Fundamentals of Operational Research Final Exam

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December 11, 2021

(1)

1.(a)

objective function: $\max \sum_{i\in\{1,2,3,4\}}^{4} q_i$

s.t. $\sum_{i=1}^{4} w_i \le W$ $(W=8)$

(b)

the state can be taken as (i,w). where w is the amount of power available for 1 to i, Define:

$F(i,w) \equiv$ total utility maximum from 1 to i with weight available w.

$F(i,w) = \max(F(i-1,w),\ q_i + F(i-1, w-w_i))$, for $i \ge 1$

$F(0,w) = 0$

the problem is solved by finding $F(n,W)$ ($n=4, W=8$ here), which can be done by solving the above recurrence in order of increasing i:

(c)

$F(i,w) = \max(F(i-1,w),\ q_i + F(i-1, w-w_i))$

$F(0,w) = 0$

$F(1, w\ge 1) = 1$ (yes)

$F(2,1) = 1$ (no)

$F(2,2) = \max\{1,6\} = 6$ (yes)

$F(2,3)\ (w\ge 3) = \max\{1,7\} = 7$ (yes)

$F(3,1) = 1$ (no)

$F(3,2) = 6$ (no)

$F(3,3) = 7$ (no)

$F(3,4) = \max\{7,18\} = 18$ (yes)

$F(3,5) = \max\{1+18, 7\} = 19$ (yes)

$F(3,6) = \max\{7, 6+18\} = 24$ (yes)

$F(3, w\ge 7) = 25$ (yes)

$F(4,8) = \max\{25, 22+7\} = 29$ (yes)

Tracking back, we include items 1,2,4 with total weight 8 and total utility 29.

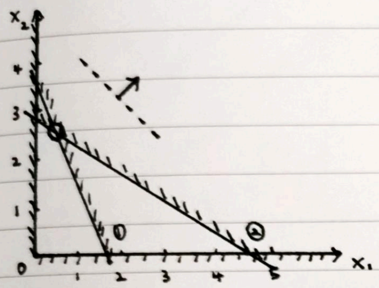
(d)

S	λ		1	2	3	4	$\hat{Q} = \sum_j q_j x_j$	$\hat{W} = \sum_j w_j x_j$	$F^{(s)}(\lambda) = \hat{Q} - \lambda(\hat{W} - W)$	Feas
		$j=$	1	2	3	4				
		$q_j=$	1	6	18	22				
		$w_j=$	1	2	4	5				
		$\lambda_j = q_j/w_j$	1	3	4.5	4.4				
1	$\lambda=4.4$ $4.4\le\lambda\le4.5$ $\lambda=4.5$ $\lambda=4.5$	$x_j=$	0	0	1	1	40	9	35.6 $40-\lambda$ 35.5 35.5	inf
2	$4.4<\lambda\le4.5$	$x_j=$	0	0	1	0	18	4	$18+4\lambda$	feas

(e) So from the table we know: $18 \le$ "optimal value in (C)" ≤ 35.5, the "best" solution is just take item 3, with utility 18, which is lower than the true optimum in (C). can't deduce true optimum.

(2)

(2) (a)

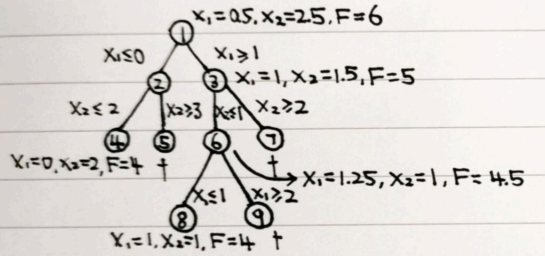


from the diagram, we can find the LP relaxation solution by:

$$\begin{cases} 4x_1+2x_2=7 & ① \\ 3x_1+5x_2=14 & ② \end{cases} \Rightarrow \begin{cases} x_1=0.5 \\ x_2=2.5 \end{cases}$$

in the case, the upper bound of original IP is $F=2\times 0.5+2\times 2.5=6$

(b)



from the B&B tree, we have two optimal integer solutions: ① $x_1=0, x_2=2$; ② $x_1=1, x_2=1$; the optimal integer programming value $F=4$.

(c)

introducing the slack variables: $\begin{cases} 4x_1+2x_2+S_3=7 \\ 3x_1+5x_2+S_4=14 \end{cases}$ $\left(\frac{5}{14}\right)$

solving the equations above, we obtain: $x_1=\frac{1}{2}-\frac{\overset{5}{2}}{14}S_3+\frac{1}{7}S_4$, $x_2=\frac{5}{2}+\frac{3}{14}S_3-\frac{2}{7}S_4$

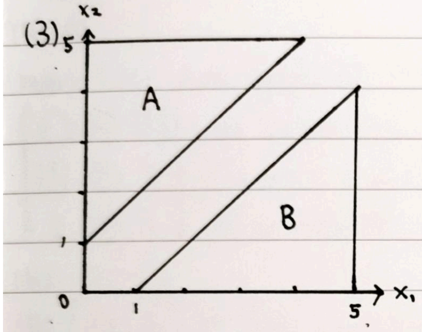
① splitting off +ve fraction from x_1: $x_1-S_4=\frac{1}{2}-\frac{\overset{5}{2}}{14}S_3-\frac{6}{7}S_4\le\frac{1}{2}\Rightarrow x_1-S_4\le 0$

Substituting $S_4=14-3x_1-5x_2$, we have $4x_1+5x_2\le 14$

② splitting off −ve fraction from x_1: $x_1+S_3=\frac{1}{2}+\frac{9}{14}S_3+\frac{1}{7}S_4\ge\frac{1}{2}\Rightarrow x_1+S_3\ge 1$

substituting $S_3=7-4x_1-2x_2$, we have $3x_1+2x_2\le 6$

(3)



(a) let δ be a 0-1 variable such that $\delta=1$, A is used; or when $\delta=0$, then B is used.

$(1-\delta) \le x_1 \le 4+(1-\delta) = 5-\delta$

$\delta \le x_2 \le 4+\delta$

$x_2 - x_1 \ge 1 - 5\cancel{\delta}(1-\delta) = 5\delta - 4$ (M=5)

$x_1 - x_2 \ge 1 - 5\delta$ (M=5)

(b) the objective function can be written as:

min $\delta(4x_1+4x_2)+(1-\delta)(2x_1+3x_2) = 2x_1+3x_2+\delta(2x_1+x_2)$, logically correct but nonlinear

rather we use an additional variable z with logical condition: $z = \begin{cases} 0, & \text{if } \delta=0 \\ 2x_1+x_2, & \text{if } \delta=1 \end{cases}$

the objective function becomes: min $2x_1+3x_2+z$

and the logical conditions on z are:

$0 \le 2x_1+x_2-z \le 14(1-\delta)$ (M=14 by $(x_1, x_2)=(5,4)$)

(4)

(4) (a)

mark best pure response: $\square$: max in columns ; O: min in rows

$$\begin{bmatrix} -3 & \boxed{4} & \boxed{5} \\ \boxed{3} & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

No saddle point.

C2 dominates C3, so C3 can be eliminated.

After that, R2 dominates R3, so R3 can be eliminated.

(b)

$$\begin{array}{cccc} & C1 & C2 & \\ R1 & -3 & 4 & p \\ R2 & 3 & 0 & 1-p \\ & q & 1-q & \end{array}$$

(R's gain (C's loss) given by: (For Mixed, don't care)

$\pi_R(p,q) = -3pq + 4p(1-q) + 3q(1-p) = -10pq + 4p + 3q = -10(p-\frac{3}{10})(q-\frac{2}{5}) + \frac{6}{5}$)

For a single pure strategy, Row player can choose R2 to guarantee 0.

(c)

$\cancel{\pi_C(p,q) = 10(p-\frac{3}{10}}$

$p=\frac{3}{10}$, $q=\frac{2}{5}$. pay off is $\frac{6}{5}$. the NE point is $(\frac{3}{10}, \frac{2}{5})$.

$$R_R(q) = \begin{cases} p=1, & \text{if } q \cancel{<\frac{3}{5}} < \frac{2}{5} \\ p\in[0,1], & \text{if } q=\frac{2}{5} \\ p=0, & \text{if } q>\frac{2}{5} \end{cases} \qquad R_C(p) = \begin{cases} q=1, & \text{if } p<\frac{3}{10} \\ q\in[0,1], & \text{if } p=\frac{3}{10} \\ q=0, & \text{if } p>\frac{3}{10} \end{cases}$$

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(d)

$p=\frac{5}{6+5}=\frac{5}{11}>\frac{3}{10}$, so Column Player should choose C2 ($q=0$)