Fundamentals of Optimization Homework 4

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(3)
From the dictionary, we can observe that x_2 and x_3 are basic variables, so we have $B = \{2, 3\}$ and $N = \{1, 4, 5\}$.
Since the dictionary is constructed from the original constraints, we set $ROW1^* = r_{11}ROW1 + r_{12}ROW2$ and $ROW2^* = r_{21}ROW1 + r_{22}ROW2$. Considering the coefficients of x_4 and x_5, we have the following relationship:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{12} \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.25 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_{21} \\ r_{22} \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.75 \end{bmatrix}$$

From which we can obtain $r_{11} = 0.5$, $r_{12} = 0.25$, $r_{21} = 0.2$ and $r_{22} = 0.75$. Then we write the $ROW1^* = 0.5ROW1 + 0.25ROW2$ and $ROW2^* = 0.5ROW1 + 0.75ROW2$ as following:

$$(0.5a_{12} + 0.25a_{22})x_2 + (0.5a_{13} + 0.25a_{23})x_3 = (6 + 0.25b_2) + (0.25 - 0.5a_{11})x_1 - 0.5x_4 - 0.25x_5$$

$$(0.5a_{12} + 0.75a_{22})x_2 + (0.5a_{13} + 0.75a_{23})x_3 = (6 + 0.75b_2) + (0.75 - 0.5a_{11})x_1 - 0.5x_4 - 0.75x_5$$

From $0.5a_{12} + 0.25a_{22} = 0$ and $0.5a_{13} + 0.25a_{23} = 1$, we obtain $a_{12} = -1$ and $a_{22} = 2$.
From $0.5a_{12} + 0.75a_{22} = 1$ and $0.5a_{13} + 0.75a_{23} = 0$, we obtain $a_{13} = 3$ and $a_{23} = -2$.
From $6 + 0.25b_2 = \hat{x}_3$ and $6 + 0.75b_2 = 12$, we obtain $b_2 = 8$ and $\hat{x}_3 = 8$.
From $0.25 - 0.5a_{11} = -0.75$ and $0.75 - 0.5a_{11} = -\bar{a}_{21}$, we obtain $a_{11} = 2$ and $\bar{a}_{21} = 0.25$.
So we rewrite the original LP problem with all known coefficients:

$$min \ -2x_1 + c_2x_2 + c_3x_3 - x_4 + c_5x_5$$

$$s.t. \ 2x_1 - x_2 + 3x_3 + x_4 = 12$$

$$-x_1 + 2x_2 - 2x_3 + x_5 = 8$$

And the dictionary is:

$$z = 52 - 4.25x_1 + \bar{c}_4x_4 - 0.75x_5$$

$$x_3 = 8 - 0.75x_1 - 0.5x_4 - 0.25x_5$$

$$x_2 = 12 - 0.25x_1 - 0.5x_4 - 0.75x_5$$

Moreover, from that:

$$A_B = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \ (A_B)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

Looking at the formula for the reduced costs:

$$\bar{c}_j = c_j - c_B^T(A_B)^{-1}A_j, \ j \in N$$

while we have deduced $(A_B)^{-1}$, we also need $c_B = [c_2, c_3]^T$ and so that we have the relationship as:

$$c_B^T\hat{x}_B = \hat{z} \to 12c_2 + 8c_3 = 52$$

as well as:

$$\bar{c}_1 = c_1 - c_B^T(A_B)^{-1}A_1 \to 0.25c_2 + 0.75c_3 = 2.25$$

From the two formulas above, we conclude $c_2 = 3$ and $c_3 = 2$.
After obtaining the value of c_2 and c_3, we can also compute the value of $\bar{c}_4$ and c_5 as:

$$\bar{c}_4 = -1 - \begin{bmatrix} 2.5 & 2.25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -3.5$$

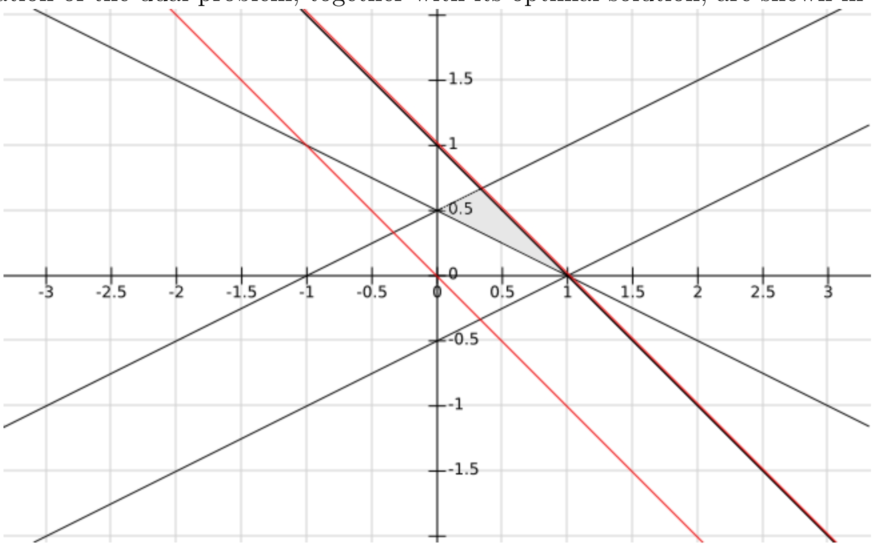
$$c_5 = -0.75 + \begin{bmatrix} 2.5 & 2.25 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2$$

In the end, we find all the unknown values as $c_2 = 3, c_3 = 2, c_5 = 2, a_{11} = 2, a_{12} = -1, a_{13} = 3, a_{22} = 2, a_{23} = -2, b_2 = 8, \bar{c}_4 = -3.5, \hat{x}_3 = 8, \bar{a}_{21} = 0.25$.

(4.1)
We start by formulating the dual problem, which will only have two variables:

$$\begin{aligned} (D)\ max\ & 2y_1 + 2y_2 \\ s.t.\ & y_1 + y_2 \leq 1 \\ & -y_1 + 2y_2 \leq 1 \\ & y_1 - 2y_2 \leq 1 \\ & -y_1 - 2y_2 \leq -1 \\ & y_1, y_2 \in R \end{aligned}$$

The graphical representation of the dual problem, together with its optimal solution, are shown in the figure below:



The optimal solution of (D) is a **line segment** with two endpoints (vertexs) as $y^* = [1, 0]^T$ and $y^{**} = [\frac{1}{3}, \frac{2}{3}]^T$ ($P^* = \{\lambda[1, 0]^T + (1 - \lambda)[\frac{1}{3}, \frac{2}{3}]^T : \lambda \in [0, 1]\}$) with unique finite optimal value $z^* = 2$. (i) $\lambda = 1$:
(Note that three of the four constraints of (D) are active at $y^* = [1, 0]^T$, which is a degenerate basic feasible solution of (D). To obtain an optimal primal solution, we write down and solve the complementary slackness conditions. (P) should satisfy primal feasibility and complementary slackness conditions together with the unique dual optimal solution. In general, two feasible solutions $\bar{x} \in R$ and $\bar{y} \in R$ of the primal and the dual problem, respectively, must satisfy:

$$\begin{aligned} \bar{x}_1(1 - y_1 - y_2) &= 0 \\ \bar{x}_2(1 + y_1 - 2y_2) &= 0 \\ \bar{x}_3(1 - y_1 + 2y_2) &= 0 \\ \bar{x}_4(-1 + y_1 + 2y_2) &= 0 \\ \bar{y}_1(2 - \bar{x}_1 + \bar{x}_2 - \bar{x}_3 + \bar{x}_4) &= 0 \\ \bar{y}_2(2 - \bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3 + 2\bar{x}_4) &= 0 \end{aligned}$$

Substituting the values of the unique dual optimal solution in the complementary slackness conditions, we get any primal optimal solution $\bar{x}$ must satisfy the following conditions:

$$\begin{aligned} \bar{x}_1 \times 0 &= 0 \\ \bar{x}_2 \times 2 &= 0 \\ \bar{x}_3 \times 0 &= 0 \\ \bar{x}_4 \times 0 &= 0 \\ 2 - \bar{x}_1 + \bar{x}_2 - \bar{x}_3 + \bar{x}_4 &= 0 \\ 2 - \bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3 + 2\bar{x}_4 &= 0 \end{aligned}$$

Note that we end up with two equations and three unknowns. To find a solution, we will treat $\bar{x}_4 = \alpha$, where $\alpha \in R$:

$$\bar{x}_1 + \bar{x}_3 = 2 + \alpha$$
$$\bar{x}_1 - 2\bar{x}_3 = 2 + 2\alpha$$

Solving this system simutaneously, we obtain:

$$\bar{x}_1 = 2 + \frac{4}{3}\alpha$$
$$\bar{x}_3 = -\frac{1}{3}\alpha$$

Therefore, we obtain an infinite number of solutions given by:

$$\bar{x} = [\frac{4\alpha + 6}{3}, 0, -\frac{\alpha}{3}, \alpha]^T$$

With the constraint $\bar{x} \geq 0$, we have $\alpha = 0$, indicating:

$$\bar{x} = [2, 0, 0, 0]^T$$

with $z^* = 2$.
(ii) $\lambda = 0$:
In this case, $y^{**} = [\frac{1}{3}, \frac{2}{3}]^T$:

$$\bar{x}_1 \times 0 = 0$$
$$\bar{x}_2 \times 0 = 0$$
$$\bar{x}_3 \times 2 = 0$$
$$\bar{x}_4 \times \frac{2}{3} = 0$$
$$2 - \bar{x}_1 + \bar{x}_2 - \bar{x}_3 + \bar{x}_4 = 0$$
$$2 - \bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3 + 2\bar{x}_4 = 0$$

Now we have $\bar{x}_3 = 0$ and $\bar{x}_4 = 0$, with two equations and two unknowns, and solving as following:

$$\bar{x}_1 - \bar{x}_2 = 2$$
$$\bar{x}_1 + 2\bar{x}_2 = 2$$

Finnaly we obtain:

$$\bar{x} = [2, 0, 0, 0]^T$$

which is the same as the result in (i) above.
(iii) $\lambda \in (0, 1)$:
In this case, $y^{***} = [\frac{2\lambda+1}{3}, \frac{-2\lambda+2}{3}]^T$:

$$\bar{x}_1 \times 0 = 0$$
$$\bar{x}_2 \times (2\lambda) = 0$$
$$\bar{x}_3 \times (2 - 2\lambda) = 0$$
$$\bar{x}_4 \times \frac{-2\lambda + 2}{3} = 0$$
$$2 - \bar{x}_1 + \bar{x}_2 - \bar{x}_3 + \bar{x}_4 = 0$$
$$2 - \bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_3 + 2\bar{x}_4 = 0$$

Since $2\lambda > 0$ and $2 - 2\lambda > 0$ and $\frac{-2\lambda+2}{3} > 0$ for $\frac{-2\lambda+2}{3}$, we obtain $\bar{x}_2 = 0$ and $\bar{x}_3 = 0$ and $\bar{x}_4 = 0$, and by solving the formulas we have:

$$\bar{x} = [2, 0, 0, 0]^T$$

Overall, with $\lambda \in [0, 1]$, we find the optimal solution for (P) as $\bar{x} = [2, 0, 0, 0]^T$.

(4.2)
(i)

(P1)

$$
\begin{aligned}
\min\ & 1a_1 + 1a_2 + \cdots + 1a_m \\
\text{s.t.}\ & A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n + a_1 = b_1 \\
& A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n + a_2 = b_2 \\
& \cdots \\
& A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n + a_m = b_m \\
& x_1, x_2, \cdots, x_n, a_1, a_2, \cdots, a_m \geq 0
\end{aligned}
$$

(D1)

$$
\begin{aligned}
\max\ & b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{s.t.}\ & A_{11}y_1 + A_{21}y_2 + \cdots + A_{m1}y_m \leq 0 \\
& A_{12}y_1 + A_{22}y_2 + \cdots + A_{m2}y_m \leq 0 \\
& \cdots \leq 0 \\
& A_{1n}y_1 + A_{2n}y_2 + \cdots + A_{mn}y_m \leq 0 \\
& y_1 \leq 1 \\
& y_2 \leq 1 \\
& y_3 \leq 1 \\
& \cdots \leq 1 \\
& y_m \leq 1
\end{aligned}
$$

$$(D1)\ max\{b^Ty : A^Ty \leq 0, y \leq 1\}$$

(ii)

$P = \emptyset \rightarrow$there exists a vector $\hat{y} \in R^m$ such that $A^T\hat{y} \leq 0$ and $b^T\hat{y} > 0$

Since $P = \emptyset$, $(P)\ min\{c^Tx : Ax = b, x \geq 0\}$ is empty (infeasible). By this, from **example 24.1**, we can obtain that $(D)\ max\{b^Ty : A^Ty \leq c\}$ is also empty (infeasible).

From **proposition 21.1**, as we already suppose that $b \geq 0$, we can be sure that $(P1)\ min\{e^Ta :\ Ax + a = b, x \geq 0, a \geq 0\}$ is nonempty (feasible), and we have a clear feasible solution of $\hat{x} = 0, \hat{a} = b$. At this feasible solution, we have objective function value $e^Ta = e^Tb \geq 0$ since $e = [1, 1, ..., 1]^T$ and $b \geq 0$. However, if $e^Tb = 0$, then every b_m should be strictly equal to 0, $b = 0$, and this will lead the original (P) to be as $(P)\ min\{c^Tx : Ax = 0, x \geq 0\}$, which is no longer infeasible, with a feasible solution $\hat{x} = 0$. So in order to keep the original (P)'s infeasibility, b cannot be strictly 0 for every item b_m, and so that $e^T\hat{a} = e^Tb > 0$.

Note that by **proposition 24.2** (Strong Duality Theorem), since $A \in R^{m\times n}$ and have nonempty feasible regions, then for $(P1)$ and $(D1)$, $z^*_{D1} = z^*_{P1} > 0$, which indicates that the optimal value of $(D1)$ $(b^T\hat{y})$ is also strictly positive: $b^T\hat{y} > 0$, under the constraint $A^T\hat{y} \leq 0$.

Since there exists such a feasible solution $\hat{y}$ leading objective function $b^T\hat{y} > 0$ under the constraint of $A^T\hat{y} \leq 0$ of $(D1)$ when $P = \emptyset$, the result is proved.

There exists a vector $\hat{y} \in R^m$ such that $A^T\hat{y} \leq 0$ and $b^T\hat{y} > 0 \rightarrow P = \emptyset$

For a contradiction, we assume that P is nonempty. In this case, let $\hat{x} \in P$ be an arbitrary element. Then, $\hat{x} > 0$. Since $-A^T\hat{y} \geq 0$, $\hat{x} \in R^n$, and $-A^T\hat{y} \in R^n$, we obtain:

$$0 \leq \hat{x}^T(-A^T\hat{y}) = -\hat{x}^T A^T \hat{y} = -\hat{y}^T(A\hat{x}) = -\hat{y}^T b = -b^T\hat{y}$$

It implies that $b^T\hat{y} \leq$, contradicting the hypothesis, indicating that there would be no feasible $\hat{x} \in P$ to satisfy. So $P = \emptyset$.