

# Continuous Probability Distributions

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# Plan

- The exponential distribution
- The normal distribution
  - The standard normal distribution
  - Quantiles/percentiles of normal distribution

# The Exponential Distribution

## The Exponential Distribution

- Exponential distribution is helpful to model a non-negative continuous random variable, e.g., the battery life of a laptop or a mobile phone
- An exponential distribution with parameter  $\lambda > 0$  has a probability density function

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

## The Exponential Distribution

- The cumulative distribution function

$$\begin{aligned} F(a) &= P(X \leq a) = \int_0^a f(x)dx \\ &= \int_0^a \lambda e^{-\lambda x} \\ &= 1 - e^{-\lambda a} \end{aligned}$$

## The Exponential Distribution

- Expectation and variance

$$E(X) = \frac{1}{\lambda} \text{ and } V(X) = \frac{1}{\lambda^2}$$

- The  $p$ th quantile

$$F(x_p) = p \Rightarrow 1 - e^{-\lambda x_p} = p \Rightarrow x_p = -(1/\lambda) \log(1 - p)$$

- E.g., median:  $F(m) = 0.5 \Rightarrow m = -(1/\lambda) \log(0.5)$

- Integration by parts

$$\int uv \, dx = u \int v \, dx - \int \frac{du}{dx} \left( \int v \, dx \right) dx$$

- Expected value

$$\begin{aligned}
 E(X) &= \int_0^\infty x f(x) dx = \lambda \int_0^\infty x e^{-\lambda x} dx \\
 \int_0^\infty x e^{-\lambda x} dx &= x \times \frac{e^{-\lambda x}}{(-\lambda)} \Big|_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{(-\lambda)} dx \\
 &= \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dx = \frac{-1}{\lambda^2} e^{-\lambda x} \Big|_0^\infty = \frac{1}{\lambda^2} \\
 E(X) &= \lambda \times \frac{1}{\lambda^2} = \frac{1}{\lambda}
 \end{aligned}$$

- Show that

$$E(X^2) = 2/\lambda^2 \text{ and } \text{Var}(X) = E(X^2) - [E(X)]^2 = 1/\lambda^2$$

## Gamma function

- Expected value and variance of exponential distribution can be easily derived using gamma function

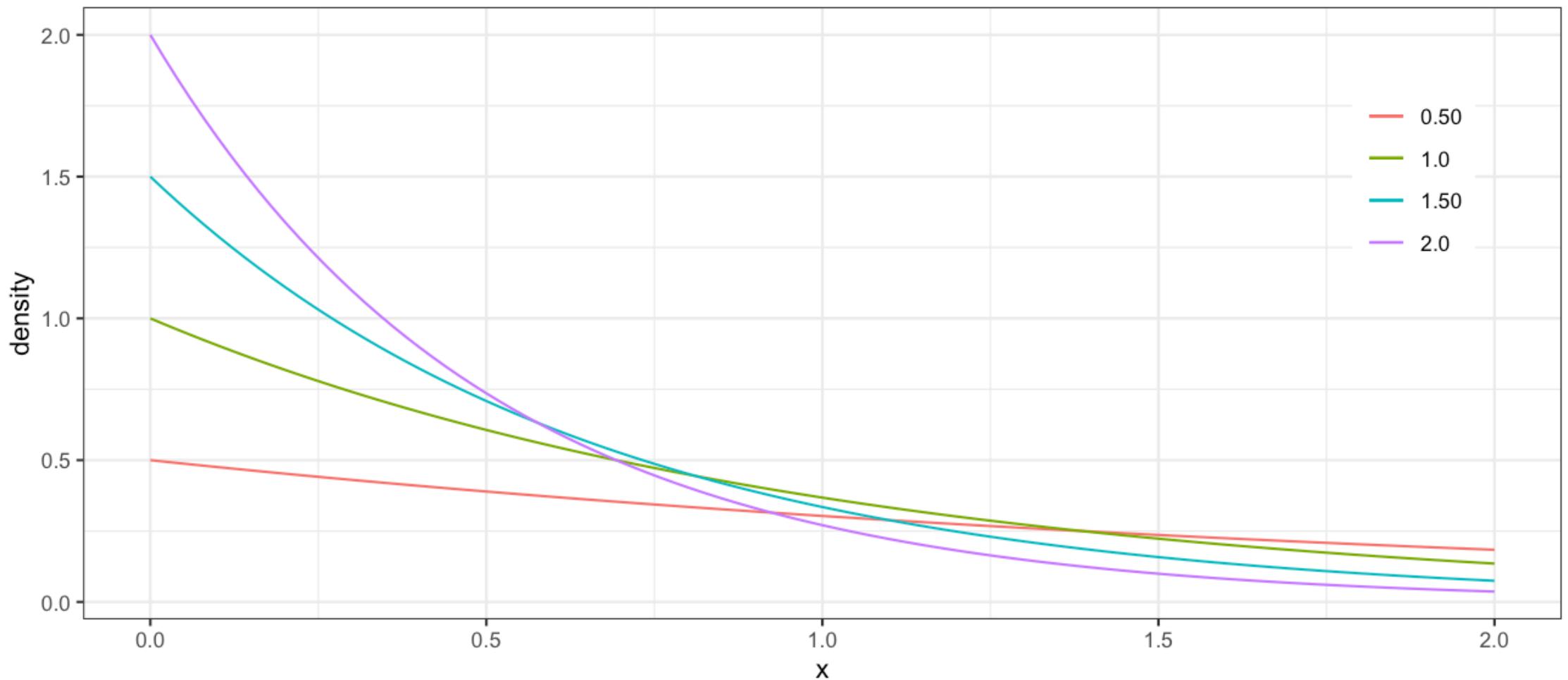
$$(k - 1)! = \Gamma k = \int_0^{\infty} x^{k-1} e^{-x} dx$$

$$E(X) = \int_0^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda} \int_0^{\infty} y^{2-1} e^{-y} dy = \frac{1}{\lambda} \quad (\text{with } y = \lambda x)$$

$$E(X^2) = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{\lambda^2}$$

## Exponential distributions for different values of $\lambda$



## Memoryless property of exponential distribution

- If a random variable  $X$  measures the time until a certain event occurs and the event has not occurred by time  $x_0$
- The additional waiting time for the event to occur beyond  $x_0$  has the same exponential distribution as  $X$

## Memoryless property of exponential distribution

$$\begin{aligned} P(X \geq x_0 + y \mid X \geq x_0) &= \frac{P(X \geq x_0 + y, X \geq x_0)}{P(X \geq x_0)} \\ &= \frac{P(X \geq x_0 + y)}{P(X \geq x_0)} \\ &= \frac{e^{-\lambda(x_0+y)}}{e^{-\lambda x_0}} \\ &= e^{-\lambda y} \\ &= P(X \geq y) \end{aligned}$$

## Homework 4A

*Problems 4.2.2:* Suppose you are waiting for a friend to call you, and the time you wait in minutes has an exponential distribution with parameter  $\lambda = 0.1$ .

- What is the expectation of your waiting time?
- What is the probability that you will wait longer than 10 minutes?
- What is the probability that you will wait less than 5 minutes?
- Suppose that after 5 minutes, you are still waiting for the call. What is the distribution of your additional waiting time? In this case, what is the probability that your total waiting time is longer than 15 minutes?

## Homework 4A

*Problems 4.2.3:* The time in days between breakdowns of a machine is exponentially distributed with  $\lambda = 0.2$ .

- What is the expected time and corresponding standard deviation between machine breakdowns?
- What is the median time between machine breakdowns?
- What is the probability that after the machine is repaired, it will last at least a week before failing again?
- If the machine has performed satisfactorily for six days, what is the probability that it lasts at least two more days before breaking down?

# The Normal Distribution

## The Normal Distribution

- Normal distribution has two parameters
  - Mean,  $E(X) = \mu$
  - variance,  $Var(X) = \sigma^2$
- A random variable  $X$  follows a normal distribution with mean  $\mu$  and variance  $\sigma^2$

$$X \sim N(\mu, \sigma^2)$$

- $-\infty < X < \infty$ ,  $-\infty < \mu < \infty$  and  $\sigma^2 > 0$

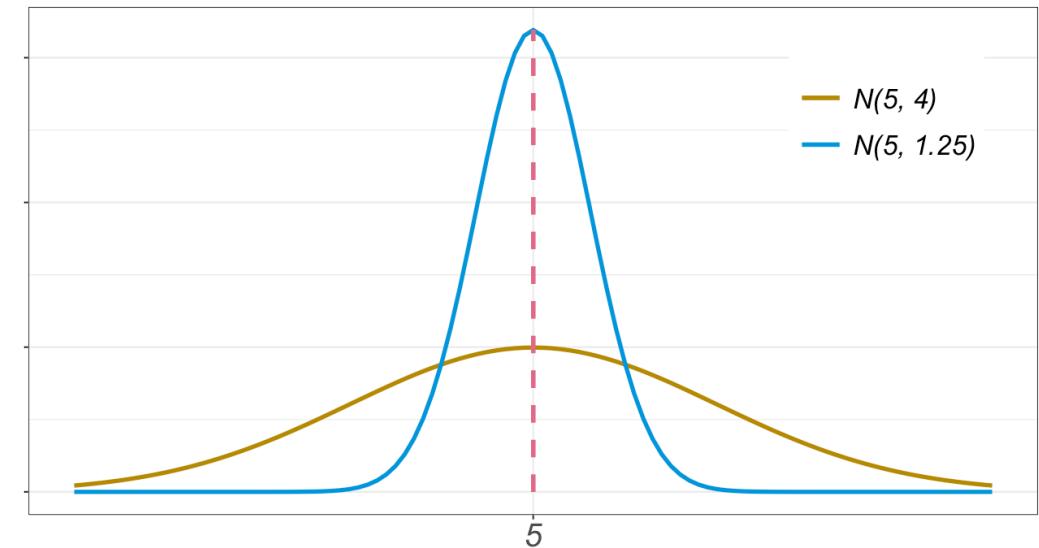
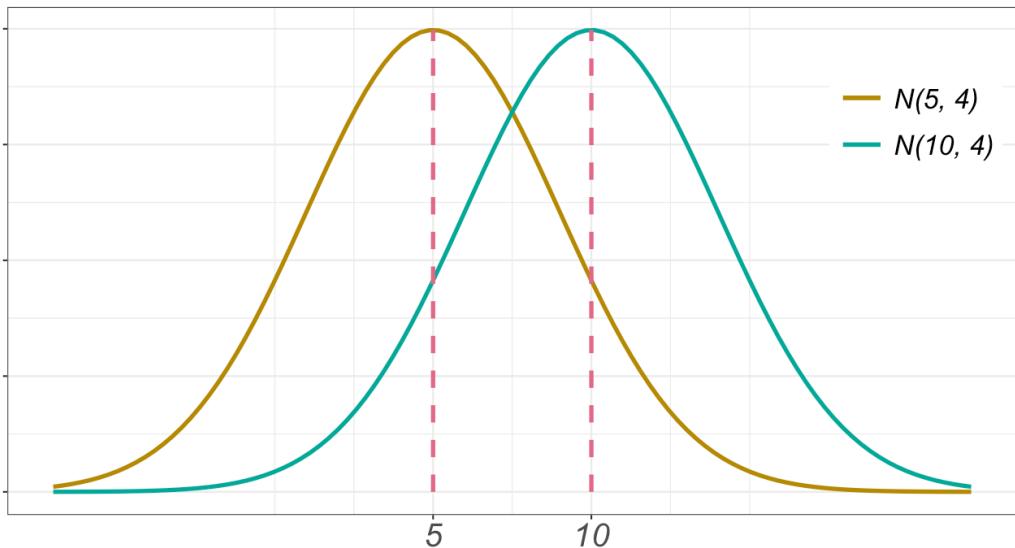
## The Normal Distribution

- The probability density function of  $X$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty$$

- The normal density function is bell-shaped and symmetric about its mean  $\mu$

# Parameters of normal distribution and its density function



# The Standard Normal Distribution

## The Standard Normal Distribution

- A normal distribution with parameters  $\mu = 0$  and  $\sigma = 1$  is known as the standard normal distribution
- The probability density function of the standard normal distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

- The standard normal distribution is symmetric about  $x = 0$
- $\phi \rightarrow$  "phi" (lowercase)

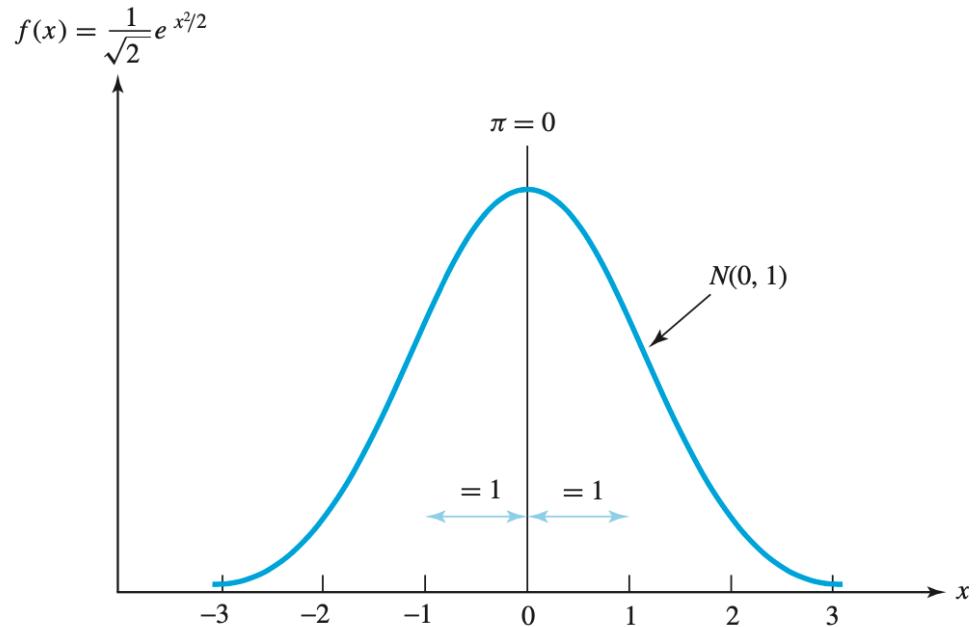
## The Standard Normal Distribution

- The cumulative distribution function (CDF) of the standard normal distribution

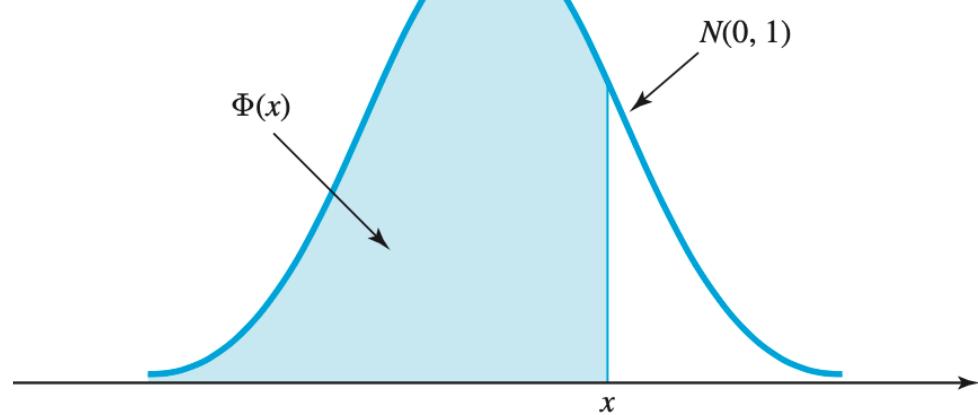
$$\Phi(x) = P(X \leq x) = \int_{-\infty}^x \phi(y) dy$$

- There is no closed-form solution of  $\Phi(x)$
- $\Phi(x)$  can be obtained using statistical software or tables of standard normal distribution
- $\Phi \rightarrow$  "phi" (uppercase)

## Probability density function



## Cumulative distribution function



- $\Phi(x) = P(X \leq x)$

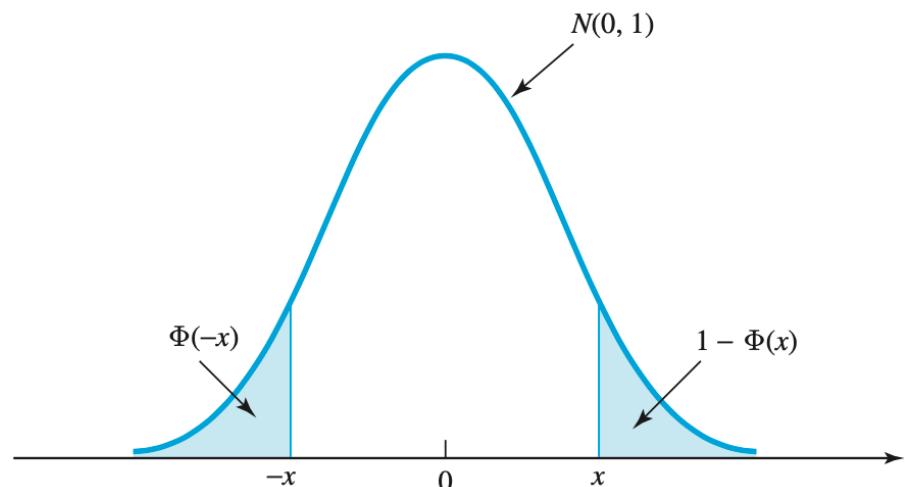
## Standard normal distribution

- Standard normal distribution is symmetric about the value 0

$$\Phi(x) = P(X \leq x)$$

$$\begin{aligned}\Rightarrow 1 - \Phi(x) &= P(X > x) \\ &= P(X \leq -x) \\ &= \Phi(-x)\end{aligned}$$

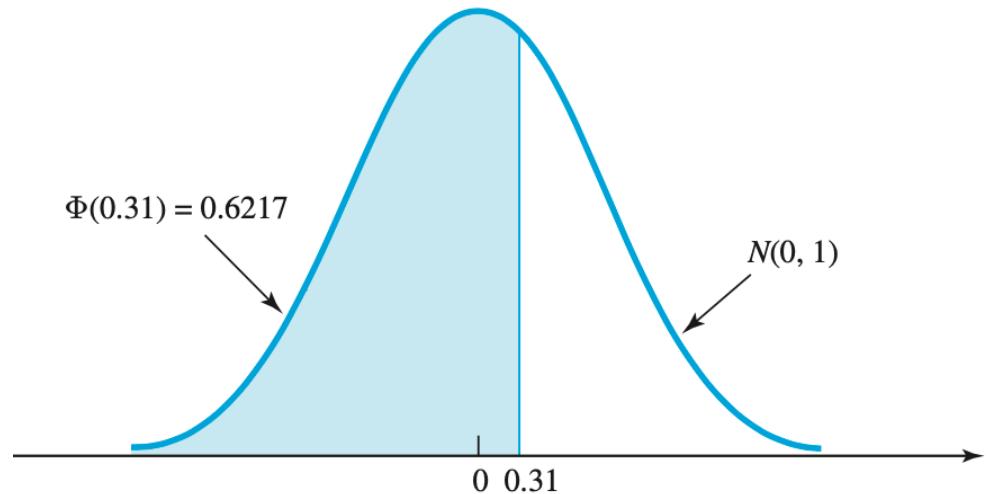
$$\Rightarrow \Phi(x) + \Phi(-x) = 1$$



## The standard normal distribution

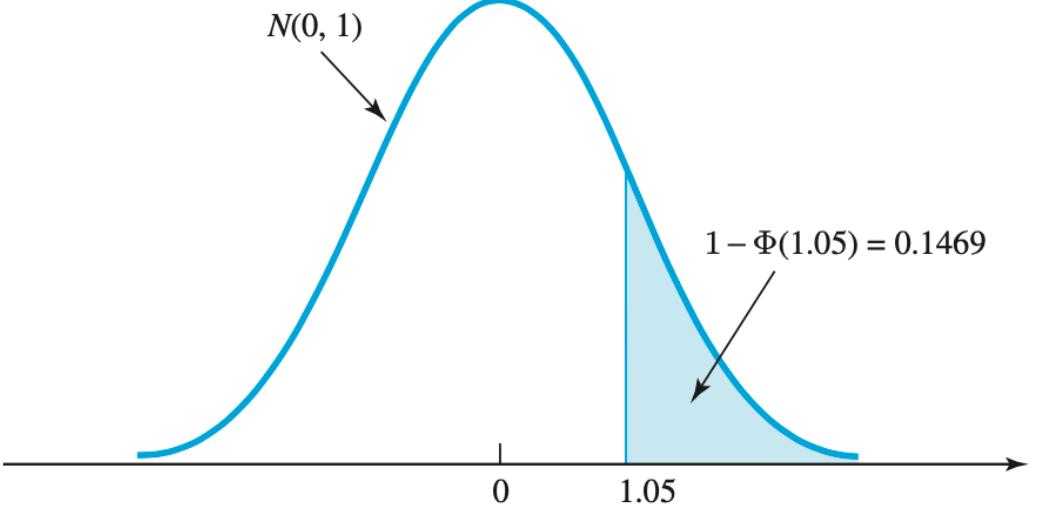
- Using standard normal table (z-table)

$$P(X \leq .31) = \Phi(.31) = 0.6217$$



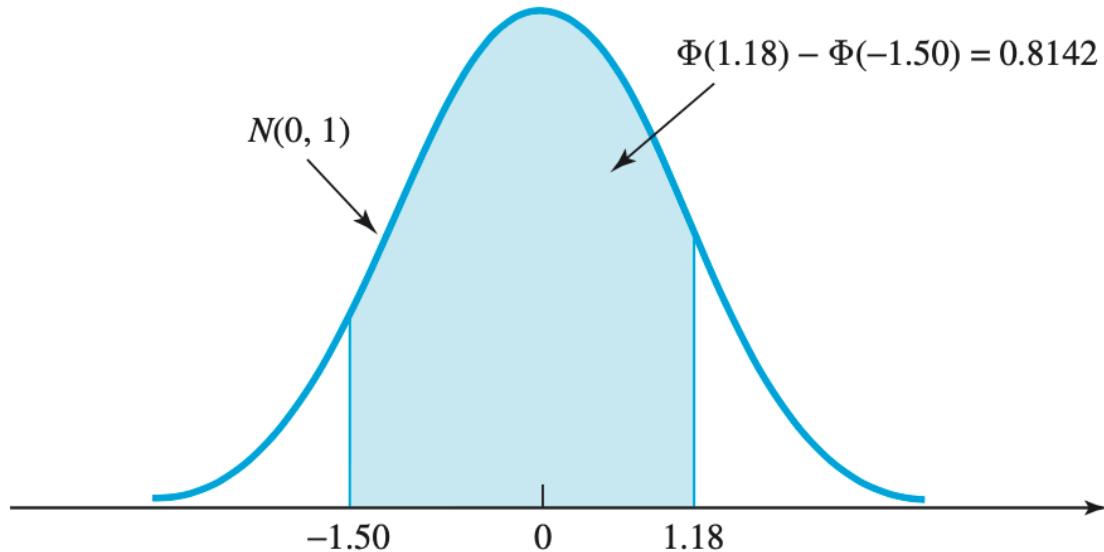
## The standard normal distribution

$$\begin{aligned}P(X \geq 1.05) &= 1 - \Phi(1.05) \\&= 1 - 0.8531 \\&= 0.1469\end{aligned}$$



## The standard normal distribution

$$\begin{aligned}P(-1.50 \leq X \leq 1.18) &= \Phi(1.18) - \Phi(-1.50) \\&= 0.8810 - 0.0668 = 0.8142\end{aligned}$$



## Homework 5A

**Problem 5.1.1** Suppose that  $X \sim N(0, 1)$ . Find:

(a)  $P(X \leq 1.34)$

(d)  $P(0.09 \leq X \leq 1.76)$

(b)  $P(X \geq -0.22)$

(e)  $P(|X| \leq 0.3)$

(c)  $P(-2.19 \leq X \leq 0.43)$

(f)  $P(|X| \geq 0.23)$

## Quantiles/percentiles

- Let  $x_p$  is the  $p$ th quantile of  $X \sim N(0, 1)$ , and  $x_p$  must satisfy

$$P(X \leq x_p) = \Phi(x_p) = p \quad \Rightarrow \quad x_p = \Phi^{-1}(p)$$

- For example, the median is defined as

$$m = \Phi^{-1}(0.5) = 0$$

## Quantiles/percentiles

- The 0.8 quantile (or 80th percentile) of  $X \sim N(0, 1)$  satisfies

$$P(X \leq x) = \Phi(x) = 0.80$$

- From the  $z$ -table

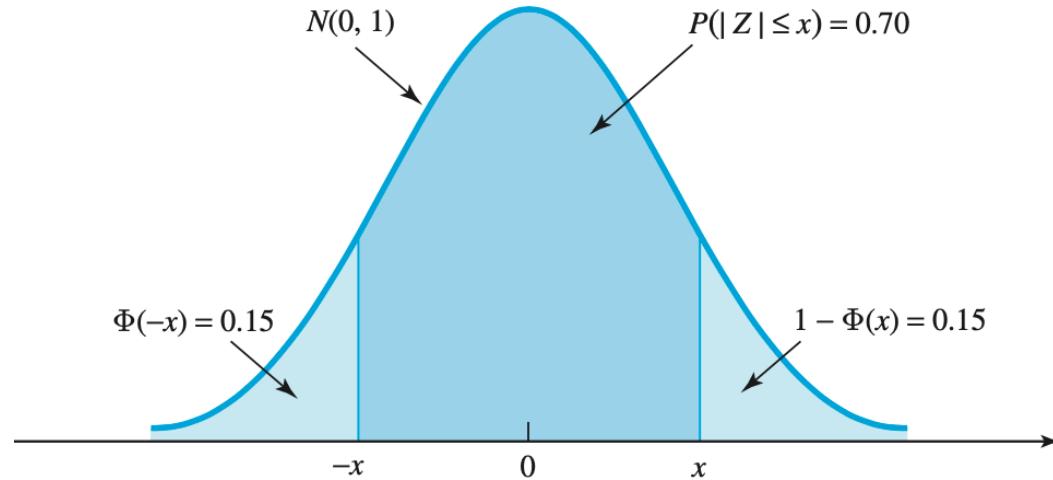
$$\Phi(0.84) = 0.7995 \text{ and } \Phi(0.85) = 0.8023$$

- The 0.8 quantile of  $X \sim N(0, 1)$  lies between the values 0.84 and 0.85

$$x \simeq (0.84 + 0.85)/2 = 0.845$$

- **Notations:**  $\Phi(x) = 0.80 \Rightarrow x = \Phi^{-1}(0.80) = 0.845$

- Find the value of  $x$  so that  $P(|X| \leq x) = 0.7$



- Since  $\Phi(-x) = 0.15$ , from the z-table we can find that  $x$  lies between 1.03 and 1.04.
  - $x \simeq 1.035$

## Quantiles/percentiles

- It can also be shown that

$$P(|X| \leq x) = P(-x < X < x) = p$$

$$\Phi(x) - \Phi(-x) = p$$

$$\Phi(x) - [1 - \Phi(x)] = p$$

$$2\Phi(x) - 1 = p$$

$$\Phi(x) = (1 + p)/2$$

$$x = \Phi^{-1}\left((1 + p)/2\right)$$

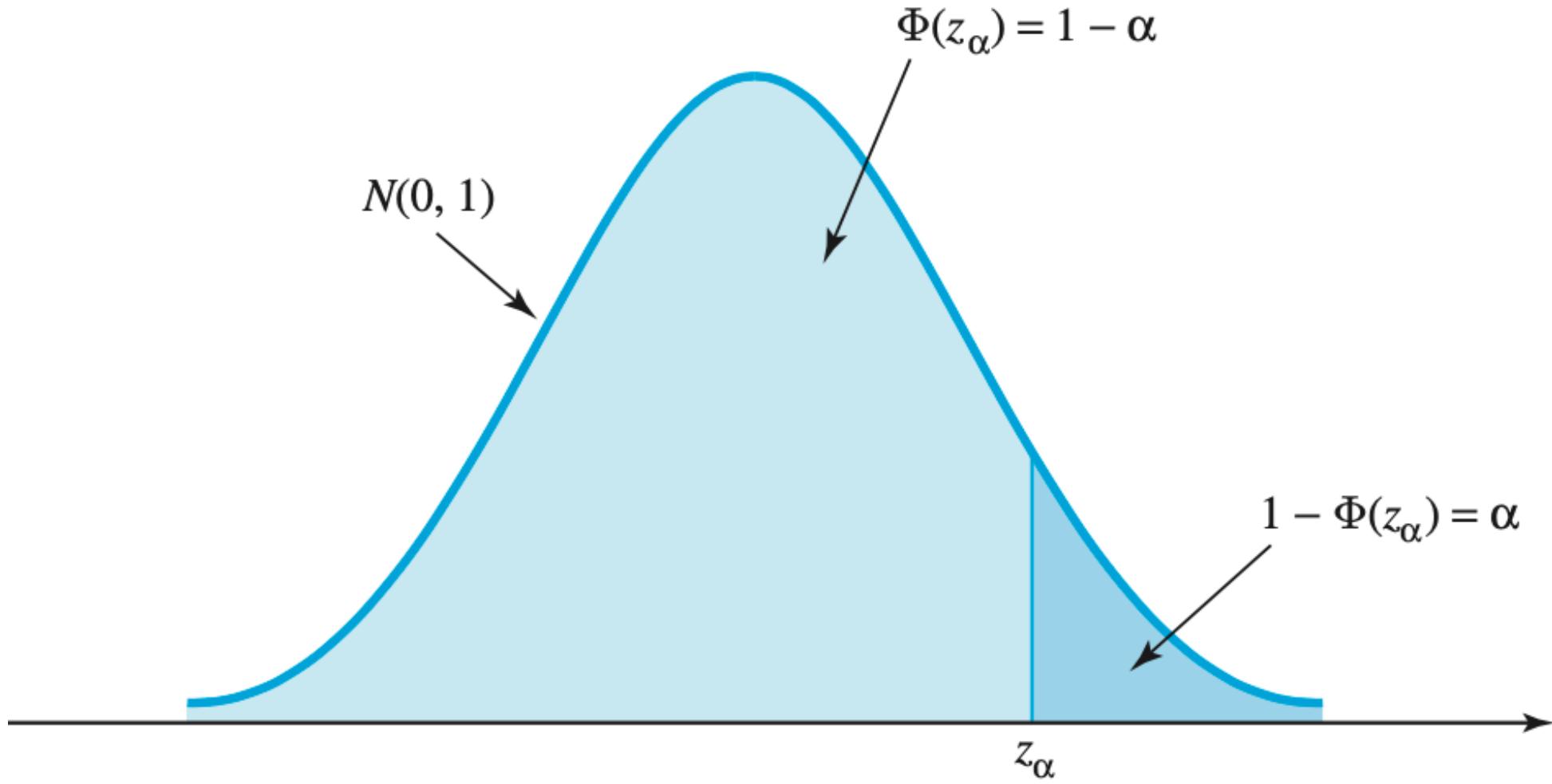
## Quantiles/percentiles

- $(1 - \alpha)100^{th}$  percentile or  $(1 - \alpha)$  quantile of the standard normal distribution is denoted by  $z_\alpha$  (for  $0 < \alpha < 0.5$ )

$$\begin{aligned} P(X \leq z_\alpha) &= 1 - \alpha \\ \Rightarrow \Phi(z_\alpha) &= 1 - \alpha \Rightarrow z_\alpha = \Phi^{-1}(1 - \alpha) \end{aligned}$$

- The quantile  $z_\alpha$  is also referred as *critical points* of the standard normal distribution, e.g.

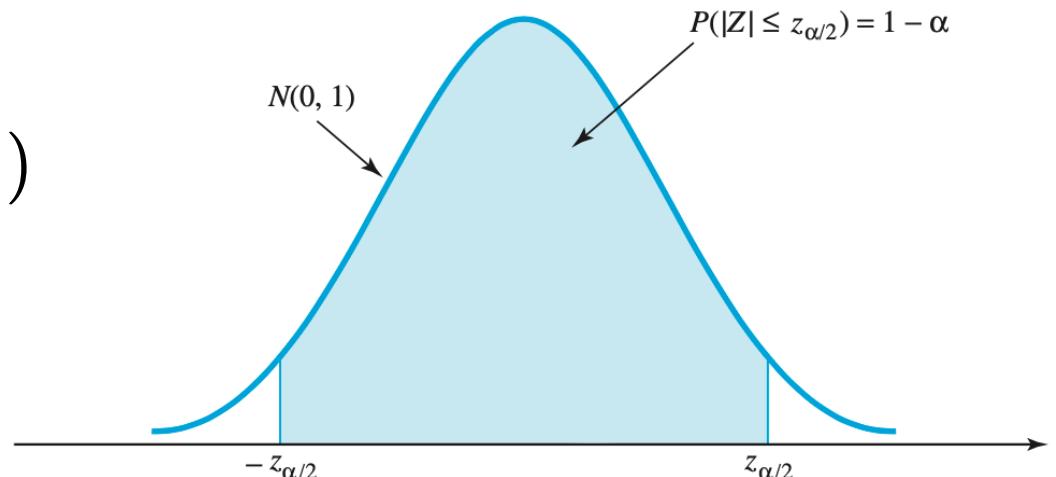
$$P(X < z_{0.05}) = 1 - 0.05 = 0.95$$



## Quantiles/percentiles

- It can be shown that

$$\begin{aligned} P(|X| \leq z_{\alpha/2}) &= P(-z_{\alpha/2} \leq X \leq z_{\alpha/2}) \\ &= \Phi(z_{\alpha/2}) - \Phi(-z_{\alpha/2}) \\ &= (1 - \alpha/2) - \alpha/2 \\ &= 1 - \alpha \end{aligned}$$



## Quantiles/percentiles

- Standard normal quantiles that are commonly used in statistical analysis

$$P(X < -1.96) = P(X > 1.96) = 0.025$$

$$\begin{aligned} P(|X| \leq 1.96) &= P(-1.96 \leq X \leq 1.96) \\ &= 1 - 2 \times 0.025 \\ &= 0.95 \end{aligned}$$

- $z_{.025} = 1.96$ ,  $z_{.05} = 1.64$ , and  $z_{.10} = 1.28$

## Homework 5A

*Problem 5.1.1:* Suppose that  $X \sim N(0, 1)$ . Find:

- (f) The values of  $x$  for which  $P(X \leq x) = 0.55$
- (g) The values of  $x$  for which  $P(X \geq x) = 0.72$
- (h) The values of  $x$  for which  $P(|X| \leq x) = 0.31$

## General normal distribution

- If  $X \sim N(\mu, \sigma^2)$  then the transformed (standardized) variable

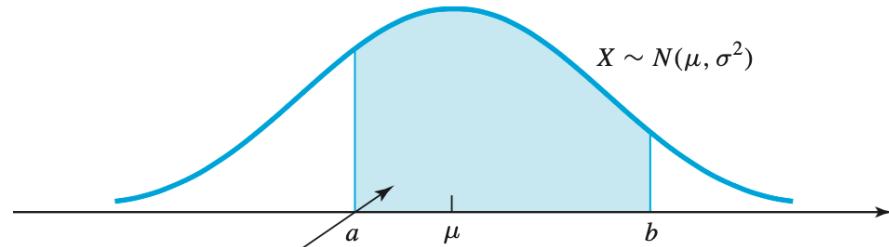
$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

- Show that
  - $E(Z) = 0$
  - $V(Z) = 1$
  - $Z$  follows a normal distribution

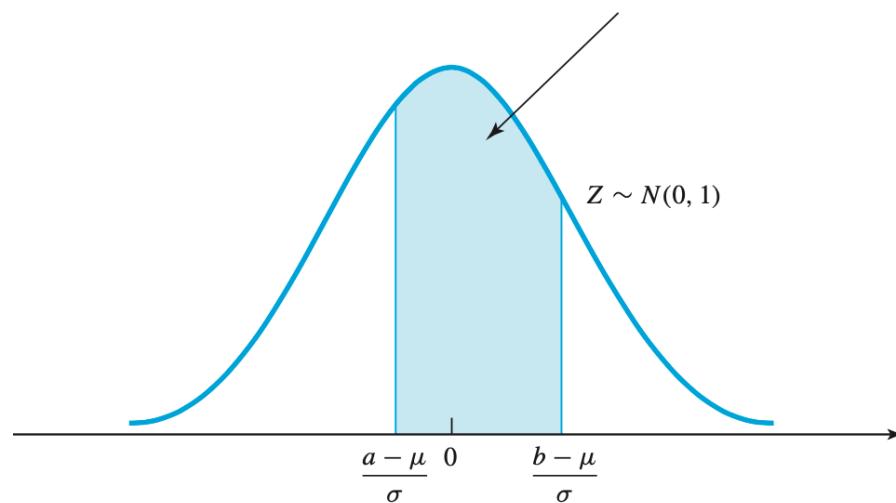
## General normal distribution

- A probability statement related to  $X \sim N(\mu, \sigma^2)$  can be expressed in terms of  $Z \sim N(0, 1)$ , i.e. in terms of  $\Phi(x) = P(X \leq x)$

$$\begin{aligned}P(a \leq X \leq b) &= P\left(\frac{a - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{b - \mu}{\sigma}\right) \\&= P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right) \\&= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)\end{aligned}$$



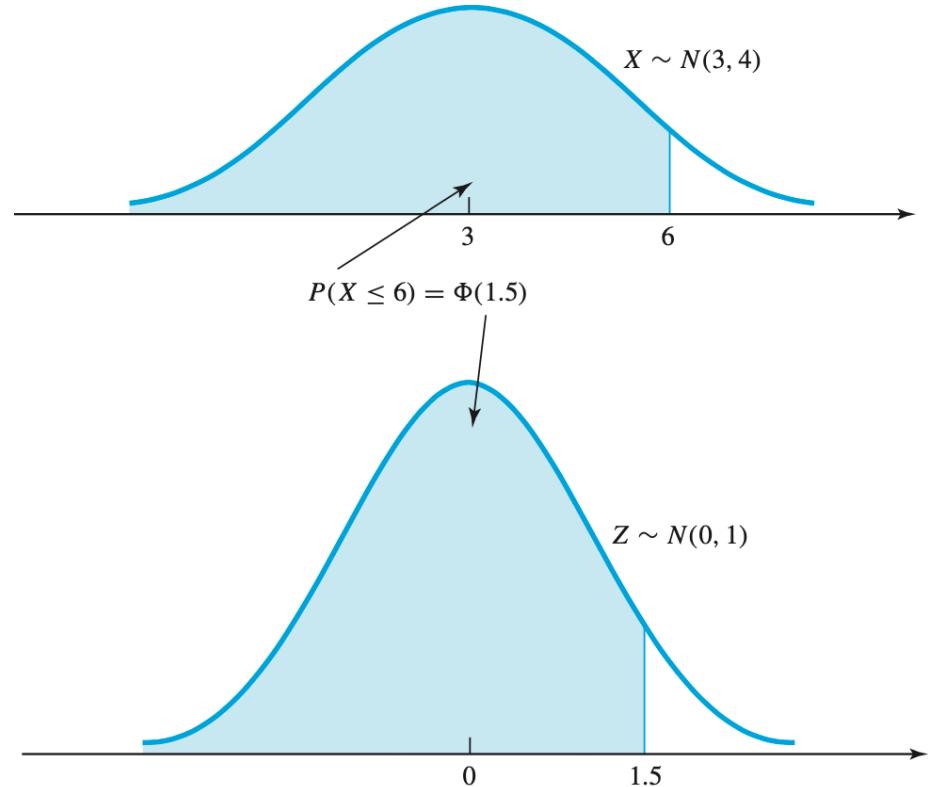
$$P(a \leq X \leq b) = P\left(\frac{a - \mu}{\sigma} \leq Z \leq \frac{b - \mu}{\sigma}\right) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$



## Probability calculations for normal distributions

Suppose  $X \sim N(3, 4)$

$$\begin{aligned} P(X \leq 6) &= \Phi\left(\frac{6 - 3}{2}\right) \\ &= \Phi(1.5) \\ &= 0.9332 \end{aligned}$$



## Probability calculations for normal distributions

Suppose  $X \sim N(3, 4)$

$$\begin{aligned} P(2 \leq X \leq 5.4) &= \Phi\left(\frac{5.4 - 3}{2}\right) - \Phi\left(\frac{2 - 3}{2}\right) \\ &= \Phi(1.2) - \Phi(-0.5) \\ &= 0.8849 - 0.3085 \\ &= 0.5764 \end{aligned}$$

## Homework 5B

**Problem 5.1.3** Suppose that  $X \sim N(10, 2)$ . Find:

- (a)  $P(X \leq 10.3)$
- (b)  $P(X \geq 11.98)$
- (d)  $P(10.88 \leq X \leq 13.22)$
- (e)  $P(|X - 10| \leq 3)$

## Percentiles of $X \sim N(\mu, \sigma^2)$

- The  $p$ th percentile of  $X \sim N(\mu, \sigma^2)$  satisfies

$$\begin{aligned} P(X \leq x_p) = p &\Rightarrow \Phi\left(\frac{x_p - \mu}{\sigma}\right) = p \\ &\Rightarrow \frac{x_p - \mu}{\sigma} = \Phi^{-1}(p) \\ &\Rightarrow x_p = \mu + \sigma \Phi^{-1}(p) \end{aligned}$$

- For  $X \sim N(16, 6)$ , find  $x$  for which  $P(X \leq x) = .78$

## Homework 5B

**Problem 5.1.3** Suppose that  $X \sim N(10, 2)$ . Find:

- (f) The value of  $x$  for which  $P(X \leq x) = 0.81$
- (g) The value of  $x$  for which  $P(X \geq x) = 0.04$
- (h) The value of  $x$  for which  $P(|X - 10| \geq x) = 0.63$

## Percentiles of $X \sim N(\mu, \sigma^2)$

$$P(Z \leq z_\alpha) = 1 - \alpha \Rightarrow z_\alpha \Rightarrow (1 - \alpha)^{\text{th}} \text{ percentile of } Z \sim N(0, 1)$$

$$P(Z \leq z_\alpha) = P(X \leq \mu + \sigma z_\alpha) = 1 - \alpha$$

$$\mu + \sigma z_\alpha \Rightarrow (1 - \alpha)^{\text{th}} \text{ percentile of } X \sim N(\mu, \sigma^2)$$

- E.g., 95th percentile of
  - $Z \sim N(0, 1) \Rightarrow z_{.05} = 1.645$
  - $X \sim N(3, 4) \Rightarrow \mu + \sigma z_{.05} = 3 + (2 \times 1.645) = 6.29$

## Percentiles of $X \sim N(\mu, \sigma^2)$

- For  $Z \sim N(0, 1)$

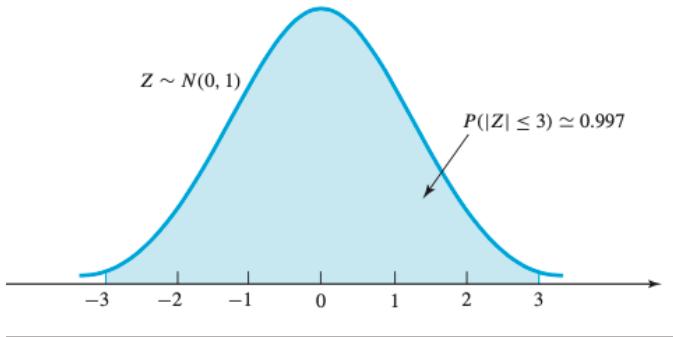
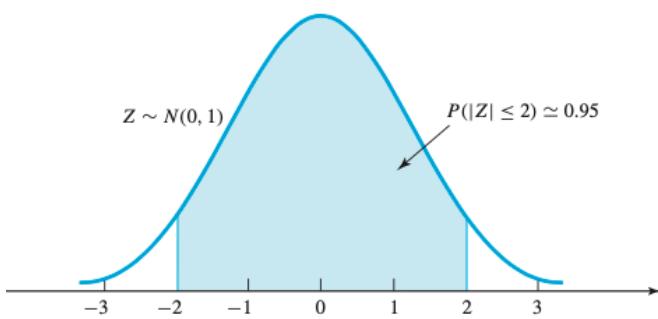
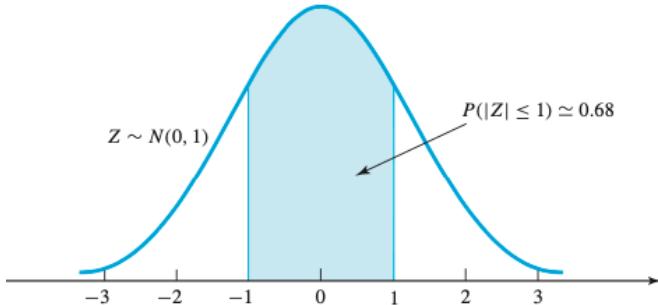
$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$$

- For  $X \sim N(\mu, \sigma^2)$

$$P(\mu - \sigma z_{\alpha/2} < X < \mu + \sigma z_{\alpha/2}) = 1 - \alpha$$

- For  $X \sim N(10, 4)$ ,  $\alpha = 0.10$ , and  $z_{0.05} = 1.645$

$$\mu - \sigma z_{\alpha/2} = 10 - 2(1.645) \text{ and } \mu + \sigma z_{\alpha/2} = 10 + 2(1.645)$$



## Example 37

- A company manufactures concrete blocks that are used for construction purposes.
- Suppose that the weights of the individual concrete blocks are normally distributed with  $\mu = 11.0\text{kg}$  and  $\sigma = 0.3\text{kg}$ .
- Obtain the interval so that the company can be 99% confident that a randomly selected concrete block weights within the interval

$$(1 - \alpha)100\% = 99\% \Rightarrow \alpha = 0.01 \Rightarrow \alpha/2 = .005$$

## Example 37

- The interval is

$$\begin{aligned}(\mu - \sigma z_{.005}, \mu + \sigma z_{.005}) &= [11 - .3 \times 2.576, 11 + .3 \times 2.576] \\&= [10.23, 11.77]\end{aligned}$$

- Obtain  $P(X \leq 10.5) = ?$

## Example 35

- A Wall Street analyst estimates that the annual return from the stock of company "A" can be considered to be an observation from a normal distribution with
  - $\mu = 8.0\%$  and  $\sigma = 1.5\%$ .

## Example 35

- The analyst's investment choices are based upon the considerations that any return greater than 5% is "satisfactory" and a return greater than 10% is "excellent."
  - Find the probability that company A's stock will prove "unsatisfactory".
  - Find the probability that company A's stock will prove "excellent".

## Homework 5A

### 5.1.8

- What are the upper and lower quartiles of a  $N(0, 1)$  distribution?
- What is the interquartile range?
- What is the interquartile range of a  $N(\mu, \sigma^2)$  distribution?

## Homework 5A

### 5.1.9

- The thicknesses of glass sheets produced by a certain process are normally distributed with  $\mu = 3.00$  mm and  $\sigma = 0.12$  mm.
  - What is the probability that a glass sheet is thicker than 3.2 mm?
  - What is the probability that a glass sheet is thinner than 2.7 mm?
  - What is the value of  $c$  for which there is a 99% the probability that a glass sheet has a thickness within the interval  $[3.00 - c, 3.00 + c]$ ?

## Homework 5C

**5.7.3** Adult salmon lengths ( $X$ ) follows a normal distribution with  $\mu = 70$  cm and  $\sigma = 5.4$  cm.

- What is the probability that an adult salmon is longer than 80 cm?
- What is the probability that an adult salmon is shorter than 55 cm?
- What is the probability that an adult salmon is between 65 and 78 cm long?
- What is the value of  $c$  for which there is a 95% probability that an adult salmon has a length within the interval  $[70 - c, 70 + c]$ ?