"Recurrent Problem"

Chapter → 1

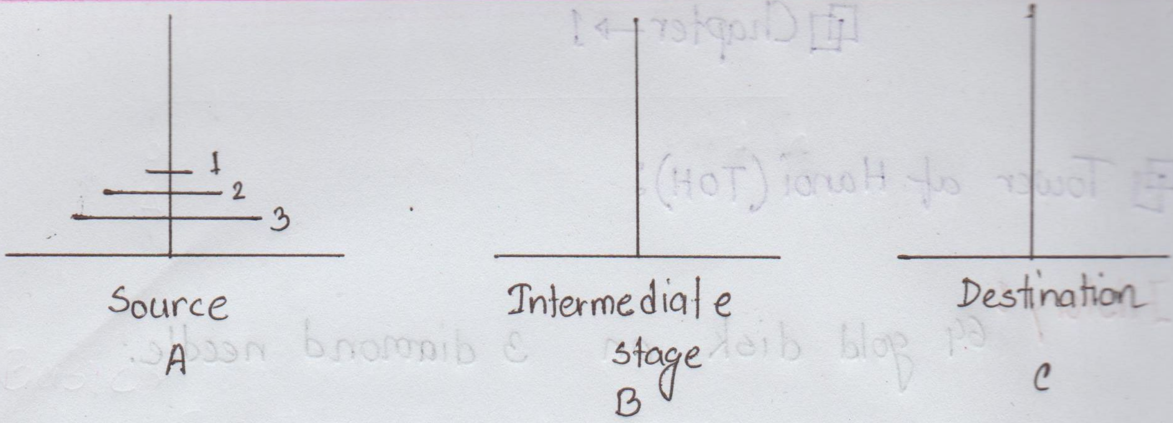
Tower of Hanoi (TOH):

History: 64 gold disk on 3 diamond needle.

Recurrent problem solve করার steps:

1. Assign Appropriate notation.
2. Find Recurrence
3. Find Recurrence Relation / equation
4. Starting from the smaller ones, find / guess sol^n.
5. If needed, break down the recurrence to get sol^n
6. Prove the sol^n to be valid.

P.T.O



1.

T_n = n সংখ্যক disk কে A থেকে C নেয়ার জন্য min^m no. of movements

$1 \rightarrow C$

$2 \rightarrow B$

$1 \rightarrow B$

$3 \rightarrow C$ **Then: 1 >> A**
2 >> C
1 >> C ...... Complete. So, for 3 disks, 2^3 - 1 = 7 Moves required

2. Recurrence বের হয়ে গেল

3. n-1 সংখ্যক disk সরানোর জন্য min^m T_{n-1} moves

n^{th} " " " 1 1 "

$$T_n \geqslant T_{n-1} + 1 + T_{n-1}$$

← recurrence relation

Eq^n এর closed & open form:

$$T_3 = 2T_2 + 1$$

$$= 2(2T_1 + 1) + 1$$

$$= 4T_1 + 2 + 1$$

$$= 4 + 2 + 1 = 7$$

[এমন হলে open form
→ যদি calculation করে করে ভাঙ্গতে ভাঙ্গতে বের করতে হয়]

[if এমন হয় → $T_n = 2n * 6 + 1$ → closed form
→ as n জানলেই direct ans পাওয়া যায়]

4. $T_0 = 0$

$T_1 = 1$

$T_2 = 3$

$T_3 = 7$

$T_4 = 15$

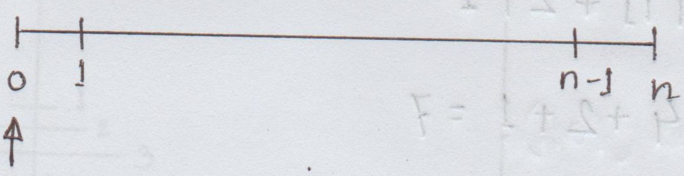
⋮

$T_n = 2^n - 1$

5. Step 5 লাগে না as guess করেই ফেলছি

6. Induction method এ prove. এখানে prove করতে চাই

$T_n = 2^n - 1$



Basic:

For the lowest value of n which is 0, we show that $T_n = 2^n - 1$ holds

$$T_0 = 2^0 - 1 = 1 - 1 = 0$$

[যে eq^n prove করতে চাই তার সবচেয়ে ছোট value এর জন্য দেখব সত্য নাকি → basis]

this is called BASIS

Hypothesis:

For all values of n between 1 to n−1 we assume that $T_n = 2^n - 1$ sol^n holds.

[1 থেকে শুরু করে n−1 পর্যন্ত যত value হতে পারে সবার জন্য eq^n টা সত্য দাবি করে আবার last value এর জন্য সত্য prove করা গেলে 0−n পর্যন্ত range এর সব সত্য। এটাই inductive method। এই দাবিটা hypothesis এ করা হয়]

Induction:

n এর জন্য recurrence relation,

$T_n = 2T_{n-1} + 1$ hypothesis এ T_{n-1} এর value guess সত্য দাবি করে এসেছি

$= 2(2^{n-1} - 1) + 1$

$= 2^n - 2 + 1$

$= 2^n - 1$

$\boxed{\therefore T_n = 2^n - 1}$ [Proved]

/* ⊛ এভাবে recurrence prob solve করা হয়। Worst case এ 5 no step use করা হয়।

⊛ TOH এর জন্য closed eq^n?

⊛ TOH এর sol^n বের করতে বলা হলে পুরোটা লিখতে হবে

⊛ শুধু inductive proof চাওয়া হলে recurrence r/tn ছাড়া direct লিখে proof এ চলে যাবো */

from PDF => MUST See also:

Double Tower of Hanoi (DTOH) and
Triple Tower of Hanoi (TTOH) (with their Inductive Proof Too!)

⊞ Lines on the plane:

n সংখ্যক line দ্বারা max^m area cover. একটা surface এ straight line দিয়ে কয়টা max region.

∴ infinity পর্যন্ত extend করা possible না line গুলো so region এই আটকাতে হবে.

Notation:

L_n = Max^m number of regions obtainable by n ~~n~~number of intersecting line on the planes.

2 lines : 4 region

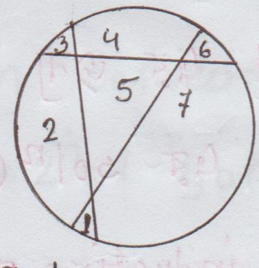
3 lines : 7 " — 3 region increased

4 " : $7 + 4 = 11$ "

5 " : $11 + 5 = 16$ region

$L_5 = L_4 + 5$

$L_6 = L_5 + 6$



3 lines

[n তম line আগের (n-1) line কে কাট করলে আগের সাথে n line add করবে]

$L_n = L_{n-1} + n$ ← open form

$L_0 = 1$

$$\begin{cases} L_{10} = L_9 + 10 \\ \quad = L_8 + 9 + 10 \\ L_0 = 1 \end{cases}$$

$$\left.\begin{aligned} L_0 &= 1 \\ L_1 &= 2 \\ L_2 &= 4 \\ L_3 &= 7 \\ L_4 &= 11 \\ L_5 &= 16 \end{aligned}\right\} \rightarrow$$

কোন pattern পাওয়া যাচ্ছে না তাই reccurence কে ভাঙতে হবে

$L_n = L_{n-1} + n$

$= L_{n-2} + (n-1) + n$

এখানে 1 বেশী

$= L_{n-3} + (n-2) + (n-1) + n$

এখানে যা

$= L_{n-4} + (n-3) + (n-2) + (n-1) + n$

$\vdots$

$$= L_0 + \underbrace{1 + 2 + 3 + \ldots\ldots + (n-3) + (n-2) + \ldots\ldots + n}_{S_n}$$

According Gauss Method,

$$S_n = 1+2+3+ \dots\dots + (n-3)+(n-2)+(n-1)+n$$

$$S_n = n + (n-1) + (n-2) + \dots\dots + 3 + 2 + 1$$

$$(+)\quad 2S_n = (n+1) + (n+1) + \dots\dots + (n+1)$$

$$= n(n+1)$$

$$\therefore S_n = \frac{n(n+1)}{2}$$

$$\boxed{L_n = 1 + \frac{n(n+1)}{2} = \frac{n(n+1)}{2} + 1}$$

closed form created

Indution method prove:

Basis:

For the lowest value of n which is 0, we show that our solution

$$l_n = \frac{n(n+1)}{2} + 1$$

$$l_0 = \frac{0(0+1)}{2} + 1$$

$$= 0 + 1 = 1$$

(*) basic & hypothesis must likhtei হবে।

→ must

Induction:

For all values bet^n 1 to n-1 we assume that our sol^n is true.

[open form থেকে শুরু করতে হবে]

$$L_n = L_{n-1} + n$$

$$= \frac{(n-1)(n-1+1)}{2} + 1 + n \quad [\text{by hypothesis}]$$

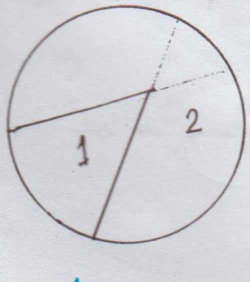
$$= \frac{n^2-n}{2} + 1 + n$$

$$= \frac{n^2-n+2+2n}{2} = \frac{n^2+n+2}{2} = \frac{n(n+1)}{2} + 2/2$$

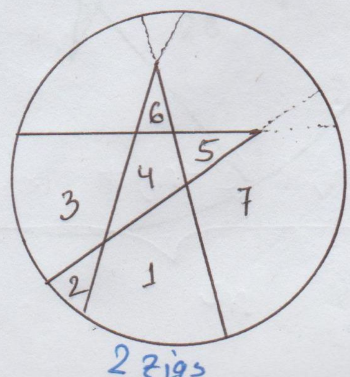
$$= \frac{n(n+1)}{2} + 1$$

[proved]

Zig: কয়টা zig নিজেদের মধ্যে cross এ max region?
(7)



1 zig



2 zigs

$= 1 + n(n+1)/2$

zig	lines	line region	zig region	lost region
1	2	4	2	2
2	4	11	7	4
3	6	22	16	6

[যত no zig তার 2 গুন lost
1 টা zig থাকলে $\rightarrow l_{2n}$]

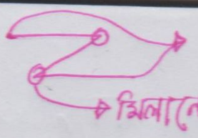
$\therefore$ zig region, $Z_n = l_{2n} - 2n$

$$= \frac{2n(2n+1)}{2} + 1 - 2n$$

$$= 2n^2 + n + 1 - 2n$$

$$\boxed{\therefore Z_n = 2n^2 - n + 1} \rightarrow \text{[Ans]}$$

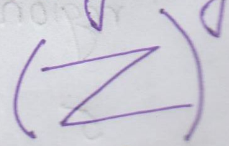
(*) Only end point গুলা circle এ মিলবে



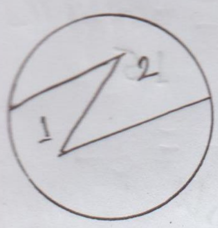
(*) দুটা zigzag এর intrsctn point দ্বারা অন্য zigzag cant pass

max^m region crt হবে

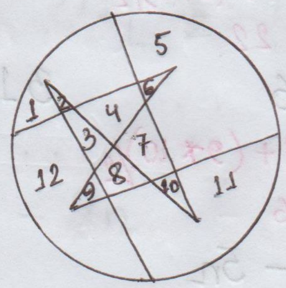
⊞ Zig-zag: → closed eq^n বের করতে হবে.



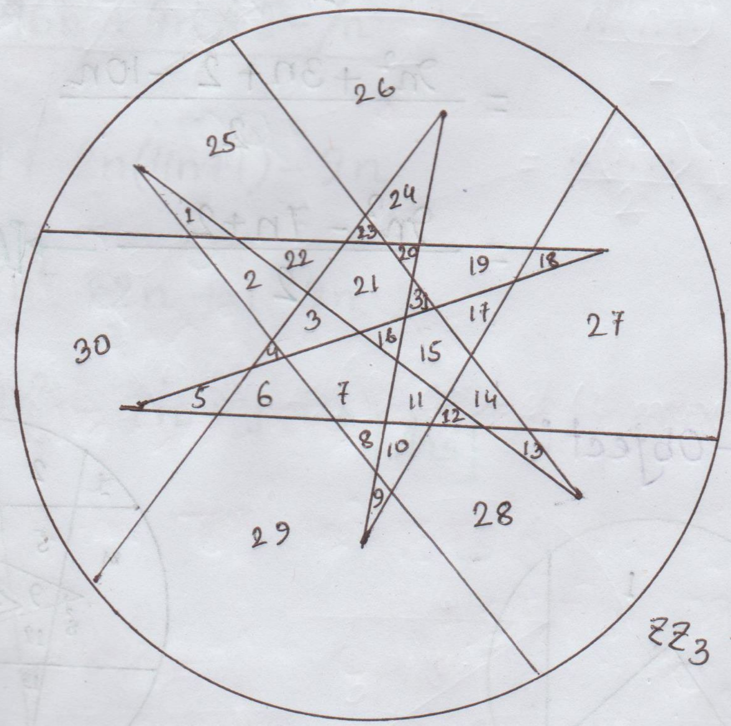
3 closed lines



$ZZ_1 = 2$
↓
3 lines



$ZZ_2 = 12$



$ZZ_3 = 31$

zigzag	lines	line region	zigzag region	lost region
1	3	$7 = 1 + \frac{n*(n+1)}{2} = 1+(3*4)/2 = 7$	2	5
2	6	$22 = 1+(6*7)/2 = 22$	12	10
3	9	$46 = 1+(9*10)/2 = 46$	31	15

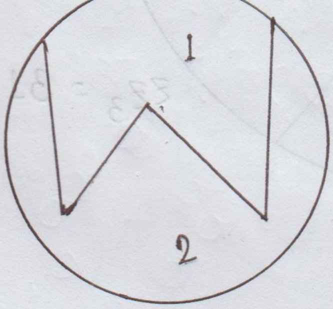
$$\therefore ZZ_n = l_{3n} - 5n$$

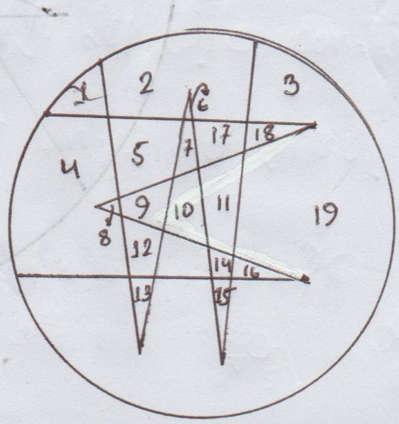
$$= \frac{3n(3n+1)}{2} + 1 - 5n$$

$$= \frac{9n^2+3n+2-10n}{2}$$

$$= \frac{9n^2-7n+2}{2} \longrightarrow [Ans]$$

⊞ W-Object:





W	lines	line region	W region	lost region
1	4	$11 = \left[\frac{4(4+1)}{2} + 1\right]$	2	9
2	8	$37 = \left[\frac{8(8+1)}{2} + 1\right]$	19	18

$$\therefore W_n = l_{4n} - 9n$$

$$= 1 + \frac{4n(4n+1)}{2} - 9n$$

$$= \cancel{\frac{16n^2 + 4n + 2 - 9n}{}}$$

$$= 1 + 2n(4n+1) - 9n$$

$$= 8n^2 + 2n + 1 - 9n$$

$$= 8n^2 - 7n + 1 \longrightarrow [\text{Ans}]$$

⊞ JOSEPHUS PROBLEM:

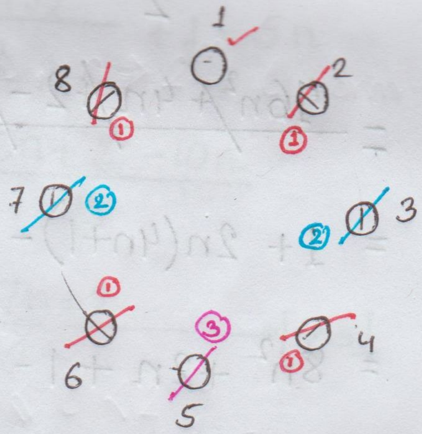
Flavious Josephus: 41 Jewis

[Prob:
Circle এ কতজন লোক ই দাঁড়াবে। প্রত্যেকে তার পরের জনকে Kill করবে। thn বের করব কোন no. মানুষ survive করবে at last]

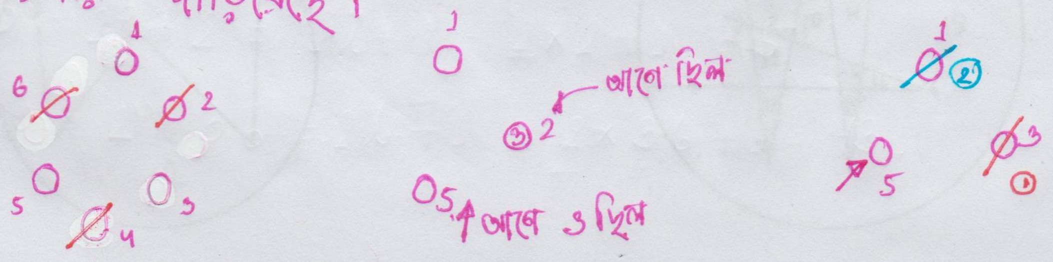
$J(1) = 1$ → কোন no. position টা safe
→ survivour's number

↑ number of people in circle.

$\therefore J(8) = 1$ →



[জোড়/বেজোড় সংখ্যক মানুষের ক্ষেত্রে ২টা case consider
যার দাঁড়ানোর (n) কথা ছিল তার 2n-1 no লোকটা সেই জায়গায় দাঁড়িয়েছে।



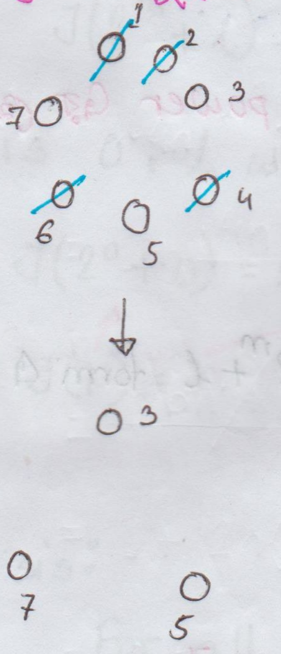
→ next step এ জোড় সংখ্যক মানুষের ক্ষেত্রে:

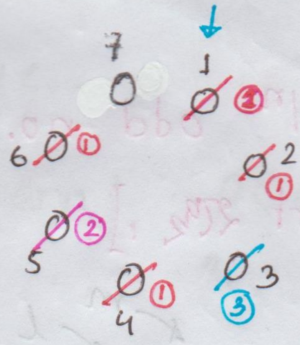
(*) $J(2n) = 2J(n) - 1$

$J(6) = J(2 \times 3)$

$= 2J(3) - 1$

Odd সংখ্যক মানুষের ক্ষেত্রে:





(*) $J(2n+1) = 2J(n) + 1$

So,

$J(1) = 1$

even → $J(2n) = 2J(n) - 1$

odd → $J(2n+1) = 2J(n) + 1$

Recurrence equation

[n কে 2^m+l ধরে নিয়ে] $J(2^m+l) = 2l+1 \longrightarrow$ closed form

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

[Solⁿ odd no. হচ্ছে, 2 এর power এর থেকে chain start হচ্ছে।]

$\underline{\underline{n}}$

$14 = 2^3 + 6$ ($m = 3$, $l = 6$)   [অর্থাৎ, n কে $2^m + l$ form এ লেখা হচ্ছে]

$15 = 2^3 + 7$

$1 = 2^0 + 0$

[*** বলা থাকবে, $m \geq 0$, $0 \leq l < 2^m$]

[মানে $100 =$ কে $2^5 + 68$ লেখা যাবে না]

$14 = 2^3 + 6 = 6 \times 2 + 1 = 13$

$10 = 2^3 + 2 = 2l + 1 = 2 \times 2 + 1 = 5$

$100 = 2^6 + 36 = 2l + 1 = 2 \times 36 + 1 = 73.$

Inductive Proof: $J(2^m + l) = 2l + 1$

Basis:

We prove the closed form on m.

for the lowest value of m which is zero, we show that

$$J(2^m + l) = 2l + 1 \text{ is true}$$

if m is 0, l will also be zero [$\because 0 \leq l < 2^m$]

$$\therefore J(2^0 + 0) = 2 \times 0 + 1 = 1$$

$$\Rightarrow J(1) = 1$$

So, $J(2^m + l) = 2l + 1$ holds for $m = 0$

Hypothesis:

For all values of m between 1 to m - 1

$J(2^m + l) = 2l + 1$ is true.

that is, assume $J(2^k + x) = 2x+1$ is true for $k = 0, 1, \ldots,$ (upto) $m-1$

So, $J(2^{m-1} + x) = 2x+1$

$J(2^{m-1} + l/2) = 2(l/2) + 1 = l + 1$

Inductive proof:

[$J(2n) = 2l + 1$ proof করতে চাই for even $(2n)$]

Even Case:

$$J(2n) = 2J(n) - 1$$

here, $2n = 2^m + l$

$\therefore n = 2^{m-1} + l/2$

ঠিক আগের Josephus prob এর

$J(2n) = 2J(2^{m-1} + l/2) - 1$ [n এর value বসিয়ে]

$= 2[2 \times l/2 + 1] - 1$ [hypothesis's অনুযায়ী]

$= 2l + 2 - 1$

$= 2l + 1$

$J(2n) = 2l + 1$

[odd case (2n+1) এর জন্য o basic, hypothesis same only proof diff]

ODD Case:

$J(2n+1) = 2J(n) + 1$

here, $2n+1 = 2^m + l$

$\Rightarrow 2n = 2^m + l - 1$

$\Rightarrow n = 2^{m-1} + \frac{l-1}{2}$

$\therefore J(2n+1) = 2J\left(2^{m-1} + \frac{l-1}{2}\right) + 1$

$$= 2\left[2 \times \frac{l-1}{2} + 1\right] + 1$$

$$= 2\left[l - \cancel{1} + \cancel{1}\right] + 1$$

$$= 2l + 1$$

$$\therefore J(2n+1) = 2l + 1$$

[So odd even দুটোর ক্ষেত্রেই proof করা গেল

Give an inductive proof বললে recurrence eq^n গুলি just লিখে proof এ চলে যেতে হবে।

problem proof বললে পুরোটা আগে eq^n কিভাবে এসেছে তাও দেখাতে হবে → odd এর জন্য ২টো dia
even " [" "] " থেকে thn recurrence eq^n

Josephus prob শুধু even/odd or even + odd এর জন্য আসতে পারে]

$$J(n) = J(2^m + l) = 2l+1$$

Binary solution of Josephus problem:

Question: Derive the Binary Property of the Josephus Problem.

$$n = (b_m b_{m-1} b_{m-2} \ldots \ldots b_2 b_1 b_0)_2$$

b_m is always 1

Since n = 2^m + l; So: l = n - 2^m => The MSB of the binary number (n-2^m) is 0, the rest of the bits are same as the bits of n. See example=>

$8 = 8 + 0 = 1000$
$ 0000$

$\vdots$

$(2^3 + l)$

$11 = 8 + 3 = 1000$
$ 0011$

$\vdots$

$(2^3 + l)$

$15 = 8 + 7 = 1000$
$ 0111$

Note: the last m bits of n is same as the bits of l. Only the MSB different. example:

if n=15(1111) then l=7 (0111)

if n=11(1011) then l=3 (0011)

$$l = (0\ b_{m-1} b_{m-2} \ldots \ldots b_2 b_1 b_0)_2$$

$$\Rightarrow 2l = (b_{m-1} b_{m-2} \ldots \ldots b_1 b_0 0)_2$$

→ left shift করা হল

[no. of people কে binary তে নিব যেন leading '0' না থাকে

$n = 2^m + l,\ l << n$

$$2l+1 = (b_{m-1} b_{m-2} \ldots \ldots b_1 b_0 1)_2$$

$$= (b_{m-1} b_{m-2} \ldots \ldots b_1 b_0 b_m)_2$$

[as b_m always 1]

n কে যত bit binary তে লিখব l কেও তত bit binary তে লিখলে l এর MSB '0' আসবে (always)

Ex →

$n = 24 = \rightarrow 11000$

$= 2^m + l$

$= 2^4 + 8$

01000

Ex:

$100 = 1100100 \qquad 2^m + l$

$2^6 + \underbrace{36}_{l}$

$1001001 = 73$

$= 36 \times 2 + 1$

$= 73$

[Josephus prob এ n কে bin এ convert করে MSB কে ঘুরিয়ে শেষে আনলে thn তার decimal বের করলেই solution বের হয়ে যাবে]

(*) যেকোন binary no কে 1 ঘর left shift করলে no. টা double হয়ে যায় *

* Ex →

$5 = 0101$) double
$10 = 1010$

*** So, n ke just 1-bit left rotate korlei solution J(n) paowa jabe see examples

Generalized version of Josephus problem:

$$f(1) = \alpha$$
$$f(2n) = 2f(n) + \beta$$
$$f(2n+1) = 2f(n) + \gamma$$

অনেকটা Josephus problem এর মত এমন recurrence দেয়া আছে, closed form বের করতে হবে
ques এ α, β, γ দেয়া থাকবে

Josephus Problem Recurrence:
J(1)=1
J(2n) = 2J(n) - 1
J(2n+1) = 2J(n) +1
So: alpha=1, *beta*= -1, *gamma*=1

$$f(1) = \alpha$$
$$f(2) = f(2\times 1) = 2f(1) + \beta = 2\alpha + \beta$$
$$f(3) = f(2\times 1 + 1) = 2f(1) + \text{gamma} = 2\alpha + \cancel{\beta}\ \text{gamma}$$

So, f(n) যাই হোক এমন solution আসবে

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \quad \ldots\ (1)$$

ques এ দেয়া থাকবে

***This whole lengthy method is called**
***** REPERTOIRE method*****

Let, $\alpha = 1, \beta = \gamma = 0,$

from eq^n (1)

$$f(n) = A(n)$$
$$A(1) = 1 \quad [\because f(n) = A(n),\ f(1) = A(1) = \alpha = 1]$$
$$A(2n) = 2A(n) + \beta = 2A(n) + 0 = 2A(n)$$

*This whole lengthy method is called
*** REPERTOIRE method***

$A(2n+1) = 2A(n)$

$A(8) = A(2\times4) = 2A(4)$

$= 2[2A(2)]$

$= 2[2[2A(1)]]$

$= 2^3$

$A(9) = A(2\times4+1) = 2A(4) = 2^3$

*A(11) = A(2*5+1)*

*= 2*A(5) = 2 [A(2*2+1)]*

= 2 [2 [A(2)]] = 2 [2 [2 [A(1)]]] = *2^3 as before*

So, n যাই হোক না কেন,

$\boxed{A(n) = 2^m}$ $n = 2^m + l$

Let, $f(n) = 1$

$f(1) = 1 = \alpha$ $[\because f(1) = \alpha]$

$f(2n) = 2f(n) + \beta$

$1 = 2 + \beta$

$\therefore \boxed{\beta = -1}$

$f(2n+1) = 2f(n) + \gamma$

$1 = 2 + \gamma$

$\boxed{\therefore \gamma = -1}$

***This whole lengthy method is called**
***** REPERTOIRE method*****

Put (α, β, γ) values in eq^n (1),

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$$\boxed{1 = A(n) - B(n) - C(n) \quad \ldots\ldots (11)}$$

We already know A(n) = 2^m

Let,

$$f(n) = n$$

$$f(1) = 1 = \alpha$$

$$f(2n) = 2f(n) + \beta$$

$$\therefore 2n = 2n + \beta$$

$$\therefore \beta = 0$$

$$f(2n+1) = 2f(n) + \gamma$$

$$2n+1 = 2n + \gamma$$

$$\therefore \gamma = 1$$

Put (α, β, γ) values in eq^n (1)

$$n = A(n) + C(n) \quad \ldots\ldots (111)$$

But, we know, A(n) = 2^m

$$2^m + l = A(n) + C(n) \quad [\because n = 2^m + l]$$

$$2^m + l = 2^m + C(n)$$

$$\therefore \boxed{C(n) = l}$$

***This whole lengthy method is called**
***** REPERTOIRE method*****

$\therefore$ from eq^n (11)

$1 = 2^m - B(n) - l$

$\therefore \boxed{B(n) = 2^m - l - 1}$

General solution:

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

where $n = 2^m + l$

$$\therefore \boxed{f(n) = 2^m\alpha + (2^m - l - 1)\beta + l\gamma}$$

$\therefore$ This is the solution of the above recurrence

Verify josephus prob accordind to

See the Josephus Problem Recurrence:
J(1)=1
J(2n) = 2J(n) - 1
J(2n) = 2J(n) +1
So: Find that in Josephus problem,
alpha=1,
beta= -1, gamma=1

$$f(n) = 2^m\alpha + (2^m - l - 1)\beta + l\gamma$$

$$= 2^m - 2^m + l + 1 + l$$

$$\boxed{f(n) = 2l + 1}$$ ← verify করা গেল

***Q: Find a closed form expression for f(n) from the following recurrence relations by using Repertoire method.**
f(1) = 2
f(2n) = 2f(n) + 2
f(2n+1) = 2f(n) + 3

~~Next~~ if $\alpha = 2$, $\beta = 2$, $\gamma = 3$ দিলে

$$f(6) = \underbrace{2^2}_{2^m} . 2 + (2^2 - 2 - 1) . 2 + 2 . \underbrace{3}_{l}$$

$2^2 + 2$ → 2^m, l

Question: Similar Recurrence deya thakte pare in exam/quiz. Repertoire method diye sheta solve korte paro, But tate error hoite pare (jehetu long procedure) ... ... EASIER hoilo general ta solve kora, then just general tar solution e alpha, beta, gamma er value boshiye deya ... as shown above ... tahole derivation korte giye error howar chance tahole thaklo na ...

$$= 8 + 3 + 6 = 16$$

Radix Based solution of Recurrence:

Given a recurrence in the following form –

$$f(j) = \alpha_j \quad ; \quad 1 \leqslant j < d$$

2 বাদেও any no. হতে পারে

$$* f(dn + j) = c\, f(n) + \beta_j \quad ; \quad 0 \leqslant j < d$$

Better to Understand from Examples below ...

$f(1) = \alpha_1$ if $d = 3$

$f(2) = \alpha_2$ $c = 10$

$j = 1, 2$

$f(3n + 0) = 10 f(n) + \beta_0$ j 0, 1, 2

$f(3n + 1) = 10 f(n) + \beta_1$

$f(3n + 2) = 10 f(n) + \beta_2$

if $d = 2$, $c = 2$,

$f(1) = \alpha_1$

$f(2n + 0) = 2 f(n) + \beta_0$

$f(2n + 1) = 2 f(n) + \beta_1$

এই recurrence টা আগের recurrence টার মত

So, এই general form theke যেকোন soln বের করা যাবে।

* এই recurrence এর direct solution:

This is the Radix based solution of the general recurrence ...

not decimal

here, base = d

$$f\left((b_m\ b_{m-1} \ldots \ldots b_1\ b_0)_d\right) = \left(\alpha_{b_m}\ \beta_{b_{m-1}}\ \beta_{b_{m-2}} \ldots \beta_{b_1}\ \beta_{b_0}\right)_c$$

Example:

Given,

$f(1) = 34 \leftarrow \alpha_1$

$f(2) = 5 \leftarrow \alpha_2$

$f(3n+0) = 10f(n) + 76 \leftarrow \beta_0$

$f(3n+1) = 10f(n) - 2 \leftarrow \beta_1$

$f(3n+2) = 10f(n) + 8 \leftarrow \beta_2$

$c = 10$

$d = 3$

$\therefore f(19) = ?$

Question:
Find f(19) by using the Radix based property of the above recurrence.

Sol^n:

base 3 তে নিলাম

$$\begin{array}{r|l} 3 & 19 \\ \hline 3 & 6-1 \\ \hline & 2-0 \end{array}$$

$\therefore (19)_3 = 201$

$$f((b_m b_{m-1} \ldots \ldots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \ldots \beta_{b_1} \beta_{b_0})_c$$

$$f(19) = f(\underset{b_m\ b_{m-1}\ b_{m-2}}{201})_3 = (\alpha_2 \beta_0 \beta_1)_{10} \quad [\because c = 10]$$

$$= (5 \quad 76 \quad -2)_{10}$$

$$= 5 \times 10^2 + 76 \times 10^1 + (-2) \times 10^0$$

[10 base এ আনা হল]

$$= 500 + 760 - 2$$

$$= 1258$$

//যেকোন no. কে তার base এর power দিয়ে গুণ করলে decimal এ convert হয়ে যায়

(*) $f(25) = ?$

$$(25)_3 = 221$$

$$\begin{array}{r|l} 3 & 25 \\ 3 & 8 - 1 \\ 3 & 2 - 2 \\ & 0 - 2 \end{array}$$

$$f(25) = f(221)_3 = (\alpha_2 \beta_2 \beta_1)_{10}$$

$b_m \quad b_{m-1} \quad b_{m-2}$ $= (5 \quad 8 \quad -2)_{10}$

$$= 578$$

in case of josephus problem:

Q: Find J(20) by Using the Radix Based Property of the Generalized Josephus Recurrence.

$$j(1) = 1,$$

$$j(2n+1) = 2j(n) + 1$$

$$j(2n) = 2j(n) - 1,$$

here, $d = 2$; $c = 2, \alpha_1 = 1$, $\beta_0 = -1$

$$\beta_1 = 1$$

$$j(20) = j(10100)_2 = (\alpha_1 \beta_0 \beta_1 \beta_0 \beta_0)_2$$

$$= (1 \ -1 \ 1 \ -1 \ -1)_2$$

like before
$2^m + 1$
$2^4 + 4$
$2^{\cancel{t}} + 1$
$= 9$

$= (1 \times 2^4 + (-1) \times 2^3 + 1 \times 2^2 + (-1) \times 2^1 + (-1) \times 2^0)$

$= 9$ [decimal এ আনা হল]

$\boxed{\boxplus}$ CHAPTER $\rightarrow$ 2

SUMS

$a_0 + a_1 + a_2 + \ldots\ldots + a_n$ // n+1 সংখ্যক term

term

$1 + 2 + 3 + \ldots\ldots + n$ // n

$1 + 2 + 4 + 8 + \ldots\ldots + 2^{n-1}$ // n

$$\sum_{k=0}^{n} a_k \equiv \sum_{0 \le k \le n} a_k$$

0 – 100

$\sum ODD^2$

$$1^2 + 3^2 + 5^2 + \ldots\ldots + 99^2 = \sum_{k=0}^{49} (2k+1)^2$$

or, $\sum_{\substack{0 \leq k \leq n \\ (k \text{ is odd})}} k^2$ $\qquad \sum_{\substack{0 \leq k \leq n \\ (k \text{ is prime})}} k^2$

* We can use all those notations interchanging

এই chap এ যেই series দেয়া থাকবে তার closed form বের করতে হবে।

This is OPEN form $\longrightarrow S_n = \sum_{k=0}^{n} (k^2 + k)$ $\quad S_{10} = ? \quad S_{100} = ?$

CLOSED form is: n(n+1)(2n+1)/6 + n(n+1)/2; so find

So closed form বের করতে হবে।

Sum এর closed form বের করার জন্য sum কে recurrence এর সাথে মিলাতে হবে then recurrence এর technique use ...

How to Convert a SUM into a RECURRENCE?

$S_n = 0 + 1 + 2 + 3 + \ldots + n - 1 + n$

$S_n = \underbrace{0 + 1 + 2 + 3 + \ldots + n-1}_{S_{n-1}} + n \Rightarrow a_0 + a_1 + \ldots + a_{n-1} + a_n$

$+ n$

$$\boxed{\begin{aligned} S_0 &= 0 \\ S_n &= S_{n-1} + n \end{aligned}}$$

here $\beta = 0, \gamma = 1$

// so recurrence এর সাথে মিলে যাবে

In Previous Example:
alpha=0
beta=0
gamma=1

general form

$$\begin{cases} R_0 = \alpha \\ R_n = R_{n-1} + \beta + \gamma n \end{cases}$$

// α, β, γ will be given in the eqn

$$S_n = \sum_{k=0}^{n} a_k$$

$$S_0 = a_0$$

$$S_n = S_{n-1} + a_n$$

Repertoire Method:

Place This Box Here

$$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma \dots (i)$$

// A, B, C coefficient
$A(n), B(n), C(n)$ বের করতে হবে।

Case 1:

Let, $\boxed{R_n = 1}$

$R_0 = 1 = \alpha$

$R_0 = \alpha$

$R_1 = R_0 + \beta + \gamma$
$= \alpha + \beta + \gamma$

Rn=1
so, R1=1

$R_1 = \alpha + \beta + \gamma$

$\Rightarrow 1 = 1 + \beta + \gamma$

$\therefore \beta + \gamma = 0 \dots\dots (ii)$

$R_2 = R_1 + \beta + 2\gamma$
$= \alpha + \beta + \gamma + \beta + 2\gamma$
$= \alpha + 2\beta + 3\gamma$

Rn=1
so, R2=1

$R_2 = R_1 + \beta + 2\gamma$

$\Rightarrow 1 = 1 + \beta + 2\gamma$

$\therefore \beta + 2\gamma = 0 \dots\dots (iii)$

(iii) − (ii) ⇒

$$\begin{array}{r} \beta + 2\gamma = 0 \\ \beta + \gamma = 0 \\ \hline \therefore \gamma = 0 \end{array}$$

$\therefore$ Put γ in eqn (iii) ⇒
$\beta = 0$

Put (α, β, γ) values in eq^n (1)

$$R_n = A(n)\alpha + \beta \cdot B(n) + C(n)\gamma$$

$$\Rightarrow 1 = A(n)$$

$$\boxed{\therefore A(n) = 1}$$

Case 2:

$$\boxed{\text{Let, } R_n = n}$$

Rn=n so, R0=0

$$R_0 = 0 = \alpha$$

$$R_1 = \alpha + \beta + \gamma$$

Rn=n so, R1=1

$$\Rightarrow 1 = \beta + \gamma \; \ldots \ldots (IV)$$

Rn=n so, R2=2

$$R_2 = R_1 + \beta + 2\gamma$$

$$\Rightarrow 2 = 1 + \beta + 2\gamma$$

$$\therefore \beta + 2\gamma = 1 \; \ldots \ldots (V)$$

(V) − (iv) ⇒

$$\beta + 2\gamma = 1$$

$$\beta + \gamma = 1$$

$$\gamma = 0$$

Put γ in eq^n (iv)

$$\beta = 1$$

Put (α, β, γ) values in eq^n (i)

$\cancel{n}\, n = \beta(n)$

$\boxed{\therefore \beta(n) = n}$

Case 3:

$\boxed{\text{Let } R_n = n^2}$

So, R0 = 0, R1 = 1, R2 = 4

$R_0 = 0 = \alpha$

$R_1 = \alpha + \beta + \gamma$

$\Rightarrow 1 = \beta + \gamma \ldots\ldots \text{(vi)}$

$R_2 = R_1 + \beta + 2\gamma$

$\Rightarrow 4 = 1 + \beta + 2\gamma$

$\therefore \beta + 2\gamma = 3 \ldots\ldots \text{(vii)}$

(vii) − (vi) $\Rightarrow$

$$\begin{array}{r} \beta + 2\gamma = 3 \\ -\beta + \gamma = 1 \\ \hline \gamma = 2 \end{array}$$

Put γ in eq^n (vii)

$\Rightarrow \beta = -1$

Put (α, β, γ) in eq^n (i)

$\Rightarrow n^2 = -\beta(n) + 2C(n)$

$$\boxed{\therefore c(n) = \frac{n^2+n}{2}}$$ [Since $B(n) = n$ shown previously]

$\therefore R_0 = \alpha$

$R_n = R_{n-1} + \beta + \gamma n$

$R_n = A(n)\alpha + B(n)\beta + c(n)\gamma$

$$\boxed{R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma}$$

$S_n = \sum_{K=0}^{n} (a+bK)$

Find the closed form for the above ~~equation~~ solution,

$$S_n = \underbrace{(a) + (a+b) + (a+2b) + \ldots \ldots \ldots + (a+(n-1)b)}_{S_{n-1}} + (a+nb)$$

put $K=0 \Rightarrow$

term(0)

$S_0 = a + b \times 0 = a$

put $k=n \Rightarrow$

term(n)

$S_n = a + bn$

$S_n = S_{n-1} + (a+bn)$

$S_0 = a$

$S_n = S_{n-1} + a + bn$

এইজন্যে general solution এ xtra β নিয়েছিলাম

Thus:

$S_0 = a$

$S_n = S_{n-1} + a + bn$

so: alpha = a

beta = a

gamma = b

here, $\alpha = a$, $\beta = a$, $\gamma = b$

$\because R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma$

$\therefore S_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma$

$= a + an + \frac{n^2+n}{2} b$

$= a(n+1) + \frac{n(n+1)}{2} b$

$$\boxed{\therefore S_n = a(n+1) + \frac{n(n+1)}{2} b}$$

$R_0 = \alpha$ recurrence এর সাথে মিলছে তাই এমন solution // এটা must বলতে হবে খাতায়

$S_n = \sum_{k=0}^{n} (3+2k)$ find $S_{80} = ?$

$\because S_n = \sum_{k=0}^{n} (a+bk)$ এর সাথে মিলেছে

Here, $a = 3$
$b = 2$

$S_{80} = a(n+1) + \frac{n(n+1)}{2} b$

$= 3(80+1) + \frac{80(80+1)}{2} \times 2$

$= 6723$

* Also, see Question 7 (Page 4) of "Practise_Questions_02.docx"

[// $S_n = \sum_{k=0}^{n} (a+bk)$ এর সাথে বা $R_n = \ldots$

এর যেকোন টার সাথে মিলিয়ে solution বের করা যেতে পারে, but যেটা বলে নিতে হবে must]

$R_0 = \alpha$

$R_n = R_{n-1} + \beta + \gamma n$

$R_n = \alpha + n\beta + \frac{(n^2+n)}{2}\gamma$

\# $0 + 1 + 2 + 3 + \ldots \ldots + n$. Find closed form for

$S_n = \sum_{k=0}^{n} k$

$S_n = 0 + \underbrace{1 + 2 + \ldots \ldots + (n-1)}_{S_{n-1}} + n$

$S_0 = 0 = \alpha$, $\boxed{\begin{array}{l} S_0 = 0 \\ S_n = S_{n-1} + n \end{array}}$

here, $\alpha = 0$

$\beta = 0$

$\gamma = 1$

$\therefore S_n = 0 + 0 + \frac{n(n+1)}{2} \cdot 1$

$S_n = \frac{n(n+1)}{2}$ // closed form

$S_n = 1 + 3 + 5 + 7 + \dots + (2n+1)$

$S_n = \sum_{k=0}^{n} (2k+1)$, $\sum a + bk$ (এভাবেও করা যাবে)

a = 1
b = 2

OR,

$R_0 = \alpha$

$R_n = R_{n-1} + \beta + \gamma n$

$R_n = \alpha + n\beta + \frac{n^2+n}{2}\gamma$ recurrence use করে

$$S_n = \underbrace{1 + 3 + 5 + \dots \dots + (2n-1)}_{S_{n-1}} + (2n+1)$$

$S_0 = 1$, $S_n = S_{n-1} + 2n + 1$

So: $S_0 = 1$
$S_n = S_{n-1} + 1 + 2n$

here, $\alpha = 1$, $\beta = 1$, $\gamma = 2$

$\therefore S_n = (n+1)^2$ [Ans]

(*) $0 + 2 + 4 + \dots$?

$S_n = 1 + n*1 + n(n+1)/2 * 2$

$= 1 + n + n(n+1)$

$= (1+n) * (1+n)$

$= (1+n)^2$

Converting a Recurrence to a sum:

//যেকোন Recurrence কে sum এ convert করলে complexity কমবে

Example:

Tower of Hanoi:

$$T_0 = 0$$

$$T_n = 2T_{n-1} + 1$$

Multiply both sides by 2^{-n} or $\frac{1}{2^n}$

↳ যেকোন recurrence কে sum এ নিতে গুন করতে হয় যা দিয়ে তাকে বলে Summation factor

$$\frac{T_0}{2^0} = \frac{0}{2^0} = 0$$

$$\frac{T_n}{2^n} = \frac{2T_{n-1}}{2^n} + \frac{1}{2^n}$$

or, $$\boxed{\frac{T_n}{2^n} = \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}} \quad \ldots\ldots (1)$$

Let, $S_n = \frac{T_n}{2^n}$

eqn (1) becomes,

$$S_n = S_{n-1} + 2^{-n}$$

implys এ this series

$2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5}$

↳ এই series এর closed form $1 - \left(\frac{1}{2}\right)^n$

There is a HUGE MISSING Part here, which you can find in "Lec4-Sums.pdf" in the "PDF Lectures" folder. You must show those Steps to get marks

$S_n = 1 - \left(\frac{1}{2}\right)^n$ ✓ //recurrence কে গুণ করে sum এ convert করে then sum এর closed form use করলাম

$\therefore \frac{T_n}{2^n} = 1 - \left(\frac{1}{2}\right)^n$

$\therefore \boxed{T_n = 2^n - 1}$

$\boxed{4}$ General form:

$a_n T_n = b_n T_{n-1} + C_n$ //recurrence এর general form

Multiply both sides by a $\boxed{\text{summation factor}}$ s_n, (small 's')

$s_n a_n T_n = s_n b_n T_{n-1} + s_n C_n \quad \ldots\ldots (1)$

Let, $s_n b_n = s_{n-1} a_{n-1}$

eqn (1) becomes

$s_n a_n T_n = s_{n-1} a_{n-1} T_{n-1} + s_n C_n \quad \ldots\ldots (11)$

Let

$S_n = s_n a_n T_n$ (capital 'S')

Small 's' = summation factor
Capital $S_n = s_n * a_n * T_n$ (the small s_n is summation factor)

eqn (11) becomes

$S_n = S_{n-1} + s_n C_n$

$= S_{n-2} + s_{n-1} C_{n-1} + s_n C_n$

$$= s_{n-3} + s_{n-2}C_{n-2} + s_{n-1}C_{n-1} + s_nC_n$$

$$\vdots$$

$$= s_0 + s_1C_1 + s_2C_2 + \cdots\cdots + s_nC_n$$

$$\therefore S_n = s_0 + \sum_{K=1}^{n} s_K C_K$$

$$S_n = s_n a_n T_n$$

$$\therefore S_0 = s_0 a_0 T_0$$

$$s_n b_n = s_{n-1} a_{n-1}$$

$$\therefore s_1 b_1 = s_0 a_0$$

$$\therefore S_0 = s_1 b_1 T_0$$

$$S_n = \underbrace{s_1 b_1 T_0}_{\text{or, } s_0a_0T_0} + \sum_{K=1}^{n} s_K C_K$$

$$\Rightarrow s_n a_n T_n = s_1 b_1 T_0 + \sum_{K=1}^{n} s_K C_K$$

$$\therefore \boxed{T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{K=1}^{n} s_K C_K\right)} \rightarrow \text{recurrence কে sum এ নেওয়া হল}$$

unknown

unknown

Summation factor

Determine the value of s_n :

$$s_n b_n = s_{n-1} a_{n-1}$$

$$\therefore s_n = \frac{s_{n-1} \cdot a_{n-1}}{b_n}$$

$$= s_{n-2} \frac{a_{n-2}\, a_{n-1}}{b_n\, b_{n-1}}$$

$$= s_{n-3} \frac{a_{n-3}\, a_{n-2}\, a_{n-1}}{b_n\, b_{n-1}\, b_{n-2}}$$

$$\vdots$$

$$= s_1 \left(\frac{a_{n-1}\, a_{n-2} \cdots a_{n-3} \cdots a_1}{b_n\, b_{n-1}\, b_{n-2} \cdots b_2} \right)$$

putting value of s_1

$\hookrightarrow$ putting $s_1 = 1$ (or any integer is OK)

$$\boxed{\therefore s_n = \frac{a_{n-1}\, a_{n-2}\, a_{n-3} \cdots a_1}{b_n\, b_{n-1}\, b_{n-2} \cdots b_2}}$$

$1 + 2 + 3 + 4 + \cdots + n$

$s_n = s_{n-1} + n$

$= s_{n-2} + (n-1) + n$

$\vdots$

$= s_1 + \cdots$

1st term of series

$s_1 = \frac{a_0}{b_1}$

$\frac{a_0}{b_1} \times \frac{a_1}{b_2} \times \frac{a_2}{b_3} + \cdots \frac{a_{n-1}}{b_n}$

// So,

$\boxed{a_n T_n = b_n T_{n-1} + c_n}$ কে sum এ নিতে গেলে s_n দিয়ে গুন করতে হয়

$$\boxed{s_n = \frac{a_{n-1}\, a_{n-2}\, a_{n-3} \cdots a_0}{b_n\, b_{n-1} \cdots b_2}}$$

যেন never '0' হয় then Total '0' হবে

The Summation Factor of THIS recurrence is THIS

$a_n T_n = b_n T_{n-1} + c_n$

For Tower of Hanoi:

$\boxed{T_n = 2T_{n-1} + 1}$ here $a_n = 1$, $b_n = 2$

$c_n = 1$

$\therefore s_n = \frac{1 \cdot 1 \cdot 1 \cdots\cdots 1}{2 \cdot 2 \cdot 2 \cdots\cdots 2} = \frac{1}{2^{n-1}} = 2^{-n+1}$

$T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{K=1}^{n} s_K c_K\right)$

$T_n = \frac{1}{2^{-n+1} * 1}\left(2^{-1+1} * 2 * 0 + \sum_{K=1}^{n} 2^{-K+1} * 1\right)$

$= 2^{n-1}\left(\sum_{K=0}^{n-1} 2^{-K}\right)$

এই sum এর closed form

$= \frac{2^{n-1}\left(1 - \left(\frac{1}{2}\right)^{n-1+1}\right)}{\left(1 - \frac{1}{2}\right)} = 2^{n-1} \frac{1 - (1/2)^n}{\frac{1}{2}}$

$= 2^n \left(1 - \left(\frac{1}{2}\right)^n\right) = 2^n - 1$

$\therefore \boxed{T_n = 2^n - 1}$

◫ Quick Sort Recurrence:

$$C_0 = 0$$

$$C_n = n+1 + \frac{2}{n}\sum_{K=0}^{n-1} C_K$$

recurrence

C_n = number of comparisons required for partitioning n-time in the array.

(recurrence এর সাথে মিলাতে একে simpifly করা হচ্ছে)

/* n = 5 হলে

$$C_5 = 5+1+\frac{2}{5}(C_0+C_1+C_2+C_3+C_4) * 1$$

$$a_n T_n = b_n T_{n-1} + C_n$$

$$S_n = \frac{a_{n-1} a_{n-2} \cdots a_0}{b_n b_{n-1} \cdots b_1}$$

$$T_n = \frac{1}{S_n a_n}\left(S_1 b_1 T_0 + \sum_{K=1}^{n} S_K C_K\right)$$

Multiply both sides by n ⇒

$$nC_n = n^2 + n + 2\sum_{K=0}^{n-1} C_K \quad \cdots \cdots (i)$$

Replace n by n-1,

$$\Rightarrow (n-1) C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{K=0}^{n-2} C_K \quad \cdots (ii)$$

(i) − (ii) ⇒

$$nC_n - (n-1)C_{n-1} = n^2 + n + 2\sum_{K=0}^{n-1} C_K - (n-1)^2 - (n-1) - 2\sum_{K=0}^{n-2} C_K$$

$$= n^2 + n + 2\sum_{K=0}^{n-1} C_K - n^2 + 2n - 1 - n + 1 - 2\sum_{K=0}^{n-2} C_K$$

$$\Rightarrow nC_n - (n-1)C_{n-1} = 2n + 2\sum_{k=0}^{n-1} C_k - 2\sum_{k=0}^{n-2} C_k$$

$C_0 + \dots + C_{n-1}$

$C_0 + \dots + C_{n-2}$

$$= 2n + 2C_{n-1}$$

At last we are now free from the Summation term in Cn :)

$$\Rightarrow nC_n = 2n + 2C_{n-1} + (n-1)C_{n-1}$$

$$= 2n + C_{n-1}(2+n-1)$$

$$\boxed{nC_n = (n+1)C_{n-1} + 2n}$$

compare with

$$a_nT_n = b_nT_{n-1} + c_n$$

// So,

$a_n = n$, $b_n = (n+1)$, $c_n = 2n$

$$\frac{a_1 a_2 \dots \dots a_{n-1}}{b_2 b_3 \dots \dots b_n}$$

$$\therefore S_n = \frac{\cancel{(n-1)}\ \cancel{(n-2)}\ (n-3) \dots \dots 3.2.1}{(n+1)\, n\, \cancel{(n-1)}\ \cancel{(n-2)} \dots \dots 3}$$

n+1, n-1 এর চেয়ে 2 বড় বলে

$$= \frac{2}{n(n+1)}$$

$$\circledast\ T_n = \frac{S_n}{S_n a_n} = \frac{1}{S_n a_n}\left(S_1 b_1 T_0 + \sum_{k=1}^{n} S_k C_k\right)$$

$$\therefore C_n = \frac{1}{\frac{2}{\cancel{n}(n+1)} \times \cancel{n}}\left(1.2.0 + \sum_{k=1}^{n} \frac{2}{\cancel{k}(k+1)} \times 2\cancel{k}\right)$$

$$= \frac{(n+1)}{2}\left(4\sum_{k=1}^{n} \frac{1}{k+1}\right)$$

recurrence কে sum এ convert

$$\therefore \boxed{C_n = 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1}}$$

Quick sort এর recurrence কে sum এ আনতে বললে Ans এটাই

*** To easily simplify the Above expression into the following Harmonic Series Expression, See the file: "Sum (QS Harmonic).jpg" - which is in the "PDF Lectures" folder

Harmonic no. এর series:

$$H_n = \sum_{k=1}^{n}\frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n}$$

এই series কে harmonic no এর series এর respect এ represent করতে চাই

$$\sum_{k=1}^{n}\frac{1}{k+1} = \sum_{1\leq k\leq n}\frac{1}{k+1}$$

Replace k by k-1

$$\Rightarrow \sum_{1\leq k-1\leq n}\frac{1}{k-1+1}$$

$$= \sum_{2\leq k\leq n+1}\frac{1}{k}$$

$\because 1\leq k-1\leq n$

$k-1=1, \quad k-1=n$

$\therefore k=2 \quad k=n+1$

k-1 কে k এর respect এ আনা হল

$$= \sum_{1\leq k\leq n}\frac{1}{k} - \frac{1}{1} + \frac{1}{n+1}$$

$$= H_n - 1 + \frac{1}{n+1}$$

$$= H_n - \frac{n-1+1}{n+1}$$

$$= H_n - \frac{n}{n+1}.$$

$$\therefore C_n = 2(n+1)\left[H_n - \frac{n}{n+1}\right]$$

$$\boxed{C_n = 2(n+1)H_n - 2n}$$

// আর যদি Cn কে $C_n = 2(n+1)H_n - 2n$ for G আনতে বলা হলে পুরাটি করে G পর্যন্ত আনতে হবে।

Finding a close form by index replacement:

$$S_n = \sum_{k=0}^{n} (a+bk)$$ → solution ছিল $a(n+1) + \frac{n(n+1)}{2} b$

$$S_n = \sum_{0 \leq n-k \leq n} (a+bk) \quad \ldots\ldots (1)$$ // index replacement এর ক্ষেত্রে আগে এই form এ আনতে হবে

Replace k by n−k ⇒

$$S_n = \sum_{0 \leq n-k \leq n} a + b(n-k)$$

$n-k=0$
$\therefore k=n$
$n-k=n$
$k=0$

$$= \sum_{0 \leq k \leq n} a + bn - bk \quad \ldots\ldots (11)$$

$(i)+(ii) \Rightarrow$ // index same তাই add করা যাবে

$$2S_n = \sum_{0\leq k\leq n} a + \cancel{bk} + a + bn - \cancel{bk}$$

$$= \sum_{0\leq k\leq n} 2a + bn$$

(*) index never বাদ দেওয়া যাবে

$$= (2a+bn)\sum_{0\leq k\leq n} 1$$ // ∵ constant, বাইরে আনা গেল

$$= (2a+bn)(n+1)$$

[xm এ method বললে সেই method এ না বললে any method]

$$\therefore S_n = \frac{(2a+bn)(n+1)}{2}$$

$$= \frac{2a(n+1)+bn(n+1)}{2}$$

$$= a(n+1) + \frac{n(n+1)}{2}b$$

(*) $\sum\limits_{0\leq k\leq n} k$, $\sum 2k$, $\sum 2k+1$ by index replacement.

a=0, b=1

a=0, b=2

a=1, b=2

For each of these,
Solve the General $\sum$ a+bk as shown above
Then put the values of a and b
this reduces chance of calculation error in exam

Question: Write the following sum into its general form, then Convert the general sum into a recurrence. Solve the general recurrence by INDEX REPLACEMENT Technique to find the value of Sn for n = 100.
Sn = 3 + 10 + 17 + … … … + 703 OR 3 + 10 + … upto 101-th term
Solution: Write Sn as general form ($\sum$ (3+7k)) with a=3, b=7 and n = 100
... thus, Prove & Use the above result of Index Replacement Technique

Q: Derive the Formula for Perturbation Technique.

Perturbation Technique:

↳ (যদি S_n দেওয়া থাকে)

$S_{n+1} = ?$

$$S_n = \sum_{0 \leq k \leq n} a_k$$

We have to find $S_{n+1} = ?$

$$S_{n+1} = S_n + a_{n+1} \quad \text{or,}$$

$$S_{n+1} = S_0 + \sum_{1 \leq k \leq n+1} a_k$$

$$\Rightarrow S_n + a_{n+1} = S_0 + \sum_{1 \leq k \leq n+1} a_k$$

$$= a_0 + \sum_{1 \leq k \leq n+1} a_k$$

replace k by $k+1$,

$$= a_0 + \sum_{1 \leq k+1 \leq n+1} a_{k+1}$$

$k+1 = 1$
$k = 0$
$k+1 = n+1$
$k = n$

$$= a_0 + \sum_{0 \leq k \leq n} a_{k+1}$$

$$\boxed{S_n + a_{n+1} = a_0 + \sum_{0 \leq k \leq n} a_{k+1}}$$ ⟹ Formula of Perturbation Technique

You have to find the $\sum a_k$.
In the Pb formula ==> **To get a_0 ==> Put k = 0**
To get a_{k+1} ==> Put k = k+1
To get a_{n+1} ==> Put k = n+1

☐ Find a closed form for

$$S_n = \sum_{0 \leq k \leq n} a x^k$$ //power চলে এসেছে

Using perturbation Technique we can write,

$$S_n + a x^{n+1} = a x^0 + \sum_{0 \leq k \leq n} a x^{(k+1)}$$

$$= a + \sum_{0 \leq k \leq n} a x^k . x$$

$$= a + x \sum_{0 \leq k \leq n} a x^k$$

$$= a + x S_n$$

$$S_n + a x^{n+1} = a + x S_n$$

$$\Rightarrow S_n (1 - x) = a - a x^{n+1}$$

$$\therefore \boxed{S_n = \frac{a - a x^{n+1}}{1 - x}}$$ ⟶ closed form

$S_n = \sum_{0 \leq k \leq n} 5 \cdot 3^k$ $\therefore S_8 = ?$

Here, $a = 5$
$x = 3$
$n = 8$

$$\therefore S_n = \frac{a - ax^{n+1}}{1-x}$$

$$= \frac{5 - 5 \cdot 3^9}{1-3}$$

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Find closed form of $S_n = \sum_{0 \leq k \leq n} k \cdot 2^k$ using Perturbation Technique

$\left[S_n + a_{n+1} = a_0 + \sum_{0 \leq k \leq n} a_{k+1} \right]$

Here, $a_k = k.2^k$
put a_{n+1}, a_0, a_{k+1} in Pb formula

$$S_n + (n+1)2^{n+1} = 0 \cdot 2^0 + \sum_{0 \leq k \leq n} (k+1)\, 2^{k+1}$$

$$= 0 + \sum_{0 \leq k \leq n} k \cdot 2^{k+1} + \sum_{0 \leq k \leq n} 2^{k+1}$$

$$= 2 \sum_{0 \leq k \leq n} k \cdot 2^k + \sum_{0 \leq k \leq n} 2 \cdot 2^k$$

resembles

$$\therefore S_n + (n+1)2^{n+1} = 2S_n + \frac{2 - 2.2^{n+1}}{1-2}$$

$$= 2S_n - 2 + 2^{n+2}$$

$$\therefore S_n = (n+1)\cdot 2^{n+1} + 2 - 2^{n+2}$$

$$= 2^{n+1}(n+1-2) + 2$$

$$\therefore \boxed{S_n = 2^{n+1}(n-1) + 2}$$

constant

Forms: i) $\sum a x^K$, ii) $\sum K 2^K$ — 2 fixed

Closed form for, $S_n = \sum_{0 \le K \le n} K x^K$

Using pb technique we can write,

$$S_n + (n+1)x^{n+1} = 0 . x^0 + \sum_{0 \le K \le n} (K+1) x^{K+1}$$

$$= \sum_{0 \le K \le n} K x^{K+1} + \sum_{0 \le K \le n} x^{K+1}$$

$[\because \sum K x^K . x = \sum x S_n]$

$$= x S_n + \sum_{0 \le K \le n} x x^K$$

$$= x S_n + \frac{x - x \cdot x^{n+1}}{1-x}$$

$$\Rightarrow S_n(1-x) = \frac{x - x^{n+2}}{1-x} - (n+1)\, x^{n+1}$$

$$= \frac{x - x^{n+2} - (n+1)x^{n+1} + x(n+1)x^{n+1}}{(1-x)}$$

$$= \frac{x - x^{n+2}(1-n-1) - (n+1)x^{n+1}}{(1-x)}$$

$$= \frac{x + n x^{n+2} - (n+1)x^{n+1}}{1-x}$$

$$\boxed{S_n = \frac{x + n x^{n+2} - (n+1)x^{n+1}}{(1-x)^2}}$$

[এই formula use করে $\sum K 2^K$ এর solution বের করা যাবে এখানে $x = 2$]

$$S_n = \frac{x + n x^{n+2} - (n+1)x^{n+1}}{(1-x)^2}$$

$$= 2 + n 2^{n+2} - (n+1)\, 2^{n+1}$$

$$= 2^{n+1}(2n - n - 1) + 2$$

$\therefore S_n = 2^{n+1}(n-1)+2$ (verified)

\# $\sum\limits_{1 \le k \le n} 2^{-k}$ এর solution $1-\left(\frac{1}{2}\right)^n$

No Need Excluded from syllabus!

\# $\sum\limits_{1 \le k \le n} 2^{-k} = S_n$ using pb technique.

Hints:

pb technique always 0 থেকে start হয়

$\therefore \sum\limits_{1 \le k \le n} 2^{-k}$

0 হলে -2^{-0}

$S_n = \sum\limits_{0 \le k \le n} 2^{-k} - 2^{-0}$ করতে হবে

\# Find closed form of $\sum\limits_{0 \le k \le n} k^2$

$S_n = \sum\limits_{0 \le k \le n} k^2$

Using pb technique

$S_n + (n+1)^2 = 0^2 + \sum\limits_{0 \le k \le n} (k+1)^2$

$$= \sum_{0 \leq k \leq n} k^2 + 2k + 1$$

$$= \sum_{0 \leq k \leq n} k^2 + \sum_{0 \leq k \leq n} 2k + \sum_{0 \leq k \leq n} 1$$

$$\cancel{S_n} + (n+1)^2 = \cancel{S_n} + \sum_{0 \leq k \leq n} 2k + (n+1)$$

$$\Rightarrow 2 \sum_{0 \leq k \leq n} k = (n+1)^2 - (n+1)$$

$$= n^2 + 2n + 1 - n - 1$$

$$\therefore \sum_{0 \leq k \leq n} k = \frac{n^2 + n}{2}$$

// চেয়েছিলাম K^2 এর solution বের করতে but পেয়েছি K এর

↓

degree 1 কমে

$\therefore$ K^2 এর solution পেতে use K^3.

Let $\sum k^2$ বের করতে

So,

$$S_n = \sum_{0 \leq k \leq n} k^3$$

$$S_n + (n+1)^3 = 0^3 + \sum_{0 \leq k \leq n} (k+1)^3$$

or, $$\cancel{S_n} + (n+1)^3 = \sum_{0 \leq k \leq n} \left(k^3 + 3k^2 + 3k + 1\right)$$

$$= \sum_{0 \leq k \leq n} k^3 + \sum_{0 \leq k \leq n} 3k^2 + \sum_{0 \leq k \leq n} 3k + \sum_{0 \leq k \leq n} 1$$

$$= \cancel{S_n} + \sum_{0 \leq k \leq n} 3k^2 + 3\,\frac{n(n+1)}{2} + (n+1)$$

$$\Rightarrow 3\sum_{0 \leq k \leq n} k^2 = (n+1)^3 - \frac{3n(n+1)}{2} - (n+1)$$

$$= (n+1)\left\{(n+1)^2 - \frac{3n}{2} - 1\right\}$$

$$= (n+1)\left\{n^2 + 2n + 1 - \frac{3n}{2} - 1\right\}$$

$$= (n+1)\left\{n^2 + 2n - 3n/2\right\}$$

$$= (n+1)\left\{n^2 + \frac{4n - 3n}{2}\right\}$$

$$= (n+1)\left\{n^2 + n/2\right\}$$

$$3\sum_{0\leq k\leq n} k^2 = (n+1)\left(n+\frac{1}{2}\right)n$$

$$\therefore \sum_{0\leq k\leq n} k^2 = \frac{(n+1)\left(n+1/2\right)n}{3}$$

$$= \frac{(n+1)(2n+1)n}{6}$$

Chapter → 3

INTEGER FUNCTION

Floor:

integer হতেও পারে নাও পারে

mathemetical def^n

Notation: $\lfloor x \rfloor$ = The gretest integer less than or equal to x

$= n$ such that $n \leq x$, $n+1 > x$ → ছোট integer

integer

must integer

Ex:

$\lfloor 2.6 \rfloor = 2$

$\lfloor 3 \rfloor = 3$

Ceiling:

Notation:

$\lceil x \rceil$ = The least integer greater than or equal to x

= integer n such that $n \geq x$, $n-1 < x$ → বড় integer

Ex:

$\lceil 2.6 \rceil = 3$

$\lceil 3 \rceil = 3$

[Kenneth E. Inverson 1960 যে ceiling & floor এর notation invent করেন]

Floor & Ceiling function:

$f(x) = x$

[— represents $\lfloor x \rfloor$
--- " $\lceil x \rceil$]

$\lceil -1.5 \rceil = -1$

$\lfloor -1.5 \rfloor = -2$

$\lceil -1.6 \rceil = -1$

$\lfloor 3 \rfloor = 3$

$\lceil 3 \rceil = 3$

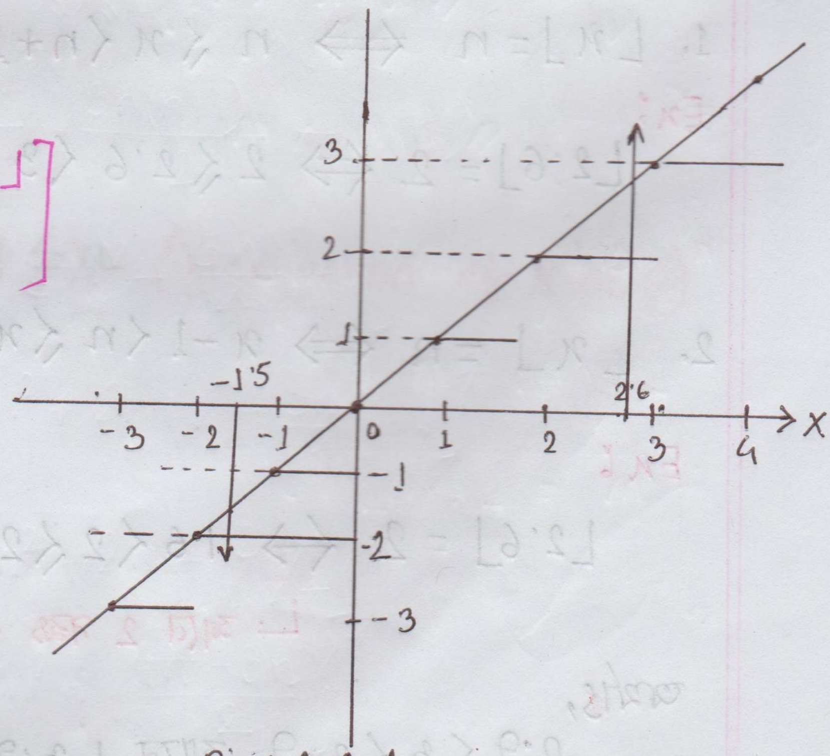

Fig: func for floor & Ceiling

Rules:

iff (if and only if)

* $\lfloor x \rfloor = x \iff x$ is integer $\iff \lceil x \rceil = x$

(Properties)

$\lceil -1.6 \rceil = -1 = -\lfloor 1.6 \rfloor$

// $x \Rightarrow y$ মানে x হলে y হবে

$\lceil -2.9 \rceil = -\lfloor 2.9 \rfloor = -2$

$\lfloor -2.9 \rfloor = -\lceil 2.9 \rceil = -3$

* $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$

(properties)

Some rules: (from graph বের করা হচ্ছে)

(In all the following rules,
x = any number - integer or real valued num
n = always an integer)

1. $\lfloor x \rfloor = n \iff n \leq x < n+1$

Ex:

$\lfloor 2.6 \rfloor = 2 \iff 2 \leq 2.6 < 3$

2. $\lfloor x \rfloor = n \iff x-1 < n \leq x$

Ex:

$\lfloor 2.6 \rfloor = 2 \iff 1.6 < 2 \leq 2.6$

$\lfloor$ মানে 2 পর্যন্ত vice verse equal

অর্থাৎ,

$2.9 < 3 \leq 3.9$ মানে $\lfloor 3.9 \rfloor = 3$

3. $\lceil x \rceil = n \Leftrightarrow \cancel{x \leq n < x+1}$ $n-1 < x \leq n$

4. $\lceil x \rceil = n \Leftrightarrow x \leq n < x+1$

5. $\boxed{\lfloor x+n \rfloor = \lfloor x \rfloor + n \text{, integer } n}$

$\lfloor 2.9+2 \rfloor$ এমন case হলে,

$$\lfloor 2.9+2 \rfloor = 2 + \lfloor 2.9 \rfloor$$
$$= 2 + 2$$
$$= 4$$

$\lfloor 3+2 \rfloor = 3+2=5$ or, $3+\lfloor 2 \rfloor$ করা যাবে

6. $n \leq x \Leftrightarrow n \leq \lfloor x \rfloor$

7. $x \leq n \Leftrightarrow \lceil x \rceil \leq n$ (Vice versa of Rule 6)

$\boxed{\text{If } n \text{ is an } m\text{-bit integer, then prove that } m = \lfloor \lg n \rfloor + 1}$

n is an m-bit integer.

So n is like $\underbrace{1XXXX\ldots X}_{m \text{ bits}}$, which means

n is <u>at least</u> $\underbrace{1000\ldots 0}_{m \text{ bits}} = \boxed{2^{m-1}}$

n is <u>at most</u> $\underbrace{111\ldots 1}_{m \text{ bits}}$

$= \underbrace{100\ldots 0}_{m+1 \text{ bits}} - 1$

$\boxed{= 2^m - 1}$

So: $2^{m-1} \leq n < 2^m$

$\Rightarrow \lg 2^{m-1} \leq \lg n < \lg 2^m$

$\Rightarrow m-1 \leq \lg n < m$

$\therefore \lfloor \lg n \rfloor = m-1$ [$\because$ rule 1: $n \leq x < n+1 \Leftrightarrow \lfloor x \rfloor = n$]

$\therefore m = 1 + \lfloor \lg n \rfloor$

[$\lg \Rightarrow$ base-2: $\log_2$
$\ln \Rightarrow$ base-e: $\log_e$
$\log \Rightarrow$ base-10: $\log_{10}$]

$\therefore \boxed{m = \lfloor \lg n \rfloor + 1}$

Example:

$n = 35 \quad m = ?$

$$\lg_2 35 = \frac{\log_{10} 35}{\log_{10} 2}$$

$$m = \lfloor \lg 35 \rfloor + 1$$
$$= 5 + 1$$
$$= 6$$

So, 6-bits are required to write 35 in binary

$n = 32, \; m = 6.$

So, 6-bits are required to write 32 in binary
But, to write 31 in binary, 5 bits required

Prove or disprove that $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$; real $x \geq 0$

1. If x is integer, then $\lfloor x \rfloor = x$,

$$\text{So LHS} = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$$
$$= \lfloor \sqrt{x} \rfloor = \text{R.H.S}$$

2. If x is not integer,

Let $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$

rule 1:
$\lfloor x \rfloor = n \Leftrightarrow n \leq x < n+1$

Since, m and m+1 are integer, we get: from rule 1 ==>

$$\Rightarrow m \leq \sqrt{\lfloor x \rfloor} < m+1$$

$$\Rightarrow m^2 \leq \lfloor x \rfloor < (m+1)^2$$

$\Rightarrow m^2 \leq n < (m+1)^2$

rule 6 & 7:

$n \leq x \Leftrightarrow n \leq \lfloor x \rfloor$

As m^2 and (m+1)^2 both integers, we get from rules 6 & 7:

x < n <=iff=> ⌊x⌋< n

$\Rightarrow m \leq \sqrt{x} < m+1$ [doing squareroot]

$\Rightarrow \lfloor \sqrt{x} \rfloor = m$ [using rule 1]

$= \lfloor \sqrt{\lfloor x \rfloor} \rfloor$

$\therefore \lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$, real $x > 0$

[Proved]

MOD:

n mod m

$$m \mid n \mid \quad n \bmod m$$

$$n = \lfloor \frac{n}{m} \rfloor * m + n \bmod m$$

$$\begin{array}{c|c|l} m & n & \\ 2 & 7 & 3 = \lfloor \frac{n}{m} \rfloor \\ & 6 & \\ \hline & 1 & \\ \end{array}$$

$\downarrow$ n mod m

$$\Rightarrow n \bmod m = n - m \lfloor \frac{n}{m} \rfloor$$

$\downarrow \qquad \downarrow$

x mod y (we'll use)

$$\boxed{x \bmod y = x - y \lfloor \frac{x}{y} \rfloor, \; y \neq 0}$$

example:
we know: 43 mod 10 = 3
how?
43 mod 10 = 43 - 10*4
= 43 - 10*⌊43/10⌋
x mod y = x - y * ⌊x/y⌋

\# $43 \bmod 10 = 43 - 10 * 4$

$= 43 - 10 * \lfloor 43/10 \rfloor$

\# $5 \bmod 3 = 5 - 3\lfloor 5/3 \rfloor = 5 - 3 = 2$

\# $5 \bmod -3 = 5 - (-3)\left\lfloor \frac{5}{-3} \right\rfloor$

$= 5 + 3\left(-\lceil 5/3 \rceil\right)$

$= 5 - 3 \times 2$ // '−' বের করলে —

$= -1$ floor $\leftrightarrow$ ceiling হয়ে যায়

\# $-5 \bmod 3 = -5 - 3\lfloor -5/3 \rfloor$

$= -5 - 3\left(-\lceil 5/3 \rceil\right)$

$= -5 + 3(2)$

$= 1$

\# $-5 \bmod -3 = -5 - (-3)\left\lfloor \frac{-5}{-3} \right\rfloor$

$= -5 + 3 * 1$

$= -2$

$3 \bmod 5 = 3$

$3 - 5\left\lfloor \frac{3}{5} \right\rfloor$

$= 3 - 5 \cdot 0$

$= 3$

$x \bmod 0 = ?$

always $\boxed{x \bmod 0 = x}$ হয়, this is predefined value.

Properties:

1. $c(x \bmod y) = cx \bmod cy$

Verification:

$L.H.S = c\,(x \bmod y)$

$= c\,(x - y\lfloor x/y \rfloor)$

$= cx - cy\lfloor x/y \rfloor$

$= cx - cy\left\lfloor \frac{cx}{cy} \right\rfloor$

$= cx \bmod cy$

$= R.H.S.$

Prove or disprove that $(x \bmod ny) \bmod y = x \bmod y$

(integer n)

example: ((((X mod 240) mod 60) mod 30) mod 10) mod 5
= X mod 5

Case 1:

if y is 0:

$ny = 0$

$L.H.S = (x \bmod 0) \bmod y$

$= x \bmod y$ **Because, x mod 0 = x, by definition**

$= R.H.S.$

if $y \neq 0$,

$L.H.S = (x \bmod ny) \bmod y$

$= \underbrace{\left(x - ny \left\lfloor \frac{x}{ny} \right\rfloor\right)}_{\text{now consider this whole thing as x}} \bmod y$

$= x - ny \left\lfloor \frac{x}{ny} \right\rfloor - y \left\lfloor \frac{x - ny \left\lfloor \frac{x}{ny} \right\rfloor}{y} \right\rfloor$

$$= x - ny\left\lfloor \frac{x}{ny} \right\rfloor - y\left\lfloor \frac{x}{y} - \frac{ny\left\lfloor \frac{x}{ny} \right\rfloor}{y} \right\rfloor$$

$$= x - ny\left\lfloor \frac{x}{ny} \right\rfloor - y\left\lfloor \frac{x}{y} - \frac{ny\left\lfloor \frac{x}{ny} \right\rfloor}{y} \right\rfloor$$

পুরোটা integer তাই বাইরে আনা হল

$$= x - ny\left\lfloor \frac{x}{ny} \right\rfloor - y\left\lfloor \frac{x}{y} - \overbrace{n\left\lfloor \frac{x}{ny} \right\rfloor} \right\rfloor$$

$$= x - ny\left\lfloor \frac{x}{ny} \right\rfloor - y\left\lfloor \frac{x}{y} \right\rfloor + ny\left\lfloor \frac{x}{ny} \right\rfloor$$

$$= x - y\left\lfloor \frac{x}{y} \right\rfloor \qquad \left[\because \lfloor 3 + 2.6 \rfloor = 3 + \lfloor 2.6 \rfloor\right]$$

$= x \bmod y$

$= R.H.S$ [Proved]

Chapter → 4

NUMBER THEORY

Some Notations:

Divisibility:

// $\frac{n}{m}$ এর ans যদি integer হয় তাহলে বলব m dėvides n [m > 0]

m divides n if m > 0 and $\frac{n}{m}$ is an integer

$m \backslash n$ ⟶ means m > 0 and n = mk, for some integer K [ex: $\frac{10}{2} = 5$]

divisibility এর notation

m divides n

Greatest Common Divisor (GCD):

9, 12 এর GCD

$$
\begin{array}{r|r|l}
9 & 12 & 1 \\
 & 9 & \\
\hline
3 & 9 & 3 \\
 & 9 & \\
\hline
 & 0 &
\end{array}
$$

We want to define GCD mathematically:

max^m

$$GCD(m,n) = \max\{k \mid k \backslash m \text{ and } k \backslash n\}$$

12, 18

$$\begin{array}{r|l|l} 12 & 18 & 1 \\ & 12 & \\ \hline 6 & 12 & 2 \\ & 12 & \\ \hline & 0 & \end{array}$$

$k = 6$

$n = 12$ $\quad \cancel{m}n = 18$

//যদিও 3 দিয়েও 12, 18 ভাগ যায়
but max^m নিতে হবে → (6)

$GCD(0, n) = n$

//GCD হল এমন no. যা দিয়ে m & n ২টাকে ভাগ করা যায় এবং সেটাই max^m যেসব no. এর মধ্যে যেসব no. দিয়ে m, n দুটিকে ভাগ করতে পারে।

***

Stein's Algorithm to find GCD(u,v):

1. $GCD(0, v) = v$

2. If u and v are both even then $gcd(u,v) = 2 \cdot gcd(u/2, v/2)$

3. If u is even and v is odd, then $gcd(u,v) = gcd(u/2, v)$

4. If u is odd and v is even, then

$$\gcd(u,v) = \gcd(u, v/2)$$

5. If u and v are both odd and if $u \geq v$, then

$$\gcd(u,v) = \gcd\left(\frac{u-v}{2}, v\right)$$

If $u < v$ then

$$\gcd(u,v) = \gcd\left(\frac{v-u}{2}, u\right)$$

6. Repeat step 2-5 until one of u, v becomes 0,

$\gcd(\overset{u}{12}, \overset{v}{18})$ using Stein's formula:

$\gcd(12, 18)$

$= 2 . \gcd\left(\frac{12}{2}, \frac{18}{2}\right)$ //according to step 2

$= 2 . \gcd(6, 9)$

$= 2 . \gcd\left(\frac{6}{2}, 9\right)$ //according to step 3

$= 2 . \gcd(3, 9)$

$= 2 \cdot gcd\left(\frac{9-3}{2}, 3\right)$ // step 5(ii)

$= 2 \cdot gcd(3, 3)$

$= 2 \cdot gcd\left(\frac{3-3}{2}, 3\right)$

$= 2 \cdot gcd(0, 3)$ // $\because gcd(0,n) = n$

$= 2 \cdot 3$

$= 6$

// যেকোন no. কে 1 বার Right shift করলে তা 2 দিয়ে ভাগ হয়ে যায়।

division by 2 = Shift Right
multiply by 2 = Shift Left

1010 = 10
0101 = 5 // SHR

// 1টা cycle এই হয়ে যাচ্ছে, so, algo টা fast.

** this is recursive formula.

Euclid's Algorithm to find gcd (m,n)*:

1. $gcd(0,n) = n$

2. $gcd(m,n) = gcd(n \bmod m, m)$; $m > 0$

(note: the 1st operand becomes the last operand)

gcd (12, 18) using Euclid's formula:

$$gcd(12, 18) = gcd(18 \bmod 12, 12)$$
$$= gcd(6, 12)$$
$$= gcd(12 \bmod 6, 6)$$
$$= gcd(0, 6)$$
$$= 6$$

// Stein's formula computationally effective
আর এটা recurrsive বলে expensive (বেশী time লাগে)

gcd (18, 12):

$$= gcd(12 \bmod 18, 18)$$
$$= gcd(12, 18)$$ //associative

* 4 exam GCD

gcd (12, 107)

$$= gcd(107 \bmod 12, 12)$$
$$= gcd(11, 12)$$
$$= gcd(12 \bmod 11, 11)$$

$= gcd(1, 11)$

$= gcd(11 \bmod 1, 1)$

$= gcd(0, 1)$

$= 1$

Least Common multiple (LCM):

LCM (12, 18)

$$\begin{array}{r|l} 2 & 12, 18 \\ \hline 3 & 6, 9 \\ \hline & 2, 3 \end{array}$$

//২টা no. এর LCM হল এমন—
২টা integer no. যা ঐ ২টা no.
দিয়েও ভাগ করা যায় & যেটা সবচেয়ে ছোট no.

$2 \times 2 \times 3 \times 3$
$= 36$
↑ LCM

$$LCM(m, n) = \min\{k \mid k > 0 \text{ and } m \backslash k \text{ and } n \backslash k\}$$

//আরেকটা process এ LCM:

7 12, 18

18

36

বড়টা ধরব
ছোটটা দিয়ে ভাগ করা গেলনা
double করা হল then ২টা দিয়েই
ভাগ করা গেল

Another formula of Euclid's:

$$m * n = gcd(m,n) * LCM(m,n)$$

Verification:

$$12*18 = \underbrace{6}_{gcd(12,18)} * \underbrace{36}_{LCM(12,18)}$$

$$LCM(m,n) = \frac{(m*n)}{gcd(m,n)}$$

Prime Numbers:

A number P is prime iff it has two divisors namely 1 and P.

(1 & P: 2 diff no. so 1 is not a prime number by def^n, so our prime no. starts from 2.)

Composite Number:

A number is composite iff it has three or more divisors

$$\begin{cases} 91 = 7 \times 13 \times 1 \\ 161 = 7 \times 23 \times 1 \end{cases}$$

// any positive integer greater than 1 is either prime or composite.

//1টা composite no. এর ভেতর কমপক্ষে 1টা prime na thakbei.

$32 = 8 \times 4$

but $32 = 16 \times 2 \rightarrow$ prime

$12 = 4 \times 3 \rightarrow$ prime.

Composite

Every ~~Prime~~ number has a prime divisor:

Proof by contradiction:

Or, you can Prove the next Proof : "Every integer is either prime Or product of Primes" ==> which Automatically Proves this proof

/* n is a min^m composite no. which has no prime divisor

$\because$ n is a composite no.

$n = 1 \times a \times \ldots \ldots \times m$ [1, a, m কোনটিই prime no. না

$21 = 7 \times 3$

m composite তাহলে যেহেতু এটা composite. একে ও divisor দিয়ে express করা যাবে যা কিনা n এর চেয়ে ছোট but শুরুতে ধরা হয়েছে n ই সবচেয়ে ছোট */

* Proof by contradiction:

Suppose there exists a number which is composite but doesn't have any prime divisor. Let n, be such smallest number.

Because, n is a composite number it has a divisor m greater than 1.

Now, m is also a composite number and is not prime according to induction. But, m is smaller than n which contradicts that n is such smallest number. So, no number can be found of such type.

[Proved]

四 Any positive integer >1 is either prime or product of primes:

$$n = P_1, P_2 \cdots\cdots P_n = \prod_{K=1}^{n} P_K$$

$12 = 4 \times 3$
$= 2 \times 2 \times 3$

Basic:

lowest integer greater than 1 is 2.

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There are infinitely many Prime Number in the universe:

// Proof by contradiction করব। এজন্য ধরে নিব prime no. এর সেট finite $\{2, 3, \ldots P_K\}$

$M = \{2 \times 3 \times 5 \times \ldots \times P_K)$

$M = \{2 \times 3 \times 5 \ldots \times P_K) + 1$ //

Let us assume that, there are finite number of primes in the universe.

Let there are K primes, The list of finite prime is $\{2, 3, 5, \ldots P_K\}$

Now, consider a number

$$M = (2 \times 3 \times 5 \times \ldots \times P_K) + 1$$

None of the k primes divides M. So, M is either a Prime number itself or there is another prime not in our list of Prime that divides M,

This contradicts our assumption that there are K number of primes.

So, there must be infinitely many.

method name.

Sieve of Eratosthenes to Find all prime prime number up to n:

[quickest way to find primes so far]

Algorithm:

1. Create a contiguous list of numbers from 2 to n
2. Strike out all numbers that are multiple of 2.
3. The next number of the list is a prime.
4. Strike all multiples of that number.
5. Repeat step 3, 4 until you reach a number Equal or Greater than the square root of n.

Let, $n = 64$ $\therefore \lceil \sqrt{n} \rceil = 8$ // 8 তে থামতেছি ($\because$ 4m step 5)

// normally আমরা e তে n/2 পর্যন্ত check করি এক্ষেত্রে $\sqrt{n}$ পর্যন্ত check করলেই prime কিনা বুঝা যাবে so the quickest way.

✓ → prime

(1) 2✓ (2) 3✓ ~~4~~ (3) 5✓ ~~6~~ (4) 7✓ ~~8~~ ~~9~~ ~~10~~

(5) 11✓ ~~12~~ 13✓ ~~14~~ ~~15~~ ~~16~~ 17✓ ~~18~~ 19✓ ~~20~~ 21

Stop

~~21~~ ~~22~~ 23✓ ~~24~~ ~~25~~ ~~26~~ ~~27~~ ~~28~~ 29✓ ~~30~~

31✓ ~~32~~ ~~33~~ ~~34~~ ~~35~~ ~~36~~ 37✓ ~~38~~ ~~39~~ ~~40~~

41✓ ~~42~~ 43✓ ~~44~~ ~~45~~ ~~46~~ 47✓ ~~48~~ ~~49~~ ~~50~~

~~51~~ ~~52~~ 53✓ ~~54~~ ~~55~~ ~~56~~ ~~57~~ ~~58~~ 59✓ ~~60~~

61✓ ~~62~~ ~~63~~ ~~64~~.

Fibonacci Number:

0 1 1 2 3 5 8 13 21 ....

→ recurssive def^n of Fibonacci number

$$\begin{cases} Fib(0) = 0 \\ Fib(1) = 1 \\ Fib(n) = Fib(n-1) + Fib(n-2) \end{cases}$$

//recurssive method DFS এর equivalent

Root – Left – right

DFS এ প্রথমে root theke left এ দিতে

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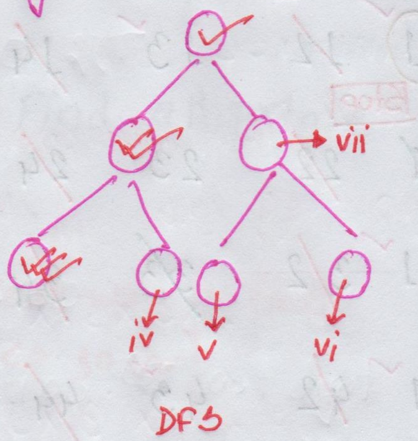
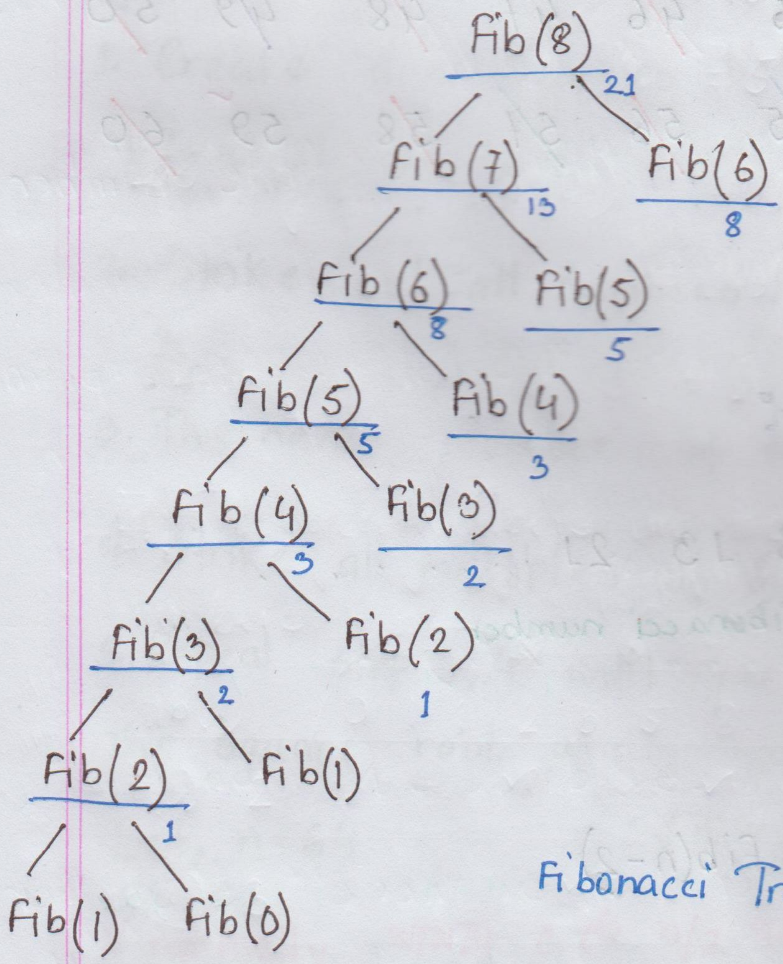
একেবারে left এ কিছু না পেলে then right.



Left Right Root



Fibonacci Tree.

**

What is the last digit of f(1600)?

0 1 1 2 3 5 8 . 13 21 34

last digits of first few fibonacci numbers : $F_0 - F_{25}$

F_0 F_1 F_2 F_3 F_4 F_5 F_6 ...

0 . 1 1 2 3 5 8 3 1 4 5 9 4 . 3 , 7, 0, 7

7, 4, 1, 5 6, 1, 7, 8, 5

60th fibonacci no.

1548008755920

Last digit of F59, F60, F61 are : 1, 0, 1 . . .

Last digit of 61th fibonacci is 1,

59th fibonacci এর পর thk last digit এর cycle আবার repeat হবে। মানে 60th fibonacci থেকে cycle start, এর পর 120th এ fibonacci thk cycle start এভাবে।

তাই 60 পর্যন্ত store করলেই হবে (programming এর ক্ষেত্রে)

f(1600) = 1600 % 60 = 40

→ 40th fibonacci এর last digit টাই এর ও last digit হবে

(*) f(1500) এর last digit? → Ans: 0

1500 % 60 = 0