Time Series Final Exam

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1.
(a) FALSE
(b) FALSE
(c) FALSE
(d) TRUE
(e) TRUE
(f) FALSE
(g) TRUE
(h) FALSE
(i) TRUE
(j) TRUE

2.

(a)(i)

$Y_t = (1+\theta B^2)W_t$, so it is ARIMA(0.0,2)

(ii)

To check whether it is invertible, consider the roots of $(1+\theta B^2)$, which is given by: $\pm\frac{\sqrt{-4\theta}}{2\theta} = \pm\frac{\sqrt{-\theta}}{\theta}$, and the modulus of roots is: $\sqrt{(\frac{\sqrt{-\theta}}{\theta})^2+(\frac{-\sqrt{-\theta}}{\theta})^2} = \sqrt{\frac{2|\theta|}{|\theta|^2}} = \sqrt{\frac{2}{|\theta|}} > 1$, so invertible

Using Wold decomposition, $Y_t = \frac{\theta(B)}{\phi(B)}W_t$, so conversely we could have $W_t = \frac{\phi(B)}{\theta(B)}Y_t = \frac{1}{1+\theta B^2}Y_t = (1-\theta B^2+\theta^2B^4-\theta^3B^6+\cdots)Y_t = \sum_{j=0}^{\infty}(-\theta B^2)^j Y_t = \sum_{j=0}^{\infty}(-\theta)^j Y_{t-2j}$ (so $\pi_j = \begin{cases} 0, & j \text{ is odd} \\ (-\theta)^j, & j \text{ is even} \end{cases}$, and so that $\sum_{j=0}^{\infty}\pi_j Y_{t-j} = W_t$)

(iii)

From (ii) we have $(1-\theta B^2+\theta^2B^4-\cdots)Y_t = W_t$, so it could be written as: $Y_t = (\theta B^2-\theta^2B^4+\cdots)Y_t+W_t$
$= \theta Y_{t-2}-\theta^2Y_{t-4}+\cdots+W_t = \sum_{j=1}^{\infty}(-1)^{j-1}\theta^j Y_{t-2j}+W_t = AR(\infty)$

(iv)

$Y^n_{n+h} = \underset{(=0)}{W_{n+h}}+\theta W_{n+h-2} = \theta W_{n+h-2}$. ~~From (iii) we know $Y^n_{n+h} = \theta Y^n_{n+h-2}$~~

$Y^n_{n+1} = \theta W_{n-1} = \theta Y_{n-1}-\theta^2 Y_{n-3}+\theta^3Y_{n-5}-\cdots$

$Y^n_{n+2} = \theta W_n = \theta Y_n-\theta^2Y_{n-2}+\theta^3Y_{n-4}-\cdots$

$Y^n_{n+h} = 0$ for $h \geqslant 3$ $(W_{n+1}, W_{n+2}, W_{n+3}\cdots = 0)$

(b)

$f(w)=f(-w)$, $f(w)=f(w+k)$, $k\in\mathbb{Z}$, so f just needs definition within $w\in[0,\frac{1}{2}]$.

$g(w) = \sum_{k=-\infty}^{\infty}\rho_k\exp(-2\pi iwk) = 1+\sum_{k=-\infty}^{-1}\rho_k\exp(-2\pi iwk)+\sum_{k=1}^{\infty}\rho_k\exp(-2\pi iwk) = 1+2\sum_{k=1}^{\infty}\rho_k\left[\frac{\exp(-2\pi iwk)+\exp(-2\pi iwk)}{2}\right] = 1+2\sum_{k=1}^{\infty}\rho_k\cos(2\pi wk)$

(c)

3.

(a)(i)

It follows AR(2). It could be rewritten as $\Phi(B)X_t = \theta(B)\varepsilon_t$ where $\Phi(B) = 1-\phi_1 B - \phi_2 B^2$, $\theta(B)=1$; Since it could be regarded as ARMA(2,0), it would be invertible if roots of $\theta(B)$ lying outside unit circle (so no effect from ϕ_1 and ϕ_2 condition). Pure AR without MA term so that would be invertible as long as ϕ_1 and ϕ_2 are not both 0.

(ii)

To be causal and stationary, the roots of $\Phi(B) = 1-\phi_1 B-\phi_2 B^2$ should be outside the unit circle. There are three parameters: ϕ_1, ϕ_2, σ

(b)(i)

In AR(1), $\gamma(k) = cov(X_t, X_{t+k}) = \frac{\sigma^2 \alpha_1^k}{(1-\alpha_1^2)}$ (if $|\alpha_1|<1$), and so that $Var(X_t) = \gamma(0) = \frac{\sigma^2}{1-\alpha_1^2}$ and $\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \alpha_1^{|k|}$. Given $\rho_1 = \frac{2}{3}$, if $\phi_2 = 0$, then the process becomes AR(1) as $X_t = \phi_1 X_{t-1} + \varepsilon_t = \phi_1^2 X_{t-2} + \phi_1 \varepsilon_t + \varepsilon_t$, so that the ρ_2 should be $\rho_1^2 = \frac{4}{9}$. Since $\rho_2 \neq \rho_1^2$, we find $\phi_2 \neq 0$.

(ii)

As AR(2), $\cancel{X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t, (1-\phi_1 B - \phi_2 B^2)X_t = \varepsilon_t, X_t = \frac{1}{1-(\phi_1 B + \phi_2 B^2)} X_t = (1+(\phi_1 B+\phi_2 B^2)+(\phi_1 B + \phi_2 B^2)^2 + \cdots) W_t}$. autocovariance function is $\gamma(k) = cov(X_t, X_{t+k}) = E[X_t X_{t+k}] - E(X_t)E(X_{t+k})$, since $E(X_t)E(X_{t+k}) = 0\cdot 0 = 0$, the $\gamma(k) = E[X_t X_{t+k}]$. $\gamma(0) = \phi_1^2 \gamma(0) + \phi_2^2 \gamma(0) + 2\cdot \phi_1 \phi_2 \gamma(1) + \sigma^2$, $\gamma(k) = \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2)$, $\gamma_1 = \frac{B_1 \gamma(0)}{1-B_2}$, and finally $Var(X_t) = \gamma(0) = \frac{(1-\phi_2)\sigma^2}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}$

(iii)

As AR(2), $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$, $X_t = \frac{1}{(1-m_1 B)} \cdot \frac{1}{(1-m_2 B)}$ where $(1-m_1 B)(1-m_2 B) = 1-\phi_1 B - \phi_2 B^2$. Then, $\frac{1}{(1-m_1 B)} \cdot \frac{1}{(1-m_2 B)} = (1+m_1 B + m_1^2 B_1^2 + \cdots)\cdot(1+m_2 B + m_2^2 B^2 + \cdots) = \sum_{i=0}^{\infty} (\sum_{j=0}^{i} m_1^{i-j} m_2^j) B^i$, so that: $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$ where $\psi_j = \sum_{i=0}^{j} m_1^{i-j} m_2^j$

(c)

For $\tilde{\rho}_1$: given $\tilde{\rho}_t = \phi_1 \tilde{\rho}_{t-1} + \phi_2 \tilde{\rho}_{t-2}$, when $t=1$, and $\tilde{\rho}_0 = 1$ (itself), we obtain $\tilde{\rho}_1 = \phi_1 \tilde{\rho}_0 + \phi_2 \tilde{\rho}_{-1}$ due to stationarity, $\tilde{\rho}_1 = \tilde{\rho}_{-1}$, so $\tilde{\rho}_1 = \phi_1 + \phi_2 \tilde{\rho}_1$, and $\tilde{\rho}_1 = \frac{\phi_1}{1-\phi_2}$

~~For $\tilde{\rho}_2$: $\tilde{\rho}_2 = \phi_1 \tilde{\rho}_1 + \phi_2 \tilde{\rho}_0 = \phi_1 \cdot \frac{\phi_1}{1-\phi_2} + \phi_2$~~

(d)

ARMA(2,2). transform X_t into $Y_t = X_t - \mu$ where $\mu = \frac{c}{1+2-1} = \frac{c}{2}$