2. System Modeling and Problem Statement

Notation / geometry (as on the figure)

- e_d: lateral deviation (distance from vehicle CG to nearest point T on desired path).
- $e_\varphi = \varphi_d - \varphi$: yaw angle error (desired yaw minus actual yaw).
- β: sideslip angle at CG (small).
- γ: yaw rate (paper's sign convention — see remark below).
- v_x: longitudinal velocity (assumed positive and large compared to lateral components).
- $\rho_T(\sigma)$: curvature of the desired trajectory at point T.
- F_{yf}, F_{yr}: front/rear lateral tire forces.
- C_f, C_r: cornering stiffnesses (front, rear).
- α_f, α_r: slip angles (front, rear).
- m: vehicle mass, I_z: yaw moment of inertia, l_f, l_r: distances from CG to front/rear axle.
- ΔM_z: direct yaw moment produced by differences in longitudinal wheel forces (actuation by hub motors).
- $I_w, w_i, T_i, F_{xi}, R_t, l_s$ — wheel inertia, wheel rotational speed, hub torque, longitudinal contact force, rolling radius, track-width related length, respectively.

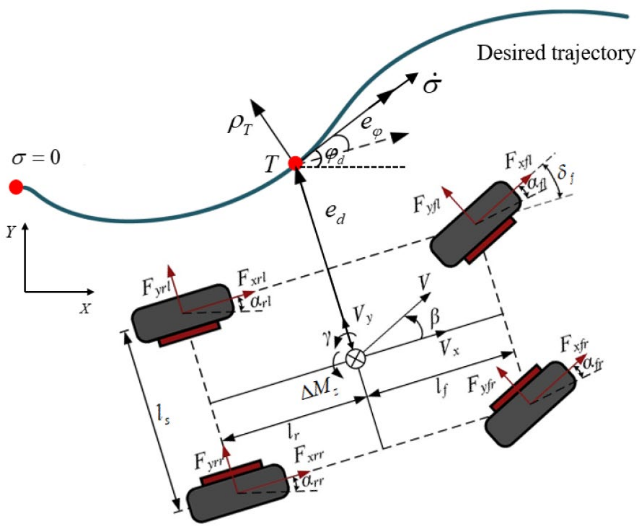

Figure 1. Trajectory tracking process of FWIA EV.

2.1. System Modeling Considering Dynamic Characteristics of Steer-by-Wire

1) Kinematic lateral/yaw error dynamics → equation (1)

Goal: obtain

$$\begin{cases} \dot{e}_d = v_x e_\varphi + v_x \beta + d_1, \\ \dot{e}_\varphi = \gamma + d_2, \end{cases} \qquad (1)$$

Derivation of $\dot{e}_d$ (lateral error rate)

Consider the vehicle velocity vector at the CG. The lateral component of CG velocity relative to the path tangent is approximately

$$v_y \approx v_x \beta \quad (\text{since } \beta \text{ small, } v_y = v_x \tan\beta \approx v_x \beta).$$

The path error angle between vehicle longitudinal axis and path tangent is e_φ. The lateral motion of the CG relative to the path is dominated by the component $v_x \sin(e_\varphi + \beta)$. For small angles,

$$\sin(e_\varphi + \beta) \approx e_\varphi + \beta.$$

Hence the lateral error derivative (ideal, no modeling error) is

$$\dot{e}_d \approx v_x (e_\varphi + \beta).$$

The paper adds d_1 to represent modeling error and external disturbances, giving the first line in (1):

$$\boxed{\dot{e}_d = v_x e_\varphi + v_x\beta + d_1.}$$

Derivation of $\dot{e}_\varphi$ (yaw error rate)

By definition $e_\varphi = \varphi_d - \varphi$. Differentiate:

$$\dot{e}_\varphi = \dot{\varphi}_d - \dot{\varphi}.$$

The desired yaw rate along the reference path equals $v_x\,\rho_T(\sigma)$ (curvature times forward speed). Denote the vehicle yaw rate by $\dot{\varphi}$. Then

$$\dot{e}_\varphi = v_x\rho_T(\sigma) - \dot{\varphi}.$$

Rewriting in the compact form $\dot{e}_\varphi = \gamma + d_2$ simply collects the terms into a vehicle yaw-rate term γ (symbol for the measured yaw rate) and a disturbance term d_2 representing the path curvature contribution (or its negative, depending on sign choice). In the paper they write $d_2 = -\rho_T(\sigma)v_x$ so that

$$\boxed{\dot{e}_\varphi = \gamma + d_2, \qquad d_2 = -\rho_T(\sigma)v_x,}$$

which reproduces the compact equation (1). (This is only a book-keeping/sign-convention choice: the physical content is $\dot{e}_\varphi = v_x\rho_T - \dot{\varphi}$.)

2) Yaw/lateral dynamics (vehicle planar dynamics) → equation (2)

$$\begin{cases}\dot{\beta} = \frac{1}{mv_x}\left(F_{yf} + F_{yr}\right) + d_3,\\ \dot{\gamma} = \frac{1}{I_z}\left(l_f F_{yf} - l_r F_{yr}\right) + \frac{1}{I_z}\Delta M_z + d_4.\end{cases} \tag{2}$$

Derivation of $\dot{\beta}$ (sideslip dynamics)

Start from lateral force balance in the vehicle body frame (standard bicycle / small-angle derivation). The lateral equation of motion about CG is

$$m\left(\dot{v}_y + v_x\dot{\varphi}\right) = F_{yf} + F_{yr} + \text{(disturbance)}.$$

Using $v_y = v_x\beta$ (small β) gives $\dot{v}_y = v_x\dot{\beta}$ (neglecting higher order terms if v_x taken as slowly varying). Substitute:

$$m\left(v_x\dot{\beta} + v_x\dot{\varphi}\right) = F_{yf} + F_{yr} + \text{(dynamics error)}.$$

Divide by mv_x:

$$\dot{\beta} + \dot{\varphi} = \frac{F_{yf} + F_{yr}}{mv_x} + \tilde{d},$$

where $\tilde{d}$ collects unmodeled terms. If we adopt the paper's grouping and sign conventions (they incorporate $-\dot{\varphi}$ into the symbol γ or into the disturbance d_3), the dynamics are written compactly as

$$\boxed{\dot{\beta} = \frac{1}{mv_x}\left(F_{yf} + F_{yr}\right) + d_3.}$$

(Again d_3 captures the $\pm\dot{\varphi}$ piece depending on the sign convention, plus unmodeled dynamics.)

Derivation of $\dot{\gamma}$ (yaw acceleration / moment balance)

Yaw moment equilibrium about CG:

$$I_z\dot{\varphi} = l_f F_{yf} - l_r F_{yr} + \Delta M_z + \text{(disturbance terms)}.$$

It denotes γ as the yaw rate (or a signed version of it) and lumps disturbances into d_4, then rearranging gives

$$\boxed{\dot{\gamma} = \frac{1}{I_z}\left(l_f F_{yf} - l_r F_{yr}\right) + \frac{1}{I_z}\Delta M_z + d_4.}$$

> **Physical meaning:** the first equation says lateral acceleration (through $\dot{\beta}$) comes from lateral tire forces; the second says yaw acceleration comes from the moment arm difference of front/rear lateral forces plus any direct yaw moment ΔM_z created by differential wheel longitudinal forces.

3) Direct yaw moment from differential longitudinal wheel forces → equation (3)

$$\Delta M_z = \textstyle\sum_{i=1}^{4}(-1)^i\, F_{xi}\,\frac{l_s}{2}. \qquad (3)$$

Derivation / explanation

- F_{xi} is the longitudinal force at wheel i (signed: forward positive).
- The wheels form two sides (left / right). A left–right imbalance in the longitudinal forces produces a net yaw moment about the CG.
- If l_s denotes the track width (distance between left and right wheel centers), then each off-center longitudinal force produces a moment $F_{xi} \times (l_s/2)$ with sign depending on wheel side.
- Summing across all four wheels and applying alternating sign yields eq. (3):

$$\boxed{\Delta M_z = \sum_{i=1}^{4}(-1)^i\, F_{xi}\,\frac{l_s}{2}.}$$

This compactly encodes the net yaw torque produced by the hub motors through unequal longitudinal forces.

4) Wheel rotational dynamics → equation (4)

$$I_w \dot{w}_i = T_i - F_{xi} R_t. \qquad (4)$$

Derivation / meaning

- For wheel i: rotational dynamics are $I_w \dot{w}_i$ = torque applied by hub motor − torque due to tire longitudinal force.
- The torque resisting wheel rotation from the ground contact is $F_{xi} R_t$ (longitudinal force times rolling radius).
- Denote motor torque by T_i. Then

$$\boxed{I_w \dot{w}_i = T_i - F_{xi} R_t.}$$

This is the standard single-wheel rotational model.

5) Lateral tire force linearization → equation (5)

$$F_{yf} = \mu C_f \alpha_f, \qquad F_{yr} = \mu C_r \alpha_r.$$

Derivation / assumptions

- For small slip angles α the lateral tire force is often approximated linearly as $F_y \approx C_\alpha\, \alpha$, where C_α is the cornering stiffness.
- The paper includes a multiplicative factor μ (road-adhesion coefficient) to account for friction scaling:

$$\boxed{F_{yf} = \mu C_f \alpha_f, \quad F_{yr} = \mu C_r \alpha_r.} \qquad (5)$$

- This is a first-order (linear) tire model valid for small slips and when a single scalar friction μ scales the maximum lateral force.

6) Slip-angle geometry → equation (6)

$$\alpha_f = \beta + \frac{l_f}{v_x}\gamma - \delta_f, \qquad \alpha_r = \beta - \frac{l_r}{v_x}\gamma.$$

Derivation

- The slip angle at the front axle equals the angle between the wheel plane and the velocity vector at the front axle. For small angles:

- lateral velocity at front axle ≈ $v_x\beta + l_f\dot{\varphi}$ (the yaw component at the front axle adds $l_f\dot{\varphi}$).
- dividing by forward speed v_x gives the effective angle contribution $\beta + (l_f/v_x)\dot{\varphi}$.
- subtract the steering angle δ_f (wheel steering reduces slip) to get α_f.

- Doing the same at the rear (no steering) gives α_r.
- we get

$$\boxed{\alpha_f = \beta + \frac{l_f}{v_x}\gamma - \delta_f, \qquad \alpha_r = \beta - \frac{l_r}{v_x}\gamma.} \tag{6}$$

7) Steering actuator dynamics — derive equation (7)

Starting point / physical idea

- The steering assembly (steer-by-wire) has a rotational inertia J_w, viscous damping b_w, a motor that produces torque proportional to motor current i_m, and it is loaded by the lateral tire force at the front wheel which produces a moment around the steering axis because the lateral force acts at a point offset from the steering axis.

  Torque balance around the steering axis (scalar)

Write rotational dynamics for the steering subsystem:

$$J_w\ddot{\delta}_f + b_w\dot{\delta}_f = M_{\text{motor}} + M_{\text{tire}} + M_{\text{dist}},$$

were

- M_{motor} is motor torque applied to steering,
- M_{tire} is torque produced by the lateral tire force about the steering axis,
- M_{dist} lumps modeling errors (paper calls this d_5).

Motor torque model

The motor torque delivered to the steering column is

$$M_{\text{motor}} = \eta_m r_s k_m\, i_m,$$

where η_m is motor efficiency, r_s is steering ratio, k_m is motor constant and i_m motor current.

Tire-generated moment

- The lateral force at the front wheel is $F_{yf} = \mu C_f \alpha_f$ (eq. (5)).
- That lateral force acts at an effective lever arm equal to the sum of the aerodynamic offset l_p and the mechanical offset l_m, so the moment about the steering axis due to the lateral force is

$$M_{\text{tire}} = -\left(l_p + l_m\right) F_{yf}.$$

(Sign: chosen so that a positive lateral force producing a restoring moment appears with a minus sign in the steering dynamics as in the paper; the paper's final sign convention produces the form in (7).)

Now substitute the linear tire model and slip-angle

Use $F_{yf} = \mu C_f \alpha_f$ and $\alpha_f = \beta + \frac{l_f}{v_x}\gamma - \delta_f$ (eqs. (5) – (6)). Substitute:

$$M_{\text{tire}} = -\left(l_p + l_m\right) \mu C_f \left(\beta + \frac{l_f}{v_x}\gamma - \delta_f\right).$$

Collect terms and divide by J_w

Plug motor & tire moments into the torque balance and divide by J_w:

$$\ddot{\delta}_f + \frac{b_w}{J_w}\dot{\delta}_f = \frac{\eta_m r_s k_m}{J_w} i_m - \frac{\left(l_p + l_m\right) \mu C_f}{J_w}\left(\beta + \frac{l_f}{v_x}\gamma - \delta_f\right) + \frac{d_5}{J_w}.$$

Rearrange to isolate $\ddot{\delta}_f$:

$$\ddot{\delta}_f = -\frac{\mu C_f\left(l_p + l_m\right)}{J_w}\beta - \frac{\mu C_f l_f\left(l_p + l_m\right)}{J_w v_x}\gamma + \frac{\mu C_f\left(l_p + l_m\right)}{J_w}\delta_f - \frac{b_w}{J_w}\dot{\delta}_f + \frac{\eta_m r_s k_m}{J_w} i_m + \frac{d_5}{J_w}.$$

This matches the eq. (7) (they include d_5 as the lumped modeling error; the paper writes d_5 directly on the right — we can keep $\frac{d_5}{J_w}$ or absorb $1/J_w$ into the definition of d_5; the paper denotes it simply d_5).

Boxed final form (eq. (7))

$$\boxed{\ddot{\delta}_f = -\frac{\mu C_f(l_p+l_m)}{J_w}\,\beta - \frac{\mu C_f l_f(l_p+l_m)}{J_w v_x}\,\gamma + \frac{\mu C_f(l_p+l_m)}{J_w}\,\delta_f - \frac{b_w}{J_w}\,\dot{\delta}_f + \frac{\eta_m r_s k_m}{J_w}\,i_m + d_5.} \tag{7}$$

2.2. Polytopic Model Establishment for System Uncertainty

8) Parameter uncertainty parameterization — equation (8)

The paper models several uncertain scalar parameters as **nominal + bounded time-varying perturbation** (multiplicative form converted to additive on those scalars). The parameters and their decompositions (paper eq. (8)):

$$\begin{aligned} \mu(t) &= \mu_n + \Delta\mu\, N_1(t), \\ C_f(t) &= C_{fn} + \Delta C_f\, N_2(t), \\ C_r(t) &= C_{rn} + \Delta C_r\, N_3(t), \\ \frac{1}{m(t)} &= \frac{1}{m_n} + \frac{1}{\Delta m}\, N_4(t), \\ \frac{1}{I_z(t)} &= \frac{1}{I_{zn}} + \frac{1}{\Delta I_z}\, N_5(t), \end{aligned} \tag{8}$$

with each scalar uncertainty multiplier $N_i(t)$ satisfying $|N_i(t)| \leq 1$.

Nominal and half-range definitions

Defining the nominal value as the midpoint (or an equivalent harmonic form for reciprocals) and the uncertainty half-range as half the difference between max and min. Specifically (paper definitions):

- For μ, C_f, C_r:

  $\mu_n = 1/2\,(\mu_{\max} + \mu_{\min}),\quad \Delta\mu = 1/2\,(\mu_{\max} - \mu_{\min}), C_{fn} = 1/2\,\big(C_{f\max} + C_{f\min}\big),\quad \Delta C_f = 1/2\,\big(C_{f\max} - C_{f\min}\big),$

  and likewise, for C_r.

- For m and I_z the paper uses expressions involving harmonic means (since the uncertainty is presented for reciprocals $1/m$ and $1/I_z$). The paper gives

  $m_n = \frac{2m_{\min}m_{\max}}{m_{\min}+m_{\max}},\qquad \Delta m = \frac{2m_{\min}m_{\max}}{m_{\max}-m_{\min}}, I_{zn} = \frac{2I_{z\min}I_{z\max}}{I_{z\min}+I_{z\max}},\qquad \Delta I_z = \frac{2I_{z\min}I_{z\max}}{I_{z\max}-I_{z\min}}.$

(These forms are taken exactly from the paper — they parameterize the reciprocals so that $1/m = 1/m_n + (1/\Delta m)N_4(t)$ holds with $|N_4| \leq 1$.)

Interpretation

- This decomposition isolates the uncertain scalars into **bounded, time-varying multipliers** $N_i(t)$ with known magnitude bound (≤1), which is convenient to express the uncertain part of the system matrices as a factorized matrix.

9) Scheduling parameters (eq. (9)) — longitudinal velocity treated as time-varying

It treats longitudinal speed v_x as a time-varying scheduling variable and selects three scheduling functions:

$$\rho_1 = v_x, \qquad \rho_2 = \frac{1}{v_x}, \qquad \rho_3 = \frac{1}{v_x^2}. \tag{9}$$

Each ρ_i ranges over a known interval (lower/upper bounds determined from expected speed range). The paper writes:

$$\rho_1 = v_x \in \left[\, \underline{v}_x,\, \overline{v}_x \,\right], \quad \rho_2 = 1/v_x \in \left[\, 1/\overline{v}_x,\, 1/\underline{v}_x \,\right], \quad \rho_3 = 1/v_x^2 \in \left[\, 1/\overline{v}_x^2,\, 1/\underline{v}_x^2 \,\right].$$

This choice is convenient because many coefficients in the linearized matrices involve factors v_x, $1/v_x$ or $1/v_x^2$; representing them as independent scheduling parameters lets the model be expressed as an LPV (linear parameter varying) system and then embedded into a polytopic (convex hull) model over the vertex combinations of the three parameters.

10) State-space LPV / polytopic model — equation (10) and matrices

State / input / disturbance definitions

Collecting the plant states and inputs as

$$x = \begin{bmatrix} e_d & e_\varphi & \beta & \gamma & \delta_f & \dot{\delta}_f \end{bmatrix}^T, \qquad u = \begin{bmatrix} \Delta M_z \\ i_m \end{bmatrix}, \qquad d(t) = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \end{bmatrix}^T.$$

Using the linearized dynamic relations (including the steering dynamics derived above and the earlier lateral/yaw dynamics), the system can be written in LPV form:

$$\dot{x}(t) = A(\rho)x(t) + B(\rho)u(t) + B_d d(t),$$

and split into nominal + uncertain parts:

$$\boxed{\dot{x}(t) = \big(A_n(\rho) + \Delta A(\rho)\big)x(t) + (B_n + \Delta B)u(t) + \bar{d}(t).} \tag{10}$$

This is equation (10) in the paper.

How the physical terms map into $A_n(\rho)$, B_n, B_d

- Rows 1–2: the kinematic error dynamics (eq. (1)):
  - row 1 gives $\dot{e}_d = v_x e_\varphi + v_x \beta + d_1$ → entries in row 1: columns for e_φ and β equal ρ_1 (since $\rho_1 = v_x$).

- row 2 gives $\dot{e}_\varphi = \gamma + d_2$ → entry in row 2 column for γ is 1.

- Rows 3–4: lateral/yaw dynamics (substitute $F_{yf} = \mu C_f \alpha_f$, $F_{yr} = \mu C_r \alpha_r$ and slip angles). After substitution and arranging, you get coefficients proportional to combinations of μC_f, μC_r, l_f, l_r, m, I_z and factors $1/v_x$ or $1/v_x^2$, which are represented using ρ_2, ρ_3 (see the explicit matrix in the paper). For example (paper notation using $c_f = \mu_n C_{fn}$, $c_r = \mu_n C_{rn}$ for nominal values):
    - The β-coefficient in $\dot{\beta}$ is $\frac{c_f + c_r}{m_n} \rho_2$.
    - The γ-coefficient contains a ρ_3 factor (from $1/v_x^2$).
    - The input ΔM_z appears in $\dot{\gamma}$ scaled by $1/I_{zn}$.
      (See the explicit shown $A_n(\rho), B_n, B_d$ block in the paper for every entry.)

- Row 5–6: steering dynamics:
    - row 5 is $\dot{\delta}_f = \dot{\delta}_f$ → a standard integrator row (0 … 0 0 1).
    - row 6 comes directly from the steering dynamics derived in (7) and produces entries in the matrix depending on μC_f and terms with a $\rho_2 = 1/v_x$ factor for the γ term, as well as the motor current i_m input in B_n with coefficient $\eta_m r_s k_m / J_w$.

Explicit nominal matrices (paper gives the numeric symbolic block)

The paper lists $A_n(\rho), B_n, B_d$ explicitly (I reproduce the structure — full matrix is in the paper):

$$A_n(\rho) = \begin{bmatrix} 0 & \rho_1 & \rho_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{c_f + c_r}{m_n}\rho_2 & \dfrac{c_f l_f - c_r l_r}{m_n}\rho_3 & -\dfrac{c_f}{m_n}\rho_2 & 0 \\ 0 & 0 & \dfrac{c_f l_f - c_r l_r}{I_{zn}} & \dfrac{c_f l_f^2 + c_r l_r^2}{I_{zn}}\rho_2 & -\dfrac{c_f l_f}{I_{zn}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{c_f(l_p + l_m)}{J_w} & -\dfrac{c_f l_f (l_p + l_m)}{J_w}\rho_2 & \dfrac{c_f(l_p + l_m)}{J_w} & -\dfrac{b_w}{J_w} \end{bmatrix}$$

$$, B_n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{I_{zn}} & 0 \\ 0 & 0 \\ 0 & \frac{\eta_m r_s k_m}{J_w} \end{bmatrix}, \qquad B_d = \begin{bmatrix} I_{5\times5} \\ 0_{1\times5} \end{bmatrix}^T$$

Here the paper defines $c_f = \mu_n C_{fn}$ and $c_r = \mu_n C_{rn}$.

Uncertain parts and factorization

- The uncertain parts $\Delta A(\rho)$ and ΔB (due to $\Delta\mu, \Delta C_f, \Delta C_r$, variations in $1/m$ and $1/I_z$) are grouped and written in the factorized form

$$[\Delta A(\rho)\ \Delta B] = H\,\Lambda\,[L_1(\rho)\ L_2], \tag{11}$$

with Λ diagonal consisting of a few repeated uncertain multipliers $N_i(t)$, and specific matrices $H, L_1(\rho), L_2$. This factorization is used later to apply the small-gain / S-procedure style LMI (Lemma 1) when deriving robust LMIs.

Polytopic embedding

Because the scheduling vector $\rho = (\rho_1, \rho_2, \rho_3)$ ranges over a box (product of intervals), the LPV system can be embedded into a **polytopic** model with 8 vertices (all combinations of lower/upper values for each scalar). The paper writes this as

$$\dot{x}(t) = \sum_{i=1}^{8} \alpha_i\,(\rho, t)\left((A_{ni} + \Delta A_i)x + (B_{ni} + \Delta B_i)u + \bar{d}(t)\right), \tag{12}$$

with barycentric weights $\alpha_i(\rho, t) \geq 0, \sum_i \alpha_i = 1$, and explicit $\alpha_i(\rho, t)$ formulas given in (13). This is a standard convex decomposition used to reduce synthesis to a finite set of vertex conditions.

$$x = \begin{bmatrix} e_d \\ e_\varphi \\ \beta \\ \gamma \\ \delta_f \\ \dot{\delta}_f \end{bmatrix}, \qquad u = \begin{bmatrix} \Delta M_z \\ i_m \end{bmatrix}, \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}.$$

I’ll define the nominal composite cornering stiffnesses used by the paper:

$$c_f \equiv \mu_n C_{fn}, \qquad c_r \equiv \mu_n C_{rn},$$

and nominal reciprocals $1/m_n,\ 1/I_{zn}$ etc. Also recall the scheduling parameters

$$\rho_1 = v_x, \quad \rho_2 = \frac{1}{v_x}, \quad \rho_3 = \frac{1}{v_x^2}.$$

i) — Intermediate substitutions (explicit)

Start from the linear tire models and slip-angle geometry:

$$F_{yf} = \mu\,C_f\,\alpha_f, \qquad F_{yr} = \mu\,C_r\,\alpha_r, \alpha_f = \beta + \tfrac{l_f}{v_x}\gamma - \delta_f, \qquad \alpha_r = \beta - \tfrac{l_r}{v_x}\gamma.$$

Take nominal (nom) values for multiplicative constants: use $c_f = \mu_n C_{fn}$, $c_r = \mu_n C_{rn}$. For the **nominal** expressions we substitute $\mu \to \mu_n,\ C_f \to C_{fn},\ C_r \to C_{rn}$. Then

$$\begin{aligned} F_{yf} &\approx c_f\left(\beta + \frac{l_f}{v_x}\gamma - \delta_f\right), \\ F_{yr} &\approx c_r\left(\beta - \frac{l_r}{v_x}\gamma\right). \end{aligned}$$

Compute the combinations that appear in the lateral/yaw dynamics:

1. Sum of lateral forces:

$$\begin{aligned} F_{yf} + F_{yr} &= c_f\left(\beta + \frac{l_f}{v_x}\gamma - \delta_f\right) + c_r\left(\beta - \frac{l_r}{v_x}\gamma\right) \\ &= (c_f + c_r)\beta + \left(\frac{c_f l_f - c_r l_r}{v_x}\right)\gamma - c_f\delta_f. \end{aligned}$$

2. Moment arm combination (front minus rear moments about CG):

$$\begin{aligned} l_f F_{yf} - l_r F_{yr} &= l_f c_f\left(\beta + \frac{l_f}{v_x}\gamma - \delta_f\right) - l_r c_r\left(\beta - \frac{l_r}{v_x}\gamma\right) \\ &= (l_f c_f - l_r c_r)\beta + \left(\frac{l_f^2 c_f + l_r^2 c_r}{v_x}\right)\gamma - l_f c_f \delta_f. \end{aligned}$$

(These two algebraic results are the core building blocks for rows 3 and 4.)

ii) — Row-by-row derivations

Row 1 — kinematic lateral error

From eq. (1): $\dot{e}_d = v_x e_\varphi + v_x\beta + d_1$.

Thus row-1 of $A_n(\rho)$ is

$$\text{row}_1 = [0 \quad \rho_1 \quad \rho_1 \quad 0 \quad 0 \quad 0],$$

and the disturbance d_1 enters through B_d (see B_d below). There is no input u term in row 1, so corresponding B_n row is zeros.

Row 2 — yaw-angle error

From eq. (1): $\dot{e}_\varphi = \gamma + d_2$.

So row-2 of $A_n(\rho)$ is

$$\text{row}_2 = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0],$$

again, no direct input term.

Row 3 — sideslip rate $\dot{\beta}$

From eq. (2):

$$\dot{\beta} = \frac{1}{m}\frac{1}{v_x}\left(F_{yf} + F_{yr}\right) + d_3.$$

Substitute the nominal expression for $F_{yf} + F_{yr}$ and pull out the dependence on v_x:

$$\begin{aligned}
\dot{\beta} \quad &= \frac{1}{m_n}\cdot\frac{1}{v_x}\left[\left(c_f + c_r\right)\beta + \left(\frac{c_f l_f - c_r l_r}{v_x}\right)\gamma - c_f\delta_f\right] + d_3 \\
&= \underbrace{\frac{c_f + c_r}{m_n}}_{\text{call } a_{\beta\beta}} \ \rho_2 \ \beta \ + \ \underbrace{\frac{c_f l_f - c_r l_r}{m_n}}_{\text{call } a_{\beta\gamma}} \ \rho_3 \ \gamma \ - \ \underbrace{\frac{c_f}{m_n}}_{\text{call } a_{\beta\delta}} \ \rho_2 \ \delta_f \ + \ d_3.
\end{aligned}$$

So row-3 of $A_n(\rho)$ (coefficient vector for states $[e_d, e_\varphi, \beta, \gamma, \delta_f, \dot{\delta}_f]$) is

$$\text{row}_3 = \left[0,\ 0,\ \frac{c_f + c_r}{m_n}\rho_2,\ \frac{c_f l_f - c_r l_r}{m_n}\rho_3,\ -\frac{c_f}{m_n}\rho_2,\ 0\right].$$

No direct input u (so corresponding row of B_n is zeros). Disturbance d_3 enters row 3 (see B_d).

Row 4 — yaw-rate acceleration $\dot{\gamma}$

From eq. (2):

$$\dot{\gamma} = \frac{1}{I_z}\left(l_f F_{yf} - l_r F_{yr}\right) + \frac{1}{I_z}\Delta M_z + d_4.$$

Substitute the nominal expression for $l_f F_{yf} - l_r F_{yr}$:

$$\begin{aligned}
\dot{\gamma} \quad &= \frac{1}{I_{zn}}\left[\left(l_f c_f - l_r c_r\right)\beta + \left(\frac{l_f^2 c_f + l_r^2 c_r}{v_x}\right)\gamma - l_f c_f \delta_f\right] + \frac{1}{I_{zn}}\Delta M_z + d_4 \\
&= \underbrace{\frac{l_f c_f - l_r c_r}{I_{zn}}}_{a_{\gamma\beta}} \ \beta \ + \ \underbrace{\frac{l_f^2 c_f + l_r^2 c_r}{I_{zn}}}_{a_{\gamma\gamma}} \ \rho_2 \ \gamma \ - \ \underbrace{\frac{l_f c_f}{I_{zn}}}_{a_{\gamma\delta}} \ \delta_f \ + \ \underbrace{\frac{1}{I_{zn}}}_{b_{\gamma,\Delta M_z}} \ \Delta M_z \ + \ d_4.
\end{aligned}$$

So row-4 of $A_n(\rho)$ is

$$\text{row}_4 = \left[0,\ 0,\ \frac{l_f c_f - l_r c_r}{I_{zn}},\ \frac{l_f^2 c_f + l_r^2 c_r}{I_{zn}}\rho_2,\ -\frac{l_f c_f}{I_{zn}},\ 0\right].$$

And the input matrix B_n row-4 (for $u = [\Delta M_z \ \ i_m]^T$) is

$$\text{row}_4(B_n) = \left[\frac{1}{I_{zn}} \quad 0\right].$$

The disturbance d_4 enters row 4 (see B_d).

Row 5 — steering angle integrator

By definition $\dot{\delta}_f = \dot{\delta}_f$ so row-5 is the integrator that picks $\dot{\delta}_f$ state:

$$\mathrm{row}_5 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1],$$

no inputs and no disturbance directly (paper uses zero in B_n row 5 and zero in B_d row 5).

Row 6 — steering actuator acceleration $\ddot{\delta}_f$

From eq. (7) (we derived in previous page):

$$\ddot{\delta}_f = -\frac{\mu C_f(l_p + l_m)}{J_w}\beta - \frac{\mu C_f l_f(l_p + l_m)}{J_w v_x}\gamma + \frac{\mu C_f(l_p + l_m)}{J_w}\delta_f - \frac{b_w}{J_w}\dot{\delta}_f + \frac{\eta_m r_s k_m}{J_w} i_m + d_5.$$

Using nominal $c_f = \mu_n C_{fn}$ we write each coefficient:

$$\begin{aligned}
a_{\delta\beta} &= -\frac{c_f(l_p + l_m)}{J_w}, \\
a_{\delta\gamma} &= -\frac{c_f l_f(l_p + l_m)}{J_w}\rho_2, \\
a_{\delta\delta} &= \frac{c_f(l_p + l_m)}{J_w}, \\
a_{\delta\dot{\delta}} &= -\frac{b_w}{J_w}, \\
b_{\delta,i_m} &= \frac{\eta_m r_s k_m}{J_w}.
\end{aligned}$$

So row-6 of $A_n(\rho)$ is

$$\mathrm{row}_6 = \left[0,\ 0,\ -\frac{c_f(l_p + l_m)}{J_w},\ -\frac{c_f l_f(l_p + l_m)}{J_w}\rho_2,\ \frac{c_f(l_p + l_m)}{J_w},\ -\frac{b_w}{J_w}\right],$$

and row-6 of B_n (for $u = [\Delta M_z\ \ i_m]^T$) is

$$\mathrm{row}_6(B_n) = \left[0 \quad \frac{\eta_m r_s k_m}{J_w}\right].$$

Disturbance d_5 enters row 6 (see B_d).

iii) — Assemble $A_n(\rho)$, B_n, B_d explicitly

Collecting all rows, the nominal state matrix $A_n(\rho)$ (6×6) is:

$$A_n(\rho) = \begin{bmatrix} 0 & \rho_1 & \rho_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{c_f + c_r}{m_n}\rho_2 & \dfrac{c_f l_f - c_r l_r}{m_n}\rho_3 & -\dfrac{c_f}{m_n}\rho_2 & 0 \\ 0 & 0 & \dfrac{l_f c_f - l_r c_r}{I_{zn}} & \dfrac{l_f^2 c_f + l_r^2 c_r}{I_{zn}}\rho_2 & -\dfrac{l_f c_f}{I_{zn}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{c_f(l_p + l_m)}{J_w} & -\dfrac{c_f l_f (l_p + l_m)}{J_w}\rho_2 & \dfrac{c_f(l_p + l_m)}{J_w} & -\dfrac{b_w}{J_w} \end{bmatrix}.$$

The nominal input matrix B_n (6×2) is:

$$B_n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{1}{I_{zn}} & 0 \\ 0 & 0 \\ 0 & \dfrac{\eta_m r_s k_m}{J_w} \end{bmatrix}.$$

Finally, the disturbance matrix B_d maps $d = [d_1, \dots, d_5]^T$ into the states. From the paper's definitions (disturbances d_1 through d_5 appear in rows 1,2,3,4,6 respectively), we have:

$$B_d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(Equivalently, the state equations include $+d_1$ in $\dot{e}_d$, $+d_2$ in $\dot{e}_\varphi$, $+d_3$ in $\dot{\beta}$, $+d_4$ in $\dot{\gamma}$, and $+d_5$ in $\ddot{\delta}_f$.)

11) — Factorization (equation (11))

The uncertain parts of the plant are written in the compact factorized form

$$[\Delta A(\rho) \quad \Delta B] = H\,\Lambda\,[L_1(\rho) \quad L_2],$$

with a diagonal uncertainty matrix Λ that contains (repeated) bounded scalar uncertainty multipliers.

i) — The matrices H and Λ

Choosing a 3×3 diagonal Λ (same scalar repeated) and a tall H that selects which state/input rows are affected:

$$\Lambda = \mathrm{diag}\{N_1,\ N_1,\ N_1\}, \qquad |N_1| \le 1, H = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Interpretation: H places the uncertain combinations into the β-equation row (state 3), the γ-equation row (state 4), and the $\ddot{\delta}_f$ row (state 6).

ii) — The $L_1(\rho)$ and L_2 blocks (explicit)

The paper gives the following explicit 3×6 block $L_1(\rho)$ (rows correspond to the 3 uncertain channels that H will map into states 3,4,6):

$$L_1(\rho) = \begin{bmatrix} 0 & 0 & (\phi_1 + \phi_2)\,\rho_2 & (\phi_1 l_f - \phi_2 l_r)\,\rho_3 & -\phi_1\,\rho_2 & 0 \\ 0 & 0 & \phi_3 l_f - \phi_4 l_r & (\phi_3 l_f^2 + \phi_4 l_r^2)\,\rho_2 & -\phi_3 l_f & 0 \\ 0 & 0 & -\dfrac{\Delta c_f\,(l_p + l_m)}{J_w} & -\dfrac{\Delta c_f l_f (l_p + l_m)}{J_w}\,\rho_2 & \dfrac{\Delta c_f (l_p + l_m)}{J_w} & 0 \end{bmatrix}.$$

The uncertain part of the input matrix is encoded by

$$L_2 = \begin{bmatrix} 0 & 0 \\ \dfrac{1}{\Delta I_z} & 0 \\ 0 & 0 \end{bmatrix}.$$

Putting these together with H and Λ produces the full $[\Delta A(\rho)\ \Delta B]$.

iii) Definitions of the composite uncertainty coefficients

The short definitions uses for the composite uncertain coefficients are:

- composite cornering-stiffness variations:

$$\Delta c_f = \mu_n \Delta C_f + \Delta\mu\, C_{fn} + \Delta\mu\, \Delta C_f, \Delta c_r = \mu_n \Delta C_r + \Delta\mu\, C_{rn} + \Delta\mu\, \Delta C_r.$$

- the ϕ-coefficients (they combine cornering-stiffness uncertainty and $1/m$, $1/I_z$ uncertainty) are

$$\phi_1 = \frac{c_f + \Delta c_f}{\Delta m} + \frac{\Delta c_f}{m_n}, \qquad \phi_2 = \frac{c_r + \Delta c_r}{\Delta m} + \frac{\Delta c_r}{m_n}, \phi_3 = \frac{c_f + \Delta c_f}{\Delta I_z} + \frac{\Delta c_f}{I_{zn}}, \qquad \phi_4 = \frac{c_r + \Delta c_r}{\Delta I_z} + \frac{\Delta c_r}{I_{zn}}.$$

Here $c_f = \mu_n C_{fn}$, $c_r = \mu_n C_{rn}$, and $m_n, I_{zn}, \Delta m, \Delta I_z$ are the nominal / half-range reciprocal parameters. These ϕ_i collect how cornering-stiffness and mass/inertia reciprocal uncertainties jointly perturb the linearized A entries.

iv) How to read this factorization (what each term changes)

- The first row of $L_1(\rho)$ (mapped by H into the $\dot{\beta}$ row) contains the coefficients that multiply the state vector $\left[e_d, e_\varphi, \beta, \gamma, \delta_f, \dot{\delta}_f\right]$ to produce the uncertain additive contributions to $\dot{\beta}$. Concretely, uncertainties in cornering stiffness and reciprocal mass produce additive terms proportional to $(\phi_1 + \phi_2)\rho_2$ (for β), $\left(\phi_1 l_f - \phi_2 l_r\right)\rho_3$ (for γ), and $-\phi_1 \rho_2$ (for δ_f).
- The second row of $L_1(\rho)$ (mapped into the $\dot{\gamma}$ row) gives uncertain contributions to $\dot{\gamma}$ from the same parametric variations (with different moment-arm combinations). The second row also produces the uncertain additive that multiplies the input ΔM_z via L_2 (the $1/\Delta I_z$ entry in L_2 together with H, Λ produces uncertainty in the ΔM_z input channel mapped into $\dot{\gamma}$).
- The third row of $L_1(\rho)$ (mapped into the steering-actuator acceleration row $\ddot{\delta}_f$) encodes how uncertainty in the *front* cornering stiffness (Δc_f) perturbs the steering dynamics coefficients (terms multiplying β, γ, δ_f in row 6). The entries contain $\Delta c_f/(J_w)$ and the same ρ_2 dependence for the γ-term as in the nominal row.

Finally, *remaining parts of the uncertainties* that are not represented in this factorized structure are lumped into the disturbance vector and treated as part of the overall disturbance $\bar{d}(t)$.

v) Expand the factorization into ΔA and ΔB

The paper gives (eq. (11))

$$[\Delta A(\rho)\ \Delta B] = H\, \Lambda\, [L_1(\rho)\ \ L_2].$$

Dimensions / shapes (for clarity)

- H is 6×3.
- $\Lambda = \mathrm{diag}\{N_1, N_1, N_1\}$ is 3×3 with scalar unknown multiplier(s) $N_1(t)$, $|N_1| \leq 1$.
- $L_1(\rho)$ is 3×6, L_2 is 3×2.
- So, product is $6 \times (6 + 2) = 6 \times 8$ which partitions as $\Delta A(6 \times 6)$ and $\Delta B(6 \times 2)$.

Because Λ is diagonal with the same scalar N_1 on the diagonal, we can simplify algebraically:

$$\Lambda = N_1 I_3 \quad \Rightarrow \quad H\Lambda = N_1 H.$$

Therefore

$$\boxed{\Delta A(\rho) = N_1\, H\, L_1(\rho), \qquad \Delta B = N_1\, H\, L_2.}$$

Now write H and L_1, L_2 explicitly (copying the paper's structure):

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, L_1(\rho) = \begin{bmatrix} 0 & 0 & (\phi_1 + \phi_2)\rho_2 & (\phi_1 l_f - \phi_2 l_r)\rho_3 & -\phi_1 \rho_2 & 0 \\ 0 & 0 & \phi_3 l_f - \phi_4 l_r & (\phi_3 l_f^2 + \phi_4 l_r^2)\rho_2 & -\phi_3 l_f & 0 \\ 0 & 0 & -\frac{\Delta c_f (l_p + l_m)}{J_w} & -\frac{\Delta c_f l_f (l_p + l_m)}{J_w}\rho_2 & \frac{\Delta c_f (l_p + l_m)}{J_w} & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 \\ \dfrac{1}{\Delta I_z} & 0 \\ 0 & 0 \end{bmatrix}.$$

Multiply H by each row of L_1 (with scalar N_1) — because of H's structure the result simply *places* the 1st row of L_1 into row 3 of ΔA, the 2nd row of L_1 into row 4 of ΔA, and the 3rd row of L_1 into row 6 of ΔA. All other rows of ΔA are zero. Same for ΔB: the 2nd row of L_2 placed into row 4 of ΔB.

So, the explicit expanded matrices are:

$\Delta A(\rho)$ **(6×6): only rows 3,4,6 nonzero**

$$\Delta A(\rho) \;=\; N_1 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\phi_1 + \phi_2)\rho_2 & (\phi_1 l_f - \phi_2 l_r)\rho_3 & -\phi_1 \rho_2 & 0 \\ 0 & 0 & \phi_3 l_f - \phi_4 l_r & (\phi_3 l_f^2 + \phi_4 l_r^2)\rho_2 & -\phi_3 l_f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{\Delta c_f (l_p + l_m)}{J_w} & -\dfrac{\Delta c_f l_f (l_p + l_m)}{J_w}\rho_2 & \dfrac{\Delta c_f (l_p + l_m)}{J_w} & 0 \end{bmatrix}.$$

ΔB **(6×2): only row 4 may be nonzero**

$$\Delta B \;=\; N_1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{1}{\Delta I_z} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

vi) Derive Δc_f, Δc_r and the ϕ_i coefficients

These are used to collect parameter uncertainty algebraically. Start from uncertain scalar decompositions (paper eq. (8)):

$$\mu = \mu_n + \Delta\mu\, N_1(t), \quad C_f = C_{fn} + \Delta C_f\, N_2(t), \quad C_r = C_{rn} + \Delta C_r\, N_3(t).$$

The *nominal composite* cornering stiffnesses are

$$c_f = \mu_n C_{fn}, \qquad c_r = \mu_n C_{rn}.$$

When both μ and C_f vary, the full perturbed front cornering stiffness is

$$\mu C_f = (\mu_n + \Delta\mu N_1)(C_{fn} + \Delta C_f N_2) = \mu_n C_{fn} + \mu_n \Delta C_f N_2 + \Delta\mu C_{fn} N_1 + \Delta\mu \Delta C_f N_1 N_2.$$

It lumps the time-varying scalars into a single uncertain multiplier (they reorder which N_i multiplies what in the factorization). For the *amplitude* (half-range) they define the deterministic scalar Δc_f equal to the maximum *additive* deviation from nominal (including cross-term).

$$\boxed{\Delta c_f = \mu_n \Delta C_f + \Delta\mu\, C_{fn} + \Delta\mu\, \Delta C_f.}$$

Likewise for the rear:

$$\boxed{\Delta c_r = \mu_n \Delta C_r + \Delta\mu\, C_{rn} + \Delta\mu\, \Delta C_r.}$$

(These $\Delta c_f, \Delta c_r$ are deterministic half-range magnitudes — they multiply the bounded scalar(s) used in the factorization.)

Now the ϕ-terms

These collect how the cornering stiffness uncertainty and reciprocal mass / reciprocal inertia uncertainty combine to perturb entries in ΔA.

$$\boxed{\phi_1 = \frac{c_f + \Delta c_f}{\Delta m} + \frac{\Delta c_f}{m_n}}, \quad \boxed{\phi_2 = \frac{c_r + \Delta c_r}{\Delta m} + \frac{\Delta c_r}{m_n}}, \quad \boxed{\phi_3 = \frac{c_f + \Delta c_f}{\Delta I_z} + \frac{\Delta c_f}{I_{zn}}, \qquad \phi_4 = \frac{c_r + \Delta c_r}{\Delta I_z} + \frac{\Delta c_r}{I_{zn}}.}$$

How these arise?

- Entries in $\dot{\beta}$ (row 3) include terms like $(c_f + c_r)/(m) \cdot (1/v_x)$. When c_f, c_r and $1/m$ are uncertain, the additive perturbation to that product can be approximated / bounded by terms proportional to combinations of $\Delta c_f, \Delta c_r, \Delta(1/m)$. If the paper uses the reciprocal-parameter decomposition

  $\frac{1}{m} = \frac{1}{m_n} + \frac{1}{\Delta m} N_4(t),$

  then the perturbation piece contributed to the coefficient multiplying β is approximately

$$\underbrace{\frac{c_f+c_r}{\Delta m}}_{\text{nominal part times } 1/\Delta m} N_4 + \underbrace{\frac{\Delta c_f+\Delta c_r}{m_n}}_{\text{stiffness half-range / nominal recip}} N_{\text{sth}}$$

The paper collects these into $(\phi_1+\phi_2)\rho_2$ etc where ϕ_1, ϕ_2 are defined as above so that the factorization becomes a linear combination of *known deterministic* matrices times bounded scalars.

- Similarly for the yaw row (row 4) the terms scale with $1/I_z$ and the ϕ_3, ϕ_4 entries collect those combinations.

You can use these ϕ_i directly in $L_1(\rho)$ as given.

viii) How $L_1(\rho)$ is evaluated at the 8 vertices: L_{1i} (and L_{2i})

The LPV scheduling vector is $\rho = [\rho_1, \rho_2, \rho_3]^T = [v_x,\ 1/v_x,\ 1/v_x^2]^T$. Each ρ_k lies in an interval $\left[\underline{\rho}_k, \overline{\rho}_k\right]$ (paper uses ρ_k and $\bar{\rho}_k$ notation for lower/upper).

To form a polytopic model we evaluate the parameter-dependent matrices at the 8 corner combinations (vertices) of the box determined by the three intervals.

Define the 8 vertex parameter triples (paper's notation — matching image):

$$(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3),\ (\rho_1, \bar{\rho}_2, \bar{\rho}_3),\ (\bar{\rho}_1, {\rho_2}', \bar{\rho}_3),\ (\bar{\rho}_1, \bar{\rho}_2, \rho_3),$$
$$(\rho_1, {\rho_2}', \bar{\rho}_3),\ (\rho_1, \bar{\rho}_2, \rho_3),\ (\bar{\rho}_1, {\rho_2}', \rho_3),\ (\rho_1, {\rho_2}', \rho_3),$$

(Where paper uses shorthand ${\rho_2}'$ etc to denote the alternate end of the interval — see your page; the exact order is not crucial as long as you consistently pair α_i with the same vertex.)

For each vertex $i \in \{1, \dots, 8\}$ compute

$$L_{1i} = L_1\big(\rho^{(i)}\big), \qquad L_{2i} = L_2.$$

So, the 8 matrices L_{1i} are formed by substituting each vertex's ρ_2, ρ_3 values into the symbolic expressions in $L_1(\rho)$. Concretely, for vertex i you substitute $\rho_2 = \rho_2^{(i)}, \rho_3 = \rho_3^{(i)}$ into the three rows of $L_1(\cdot)$.

Example: at vertex where $\rho_2 = \underline{\rho}_2, \rho_3 = \underline{\rho}_3$, the first row of L_{1i} becomes

$$\left[0,\ 0,\ (\phi_1+\phi_2)\underline{\rho}_2,\ \left(\phi_1 l_f - \phi_2 l_r\right)\underline{\rho}_3,\ -\phi_1\underline{\rho}_2,\ 0\right].$$

Repeat for all 8 vertices — the paper lists the mapping $L_{11} = L_1(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3)$, $L_{12} = L_1(\rho_1, \bar{\rho}_2, \bar{\rho}_3)$, …, $L_{18} = L_1(\rho_1, {\rho_2}', \rho_3)$.

L_2 is constant (no ρ dependence) so $L_{2i} = L_2$ for all i.

ix) Barycentric coordinates $\alpha_i(\rho, t)$ and $\Delta\rho$

The paper uses barycentric coordinates over the 3D parameter box to write any ρ as a convex combination of the 8 vertices. The weights are

$$\alpha_i(\rho, t) = \frac{\prod_{k=1}^{3}\left|\rho_k - \tilde{\rho}_k^{(i)}\right|}{\Delta\rho}, \tag{13}$$

where $\tilde{\rho}_k^{(i)}$ is the value of the k-th parameter at the *opposite* vertex coordinate used to define each α_i in the paper's particular pattern (the page shows the explicit absolute-differences product form for each α_i). The paper writes them concretely as (copied style):

$$\begin{aligned}
\alpha_1(\rho, t) &= \frac{|\rho_1 - \bar{\rho}_1|\,|\rho_2 - \bar{\rho}_2|\,|\rho_3 - \bar{\rho}_3|}{\Delta\rho}, \\
\alpha_2(\rho, t) &= \frac{|\rho_1 - \rho_1|\,|\rho_2 - \bar{\rho}_2|\,|\rho_3 - \bar{\rho}_3|}{\Delta\rho}, \\
&\vdots \\
\alpha_8(\rho, t) &= \frac{|\rho_1 - \rho_1|\,|\rho_2 - \rho_2|\,|\rho_3 - \rho_3|}{\Delta\rho},
\end{aligned}$$

with normalization denominator

$$\Delta\rho = |\bar{\rho}_1 - \rho_1|\,|\bar{\rho}_2 - \rho_2|\,|\bar{\rho}_3 - \rho_3|.$$

These α_i satisfy $\alpha_i \geq 0$ and $\sum_{i=1}^{8} \alpha_i = 1$. They are used to write the LPV system as the convex combination in eq. (12):

$$\dot{x}(t) = \sum_{i=1}^{8} \alpha_i(\rho, t)\Big((A_{ni} + \Delta A_i)x(t) + (B_{ni} + \Delta B_i)u(t) + \bar{d}(t)\Big). \tag{14}$$

x) Vertex mapping for A_{ni} (explicit list)

The paper defines each vertex nominal A_{ni} by substituting the vertex parameter triple into $A_n(\rho)$. Using their indexing:

$$\begin{aligned}
A_{n1} &= A_n(\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3), \\
A_{n2} &= A_n(\rho_1, \bar{\rho}_2, \bar{\rho}_3), \\
A_{n3} &= A_n(\bar{\rho}_1, {\rho_2}', \bar{\rho}_3), \\
A_{n4} &= A_n(\bar{\rho}_1, \bar{\rho}_2, \rho_3), \\
A_{n5} &= A_n(\rho_1, {\rho_2}', \bar{\rho}_3), \\
A_{n6} &= A_n(\rho_1, \bar{\rho}_2, \rho_3), \\
A_{n7} &= A_n(\bar{\rho}_1, {\rho_2}', \rho_3), \\
A_{n8} &= A_n(\rho_1, {\rho_2}', \rho_3).
\end{aligned}$$

Same for $L_{1i} = L_1\left(\rho^{(i)}\right)$ and $L_{2i} = L_2$. The input matrices $B_{ni} = B_n$ are constant for all vertices in the paper.

2.3 Control Problem Formulation

For the trajectory tracking control layer of the FWIA EV, the full state vector is

$$x = \left[e_d \ e_\varphi \ \beta \ \gamma \ \delta_f \ \dot{\delta}_f\right]^T.$$

Ideally, state-feedback control requires that **all six states** be measured accurately. However, in practical FWIA EVs:

- e_d and e_φ can be measured using low-cost sensors (e.g., GPS + IMU, camera-based localization).
- The yaw rate γ can be measured directly using onboard gyroscopes or indirectly reconstructed from inertial sensors.
- The front steering angle δ_f and its rate $\dot{\delta}_f$ are available from the steer-by-wire motor encoders.
- **But the sideslip angle β is difficult to measure accurately and cheaply**, because:
    - It requires expensive high-grade IMUs or GPS-RTK slip-angle estimation,
    - Or sophisticated nonlinear observers,
    - And is highly sensitive to noise and tire parameter variation.

Because β is difficult to measure with inexpensive sensors, **a full state-feedback controller is not practical**.

⇒ Therefore, the paper adopts a robust output-feedback control approach.

Only the following **measurable states** are used as output:

$$y(t) = C_1 x(t),$$

Where,

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, the **measured outputs** are:

$$y(t) = \left[e_d \ e_\varphi \ \gamma \ \delta_f \ \dot{\delta}_f\right]^T.$$

Meanwhile, for performance evaluation the paper treats **all states** as performance output:

$$z(t) = C_2 x(t), \qquad C_2 = I_{6\times 6}.$$

LPV Polytopic Output-Feedback Plant Model

The plant model including:

- Time-varying parameters $\rho = [\rho_1, \rho_2, \rho_3]^T$,
- Polytopic blending coefficients $\alpha_i(\rho, t)$,
- Nominal + uncertain matrices at each vertex.

The closed-form plant model is:

$$\dot{x}(t) = \sum_{i=1}^{8} \alpha_i\,(\rho, t)\big[(A_{ni} + \Delta A_i)x(t) + (B_{ni} + \Delta B_i)u(t) + \bar{d}(t)\big],$$

with measurement and evaluation outputs:

$$y(t) = C_1 x(t), \qquad z(t) = C_2 x(t).$$

This is the final LPV/polytopic model on which the output-feedback H∞ controller will be designed.

Control Objective (H∞ Performance Requirement)

The objective of the output-feedback controller is to ensure:

1. **Robust asymptotic stability** under parameter uncertainty and disturbances.
2. **Guaranteed H∞ disturbance attenuation**, i.e. the tracking error energy is bounded relative to the disturbance energy.

The H∞ performance inequality is:

$$\int_0^t z^T\,(\tau) z(\tau)\, d\tau \;\le\; \upsilon^2 \int_0^t \bar{d}^T\,(\tau)\bar{d}(\tau)\, d\tau, \qquad (15)$$

where $\upsilon > 0$ is the desired attenuation level.

Control Input / Actuator Constraints

Both control inputs are:

1. Steering motor current i_m,
2. Direct yaw-moment generating torque difference ΔM_z.

Let

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \Delta M_z(t) \\ i_m(t) \end{bmatrix}.$$

The actuators must satisfy the physical constraints:

$$|u_s(t)| \le u_{max,s}, \qquad s = 1,2, \tag{16}$$

where:

- $u_{\max,1}$ is the maximum allowable yaw-moment command (determined by motor torque limits and tire–road conditions),
- $u_{\max,2}$ is the maximum steering-motor current limit.

These constraints must be enforced in the LMI synthesis.

In summary:

The controller must simultaneously ensure:

- robustness,
- trajectory tracking performance,
- disturbance attenuation,
- and actuator safety limits.

Why Dynamic Output-Feedback?

The paper explains that dynamic output-feedback is chosen because:

- Not all states (especially sideslip angle β) can be measured,
- Static output-feedback is too restrictive and often infeasible,
- Dynamic output-feedback introduces controller internal state x_c allowing full LMI-based H∞ synthesis.

This is the motivation leading directly into the controller structure

Formulation (Reproduced)

For trajectory tracking control of the FWIA EV, the full state vector

$$x = \begin{bmatrix} e_d & e_\varphi & \beta & \gamma & \delta_f & \dot{\delta}_f \end{bmatrix}^T$$

is ideally required for feedback. The states e_d and e_φ can be measured using inexpensive positioning equipment, while the yaw rate γ can be obtained from onboard inertial sensors. The steering angle δ_f and its rate $\dot{\delta}_f$ are available from the steer-by-wire motor sensors. However, the sideslip angle β is difficult to measure accurately with low-cost sensors. Therefore, a state-feedback controller is impractical, and an output-feedback-based control scheme is more suitable for FWIA EVs.

Accordingly, the measurable states $e_d, e_\varphi, \gamma, \delta_f, \dot{\delta}_f$ are selected as the output

$$y(t) = C_1 x(t), \qquad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and all states are used as the evaluation output

$$z(t) = C_2 x(t), \qquad C_2 = I_{6\times 6}.$$

Using the polytopic LPV model and the parameter-dependent matrices, the control-oriented FWIA EV model becomes

$$\dot{x}(t) = \sum_{i=1}^{8} \alpha_i\,(\rho, t)\big[(A_{ni} + \varDelta A_i)x(t) + (B_{ni} + \varDelta B_i)u(t) + \bar{d}(t)\big],$$

with outputs $y(t)$ and $z(t)$ as defined above.

The control objective is to design a robust H∞ dynamic output-feedback controller such that the closed-loop system is robustly asymptotically stable under parameter uncertainty and external disturbance, and satisfies the H∞ disturbance attenuation condition

$$\int_0^t z^T\,(\tau) z(\tau)\,d\tau \le v^2 \int_0^t \bar{d}^T\,(\tau)\bar{d}(\tau)\,d\tau,$$

where $v > 0$ is the prescribed attenuation level.
Additionally, actuator safety requires the control inputs

$$|u_s(t)| \le u_{\max,s}, \qquad s = 1{,}2,$$

where $u_{\max,s}$ reflects motor torque and road-friction limits.

3. Robust H∞ Output-Feedback Trajectory Tracking Control (Reproduced)

To enhance the trajectory tracking performance under parameter uncertainty and external disturbances, the study employs a **robust H∞ dynamic output-feedback controller** of the form:

$$\begin{cases} \dot{x}_c(t) = \sum_{i=1}^{8} \sum_{j=1}^{8} \alpha_i(\rho, t)\alpha_j(\rho, t)\, A_{cij}\, x_c(t) + \sum_{j=1}^{8} \alpha_j(\rho, t)\, B_{cj}\, y(t), \\ u(t) = \sum_{j=1}^{8} \alpha_j(\rho, t)\, C_{cj}\, x_c(t) + D_c\, y(t), \end{cases} \tag{17}$$

where:

- $x_c(t) \in \mathbb{R}^6$ is the **controller state**,
- A_{cij}, B_{cj}, C_{cj} (for $i, j = 1, \ldots, 8$) and D_c are controller gain matrices with appropriate dimensions,
- $\alpha_i(\rho, t)$ are the previously defined barycentric weights of the polytopic LPV model.

Closed-loop System Formulation

Define the **augmented state vector**

$$\bar{x} = \begin{bmatrix} x \\ x_c \end{bmatrix},$$

and let the augmented input disturbance mapping be

$$\bar{B} = \begin{bmatrix} I_{6\times 6} \\ 0_{2\times 6} \end{bmatrix}.$$

Also define the extended uncertainty mapping:

$$\bar{H} = \begin{bmatrix} H \\ 0_{2\times 3} \end{bmatrix}, \qquad \bar{C} = [C_2 \quad 0_{6\times 2}].$$

The closed-loop system combining plant and controller dynamics becomes:

$$\dot{\bar{x}}(t) = \sum_{i=1}^{8} \sum_{j=1}^{8} \alpha_i(\rho, t)\alpha_j(\rho, t) \left((\bar{A}_{ij} + \bar{H}\Lambda L_{ij})\bar{x}(t) + \bar{B}\, \bar{d}(t) \right), z(t) = \bar{C}\, \bar{x}(t), \tag{18}$$

where the augmented vertex matrices are

$$\bar{A}_{ij} = \begin{bmatrix} A_{ni} + B_{ni} D_c C_1 & B_{ni} C_{cj} \\ B_{cj} C_1 & A_{cij} \end{bmatrix}, \qquad L_{ij} = [L_{1i} + L_{2i} D_c C_1 \quad L_{2i} C_{cj}].$$

This structure captures:

- the plant uncertainty via $H, \Lambda, L_{1i}, L_{2i}$,

- the controller influence via $A_{cij}, B_{cj}, C_{cj}, D_c$,
- and the polytopic dependence via $\alpha_i \alpha_j$.

Lemma 1 (Uncertainty Elimination via S-Procedure)

Lemma 1 (from [11]):
Let $Y = Y^T$, and let H, L be matrices of compatible dimensions.
Then the inequality

$$Y + H\Lambda L + L^T \Lambda^T H^T < 0 \qquad (19)$$

holds for all diagonal uncertainty matrices Λ satisfying $\Lambda^T \Lambda \leq I$,
if and only if there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} Y & \varepsilon H & L^T \\ * & -\varepsilon I & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0. \qquad (20)$$

This lemma is essential because it converts the nonlinear uncertain inequality (19) into a **linear matrix inequality (LMI)** in terms of ε.

Theorem 1 — LMI Conditions for Robust H∞ Performance

Let $v > 0$ be the desired H∞ attenuation level and $\vartheta > 0$ the allowable bound on the controller state energy.
Assume scalars $\varepsilon_{ij}, \varepsilon_{ji} > 0$ and a symmetric positive-definite matrix $P = P^T > 0$.

If the following LMIs hold:

$$\bar{Y}_{1ij} + \bar{Y}_{1ji} < 0, \qquad 1 \leq i \leq j \leq 8, \qquad (21)$$
$$\bar{Y}_{2sj} < 0, \qquad s = 1,2, \quad j = 1, \dots, 8, \qquad (22)$$

then the closed-loop system (18) is robustly asymptotically stable for all $\bar{d}(t) = 0$, and satisfies the H∞ performance condition (15) for all disturbances.

Definition of the LMI matrices

The matrices appearing in (21) and (22) are:

(a) Performance LMIs

$$\bar{Y}_{1ij} = \begin{bmatrix} \bar{A}_{ij}^T P + P\bar{A}_{ij} & PB & \bar{C}^T & PH & L_{ij}^T \\ * & -I & 0 & 0 & 0 \\ * & * & -v^2 I & 0 & 0 \\ * & * & * & -\varepsilon_{ij} I & 0 \\ * & * & * & * & -\varepsilon_{ji} I \end{bmatrix}.$$

$$\bar{Y}_{1ji} = \begin{bmatrix} \bar{A}_{ji}^T P + P\bar{A}_{ji} & PB & \bar{C}^T & PH & L_{ji}^T \\ * & -I & 0 & 0 & 0 \\ * & * & -v^2 I & 0 & 0 \\ * & * & * & -\varepsilon_{ji} I & 0 \\ * & * & * & * & -\varepsilon_{ij} I \end{bmatrix}.$$

These LMIs enforce:

- robust stability,
- H∞ disturbance attenuation,
- uncertainty elimination via Lemma 1.

(b) Actuator Saturation LMIs

Define

$$K_{hj} = [D_c C_1 \quad C_{cj}].$$

Then

$$\bar{Y}_{2sj} = \begin{bmatrix} -u_{\max,s}^2 P & K_{hj}^T \\ * & -\vartheta^{-1} I \end{bmatrix} < 0, \qquad s = 1,2,\ j = 1,\dots,8.$$

These ensure that the control inputs satisfy:

$$|u_s(t)| \le u_{\max,s}, \qquad s = 1,2.$$

Concise Proof of Theorem 1

To analyze stability and H∞ performance of the closed-loop system (18), consider the Lyapunov function candidate

$$V(\bar{x}) = \bar{x}^T P\, \bar{x}, \qquad P = P^T > 0. \tag{23}$$

Now compute the time derivative of $V(\bar{x})$ and add the H∞ performance term $z^T(t)z(t) - v^2 \bar{d}^T(t)\bar{d}(t)$.
For any attenuation level $v > 0$, we obtain

$$\dot{V}(\bar{x}) + z^T(t)z(t) - v^2\bar{d}^T(t)\bar{d}(t) \quad = \sum_{i=1}^{8}\sum_{j=1}^{8} \alpha_i(\rho,t)\alpha_j(\rho,t)\left(\bar{x}^T\Theta_{ij}\bar{x} + 2\bar{x}^T P\bar{B}\bar{d} + z^T z - v^2\bar{d}^T\bar{d}\right)$$
$$= \sum_{i=1}^{8}\sum_{j=1}^{8} \alpha_i(\rho,t)\alpha_j(\rho,t) \begin{bmatrix}\bar{x}\\ \bar{d}\end{bmatrix}^T Y_{1ij}\begin{bmatrix}\bar{x}\\ \bar{d}\end{bmatrix}. \tag{24}$$

Here:

$$\Theta_{ij} \; = \; \bar{A}_{ij}^T P + P\bar{A}_{ij} + P\bar{H}\Lambda L_{ij} + L_{ij}^T \Lambda^T \bar{H}^T P,$$

and the combined matrix Y_{1ij} is

$$Y_{1ij} = \begin{bmatrix}\Theta_{ij} + \bar{C}^T\bar{C} & P\bar{B}\\ \bar{B}^T P & -v^2 I\end{bmatrix}.$$

Main Inequality

From (24), if

$$\sum_{i=1}^{8}\sum_{j=1}^{8} \alpha_i(\rho,t)\alpha_j(\rho,t)\; Y_{1ij} < 0, \tag{25}$$

then
$\dot{V} + z^T z - v^2\bar{d}^T\bar{d} < 0,$
which implies robust asymptotic stability when $\bar{d}(t) = 0$ and satisfaction of the H∞ inequality (15) for all disturbances.

Vertex Decomposition

Using the identities

$$\alpha_i\alpha_j = \begin{cases}\alpha_i^2, & i = j,\\ \alpha_i\alpha_j, & i < j,\end{cases}$$

the inequality (25) can be rewritten as

$$\sum_{i=1}^{8} \alpha_i^2(\rho,t)\, Y_{1ii} \; + \; \sum_{i=1}^{7}\sum_{j=i+1}^{8} \alpha_i(\rho,t)\alpha_j(\rho,t)\left(Y_{1ij} + Y_{1ji}\right) < 0. \tag{26}$$

A sufficient condition for (26) is:

$$Y_{1ij} + Y_{1ji} < 0, \qquad 1 \le i \le j \le 8. \tag{27}$$

This means checking only **36 LMIs** instead of continuously over ρ.

Using Schur Complement

Applying Schur's complement to (27) yields:

$$\begin{bmatrix} \Theta_{ij} & PB & \bar{C}^T \\ * & -v^2 I & 0 \\ * & * & -I \end{bmatrix} + \begin{bmatrix} \Theta_{ji} & PB & \bar{C}^T \\ * & -v^2 I & 0 \\ * & * & -I \end{bmatrix} < 0. \qquad (28)$$

Applying Lemma 1

Recall that

$$\Theta_{ij} = \bar{A}_{ij}^T P + P\bar{A}_{ij} + P\bar{H}\Lambda L_{ij} + L_{ij}^T \Lambda^T \bar{H}^T P.$$

Using Lemma 1, the inequality (28) is satisfied if there exist $\varepsilon_{ij}, \varepsilon_{ji} > 0$ such that:

$$\begin{bmatrix} \bar{A}_{ij}^T P + P\bar{A}_{ij} & PB & \bar{C}^T & \varepsilon_{ij} P\bar{H} & L_{ij}^T \\ * & -v^2 I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\varepsilon_{ij} I & 0 \\ * & * & * & * & -\varepsilon_{ji} I \end{bmatrix} + \begin{bmatrix} \bar{A}_{ji}^T P + P\bar{A}_{ji} & PB & \bar{C}^T & \varepsilon_{ji} P\bar{H} & L_{ji}^T \\ * & -v^2 I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\varepsilon_{ji} I & 0 \\ * & * & * & * & -\varepsilon_{ij} I \end{bmatrix} < 0.$$

(29)

These are exactly the LMIs stated earlier in Theorem 1 as conditions (21) – (22).

By applying a congruence transformation with

$$\mathrm{diag}\{I,\ I,\ I,\ \left(\varepsilon_{ij} + \varepsilon_{ji}\right)^{-1} I,\ I\},$$

inequality (29) becomes equivalent to (21).

Thus, the LMI conditions of Theorem 1 guarantee robust stability and the H∞ bound.

Final H∞ Bound

From inequality (30):

$$\dot{V}(\bar{x}) + z^T z - v^2 \bar{d}^T \bar{d} \leq 0, \qquad (30)$$

integrating both sides from 0 to t yields:

$$V(t) \leq v^2 \int_0^t \bar{d}^T(\tau)\bar{d}(\tau)d\tau + V(0) \leq v^2 \parallel \bar{d}(t) \parallel_2^2 + V(0) \leq \vartheta, \quad \forall t \geq 0, \qquad (31)$$

where

- $\vartheta := v^2 \bar{d}_{\max} + V(0)$,

- $\bar{d}_{\max} := \max_t \| \bar{d}(t) \|_2^2$.

This shows that the Lyapunov function remains bounded and ensures robust H∞ performance for all disturbances.

To verify the actuator constraint (16), consider the controller (17) rewritten compactly as

$$u = K_h \bar{x}(t),$$

where

$$K_h := \left[D_c C_1 \quad \sum_{j=1}^{8} \alpha_j (\rho, t) C_{cj} \right].$$

Define the transformed variable

$$\tilde{x} = P^{1/2} \bar{x},$$

so that

$$\tilde{x}^T(t)\tilde{x}(t) = \bar{x}^T(t) P \bar{x}(t) \leq \vartheta,$$

as established earlier from the Lyapunov inequality.

Then the control input satisfies

$$\begin{aligned} |u_s(t)|^2 &= \|K_h \bar{x}(t)\|_2^2 \leq \max_{t\geq 0} \|\tilde{x}^T(t) K_h^T K_h \tilde{x}(t)\|_2 \\ &= \max_{t\geq 0} \left\| \tilde{x}^T(t) P^{-\frac{1}{2}} K_h^T K_h P^{-\frac{1}{2}} \tilde{x}(t) \right\|_2 \end{aligned} \tag{32}$$

which gives the bound:

$$|u_s(t)|^2 \leq \vartheta \, \lambda_{\max}\left(P^{-1/2} K_h^T K_h P^{-1/2}\right), \qquad s = 1,2,$$

where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of its argument.

Actuator Constraint Satisfaction

To ensure the actuator constraint

$$|u_s(t)| \leq u_{\max,s}, \qquad s = 1,2,$$

it is sufficient that

$$\vartheta P^{-1/2} K_h^T K_h P^{-1/2} < u_{\max,s}^2 I.$$

Using Schur's complement and the identity

$$\sum_{j=1}^{8} \alpha_j(\rho, t) = 1,$$

we obtain:

$$\begin{aligned} \begin{bmatrix} -u_{max,s}^2 P & K_{hj}^T \\ * & -\vartheta^{-1} I \end{bmatrix} &= \sum_{j=1}^{8} \alpha_j(\rho, t) \begin{bmatrix} -u_{max,s}^2 P & K_{hj}^T \\ * & -\vartheta^{-1} I \end{bmatrix} \\ &= \sum_{j=1}^{8} \alpha_j(\rho, t)\, \bar{Y}_{2sj} < 0. \end{aligned} \tag{33}$$

Therefore, when the LMI condition (22) of Theorem 1 holds, constraint (16) is satisfied, completing the proof. ☐

Mathematical Transformation for Controller Synthesis

To derive solvable LMIs for the controller matrices, the matrix P and its inverse are partitioned as:

$$P = \begin{bmatrix} R & F \\ F^T & W \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} S & E \\ E^T & V \end{bmatrix}, \tag{34}$$

where:

- R and S are symmetric positive definite matrices,
- F and E are full-rank matrices,
- they satisfy

  $EF^T = I - SR.$

Define the matrices:

$$Q_1 = \begin{bmatrix} S & I \\ E^T & 0 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} I & R \\ 0 & F^T \end{bmatrix}. \tag{35}$$

These matrices will be used to transform the nonlinear LMIs of Theorem 1 into the linear LMIs of Theorem 2, allowing controller matrices $\left(A_{cij}, B_{cj}, C_{cj}, D_c\right)$ to be solved explicitly.

Detailed Prove of Theorem 1:

The Setup: System and Stability

We start with the augmented closed-loop system dynamics

$$\dot{\bar{x}}(t) = \left(\bar{A}_{ij} + \Delta\bar{A}_{ij}\right)\bar{x}(t) + \bar{B}\bar{d}(t)$$

$$z(t) = \bar{C}\bar{x}(t)$$

We use the Lyapunov function:

$$V(\bar{x}) = \bar{x}^T P \bar{x}$$

We aim to satisfy the H_∞ performance criterion:

$$\dot{V} + z^T z - \omega^2 \bar{d}^T \bar{d} < 0$$

2. Expanding the Time Derivative ($\dot{V}$)

First, calculate the time derivative of the Lyapunov function along the system trajectories.

$$\dot{V} = \dot{\bar{x}}^T P \bar{x} + \bar{x}^T P \dot{\bar{x}}$$

Substitute $\dot{\bar{x}} = (\bar{A}_{ij} + \Delta\bar{A}_{ij})\bar{x} + \bar{B}\bar{d}$ into the equation:

$$\dot{V} = \left[(\bar{A}_{ij} + \Delta\bar{A}_{ij})\bar{x} + \bar{B}\bar{d}\right]^T P\bar{x} + \bar{x}^T P\left[(\bar{A}_{ij} + \Delta\bar{A}_{ij})\bar{x} + \bar{B}\bar{d}\right]$$

Expand the transpose terms (using $(AB)^T = B^T A^T$):

$$\dot{V} = \left(\bar{x}^T(\bar{A}_{ij} + \Delta\bar{A}_{ij})^T + \bar{d}^T\bar{B}^T\right)P\bar{x} + \bar{x}^T P(\bar{A}_{ij} + \Delta\bar{A}_{ij})\bar{x} + \bar{x}^T P\bar{B}\bar{d}$$

Distribute the multiplication:

$$\dot{V} = \bar{x}^T(\bar{A}_{ij} + \Delta\bar{A}_{ij})^T P\bar{x} + \bar{d}^T\bar{B}^T P\bar{x} + \bar{x}^T P(\bar{A}_{ij} + \Delta\bar{A}_{ij})\bar{x} + \bar{x}^T P\bar{B}\bar{d}$$

Combine the scalar cross terms (since $\bar{d}^T\bar{B}^T P\bar{x} = (\bar{x}^T P\bar{B}\bar{d})^T$ and the result is a scalar, they are equal):

$$\dot{V} = \bar{x}^T\left((\bar{A}_{ij} + \Delta\bar{A}_{ij})^T P + P(\bar{A}_{ij} + \Delta\bar{A}_{ij})\right)\bar{x} + 2\bar{x}^T P\bar{B}\bar{d}$$

3. Formulating the Matrix Inequality

Substitute the expanded $\dot{V}$ and $z = \bar{C}\bar{x}$ into the H_∞ condition $\dot{V} + z^T z - \omega^2\bar{d}^T\bar{d} < 0$:

$$\bar{x}^T\left(\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)^T P+P\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)\right)\bar{x}+2\bar{x}^T P\bar{B}\bar{d}+\bar{x}^T\bar{C}^T\bar{C}\bar{x}-\bar{d}^T\omega^2 I\bar{d}<0$$

Group the terms into a block matrix multiplied by the vector $\begin{bmatrix}\bar{x}\\ \bar{d}\end{bmatrix}$:

$$\begin{bmatrix}\bar{x}\\ \bar{d}\end{bmatrix}^T\begin{bmatrix}\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)^T P+P\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)+\bar{C}^T\bar{C} & P\bar{B}\\ \bar{B}^T P & -\omega^2 I\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{d}\end{bmatrix}<0$$

For this inequality to hold for all states, the matrix itself must be negative definite:

$$\Pi=\begin{bmatrix}\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)^T P+P\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)+\bar{C}^T\bar{C} & P\bar{B}\\ \bar{B}^T P & -\omega^2 I\end{bmatrix}<0$$

4. Applying Schur Complement

The term $\bar{C}^T\bar{C}$ makes the inequality quadratic in $\bar{C}$. We use the **Schur Complement** to linearize it. The lemma states that $\Phi+S^T Q^{-1}S<0$ is equivalent to $\begin{bmatrix}\Phi & S^T\\ S & -Q\end{bmatrix}<0$. (Here $Q=I$).

Applying this to remove $\bar{C}^T\bar{C}$:

$$\begin{bmatrix}\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)^T P+P\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right) & P\bar{B} & \bar{C}^T\\ \bar{B}^T P & -\omega^2 I & 0\\ \bar{C} & 0 & -I\end{bmatrix}<0$$

5. Isolating the Uncertainty

Expand the term involving $\Delta\bar{A}_{ij}$:

$$\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)^T P+P\left(\bar{A}_{ij}+\Delta\bar{A}_{ij}\right)=\left(\bar{A}_{ij}^T P+P\bar{A}_{ij}\right)+\left(\Delta\bar{A}_{ij}^T P+P\Delta\bar{A}_{ij}\right)$$

Substitute the uncertainty definition $\Delta\bar{A}_{ij}=\bar{H}\Lambda\bar{L}_{ij}$:

$$\text{Uncertain Term}=\left(\bar{H}\Lambda\bar{L}_{ij}\right)^T P+P\left(\bar{H}\Lambda\bar{L}_{ij}\right)=\bar{L}_{ij}^T\Lambda^T\bar{H}^T P+P\bar{H}\Lambda\bar{L}_{ij}$$

Now, rewrite the full matrix inequality as the sum of a **Nominal Matrix (Y)** and the **Uncertain Terms**:

$$\underbrace{\begin{bmatrix} \bar{A}_{ij}^T P + P\bar{A}_{ij} & P\bar{B} & \bar{C}^T \\ \bar{B}^T P & -\omega^2 I & 0 \\ \bar{C} & 0 & -I \end{bmatrix}}_{Y} + \underbrace{\begin{bmatrix} P\bar{H}\Lambda\bar{L}_{ij} + \bar{L}_{ij}^T\Lambda^T\bar{H}^T P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Uncertainty}} < 0$$

6. Structuring for Lemma 1

We express the uncertainty in the form $U\Lambda V + V^T\Lambda^T U^T$ to match Lemma 1.

Define column vector U and row vector V:

$$U = \begin{bmatrix} P\bar{H} \\ 0 \\ 0 \end{bmatrix}, \quad V = \begin{bmatrix} \bar{L}_{ij} & 0 & 0 \end{bmatrix}$$

Then the inequality becomes:

$$Y + \begin{bmatrix} P\bar{H} \\ 0 \\ 0 \end{bmatrix} \Lambda \begin{bmatrix} \bar{L}_{ij} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \bar{L}_{ij}^T \\ 0 \\ 0 \end{bmatrix} \Lambda^T \begin{bmatrix} \bar{H}^T P & 0 & 0 \end{bmatrix} < 0$$

7. Applying Lemma 1 (Theorem 1 Result)

Lemma 1 states that $Y + U\Lambda V + V^T\Lambda^T U^T < 0$ is true if and only if there exists $\epsilon > 0$ such that:

$$\begin{bmatrix} Y & \epsilon U & V^T \\ \epsilon U^T & -\epsilon I & 0 \\ V & 0 & -\epsilon I \end{bmatrix} < 0$$

Substituting our specific matrices (Y, U, V) into this block structure results in the final 5×5 matrix inequality.

The Full Matrix:

$$\begin{bmatrix} \bar{A}_{ij}^T P + P\bar{A}_{ij} & P\bar{B} & \bar{C}^T & \epsilon P\bar{H} & \bar{L}_{ij}^T \\ \bar{B}^T P & -\omega^2 I & 0 & 0 & 0 \\ \bar{C} & 0 & -I & 0 & 0 \\ \epsilon \bar{H}^T P & 0 & 0 & -\epsilon I & 0 \\ \bar{L}_{ij} & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0$$

Theorem 2

Theorem 2.
Given the positive constants v and ϑ, suppose there exist symmetric positive definite matrices

$$R > 0, \qquad S > 0,$$

positive scalars ε_{ij} and ε_{ji}, and matrices

$$\hat{A}_{cij}, \quad \hat{B}_{cj}, \quad \hat{C}_{cj}, \quad \hat{D}_c,$$

such that the following LMIs are satisfied:

$$\begin{aligned} &\Xi_{1ij} + \Xi_{1ji} < 0, \qquad 1 \le i \le j \le 8, \qquad\qquad (36) \\ &\Xi_{2sj} < 0, \qquad s = 1,2, \quad j = 1, \dots \quad (37) \end{aligned}$$

then the closed-loop system (18) is asymptotically stable under $\bar{d}(t) = 0$, and satisfies the H∞ performance condition (15) for all disturbances $\bar{d}(t)$.

LMI Matrices Ξ_{1ij}, Ξ_{1ji}, and Ξ_{2sj}

The matrices in (36)–(37) are defined as follows:

(a) LMI for robust H∞ stability:

$$\Xi_{1ij} = \begin{bmatrix} G_{ij} + G_{ij}^T & \hat{A}_{cij} + K_i & I & SC_2^T & H & SL_{1i}^T + \hat{C}_{cj}^T L_{2i}^T \\ * & J_{ij} + J_{ij}^T & R & C_2^T & RH & L_{1i}^T + C_1^T \hat{D}_c^T L_{2i}^T \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -v^2 I & 0 & 0 \\ * & * & * & * & -\varepsilon_{ij} I & 0 \\ * & * & * & * & * & -\varepsilon_{ji} I \end{bmatrix},$$

$$
\Xi_{1ji} = \begin{bmatrix} G_{ji} + G_{ji}^T & \hat{A}_{cji} + K_i & I & SC_2^T & H & SL_{1j}^T + \hat{C}_{cj}^T L_{2i}^T \\ * & J_{ji} + J_{ji}^T & R & C_2^T & RH & L_{1j}^T + C_1^T \widehat{D}_c^T L_{2i}^T \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -\nu^2 I & 0 & 0 \\ * & * & * & * & -\varepsilon_{ji} I & 0 \\ * & * & * & * & * & -\varepsilon_{ij} I \end{bmatrix}.
$$

Here

$$
G_{ij} = A_{ni} S + B_{ni} \hat{C}_{cj}, \qquad K_i = A_{ni} + B_{ni} \widehat{D}_c C_1, J_{ij} = R A_{ni} + \hat{B}_{cj} C_1.
$$

(b) LMI for actuator constraints:

$$
\Xi_{2sj} = \begin{bmatrix} -u_{max,s}^2 S & -u_{max,s}^2 I & C_{cj}^T & C_1^T \widehat{D}_c^T \\ * & -u_{max,s}^2 R & C_{cj}^T & C_1^T \widehat{D}_c^T \\ * & * & -\vartheta^{-1} I & 0 \\ * & * & * & -\vartheta^{-1} I \end{bmatrix}. \tag{38}
$$

These LMIs ensure that the control inputs satisfy the magnitude constraints (16) at every vertex.

Controller Reconstruction (Recovering $A_{cij}, B_{cj}, C_{cj}, D_c$)

Once the LMIs of Theorem 2 are solved, the actual controller matrices for (17) are recovered using the following transformations:

$$
\boxed{\begin{aligned} A_{cij} &= F^{-1}\left[\hat{A}_{cij} - R(A_{ni} + B_{ni} D_c C_1) S\right] E^{-T} - B_{cj} C_1 S E^{-T} - F^{-1} R B_{ni} C_{cj}, \\ B_{cj} &= F^{-1}\left(\hat{B}_{cj} - R B_{ni} D_c\right), \\ C_{cj} &= \left(\hat{C}_{cj} - D_c C_1 S\right) E^{-T}, \\ D_c &= \widehat{D}_c. \end{aligned}} \tag{39}
$$

These equations convert the **transformed controller variables**

$$
\hat{A}_{cij}, \hat{B}_{cj}, \hat{C}_{cj}, \widehat{D}_c
$$

into the **actual controller matrices** used in the output-feedback law (17).

Note that R, S, E, F come from the block-partitioned inverse of matrix P:

$$
P = \begin{bmatrix} R & F \\ F^T & W \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} S & E \\ E^T & V \end{bmatrix},
$$

with

$$
E F^T = I - SR.
$$

Concise Proof of Theorem 2

Proof.
To derive Theorem 2 from Theorem 1, perform a congruence transformation on inequality (21) using

$$\mathrm{diag}\{Q_1,\ I,\ I,\ I,\ I\},$$

and on inequality (22) using

$$\mathrm{diag}\{Q_1,\ I\}.$$

Under these transformations, the inequalities $\bar{Y}_{1ij}/\bar{Y}_{1ji}$ and $\bar{Y}_{2sj}$ become the following matrices:

$$\boxed{\begin{aligned}
\Xi_{1ij} &= \begin{bmatrix} Q_1^T\bar{A}_{ij}^T PQ_1 + Q_1^T P\bar{A}_{ij}Q_1 & Q_1^T P\bar{B} & Q_1^T\bar{C}^T & Q_1^T P\bar{H} & Q_1^T L_{ij}^T \\ * & -v^2 I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\varepsilon_{ij}^{-1} I & 0 \\ * & * & * & * & -\varepsilon_{ji} I \end{bmatrix}, \\
\Xi_{1ji} &= \begin{bmatrix} Q_1^T\bar{A}_{ji}^T PQ_1 + Q_1^T P\bar{A}_{ji}Q_1 & Q_1^T P\bar{B} & Q_1^T\bar{C}^T & Q_1^T P\bar{H} & Q_1^T L_{ji}^T \\ * & -v^2 I & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\varepsilon_{ji}^{-1} I & 0 \\ * & * & * & * & -\varepsilon_{ij} I \end{bmatrix}, \\
\Xi_{2sj} &= \begin{bmatrix} -u_{max,s}^2 Q_1^T PQ_1 & Q_1^T K_{hj}^T \\ * & \vartheta^{-1} I \end{bmatrix}.
\end{aligned}} \quad (40)$$

These expressions now match the structure required by inequalities (36)–(37) in Theorem 2.

Substitution Using Definitions of Q_1 and Q_2

From the definitions in (35), the matrices satisfy:

$$PQ_1 = Q_2.$$

Using this identity, we compute the following intermediate results:

$$
\boxed{\begin{aligned}
Q_1^T P\bar{A}_{ij}Q_1 &= Q_2^T\bar{A}_{ij}Q_1 = \begin{bmatrix} A_{ni}S + B_{ni}\hat{C}_{cj} & A_{ni} + B_{ni}\widehat{D}_c C_1 \\ \hat{A}_{cij} & RA_{ni} + \widehat{B}_{cj}C_1 \end{bmatrix}, \\
Q_1^T P\bar{B} &= Q_2^T\begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ R \end{bmatrix}, \\
Q_1^T\bar{C}^T &= \begin{bmatrix} SC_2^T \\ C_2^T \end{bmatrix}, \\
Q_1^T P\bar{H} &= Q_2^T\begin{bmatrix} H \\ 0 \end{bmatrix} = \begin{bmatrix} H \\ RH \end{bmatrix}, \\
Q_1^T L_{ij}^T &= \begin{bmatrix} SE^T & I \\ E^T & 0 \end{bmatrix}\begin{bmatrix} L_{1i}^T + (L_{2i}D_cC_1)^T \\ \left(L_{2i}C_{cj}\right)^T \end{bmatrix} = \begin{bmatrix} SL_{1i}^T + \hat{C}_{cj}^T L_{2i}^T \\ L_{1i}^T + C_1^T\widehat{D}_c^T L_{2i}^T \end{bmatrix}, \\
Q_1^T P Q_1 &= \begin{bmatrix} S & I \\ R & S \end{bmatrix}, \\
K_{hj}Q_1 &= \begin{bmatrix} D_cC_1 & C_{cj} \end{bmatrix}\begin{bmatrix} S & I \\ E^T & 0 \end{bmatrix} = \begin{bmatrix} \hat{C}_{cj} & \widehat{D}_cC_1 \end{bmatrix}.
\end{aligned}} \tag{41}
$$

Recovered Controller Variable Relations

To convert the transformed controller variables

$$\left(\hat{A}_{cij}, \widehat{B}_{cj}, \hat{C}_{cj}, \widehat{D}_c\right)$$

to the original controller matrices for (17), the following identities hold:

$$
\boxed{\begin{aligned}
\hat{A}_{cij} &= R(A_{ni} + B_{ni}D_cC_1)S + FB_{cj}C_1S + RB_{ni}C_{cj} + FA_{cij}E^T, \\
\widehat{B}_{cj} &= RB_{ni}D_c + FB_{cj}, \\
\hat{C}_{cj} &= D_cC_1S + C_{cj}E^T, \\
\widehat{D}_c &= D_c, \qquad i, j = 1, \dots, 8.
\end{aligned}} \tag{42}
$$

Substituting the relations in (41) into (40) yields exactly the LMIs (38).
Similarly, inverting (42) produces the reconstruction formulas (39).
This completes the proof.

Detailed Prove of Theorem 2:

1. The BMI Obstacle

From Theorem 1, the robust stability condition:

$$\Upsilon_{ij} + \Upsilon_{ji} < 0$$

where Υ_{ij} contains the bilinear term $P\bar{A}_{ij}$ with:

$$\bar{A}_{ij} = \begin{bmatrix} A_{ni} + B_{ni}D_cC_1 & B_{ni}C_{cj} \\ B_{cj}C_1 & A_{cj} \end{bmatrix}$$

This is a **Bilinear Matrix Inequality (BMI)** since P multiplies controller matrices, creating non-convex terms.

2. Partitioning the Lyapunov Matrix

$$P = \begin{bmatrix} R & F \\ F^T & W \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} S & E \\ E^T & V \end{bmatrix}$$

with $R, S \in \mathbb{R}^{n\times n}$ symmetric positive definite. From $PP^{-1} = I$, the $(1,1)$ block yields:

$$\boxed{RS + FE^T = I}$$

3. Defining Transformation Matrices

$$Q_1 = \begin{bmatrix} S & I \\ E^T & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} I & R \\ 0 & F^T \end{bmatrix}$$

Verification ($PQ_1 = Q_2$):

$$PQ_1 = \begin{bmatrix} R & F \\ F^T & W \end{bmatrix}\begin{bmatrix} S & I \\ E^T & 0 \end{bmatrix} = \begin{bmatrix} RS + FE^T & R \\ F^T S + W E^T & F^T \end{bmatrix} = Q_2$$

4. Change of Variables (Linearization)

$$\boxed{\begin{aligned} \widehat{D}_c &= D_c \\ \hat{C}_{cj} &= D_c C_1 S + C_{cj} E^T \\ \hat{B}_{cj} &= F B_{cj} + R B_{ni} D_c \\ \hat{A}_{cij} &= R(A_{ni} + B_{ni} D_c C_1)S + F B_{cj} C_1 S + R B_{ni} C_{cj} E^T + F A_{cj} E^T \end{aligned}}$$

5. Congruence Transformation Setup

Apply $\mathcal{T} = \mathrm{diag}(Q_1, I, I, I, I)$ to $\Upsilon_{ij} < 0$. Then:

$$Q_1^T \left(P\bar{A}_{ij}\right) Q_1 = Q_2^T \bar{A}_{ij} Q_1$$

6. Algebraic Expansion (Explicit Linearization)

$$\Pi = Q_2^T \bar{A}_{ij} Q_1 = \begin{bmatrix} I & 0 \\ R & F \end{bmatrix}\begin{bmatrix} A_{ni} + B_{ni} D_c C_1 & B_{ni} C_{cj} \\ B_{cj} C_1 & A_{cj} \end{bmatrix}\begin{bmatrix} S & I \\ E^T & 0 \end{bmatrix}$$

6.1 Top-Left Block (1,1):

$$\Pi_{11} = (A_{ni} + B_{ni} D_c C_1) S + B_{ni} C_{cj} E^T = A_{ni} S + B_{ni} \hat{C}_{cj}$$

6.2 Bottom-Left Block (2,1):

$$\Pi_{21} = R(A_{ni} + B_{ni} D_c C_1) S + F B_{cj} C_1 S + R B_{ni} C_{cj} E^T + F A_{cj} E^T = \hat{A}_{cij}$$

6.3 Top-Right Block (1,2):

$$\Pi_{12} = A_{ni} + B_{ni} D_c C_1 = A_{ni} + B_{ni} \widehat{D}_c C_1$$

6.4 Bottom-Right Block (2,2):

$$\Pi_{22} = RA_{ni} + \left(RB_{ni}D_c + FB_{cj}\right)C_1 = RA_{ni} + \widehat{B}_{cj}C_1$$

7. Transforming Auxiliary Blocks

1. Disturbance: $Q_1^T P\bar{B} = Q_2^T\bar{B} = [B_d; RB_d]^T$
2. Output: $\bar{C}Q_1 = [C_2S \quad C_2]$
3. Uncertainty: $Q_1^T P\bar{H} = [H; RH]^T$
4. Bounds: $\bar{L}_{ij}Q_1 = \left[SL_{1i} + L_{2i}\hat{C}_{cj} \quad L_{1i} + L_{2i}\widehat{D}_c C_1\right]$

8. Final LMI Assembly

$$\Xi_{1ij} = \begin{bmatrix} G_{ij} + G_{ij}^T & K_i + \hat{A}_{cij}^T & I & SC_2^T & H & SL_{1i}^T + \hat{C}_{cj}^T L_{2i}^T \\ * & J_{ij} + J_{ij}^T & R & C_2^T & RH & L_{1i}^T + C_1^T \widehat{D}_c^T L_{2i}^T \\ * & * & -2\omega^2 I & 0 & 0 & 0 \\ * & * & * & -2I & 0 & 0 \\ * & * & * & * & -2\varepsilon_{ij}^{-1} I & 0 \\ * & * & * & * & * & -2\varepsilon_{ij} I \end{bmatrix}$$

where $G_{ij} = A_{ni}S + B_{ni}\hat{C}_{cj}$, $K_i = A_{ni} + B_{ni}\widehat{D}_c C_1$, and $J_{ij} = RA_{ni} + \widehat{B}_{cj}C_1$.

Final stability condition:

$$\boxed{\Xi_{1ij} + \Xi_{1ji} < 0, \quad 1 \le i \le j \le 8}$$

9. Controller Reconstruction

Once the solver determines the linear variables:

5. Find non-singular F, E using SVD such that $FE^T = I - RS$
6. Invert the definitions:

$$\boxed{\begin{aligned} D_c &= \widehat{D}_c \\ C_{cj} &= \left(\hat{C}_{cj} - D_c C_1 S\right)(E^T)^{-1} \\ B_{cj} &= F^{-1}\left(\widehat{B}_{cj} - RB_{ni}D_c\right) \\ A_{cj} &= F^{-1}\left[\hat{A}_{cij} - R(A_{ni} + B_{ni}D_c C_1)S - FB_{cj}C_1S - RB_{ni}C_{cj}E^T\right](E^T)^{-1} \end{aligned}}$$