Inferences on a Population Mean

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Plan

- Estimation
  - Point estimation
  - Confidence intervals
- Hypothesis testing
  - p-value approah
  - Critical-value approach

Statistical inference

- Descriptive statistics deal with describing observations of a sample, e.g. sample men, sample variance, etc. are tools of descriptive statistics
- Inferential statistics deal with making a statement (conclusion) about a population characteristic using the infromation obtained from a sample
- There are two methods of statistical inference
    - Estimation and test of hypothesis
- Two methods of estimation
    - point estimation
    - interval estimation (confidence interval)

Hypothesis Testing

Hypothesis Tests of a Population Mean

- We have discussed inference procedure about unknown population mean μ using point estimate and interval estimation (confidence interval)

- Hypothesis testing is another method of statistical inference and it deals with plausibility of a specific statement about population mean μ

  - E.g. an experimenter may be interested in plausibility of a statement $\mu = 20$

Hypothesis Tests of a Population Mean

- A null hypothesis H_0 for a population mean μ is a statement that designates possible values for the population mean

$$H_0 : \mu = \mu_0$$

- The associated alternative hypothesis H_1 is the opposite of it and an alternative hypothesis could be either two-sided or one-sided

$$H_1 : \mu \neq \mu_0 \quad \rightarrow \quad \text{two-sided alternative}$$

$$H_1 : \mu > \mu_0 \quad \rightarrow \quad \text{one-sided alternative}$$

$$H_1 : \mu < \mu_0 \quad \rightarrow \quad \text{one-sided alternative}$$

Example 47 (Graphite-Epoxy composites)

- A supplier claims that its products made from a graphite-epoxy composite material have a tensile strength of 40.
- An experimenter may test this claim by collecting a random sample of products and measuring their tensile strengths.
- The experimenter is interested in testing the hypothesis

$$H_0 : \mu = 40 \ \text{ versus } \ H_1 : \mu \neq 40$$

  - The null hypothesis states that the supplier's claim concerning the tensile strength is correct.

Example 48 (Car fuel efficiency)

- A manufacturer claims that its cars achieve an average of at least 35 miles per gallon in highway driving.
- A consumer interest group tests this claim by driving a random selection of the cars in highway conditions and measuring their fuel efficiency.
- If μ denotes the true average miles per gallon achieved by the cars

$$H_0 : \mu = 35 \quad \text{versus} \quad H_1 : \mu < 35$$

  - The null hypothesis states that the manufacturer's claim regarding the fuel efficiency of its cars is correct.

Example 45 (Fabric Water Absorption Properties)

- Suppose that a fabric is unsuitable for dyeing if its water pickup is less than 55%.
- Is the cotton fabric under consideration suitable for dyeing?
- This question can be formulated as a set of one-sided hypotheses about μ, the mean water pickup of the cotton fabric

$$H_0 : \mu = 55\% \ \text{ versus } \ H_1 : \mu > 55\%$$

- These hypotheses have been chosen so that the null hypothesis corresponds to the fabric being unsuitable for dyeing and the alternative hypothesis corresponds to the fabric being suitable for dyeing

Two-sided t-test

- We have a sample $x_1, \ldots, x_n$ from a population with mean μ and variance σ^2, and $\bar{x}$ and s are corresponding sample mean and standard deviation

- The null and alternative hypotheses $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$

- Test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{(n-1)} \; under \; H_0$$

- p-value of the test $\rightarrow$ $p = P(X > |t|), \; where \; X \sim t_{(n-1)}$

- **Decision**: If $p < 0.10$ reject the H_0, otherwise don't reject the H_0

p-value

- A p-value for a particular null hypothesis H_0 based on an observed data set is defined to be "the probability of obtaining the data set or worse when the null hypothesis is true."

- A "worse" data set is one that has less affinity with the null hypothesis

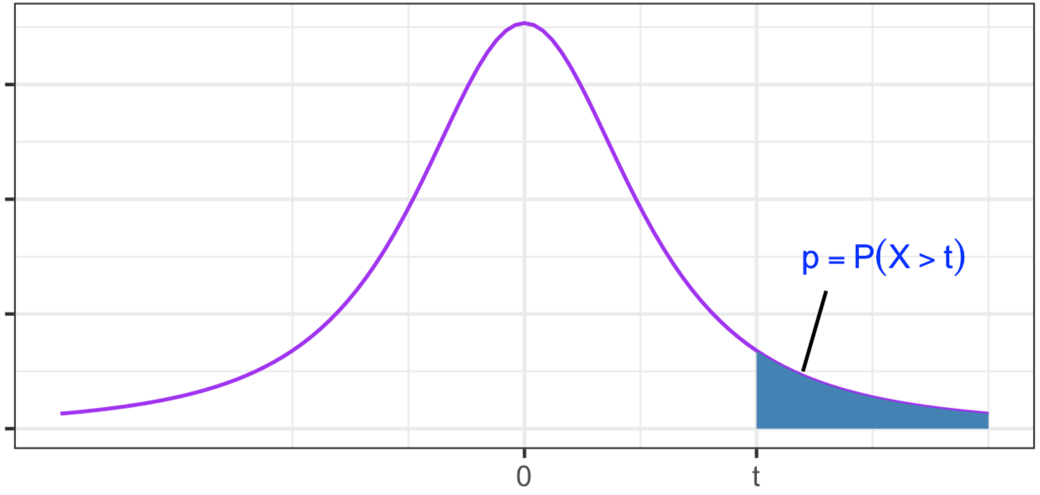
- Null hypothesis

$$H_0 : \mu = \mu_0$$

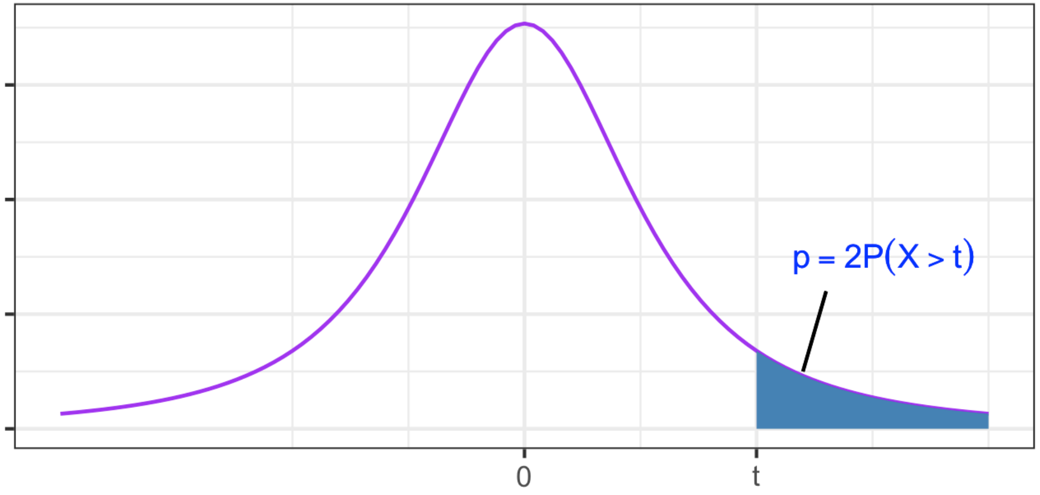
- Test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{(n-1)} \; under \; H_0$$

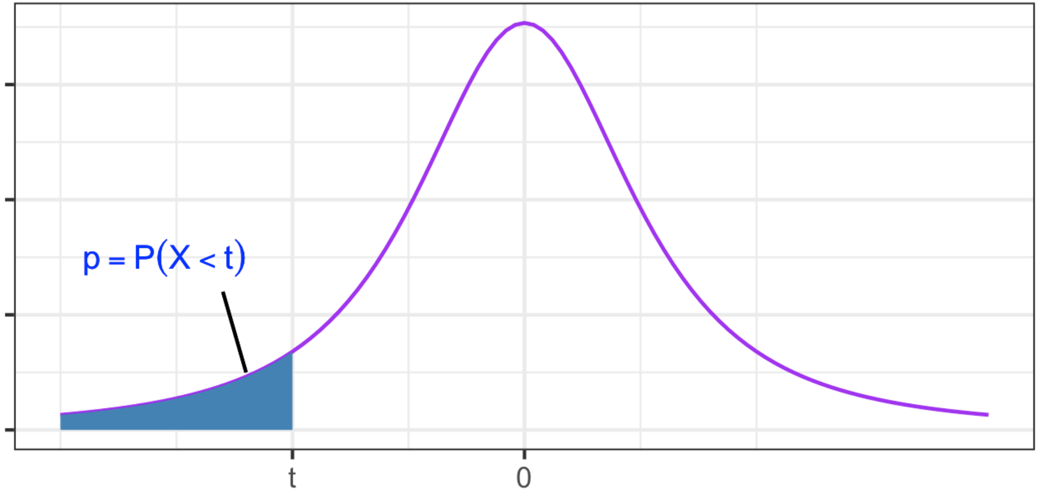
$H_1 : \mu > \mu_0$



$H_1 : \mu \neq \mu_0$



$H_1 : \mu < \mu_0$



Example 47 (Graphite-Epoxy Composite)

- When the tensile strengths of 30 randomly selected products are measured, a sample mean of $\bar{x} = 38.518$ and a sample standard deviation of $s = 2.299$ are obtained.
- Test the null hypothesis $H_0 : \mu = 40$ against $H_1 : \mu \neq 40$, where μ is unknown mean tensile strengths.
    - The t-statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{38.518 - 40.0}{2.299/\sqrt{30}} = \frac{-1.482}{0.42} = -3.53$$

    - The p-value (comparing with the t-distribution with 29 df)

$$p = P(X \geq |t|) = 2 \times P(X \geq 3.53) \in (0.001, 0.002)$$

Example 47 (Graphite-Epoxy Composite)

- Since p-value is smaller than 0.10, we can reject the null hypothesis and here is sufficient evidence to conclude that the mean tensile strength cannot be equal to the claimed value of 40.
    - i.e. population mean tensile is significantly different than 40.

Example 48 (Car fuel efficiency)

- A sample of $n = 20$ cars driven under varying highway conditions achieved fuel efficiencies with a sample mean of $\bar{x} = 34.271$ and sample standard deviation of $s = 2.915$

- Want to test the hypothesis $H_0 : \mu = 35$ against $H_1 : \mu < 35$

- The test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{34.271 - 35}{2.915/\sqrt{20}} = -1.12$$

Example 48 (Car fuel efficiency)

- The p-value

$$p = P(X \leq t) = P(X \leq -1.12) > 0.10$$

  - X follows a t-distribution with 19 degrees of freedom

- Since p-value is greater than 0.10, we cannot reject the null hypothesis $\mu \geq 35$.

Example 45 (Fabric water absorption properties)

- The data set has $n = 15$, $\bar{x} = 59.81\%$, and $s = 4.94\%$
- Want to test the $H_0 : \mu = 55\%$ against $H_1 : \mu > 55\%$
- The test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{59.81 - 55}{4.94/\sqrt{15}} = 3.77$$

Example 45 (Fabric water absorption properties)

- The p-value

$$p = P(X \geq t) = P(X \geq 3.77) \in (.001, .005)$$

  - X follows a t-distribution with 14 degrees of freedom

- This small p-value indicates that the null hypothesis can be rejected and that there is sufficient evidence to conclude that $\mu > 55\%$.

- Therefore the cotton fabric under consideration has been shown to be suitable for dyeing

8.6.27

- Show how t-table can be used to put bounds on the p-values for these hypothesis tests.
  - $n = 24, \bar{x} = 2.39, s = 0.21, H_0 : \mu = 2.5, H_1 : \mu \neq 2.5$
  - $n = 30, \bar{x} = 0.538, s = 0.026, H_0 : \mu = 0.540, H_1 : \mu < 0.540$
  - $n = 10, \bar{x} = 143.6, s = 4.8, H_0 : \mu = 135.0, H_1 : \mu > 135.0$

Problem 8.6.18

- Consider the data set

$$34, 54, 73, 38, 89, 52, 75, 33, 50, 39, 42, 42, 40, 66, 72, 85, 28, 71$$

  which is a random sample from a distribution with an unknown mean μ. Calculate the following.

  - The sample size, the sample median, the sample mean, the sample standard deviation, the sample variance, the standard error of the sample mean
  - A two-sided 99% confidence interval for μ
  - Consider the hypothesis test $H_0 : \mu = 50$ versus $H_1 : \mu \neq 50$. What bounds can you put on the p-value using t-table?

8.6.23

- Twelve samples of a metal alloy are tested. The flexibility measurements had a sample average of 732.9 and a sample standard deviation of 12.5.
  - Is there sufficient evidence to conclude that the flexibility of this kind of metal alloy is smaller than 750? Use an appropriate hypothesis test to investigate this question.
  - Construct a 99% confidence interval that provides an upper bound on the flexibility of this kind of metal alloy.

Relationship between confidence intervals and hypothesis tests

- The $(1-\alpha)100\%$ confidence interval for population mean μ

$$\left(\bar{x} - (s/\sqrt{n})\, t_{\alpha/2},\ \bar{x} + (s/\sqrt{n})\, t_{\alpha/2}\right)$$

- The hypotheses $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ can be performed using the $(1-\alpha)100\%$ confidence interval
    - If μ_0 is included in the confidence interval, the null hypothesis cannot be rejected
    - If μ_0 is not included in the confidence interval, the null hypothesis can be rejected at α level of significance

Relationship between confidence intervals and hypothesis tests

Revisit (Example 17)

- The 95% confidence interval for average milk contents μ is $(1.8705, 2.2749)$
  - We can reject a null hypothesis $H_0 : \mu = \mu_0$ at 5% level of significance if $\mu_0 \notin (1.8705, 2.2749)$, e.g. $H_0 : \mu = 2.3$
  - We cannot reject a null hypothesis $H_0 : \mu = \mu_0$ at 5% level of significance if $\mu_0 \in (1.8705, 2.2749)$, e.g. $H_0 : \mu = 2.25$

Critical-value approach of hypothesis tests

Errors in hypothesis tests

- Type I error
  - Rejecting a null hypothesis when it is true, i.e. False Positive
  - Probability of type I error is known as the significance level and is denoted by α

$$\alpha = P(reject\ H_0 \,|\, when\, H_0\, is\, true)$$

- Type II error
  - Not rejecting a null hypothesis when it is false, i.e. False negative
  - Probability of type II error is denoted by β

$$\beta = P(not\, reject\ H_0 \,|\, when\, H_0\, is\ false)$$

Significance level of a hypothesis test

- Significance level α is the probability of rjecting a null hypothesis when it is true
- A hypothesis test with a significance level α rejects the null hypothesis H_0 if a p-value smaller than α is obtained and otherwise, accepts the null hypothesis H_0.
- In this case, the probability of a Type I error, that is, the probability of rejecting the null hypothesis when it is true, is no larger than α
- Based on pre-specified significance level, one can define rejection region and we reject a null hypothesis if corresponding test statistic falls in the rejection region, otherwise don't reject the null hypothesis

Rejection regions

Two-sided alternative

- $H_0 : \mu = \mu_0$ against $\mu \neq \mu_0$ and corresponding rejection region $|t| \geq t_{\alpha/2}$

One-sided alternative I

- $H_0 : \mu = \mu_0$ against $\mu > \mu_0$ and corresponding rejection region $\{t \geq t_\alpha\}$

One-sided alternative II

- $H_0 : \mu = \mu_0$ against $\mu < \mu_0$ and corresponding rejection region $\{t \leq -t_\alpha\}$

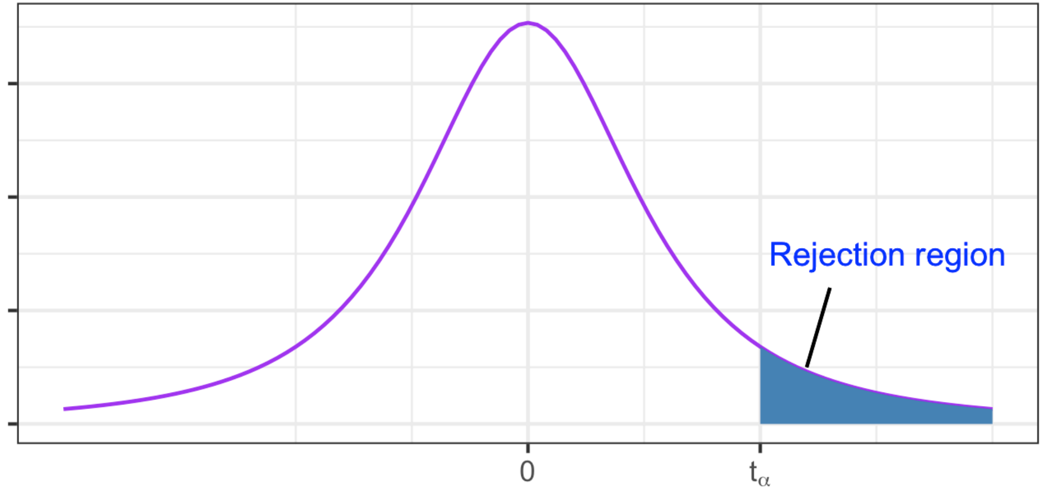
- Null hypothesis

$$H_0 : \mu = \mu_0$$

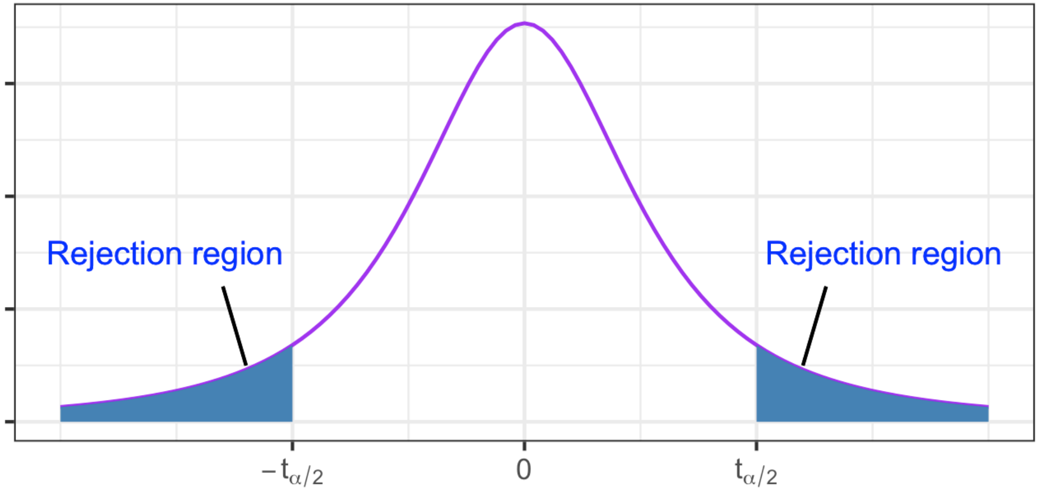
- Test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim N(0,1) \; under \; H_0$$

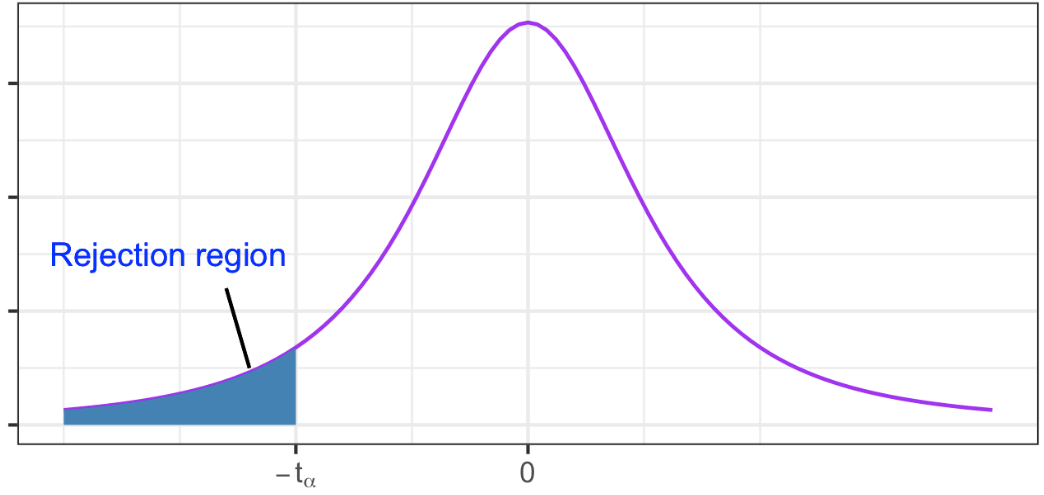
$H_1 : \mu > \mu_0$



$H_1 : \mu \neq \mu_0$



$H_1 : \mu < \mu_0$



Examples

- $n = 10$, $\alpha = .05$, $t = 2.01$, test the following hypotheses
  - $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$
  - $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$

Example 48 (Car fuel efficiency)

- A sample of $n = 20$ cars driven under varying highway conditions achieved fuel efficiencies with a sample mean of $\bar{x} = 34.271$ and sample standard deviation of $s = 2.915$
- Want to test the hypothesis $H_0 : \mu = 35$ against $H_1 : \mu < 35$
  - The test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{34.271 - 35}{2.915/\sqrt{20}} = -1.12$$

  - The rejection region at 5% significance level

$$t < -t_{.05} = -1.729$$

  - Since $t = -1.12 > -1.729$, we cannot reject the null hypothesis at 5% level of significance

Problems 8.2.1

- A sample of $n = 18$ observations has a sample mean of $\bar{x} = 57.74$ and a sample standard deviation of $s = 11.20$. Consider the hypothesis testing problems:
  - $H_0 : \mu = 55.0$ versus $H_1 : \mu \neq 55.0$
  - $H_0 : \mu = 65.0$ versus $H_1 : \mu < 65.0$
- In each case, write down an expression for the p-value.
- What do the critical points in t-table tell you about the p-values?

Problems 8.2.3

- A sample of $n = 13$ observations has a sample mean of $\bar{x} = 2.879$. If an assumed known standard deviation of $s = 0.325$ is used, calculate the p-values for the hypothesis testing problems:
  - $H_0 : \mu = 3.0$ versus $H_1 : \mu \neq 3.0$
  - $H_0 : \mu = 3.1$ versus $H_1 : \mu < 3.1$

Problems 8.2.6

- An experimenter is interested in the hypothesis testing problem

$$H_0 : \mu = 430 \;\; against \;\; H_1 : \mu \neq 430$$

  where μ is the average breaking strength of a bundle of wool fibers. Suppose that a sample of $n = 20$ wool fiber bundles is obtained and their breaking strengths are measured.

  - For what values of the t-statistic does the experimenter accept the null hypothesis with a size $\alpha = 0.10$?
  - For what values of the t-statistic does the experimenter reject the null hypothesis with a size $\alpha = 0.01$?
  - Suppose that the sample mean $\bar{x} = 436.5$ and the sample standard deviation is $s = 11.90$. Is the null hypothesis accepted or rejected with $\alpha = .10$. With $\alpha = 0.01$?
  - Obtain the corresponding p-values.

Sample size calculations

Sample size calculation

- The $(1-\alpha)100\%$ confidence interval for μ when population variance is known:

$$\bar{x} \pm z_{\alpha/2}(\sigma/\sqrt{n})$$

- The length of the confidence interval:

$$L = 2z_{\alpha/2}(\sigma/\sqrt{n})$$

- The required sample size to estimate μ with an $(1-\alpha)$ confidence interval no larger than a pre-specified value L_0

$$L \leq L_0 \;\Rightarrow\; n \geq \frac{4z_{\alpha/2}^2\sigma^2}{L_0^2}$$

  - Given the values of α and σ^2, one can estimate the required sample size to estimate μ

Sample size calculation

- A boot manufacturer is testing the quality of leather provided by a potential supplier.
- The manufacturer wants to construct a two-sided confidence interval with a confidence level 99% that has length no larger than 0.1.
- From the previous experience it is believed that the variability in the leather is such that standard deviation is 0.2031.
- What sample size would you recommend?