that it is "**exact**" on straight lines.

(b)

When the trend is cubic, we assume $u_t = a + bt + ct^2 + dt^3$, and following the similar process as above, we have following conditions:

$$\sum_{j=\alpha}^{\beta} K_j = 1,\ \sum_{j=\alpha}^{\beta} jK_j = 0,\ \sum_{j=\alpha}^{\beta} j^2 K_j = 0,\ \sum_{j=\alpha}^{\beta} j^3 K_j = 0$$

The first and second conditions are already proved previously, and the symmetric property indicates that the second and fourth conditions are satisfied, so we only need to test the third condition:

$$\sum_{j=\alpha}^{\beta} j^2 K_j = (-2)^2 \times \frac{1}{9} + (-1)^2 \times \frac{2}{9} + (0)^2 \times \frac{3}{9} + (1)^2 \times \frac{2}{9} + (2)^2 \times \frac{1}{9} = \frac{4}{3} \neq 0$$

The condition is not satisfied, so this linear filter is **not exact** on cubics.

2. ARCH(1) process

(a)

From the structure of ARCH(1), we know that $\sigma_t = (\alpha_0 + \alpha_1 Y_{t-1})^{1/2}$. Define $H_t = \{\dots, Y_{t-2}, Y_{t-1}, Y_t\}$ as the history of process up to time t. Under the law of total expectation, we have:

$$\begin{aligned} E(Y_t) &= E(E(Y_t|H_{t-1})) \\ &= E(\sigma_t E(\epsilon_t|H_{t-1})) \\ &= E(\sigma_t E(\epsilon_t)) \\ &= 0 \end{aligned}$$

Note that, given H_{t-1}, then σ_t is constant, and ϵ_t is independent of H_{t-1}.

Now, before consider the correlation function, we first calculate the variance and (auto)covariance:

$$\text{var}(Y_t) = E(Y_t^2) - E(Y_t)^2 = E(Y_t^2)$$

Without further information, if $E(Y_t^2)$ is constant, then the variance would be a constant; if not, then the constant variance condition will be violated.

As for the autocovariance ($h > 0$), we have:

$$\begin{aligned} \text{cov}(Y_t, Y_{t+h}) &= E[(Y_t - E(Y_t))(Y_{t+h} - E(Y_{t+h}))] \\ &= E(Y_t \cdot Y_{t+h}) \\ &= E(E(Y_t \cdot Y_{t+h}|H_{t+h-1})) \\ &= E(Y_t \cdot E(Y_{t+h}|H_{t+h-1})) \\ &= E(Y_t \cdot \sigma_{t+h} \cdot E(\epsilon_{t+h})) \\ &= 0 \end{aligned}$$

If $\text{var}(Y_t)$ is constant, then the process satisfies all the definitions of white noise in lecture slides, that "a stochastic process $\{Y_t\}$ is white noise if its elements are uncorrelated, mean $E(Y_t) = 0$ and variance is constant", which indicating the ARCH(1) process is a white noise process; if not, then the variance is varying, and it is not a white noise process. Further reseach on $\text{var}(Y_t)$ is shown in the next subsection.

(b)

Following the relationship $\text{var}(Y_t) = E(Y_t^2)$, let $V_t = Y_t^2 - \sigma_t^2$, so that V_t is white noise ($E(V_t) = 0$). Then, based on $Y_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 + V_t$, we have $Y_t^2 \sim AR(1)$. Remind that an AR(1) process is weakly stationary if $|\alpha_1| < 1$ (specifically, this condition appears due to restriction of positive variance). As $\alpha > 0$, now it is only necessary to set additional condition of $\alpha_1 < 1$ and then we obtain the stationarity, so that we have $E(Y_t^2) = \alpha_0 + \alpha_1 E(Y_{t-1}^2) = \alpha_0 + \alpha_1 E(Y_t^2)$. And the variance is:

$$\text{var}(Y_t) = E(Y_t^2) = \frac{\alpha_0}{1 - \alpha_1}$$

3. ARMA representation

(a)

Let $V_t = Y_t^2 - \sigma_t^2$, then we have:

$$\begin{aligned}
\sigma_t^2 &= \alpha_0 + \sum_{j=1}^{m} \alpha_j Y_{t-j}^2 + \sum_{j=1}^{r} \beta_j \sigma_{t-j}^2 \\
\sigma_t^2 + (Y_t^2 - \sigma_t^2) &= \alpha_0 + \sum_{j=1}^{m} \alpha_j Y_{t-j}^2 + \sum_{j=1}^{r} \beta_j \sigma_{t-j}^2 + V_t \\
Y_t^2 &= \alpha_0 + \sum_{j=1}^{m} \alpha_j Y_{t-j}^2 + \sum_{j=1}^{r} \beta_j (Y_{t-j}^2 - V_{t-j}) + V_t \\
Y_t^2 &= \alpha_0 + \sum_{j=1}^{m'} (\alpha_j + \beta_j) Y_{t-j}^2 + V_t - \sum_{j=1}^{r'} \beta_j V_{t-j}
\end{aligned}$$

Note that $m' = \max(m, r)$ and $r' = r$. Also, within the first sum notation, there is the risk that subscript is not defined if $m < m'$ or $r < m'$, so in all such cases, we assume that these "overflowed" coefficients are set as 0. Now, the GARCH(m, r) is represented as ARMA$(\max(m, r), r)$.

(b)

$$\begin{aligned}
L(\alpha_0, \alpha_1 | y_1, \ldots, y_n) &= p(y_1, \ldots, y_n | \alpha_0, \alpha_1) \\
&= p(y_1 | \alpha_0, \alpha_1) p(y_2 | y_1, \alpha_0, \alpha_1) \ldots p(y_n | y_1, \ldots, y_{n-1}, \alpha_0, \alpha_1) \\
&= f_0(y_0) \prod_{2}^{n} \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{y_n^2}{2\sigma_t^2}}
\end{aligned}$$

4. Heavy-tails

(a)

Since ϵ_t and σ_t are independent with each other, $E(Y_t) = E(\sigma_t)E(\epsilon_t)$. Under the $\mathcal{N}(0,1)$, we know that $E(\epsilon_t) = 0$, and so that $E(Y_t) = 0$.

Now we explore its variance first:

$$\text{var}(Y_t) = E(Y_t^2) - E(Y_t)^2 = E(Y_t^2)$$

Set $V_t = \sigma_t^2(\epsilon_t^2 - 1)$ (so $\{V_t\}$ are non-Gaussian white noise), the original formula could be written as:

$$Y_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 + V_t - \alpha_1 V_{t-1}$$

Assuming the stationarity, the equation should hold:

$$\begin{aligned}
E(Y_t^2) &= E(\alpha_0 + \alpha_1 Y_{t-1}^2 + V_t - \alpha_1 V_{t-1}) \\
&= E(\alpha_0) + E(\alpha_1 Y_{t-1}^2) + E(V_t) - E(\alpha_1 V_{t-1}) \\
&= E(\alpha_0) + \alpha_1 E(Y_{t-1}^2) \\
&= \alpha_0 + \alpha_1 E(Y_t^2)
\end{aligned}$$

So that the variance is:

$$var(Y_t) = E(Y_t^2) = \frac{\alpha_0}{1 - \alpha_1}$$

To ensure the variance is positive, $\frac{\alpha_0}{1-\alpha_1} > 0$, we obtain the **necessary condition** is $(0 \leq)\alpha_1 < 1$, which leads $\{Y_t\}$ to be stationary.

As for the autocovariance:

$$\begin{aligned}
\text{cov}(Y_t, Y_{t+h}) &= E[(Y_t - E(Y_t))(Y_{t+h} - E(Y_{t+h}))] \\
&= E(Y_t \cdot Y_{t+h}) \\
&= E(E(Y_t \cdot Y_{t+h}|\mathcal{H}_{t+h-1})) \\
&= E(Y_t \cdot E(Y_{t+h}|\mathcal{H}_{t+h-1})) \\
&= E(Y_t \cdot \sigma_{t+h} \cdot E(\epsilon_{t+h})) \\
&= 0
\end{aligned}$$

(b)

Based on the normal distribution's kurtosis property, we have:

$$E(Y_t^4|\mathcal{H}_{t-1}) = 3\{E(Y_t^2|\mathcal{H}_{t-1})\}^2 = 3(\alpha_0 + \alpha_1 Y_{t-1}^2)^2$$

Thus:

$$\begin{aligned}
E(Y_t^4) &= E[E(Y_t^4|\mathcal{H}_{t-1})] \\
&= 3E(\alpha_0 + \alpha_1 Y_{t-1}^2)^2 \\
&= 3E[\alpha_0^2 + 2\alpha_0\alpha_1 Y_{t-1}^2 + \alpha_1^2 Y_{t-1}^4] \\
&= 3(\alpha_0^2 + 2\alpha_0\alpha_1 E(Y_{t-1}^2) + \alpha_1^2 E(Y_{t-1}^4)) \\
&= 3(\alpha_0^2 + 2\alpha_0\alpha_1 \frac{\alpha_0}{1-\alpha_1} + \alpha_1^2 E(Y_t^4)) \text{ } (\alpha_1 < 1 \text{ \& assuming stationarity})
\end{aligned}$$

By reconstructing:

$$E(Y_t^4) = \frac{3\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}$$

Known that $var(Y_t) = \frac{\alpha_0}{1-\alpha_1}$, now we could calculate the kurtosis of ARCH(1):

$$\frac{E(Y_t^4)}{var(Y_t)^2} = 3\frac{1-\alpha_1^2}{1-3\alpha_1^2} \geq 3$$

The inequality above is proved due to the fact that $1-\alpha_1^2 > 1-3\alpha_1^2 > 0$. Since the fourth moment should be positive, and we already have $0 < \alpha_1^2 < 1$ and $\alpha_0 > 0$, so that we need $(1-3\alpha_1^2) > 0$, also indicating the requirement of $\alpha_1^2 < \frac{1}{3}$.

Since the kurtosis of a normal random variable under normal distribution is exact 3, we obtain that the kurtosis of Y_t is greater than normal kurtosis, which leads to the heavier tails (leptokurtic, a thin bell shape with a high peak) than does the normal.