Fundamentals of Optimization Final Exam

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(1)

1.(a)

we have $M_1 = \{1,2,3,4,5\}$, and $I(\hat{x}) = \{1,5\}$.

as $|I(\hat{x})| = 2 < 3$, the set $\{a^i : i \in I(\hat{x})\}$ cannot span $\mathbb{R}^3$.

since all constraints satisfied, $\hat{y}$ is feasible solution but not vertex

(b)

Solving the standard form of LP: (X)

$$\begin{aligned} \min\ & 4y_1^- + 3y_2 + 2y_3^+ - 2y_3^- \\ & -2y_1^- + y_2 - y_3^+ + y_3^- + s_1 = -3 \\ & y_1^- + y_2 - 2y_3^+ + 2y_3^- + s_2 = 2 \\ & -3y_1^- - 2y_2 - y_3^+ + y_3^- + s_3 = -1 \\ & y_1^- \geq 0 \quad y_2 \geq 0 \quad y_3^+, y_3^-, s_1, s_2, s_3 \geq 0 \end{aligned} \qquad (X)$$

~~Or by solving the corresponding dual LP, and I assume $c^T x$'s optimal value is also~~
~~by setting y2~~

Having a feasible solution $(-\frac{4}{7}, 0, -\frac{5}{7})$, the value is $-\frac{6}{7}$, $>$ value -6 by $\hat{y}$, so $\hat{y}$ is not optimal solution.

(2)

2.(a)

Note that x_2 is a nonbasic variable, so the changes in the cost coefficient of x_2 only affect the reduced cost of x_2. So, primal feasibility is maintained, only to check for dual feasibility. $c_2' = c_2 + \delta = -1 + \delta$. to ensure $\delta \geq -\bar{c}_2 = -12$. Thus, we can decrease c_2 by at most 12.

(b)

from (a) we know that if $c_2' = -15 \leq -13$ then we lose dual feasibility, hence we have:

$z = -52 - 2x_2 + 4x_4$ (as x_2 is nonbasic variable, only $\bar{c}_2$ is affected)

$x_1 = 8 - 2x_2 - x_3 - x_4$

$x_5 = 12 - 3x_2 + x_3$

we only have one negative reduced cost, so $j^* = 2$, x_2 will enter the basis; $k^* = 1$ and x_1 will leave

$z = -60 + x_1 + x_3 + 5x_4$

$x_2 = 4 - \frac{1}{2}x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_4$

$x_5 = \cancel{12}\ 0 + \frac{3}{2}x_1 + \frac{5}{2}x_3 + \frac{3}{2}x_4$

We have $B = \{2, 5\}$, $N = \{1, 3, 4\}$, $\hat{x} = [0, 4, 0, 0, 0]^T$, $\bar{c} = [1, 0, 1, 5, 0]^T$, $\hat{z} = -60$

Observe that this solution is now dual feasible, and hence, optimal (primal feasibility (also) retained)

(c)

Note that x_5 is a basic variable, and the changes in the cost coefficient of x_5 affect the reduced costs of all nonbasic variables and current objective function, so, primal feasibility is maintained, only need to check dual feasibility. Define $c_5' = c_5 + \delta$. x_5 is the basic variable in ROW 2, so $l = 2$. To retain dual feasibility we have to ensure for every $j \in N$, that $\delta (e^2)^T (A_B)^{-1} A^j \leq \bar{c}_j$.

$\cancel{\delta}(e^2)^T (A_B)^{-1} = [0, 1]\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = [1, 1]$, we get:

$\delta [1, 1] A^2 = \delta [1, 1]\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3\delta \leq \bar{c}_2 = 12 \Leftrightarrow \delta \leq 4$

$\delta [1, 1] A^3 = \delta [1, 1]\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -\delta \leq \bar{c}_3 = 0 \Leftrightarrow \delta \geq 0$

$\delta [1, 1] A^4 = \delta [1, 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \leq \bar{c}_4 = 4$

Hence, the current dictionary will remain optimal if and only if $0 \leq \delta \leq 4$

(d)

changes in b only affect values of basis variables and current objective function, so dual feasibility is maintained, only to check primal feasibility:

Define $b_1' = b_1 + \delta$ and $b_2' = b_2 - 2\delta$, to retain primal feasibility, to ensure $\delta A_B^{-1} e^1 \geq -x_B^*$ and $-2\delta A_B^{-1} e^2 \geq -x_B^*$

$$\left.\begin{array}{l} \delta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \delta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq \begin{bmatrix} -8 \\ -12 \end{bmatrix} \Leftrightarrow \delta \geq -8,\ \delta \geq -12 \\ -2\delta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -2\delta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \geq \begin{bmatrix} -8 \\ -12 \end{bmatrix} \Leftrightarrow \delta \leq 6 \end{array}\right\} -8 \leq \delta \leq 6$$

Hence, the current dictionary will remain optimal if and only if $-8 \leq \delta \leq 6$

(e)

since in this case, $\delta=14$, not in the range $-8\leq\delta\leq 6$ from (d), the dictionary will lose primal feasibility, so to re-optimize, we have to use the dual simplex method.

Concerning the values of x_B, substituting $\delta=14$, we get: (and then objective function value)

$x_B^*(\delta)=(A_B)^{-1}(b+\delta e^1-2\delta e^2)=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}22\\-24\end{bmatrix}=\begin{bmatrix}22\\-2\end{bmatrix}$

$z(\delta)=c_B^T x_B^*(\delta)=[-2\ -3]\begin{bmatrix}22\\-2\end{bmatrix}=-38$, hence we have:

$z=-38+12x_2+4x_4$

$x_1=22-2x_2-x_3-x_4$

$x_5=-2-3x_2+x_3$

there is only one negative basic variable, so we get $p=5$, and x_5 will leave the basis

to obtain index $q\in N$ of entering variable, the minimum ratio test:

$\min_{j\in N:\bar{a}_{pj}>0}\frac{-\bar{c}_j}{\bar{a}_{pj}}=\min\{\frac{-0}{1}\}=0$, hence, $q=3$, and x_3 will enter the basis:

$z=-38+12x_2+4x_4$

$x_1=20-5x_2-x_3-x_4-x_5$

$x_3=2+3x_2+x_5$

We have $B=\{1,3\}$, $N=\{2,4,5\}$, $\hat{x}=[20,0,2,0,0]^T$, $\bar{c}=[0,12,0,4,0]^T$, $\hat{z}=-38$

We observe that now the solution is primal feasible, and with dual feasibility retained, now the dictionary is optimal.

(f)

We add x_6 to set $N=\{2,3,4,6\}$, as a result, the dictionary will remain feasibility, but may no longer be dual feasibility, so we have to calculate reduced cost $\bar{c}_6$ to check. So, the current dictionary remains optimal if:

$\bar{c}_6=c_6-c_B^T(A_B)^{-1}A^6\geq 0 \Leftrightarrow c_B^T(A_B)^{-1}A^6\leq c_6$, we get:

$[-2\ -3]\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=[-2\ -3]\begin{bmatrix}1\\3\end{bmatrix}=-11\leq c_6$

Hence, the dictionary remains optimal if and only if $c_6\geq -11$

(3)

3.(a)

From $-\infty < z_2 < z_1 < +\infty$, we know that both (P1) and (P2) have finite optimal values.

Definethat, x_1^*, x_2^* are (one of) the optimal solution(s) of (P1) and (P2) respectively, and x_1 and x_2 are arbitrary feasible solutions of (P1) and (P2), respectively, so we have:

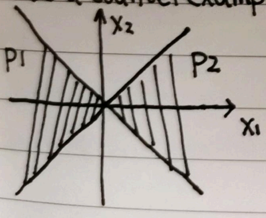
$-\infty < \hat{c}^T x_2^* < \hat{c}^T x_1^* < +\infty$; and in (P1), $\hat{c}^T x_1^* \leq \hat{c}^T x_1$; in (P2), $\hat{c}^T x_2^* \geq \hat{c}^T x_2$, so we have:

$-\infty < \hat{c}^T x_2 \leq \hat{c}^T x_2^* < \hat{c}^T x_1^* \leq \hat{c}^T x_1 < +\infty$ for the whole feasible region of (P1) and (P2), as:

$\hat{c}^T (x_1 - x_2) > 0$, strictly not equal to 0, so $P1 \cap P2 = \emptyset$

(b)

Rise a counterexample: $P1 = \{x_1 \leq -x_2, x_1 \leq x_2\}$; $P2 = \{x_1 \geq x_2; x_1 \geq -x_2\}$; $\hat{c}^T x = 0x_1 + 1x_2$ $(c = [0, 1]^T)$,



and $P1 \cap P2 = \{(0,0)\}$

consider a sequence x_2^k, then with $x_2^1, x_2^2, \cdots$, the (P1) and (P2) are both unbounded, while their intersection set is discrete and finite, (in this example, it's only a point), so $\min\{\hat{c}^T x : x \in P_1 \cap P_2\}$ has a finite optimal value, corresponding to the only one point feasible region, so the proposition is not true.

(4)

4.(a) $A^T y \le c = \emptyset \to A^T y > c$

For a contradiction, suppose there exists a $P(b)$ has a finite optimal value with optimal solution x^*. Then, $A^T y > c \to (A^T y)^T > c^T \to y^T A > c^T$. Since x^* is feasible, indicating $x \geq 0$, so we get: $y^T A x > c^T x$. As $Ax = b$, we get: $y^T b > c^T x$. However, from Proposition 23.1 (Weak Dual Theorem) we have $b^T y \le c^T x$, so there doesn't exist such a $P(b)$, so $B = \emptyset$.

(b)

To prove that B is convex, we need to show that for $b_1, b_2 \in B$, there is $b_1 + \lambda(b_2 - b_1) \in B$ for $\lambda \in [0,1]$. From (a) we know that $A^T y \le c$, so $(A^T y)^T \le c^T$, $y^T A \le c^T$, $y^T A x \le c^T x$, $y^T b \le c^T x$: as b_1, b_2 we assume to have $b_1, b_2 \in B$, we get $y^T b_1 \le c^T x$, $y^T b_2 \le c^T x$. by this we have: $(1-\lambda) y^T b_1 \le (1-\lambda) c^T x$ and $\lambda y^T b_2 \le \lambda c^T x$, since $\lambda \in [0,1]$ and $(1-\lambda) \in [0,1]$, by adding them:

$(1-\lambda) y^T b_1 + \lambda y^T b_2 \le (1-\lambda) c^T x + \lambda c^T x \Leftrightarrow y^T(b_1 + \lambda(b_2 - b_1)) \le c^T x$, so $b_1 + \lambda(b_2 - b_1) \in B$, so B is a convex set.