Probability Theory

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Plan

- Probability and probability values
- Events and combination of events
- Conditional probability and independent events
- Posterior probabilities

Probability and probability values

Probabilities

- Probability theory is a branch of mathematics that has been developed to deal with uncertainty
- Initially, mathematicians involved in analyzing gambling and mortality tables developed the theory of probability
- Probability is a scientific tool for dealing with *chance*
- Probability theory is now recognized as one of the most interesting and useful areas of mathematics

Probabilities

- It provides the basis for the science of statistical inference through experimentation and data analysis
- Probability theory is particularly relevant to the engineering sciences
  - assessment of system reliability, maintenance of suitable quality controls, etc.
- There are three ways to calculate probability in practice
  - subjective, empirical, and axiomatic

Experiment

- An experiment can be thought of as any process or procedure for which more than one outcome is possible
- E.g., the procedure of "tossing a coin" is an experiment as it leads to two possible outcomes
    - Either a "Head" or a "Tail"
- The goal of probability theory is to provide a mathematical structure for understanding the chances of the various outcomes of an experiment

Sample Space

- The *sample space* of an experiment is a set consisting of all of the possible experimental outcomes
  - A sample space is denoted by $\mathcal{S}$
  - For the experiment "tossing a coin", the sample space consists of two elements

$$\mathcal{S} = \{head, tail\}$$

Example 1: *Machine Breakdowns*

- A maintenance engineer notices that machine breakdown can be characterized as due to one of the following three causes:
  - An electrical failure within the machine
  - A mechanical failure of some component of the machine
  - Operator misuse

Example 1: *Machine Breakdowns*

- When the machine is running, the engineer is uncertain what will be the cause of the following breakdown

- The problem can be thought of as an experiment with the sample space

$$\mathcal{S} = \{\text{electrical}, \text{mechanical}, \text{misuse}\}$$

Example 2: *Defective Computer Chips*

- A company sells computer chips in boxes of 500, and each chip can be classified as either *satisfactory* or *defective*

- *The number of defective chips in a particular box of 500 chips* is uncertain

- The corresponding sample space is defined as

$$\mathcal{S} = \{0 \text{ defectives}, 1 \text{ defective}, \ldots, 500 \text{ defectives}\}$$

Example 3: *Software Errors*

- The outcome of interest is the number of separate errors in a particular piece of software

- The sample space of this experiment would be the set of all possible integers

$$\mathcal{S} = \{0, 1, 2, 3, \ldots\}$$

Example 4: *Power Plant Operation*

- A manager supervises the operation of three power plants: X, Y, and Z
- At any given time, each plant can be classified as
  - either *generating electricity* (1) or being *idle* (0)
- The notation (x, y, z) is used to denote the outcome of the experiment
  - x, y, and z represent the status of the plant X, Y, and Z, respectively

Example 4: *Power Plant Operation*

- E.g., the outcome $(0, 1, 0)$ indicates that the plant Y is generating electricity but plants X and Z are idle

- The sample space for the status of the three plants at a particular point in time is defined as

$$\mathcal{S} = \Big\{(0,0,0), (0,0,1), (0,1,0), (1,0,0), \\ (0,1,1), (1,0,1), (1,1,0), (1,1,1)\Big\}$$

Games of chance

- *Toss a single coin*

$$\mathcal{S} = \{\text{head}, \text{tail}\}$$

- *Toss of two coins*

$$\mathcal{S} = \Big\{(\text{head}, \text{head}), (\text{head}, \text{tail}), (\text{tail}, \text{head}), (\text{tail}, \text{tail})\Big\}$$

  - *(head, tail)* denotes the outcome that the first coin resulted in a "head" and the second one resulted in a "tail"
  - *(head, tail)* and *(tail, head)* represent different outcomes of the experiment

Games of chance

- *Roll a six-sided die:* $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$
- *Rolling two dice:* $S = \{(1, 1), \ldots, (6, 6)\}$

(1 , 1)	(1 , 2)	(1 , 3)	(1 , 4)	(1 , 5)	(1 , 6)
(2 , 1)	(2 , 2)	(2 , 3)	(2 , 4)	(2 , 5)	(2 , 6)
(3 , 1)	(3 , 2)	(3 , 3)	(3 , 4)	(3 , 5)	(3 , 6)
(4 , 1)	(4 , 2)	(4 , 3)	(4 , 4)	(4 , 5)	(4 , 6)
(5 , 1)	(5 , 2)	(5 , 3)	(5 , 4)	(5 , 5)	(5 , 6)
(6 , 1)	(6 , 2)	(6 , 3)	(6 , 4)	(6 , 5)	(6 , 6)

Games of chance

- *Rolling a die and a coin*

(1 , head)	(2 , head)	(3 , head)	(4 , head)	(5 , head)	(6 , head)
(1 , tail)	(2 , tail)	(3 , tail)	(4 , tail)	(5 , tail)	(6 , tail)

General rule of counting

- Suppose there are m possible outcomes of an experiment A and n possible outcomes of an experiment B
- There there are $(m \times n)$ possible outcomes of an experiment which involves both experiments A and B
- For example, if an experiment involves rolling two dice, then there will be 36 $(= 6 \times 6)$ elements in the sample space

Probability Values

Probability Values

- Each outcome of an experiment is assigned to a probability value
- Consider a sample space with n element

$$\mathcal{S} = \{O_1, O_2, \ldots, O_n\}$$

- The corresponding probability values are $p_1, p_2, \ldots, p_n$
    - p_1 is the probability of observing the outcome O_1, i.e.,

$$P(O_1) = p_1$$

    - Similarly, $P(O_2) = p_2$, $P(O_3) = p_3$, and so on

Probability Values

- The probability values must satisfy the following two conditions

$$(i) \quad 0 \leq p_1 \leq 1, \ldots, 0 \leq p_n \leq 1$$

$$(ii) \quad p_1 + p_2 + \cdots + p_n = \sum_{i=1}^{n} p_i = 1$$

- The probabilities are chosen so that the sum of the probability values over all of the elements in the sample space is one

Probability Values

- The larger the probability value of a particular outcome, the more likely it is to happen

- If two outcomes have identical probability values assigned to them, then they can be thought of as being equally likely to occur

- If one outcome has a larger probability value assigned to it than another outcome, then the first outcome can be thought of as being more likely to occur

- If all elements of a sample space with n elements are *equally likely*

$$p_i = \frac{1}{n}, \quad i = 1, \ldots, n$$

Probability Values

Electrical	Mechanical	Operator misuse
0.2	0.5	0.3

- All probability values lie between 0 and 1, and the sum of these three probability values is equal to 1
- Mechanical failures are more likely compared to the failures related to electrical and operator misuse
  - $P(mechanical) = 0.5 \rightarrow$ about half of the failures will be due to mechanical causes

Toss of a coin

- Sample space

$$\mathcal{S} = \{\text{head}, \text{tail}\}$$

- Corresponding probability values

$$P(\text{head}) = p \;\; \text{and} \;\; P(\text{tail}) = 1 - p$$

  - $0 \leq p \leq 1$

Toss of a coin

- For a fair coin, $p = 0.5$
  - There is an equal chance of observing "head" or "tail"
  - $P(head) = P(tail)$
- For a biased coin, $p \neq 0.5$
  - $p = 0.6 \ \rightarrow$ "head" is more likely to be observed compared to a "tail"

Roll of a die

- For a fair die, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, and the corresponding probability values

$$P(1) = P(2) = \cdots = P(6) = \frac{1}{6}$$

  - *Fair die* $\Longleftrightarrow$ *All elements of the sample space are equally likely*

Roll of a die

- For a biased die, one of the probability values will not be equal to 1/6

1	2	3	4	5	6
0.1	0.15	0.15	0.15	0.15	0.3

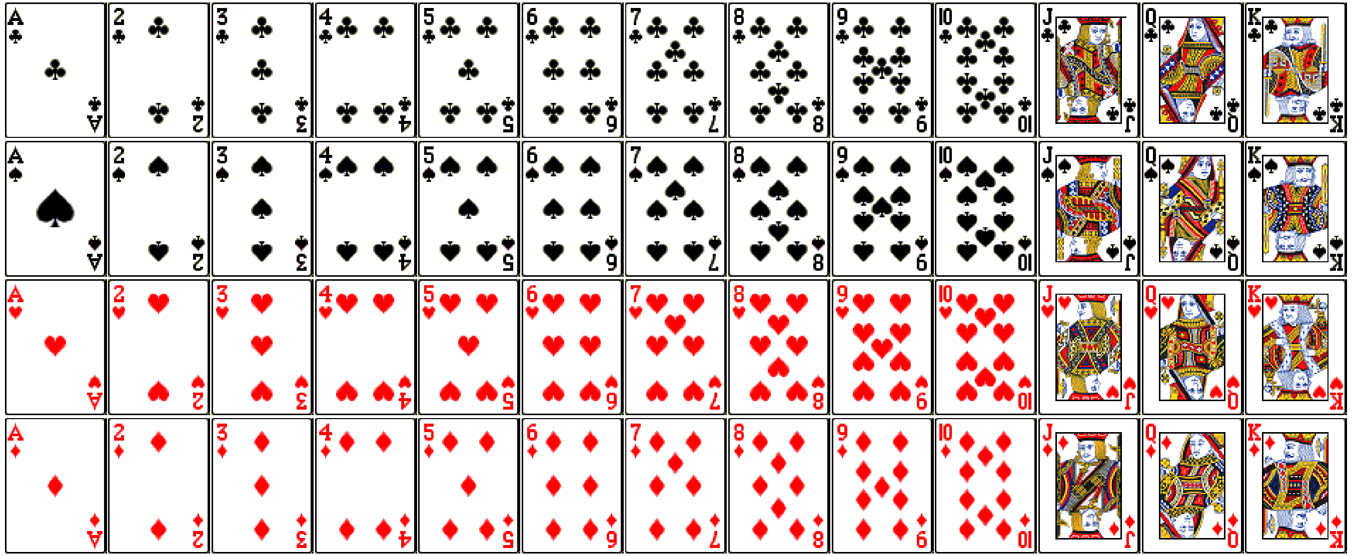
- Why are these probability values?

Rolling two fair dice

(1 , 1)	(1 , 2)	(1 , 3)	(1 , 4)	(1 , 5)	(1 , 6)
(2 , 1)	(2 , 2)	(2 , 3)	(2 , 4)	(2 , 5)	(2 , 6)
(3 , 1)	(3 , 2)	(3 , 3)	(3 , 4)	(3 , 5)	(3 , 6)
(4 , 1)	(4 , 2)	(4 , 3)	(4 , 4)	(4 , 5)	(4 , 6)
(5 , 1)	(5 , 2)	(5 , 3)	(5 , 4)	(5 , 5)	(5 , 6)
(6 , 1)	(6 , 2)	(6 , 3)	(6 , 4)	(6 , 5)	(6 , 6)

- What would be the probability values of each of these 36 outcomes?

Playing cards



- A deck of 52 cards consists of 4 suits, each of which has 13 cards
  - Two black suits (Clubs and Spades)
  - Two red suits (Hearts and Diamonds)

Homework 1A

1.1.1

- What is the sample space when a coin is tossed three times?

1.1.2

- What is the sample space for counting the number of females in a group of n people?

1.1.3.

- What is the sample space for the number of aces in a hand of 13 playing cards?

Homework 1A

1.1.4

- What is the sample space for a person's birthday?

1.1.5

- A car repair is performed on time or late, either satisfactorily or unsatisfactorily. What is the sample space for a car repair?

1.1.6

- A bag contains either red or blue balls, dull or shiny balls. What is the sample space when a ball is chosen from the bag?

Homework 1A

1.1.8

- An experiment has five outcomes: I, II, III, IV, and V.
- If $P(I) = .13$, $P(II) = .24$, $P(III) = .07$, and $P(IV) = .38$
  - What is $P(V)$?

Homework 1A

1.1.9

- An experiment has five outcomes: I, II, III, IV, and V.
- If $P(I) = .08$, $P(II) = .20$, $P(III) = .33$
  - What are the possible values of $P(V)$?
- If $P(IV) = P(V)$, what are their probability values?

Homework 1A

1.1.10

- An experiment has three outcomes: I, II and III.
- If I is twice as likely to II and II is three times as likely as III
    - What are the probability values of three outcomes?

Homework 1A

1.1.11

- A company's advertisement costs are either low with a probability of .28, average with a probability of .55, or high with a probability p.
    - What is p?

Events

Events

- Any subset of a sample space $\mathcal{S}$ is known as an event, and events are usually denoted by capital letters
  - E.g., A, B, etc. can be used to denote an event
- For an experiment of rolling a die, a subset of the sample space

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\} \text{ is } B = \{1, 3, 5\}$$

  - B is an event for observing an odd value of a die

Events

- An event is said to occur if one of the outcomes contained within the event occurs
    - E.g., the event $B = \{1, 3, 5\}$ occurs if either 1, 2, or 3 shows up and

$$P(B) = P(1) + P(3) + P(5)$$

- If elements of the sample space are equally likely

$$P(B) = \frac{n(B)}{n(S)}$$

    - $n(B)$ and $n(S)$ are number of elements in B and S, respectively

Complement of an event

- The complement of an event A, is the event consisting of everything in the sample space $\mathcal{S}$ that is not contained within the event A

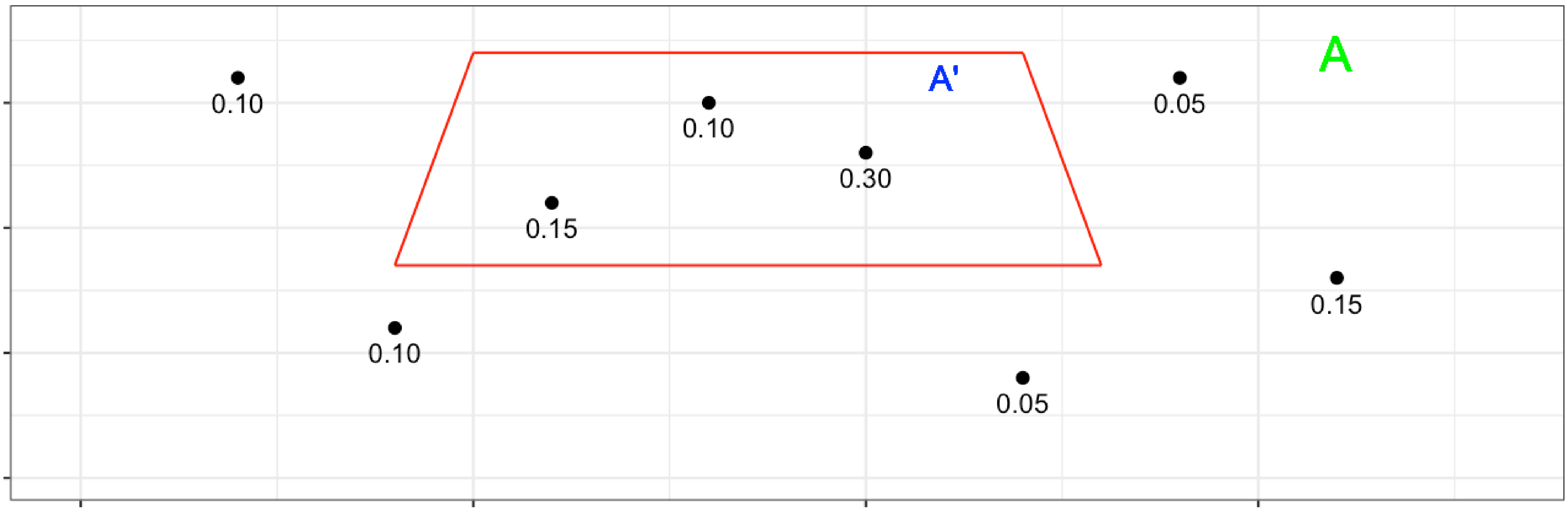
- The complement of A is denoted by A'

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{1, 3, 5\} \;\Rightarrow\; A' = \{2, 4, 6\}$$

- It can be shown that $P(A) + P(A') = 1$

- Events that consist of an individual outcome are sometimes referred to as elementary events or simple events



- $P(A') = .15 + .10 + .30 = .55$

Example 4 (Power Plant Operation)

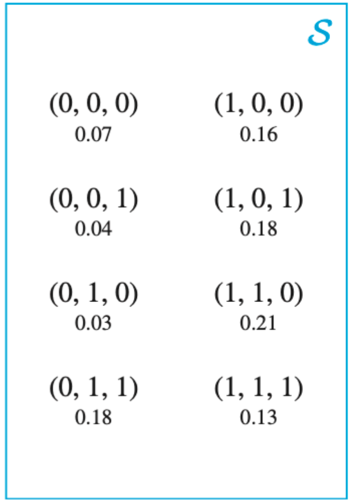


FIGURE 1.15

Probability values for power plant example

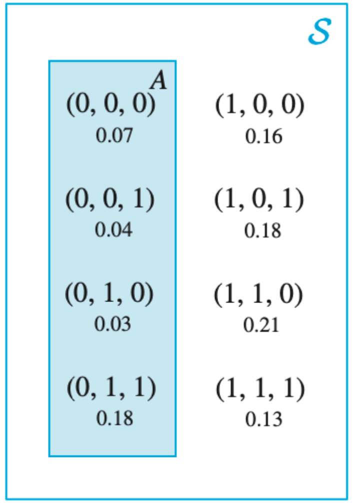


FIGURE 1.16

Event A: plant X idle

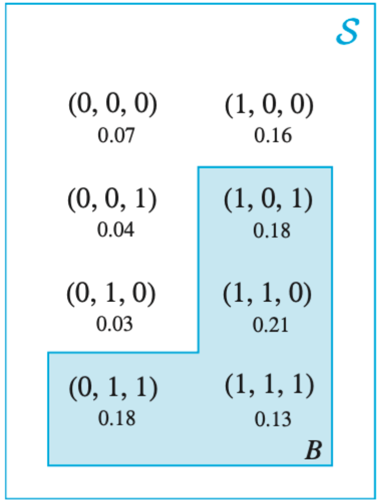


FIGURE 1.17

Event B: at least two plants generating electricity

- Calculate $P(A)$ and $P(B)$, and also for corresponding complement events.

FIGURE 1.18

Event A: sum equal to 6

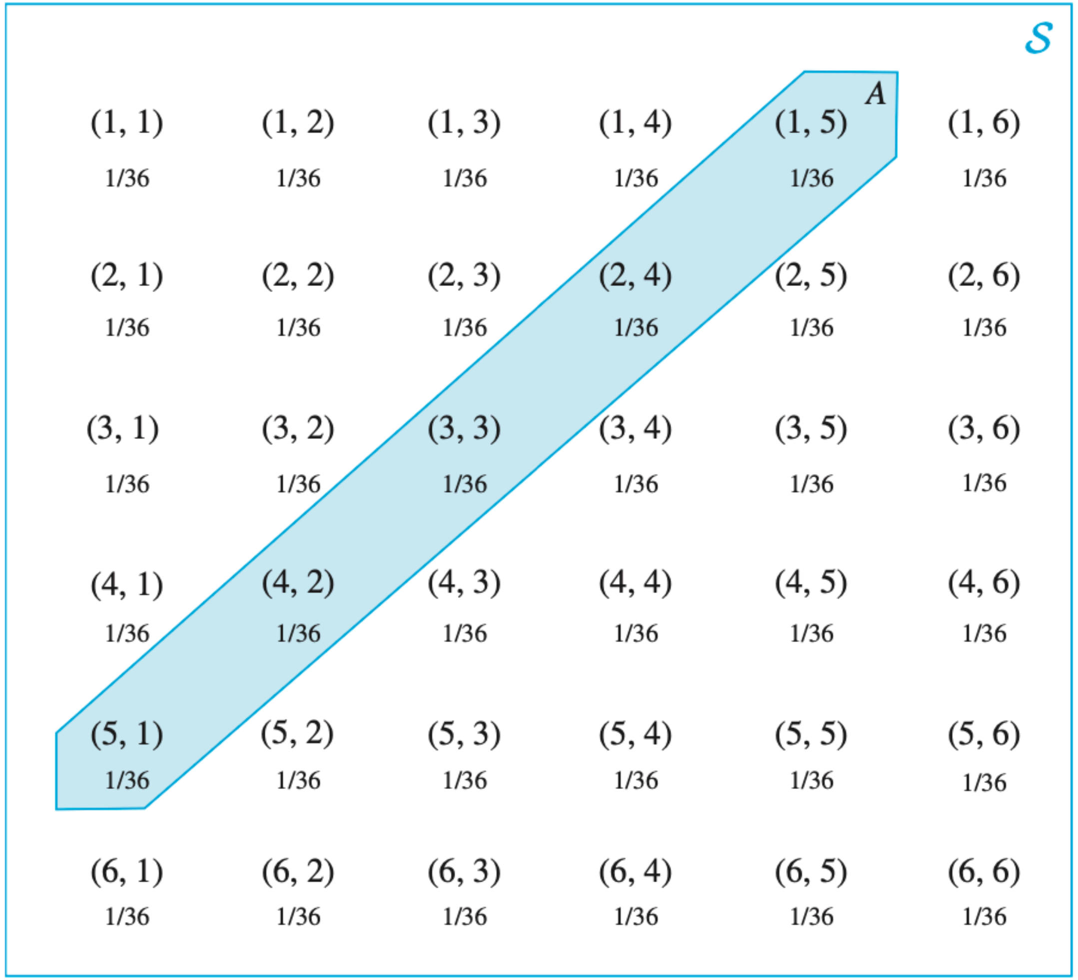
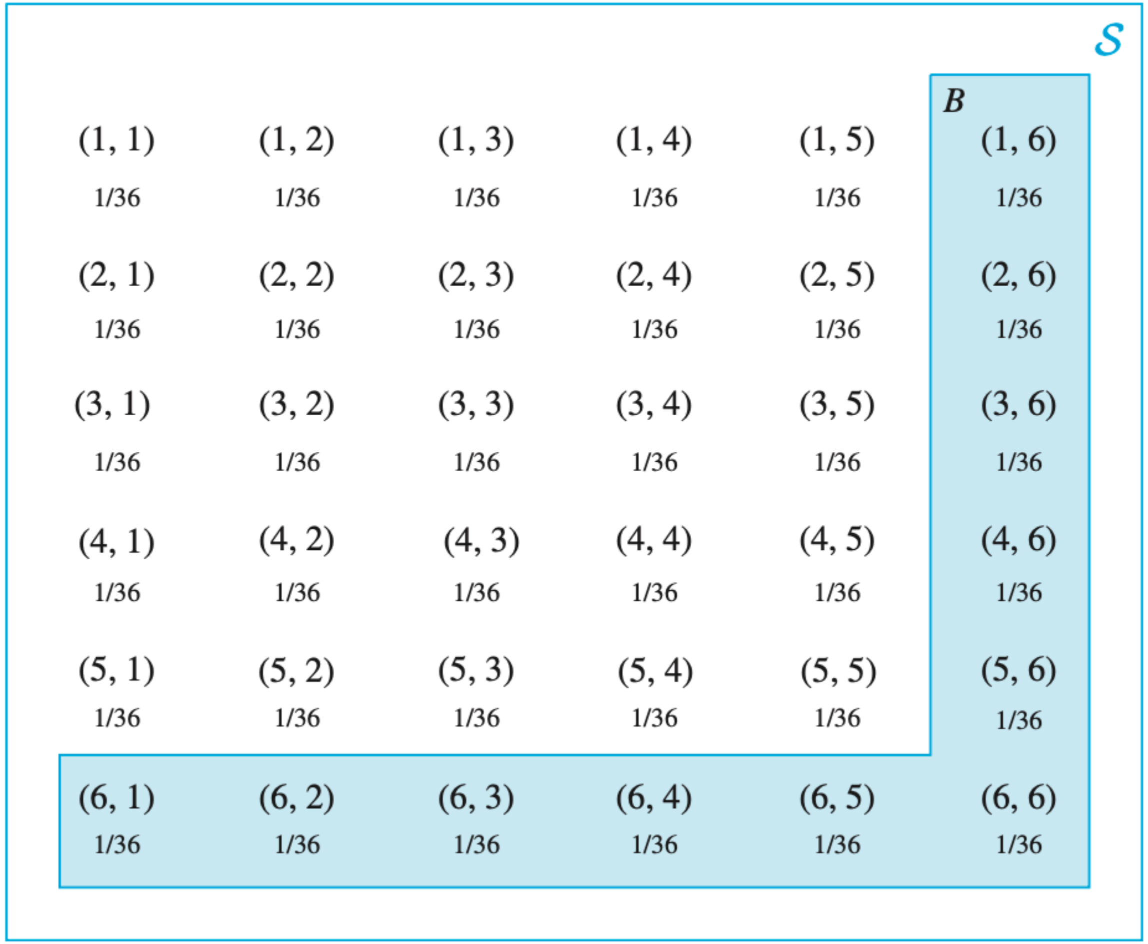


FIGURE 1.19

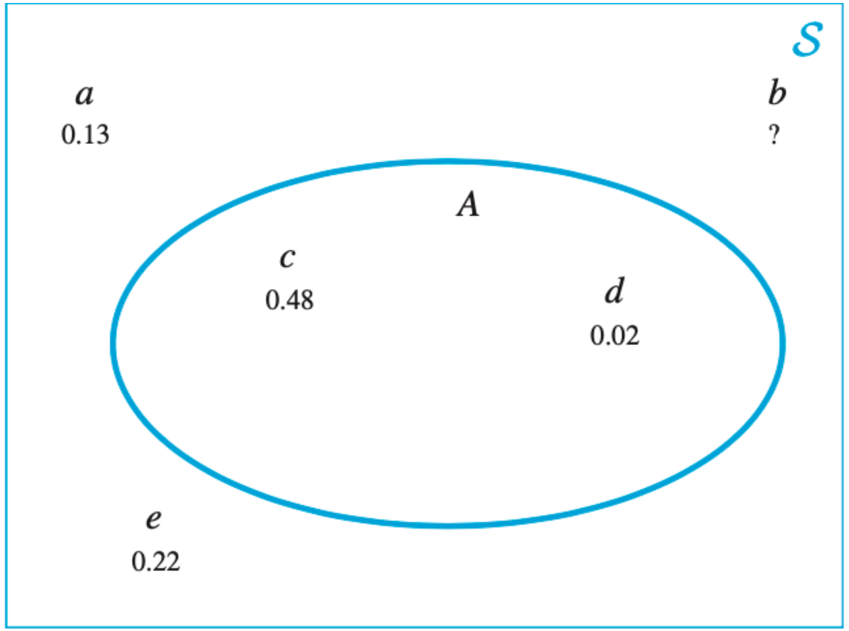
Event B: at least one 6 recorded



Homework 1B

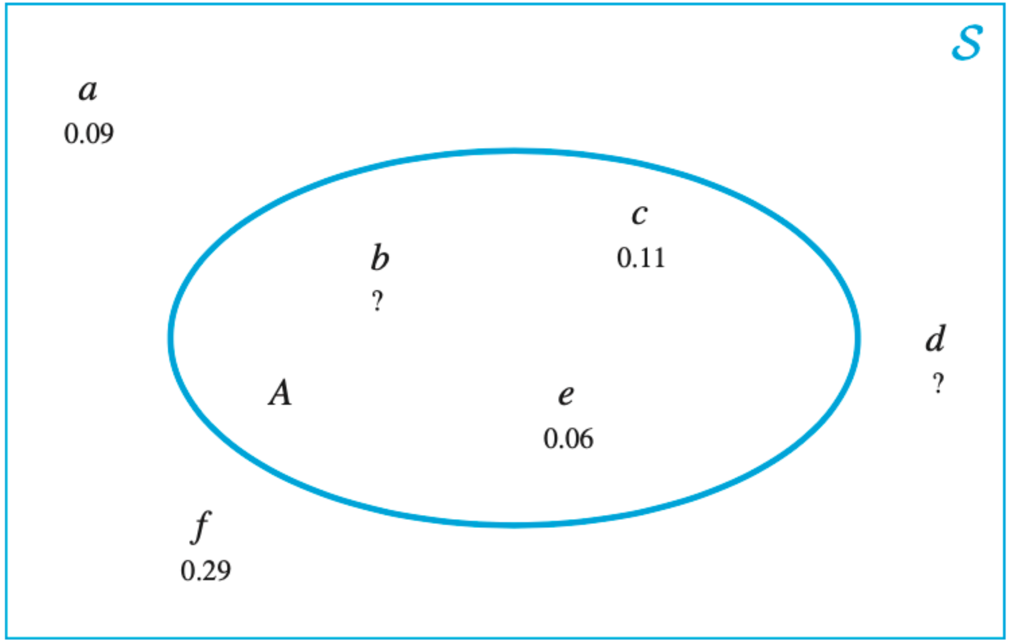
1.2.1 Consider the sample space of the following figure with outcomes a, b, c, and d.

- Calculate
  - $P(b)$
  - $P(A)$
  - $P(A')$



Homework 1B

1.2.2 Consider the sample space of the following figure with outcomes a, b, c, d, e, and f. If $P(A) = .27$. Calculate $(i)\ P(b),\ (ii)\ P(A'),\ (iii)\ P(d)$



Homework 1B

1.2.4

- When a company introduces initiatives to reduce its carbon footprint, its costs will either increase, stay the same, or decrease.
- Suppose that the probability that the costs increase is 0.03, and the probability that the costs stay the same is 0.18.
- What is the probability that costs will decrease?
- What is the probability that costs will not increase?

Homework 1B

1.2.5 An investor is monitoring stocks from Company A and B, which each either increase or decrease each day.

- On a given day, suppose the probability of 0.38 that both stocks will increase in price and a probability of 0.11 that both stocks will decrease in price. Also, there is a probability of 0.16 that the stock from Company A will decrease while the stock from Company B will increase.
  - What is the probability that the stock from Company A will increase while the stock from Company B will decrease?
  - What is the probability that at least one Company will have an increase in the stock price?

Homework 1B

1.2.7

- If a card is chosen at random from a pack of cards, what is the probability that the card is from one of the two black suits?

1.2.8

- If a card is chosen at random from a pack of cards, what is the probability that it is an ace?

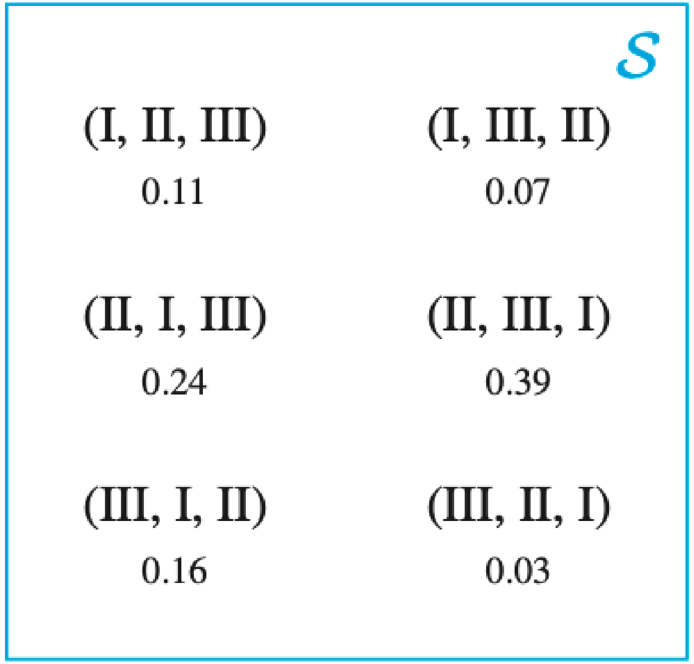
1.2.12

- A fair coin is tossed three times. What is the probability that two heads will be obtained in succession?

Homework 1B

1.2.10

- Three types of batteries are being tested, type I, type II, and type III.
- The outcome (I, II, III) denotes that the battery of type I fails first, the battery of type II next, and the battery of type III lasts the longest.



- What is the probability that a) a type I battery lasts the longest? (b) a type I battery lasts the shortest? (c) a type I battery does not last the longest? (d) a type I battery lasts longer than a type II battery?

Combinations of events

Intersections of events

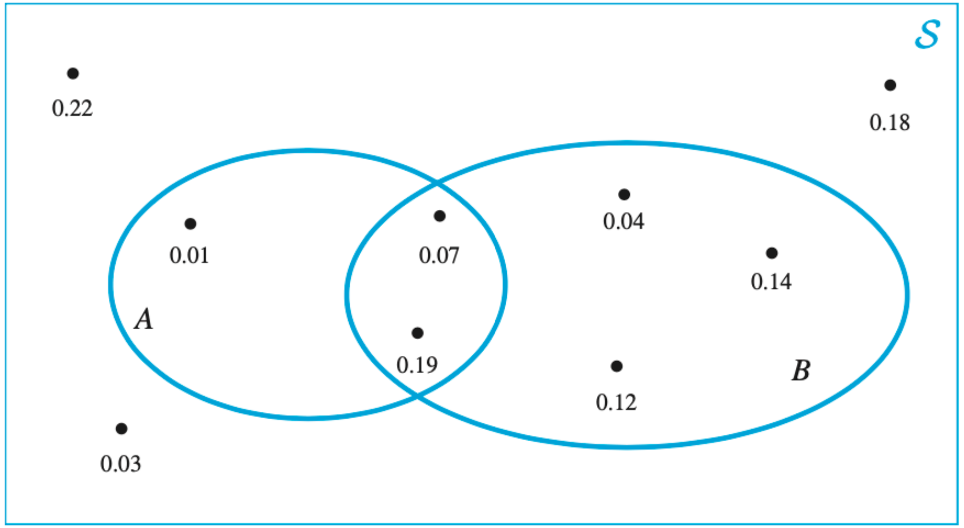
- The event $A \cap B$ is the intersection of the events A and B and consists of the outcomes that are contained within both events A and B, e.g.

$$A = \{1, 2, 3\} \text{ and } B = \{2, 3, 4, 6\}$$
$$\Rightarrow \quad A \cap B = \{2, 3\}$$

  - $P(A \cap B)$ is the probability that events A and B occur simultaneously

FIGURE 1.26

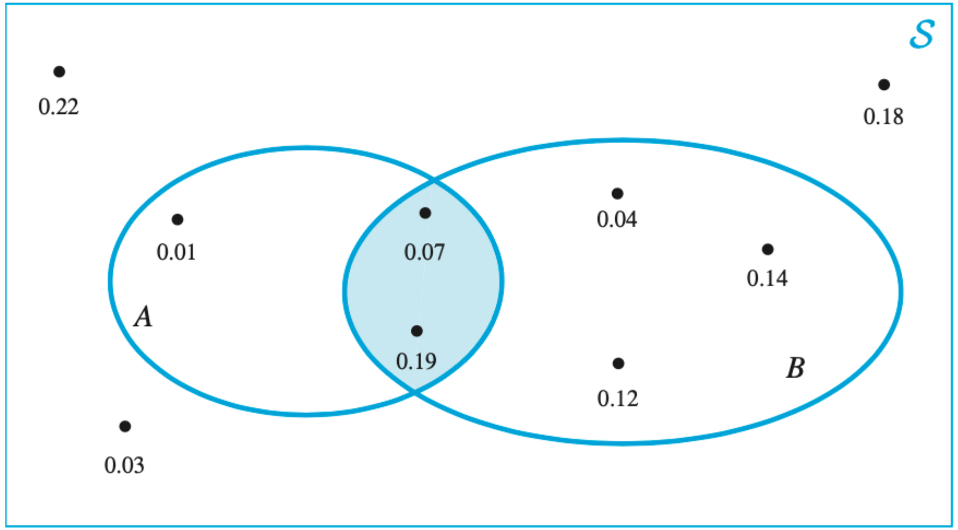
Events A and B



- $P(A \cap B) = 0.07 + 0.19 = 0.26$
- $A \cap A' = \phi \;\Rightarrow\; P(A \cap A') = 0$
- $P(A \cap B') = ?$
- $P(A' \cap B) = ?$

FIGURE 1.27

The event $A \cap B$



- Check
  - $P(A \cap B) + P(A \cap B') = P(A)$
  - $P(A \cap B) + P(A' \cap B) = P(B)$

Mutually Exclusive Events

- Two events A and B are said to be *mutually exclusive* if $A \cap B = \phi$ so that they have no outcomes in common.
    - Two mutually exclusive events cannot happen at the same time
- $A \subset B \ \rightarrow A$ is contained within an event B

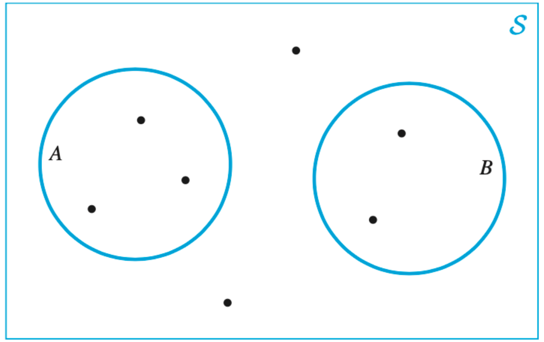


FIGURE 1.30

A and B are mutually exclusive events

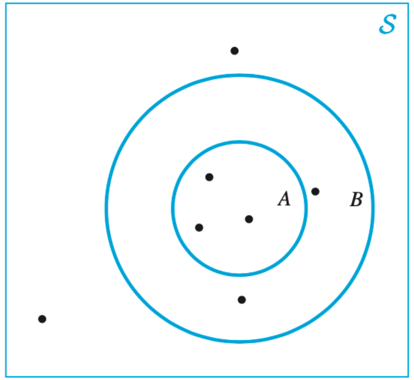


FIGURE 1.31

$A \subset B$

Unions of Events

- The event $A \cup B$ is the union of events A and B and consists of the outcomes that are contained within at least one of the events A and B.

$$A = \{1, 2, 3\} \text{ and } B = \{2, 3, 4, 6\}$$
$$\Rightarrow \quad A \cup B = \{1, 2, 3, 4, 6\}$$

- $P(A \cup B)$ is the probability that at least one of the events A and B occurs

Unions of Events

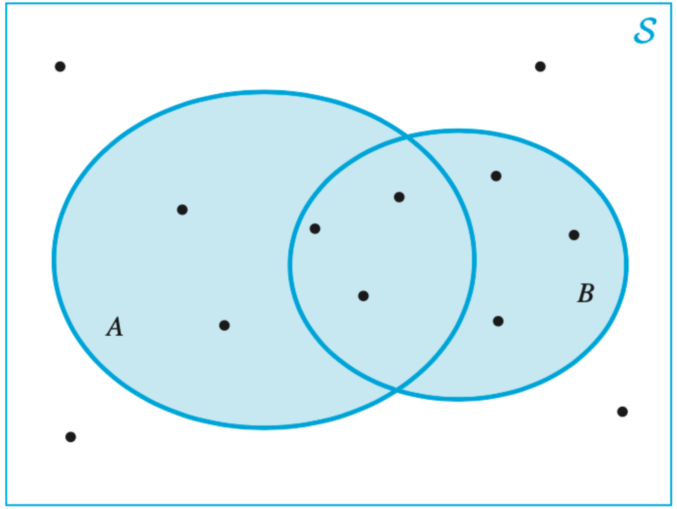


FIGURE 1.32

The event $A \cup B$

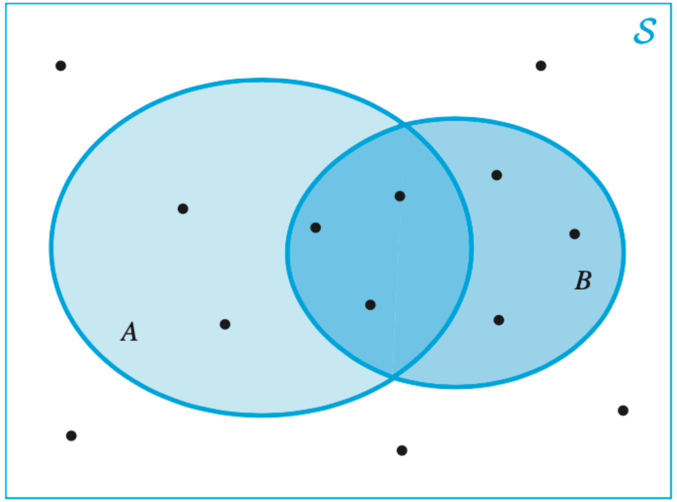


FIGURE 1.33

Decomposition of the event $A \cup B$

Unions of Events

- If A and B are mutually exclusive $\{\text{i.e.}, P(A \cap B) = 0\}$ then

$$P(A \cup B) = P(A) + P(B)$$

- It can be shown

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Homework 1C

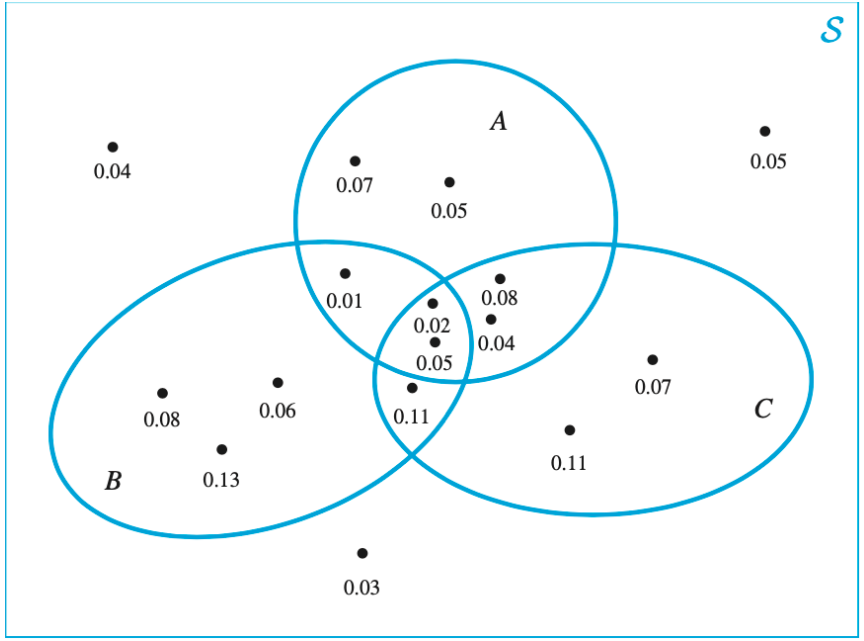
1.3.1 Consider the sample space $S = \{0, 1, 2\}$ and the event $A = \{0\}$.

- Explain why $A \neq \phi$.

1.3.5 A card is chosen from a pack of cards.

- Are the events that a card from one of the two red suits is chosen and a card from one of the two black suits mutually exclusive?

- What about the events that an ace is chosen and that a heart is chosen?



1.3.2 Consider the sample space and events in the Figure. Calculate the probabilities of the events:

$$B,\ B \cup C,\ A \cup C,\ and\ A \cap B \cup C$$

Homework 1C

1.3.6

- If $P(A) = 0.4$ and $P(A \cap B) = 0.3$
  - What are the possible values for $P(B)$?

1.3.7

- If $P(A) = 0.5$, $P(A \cap B) = 0.1$, and $P(A \cup B) = 0.8$, what is $P(B)$?

Homework 1C

1.3.12

- A bag contains 200 balls that are either red or blue, or dull or shiny. There are 55 shiny red balls, 91 shiny balls, and 79 red balls. If a ball is chosen at random:
    - What is the probability that it is either a shiny or red ball?
    - What is the probability that it is a dull blue ball?

Conditional Probability

Definition of conditional probability

- The conditional probability of an event A given an event B is defined as

$$P(A\,|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided} \quad P(B) > 0$$

  - It measures the probability that event A occurs when it is known that event B occurs.
  - Conditional probabilities are important and very useful since they provide appropriate updates of a set of probabilities once a particular event is known to have occurred

Definition of conditional probability

- If the events A and B are mutually exclusive

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0$$

- If $B \subset A$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

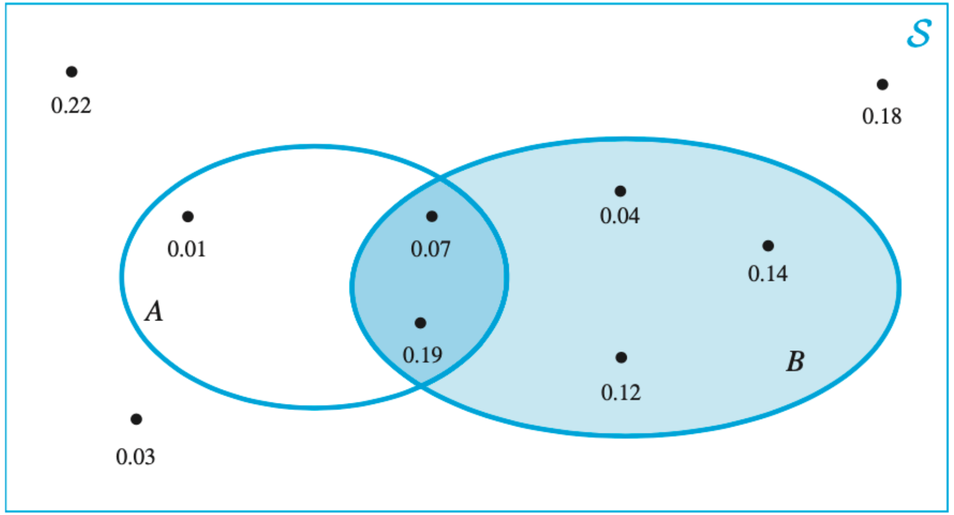
Definition of conditional probability

- Similar to $P(A) + P(A') = 1$

$$P(A \mid B) + P(A' \mid B) = 1$$

FIGURE 1.56

$P(A|B) = P(A \cap B)/P(B)$



- $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{.26}{.56} = .464$
- $P(A \mid B') = ?$

Example 4 (Power plant operation)

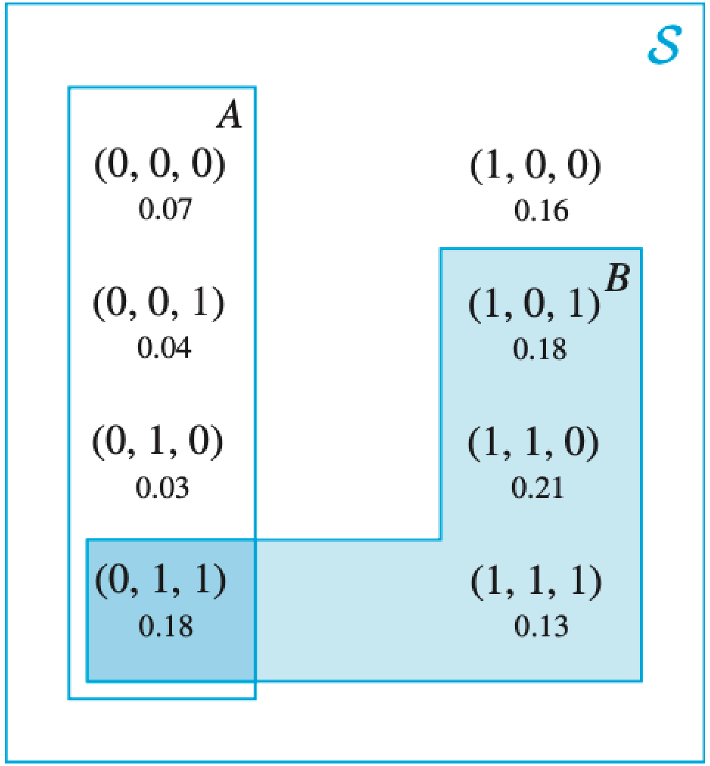


FIGURE 1.58

$P(A|B) = P(A \cap B)/P(B)$

Example 4 (Power plant operation)

- Define two events related to the power plant example
  - $A \rightarrow$ the plant X is idle $\Rightarrow \; P(A) = 0.32$
  - $B \rightarrow$ at least two plants generating electricity $\Rightarrow \; P(B) = 0.70$

Example 4 (Power plant operation)

- The probability that the plant X is idle (event A), conditional on at least two out of the three plants generating electricity (event B)

$$P(A \mid B) = \frac{P(B \cap A)}{P(B)} = \frac{0.18}{0.70} = 0.257$$

- Whereas plant X is idle about 32% of the time, it is idle only about 25.7% of the time when at least two plants generate electricity.

Rolling a die

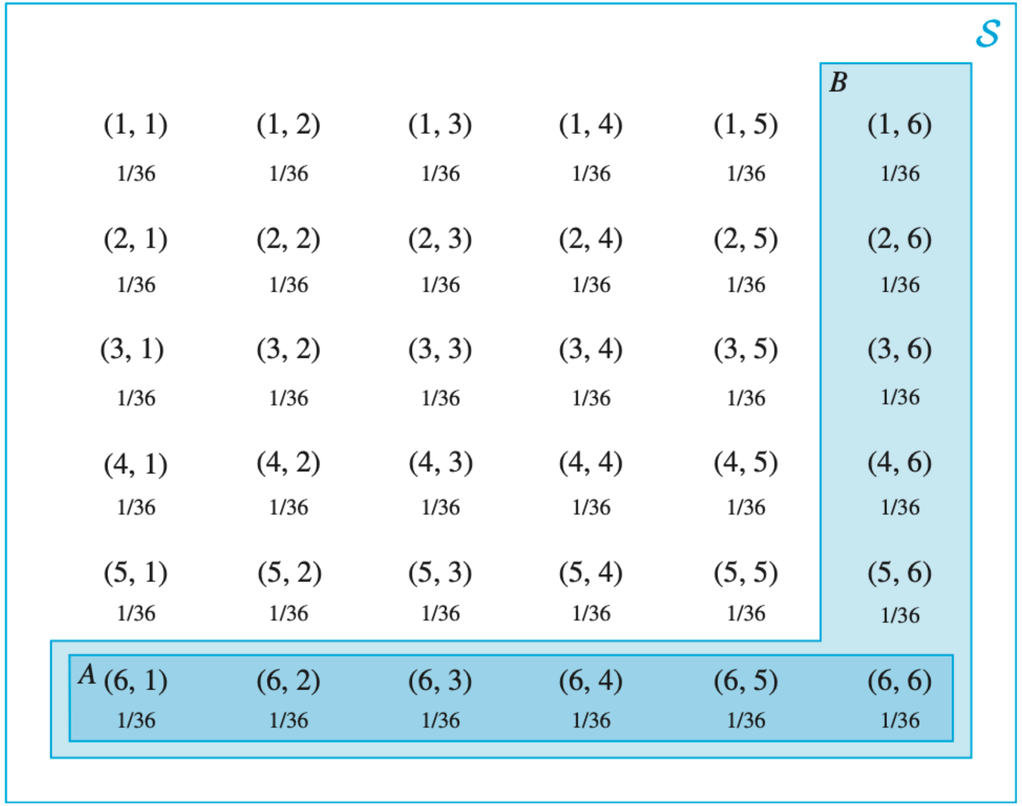
- For a fair die, the probability of scoring a 6 is $P(6) = 1/6$
- If someone rolls a die without showing you but announces that the result is even, then intuitively, the chance that a six has been obtained is $1/3$ (Why?)
- Using the concept of conditional probability

$$P(6 \mid even) = \frac{P(6 \cap even)}{P(even)} = \frac{P(6)}{P(even)} = \frac{(1/6)}{(1/2)} = \frac{1}{3}$$

Rolling two dice

FIGURE 1.60

$P(A|B) = P(A \cap B)/P(B)$



Rolling two dice

- Two dice (red and blue) are thrown and define two events
  - $A \rightarrow$ red die scores a 6 $\Rightarrow P(A) = 1/6$
  - $B \rightarrow$ at least one 6 is obtained in two dice $\Rightarrow P(B) = 11/36$

Rolling two dice

- Suppose somebody rolls the two dice without showing you, but announces that at least one six has been scored

- What is the probability that the red die scored a 6?

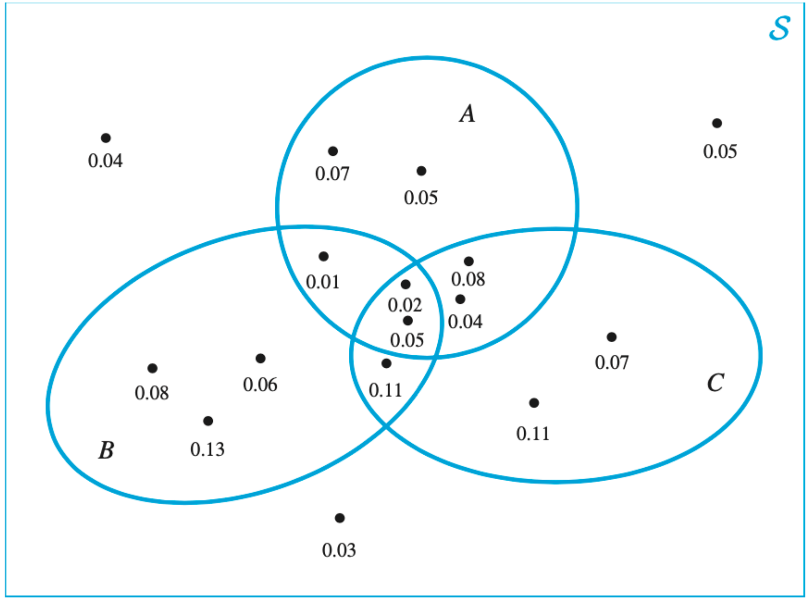
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/6}{11/36} = \frac{6}{11}$$

- As expected, this conditional probability is larger than $P(A) = 1/6$

Homework 1D

1.4.1 Consider the following Figure and calculate the probabilities.

- $P(A \mid B)$
- $P(C \mid A)$
- $P(B \mid A \cap B)$
- $P(B \mid A \cup B)$



Homework 1D

1.4.3 A card is drawn at random from a pack of cards. Calculate:

1. $P(\text{Heart } A \mid \text{card from red suit})$
2. $P(\text{heart} \mid \text{card from red suit})$
3. $P(\text{card from red suit} \mid \text{heart})$
4. $P(\text{heart} \mid \text{card from black suit})$
5. $P(\text{king} \mid \text{card from red suit})$
6. $P(\text{king} \mid \text{red picture card})$

Homework 1D

1.4.5

- A ball is chosen at random from a bag containing 150 balls that are either red or blue and dull or shiny.
- There are 36 red shiny balls and 54 blue balls.
  - What is the probability of the chosen ball being shiny, conditional on it being red?
  - What is the probability of the chosen ball being dull, conditional on it being red?

Homework 1D

1.4.6

- A car repair is either on time or late and satisfactory or unsatisfactory.
- If a repair is made on time, then there is a probability of 0.85 that it is satisfactory.
- There is a probability of 0.77 that a repair will be made on time.
- What is the probability that a repair is made on time and is satisfactory?

Homework 1D

1.4.8

- Suppose that births are equally likely to be on any day.
  - What is the probability that somebody chosen at random has a birthday on the first day of a month?
  - How does this probability change conditional on the knowledge that the person's birthday is in March?
  - In February?

Homework 1D

1.4.15

- There is a 4% probability that the plane used for a commercial flight has technical problems, and this causes a delay in the flight.
- If there are no technical problems with the plane, then there is still a 33% probability that the flight is delayed due to all other reasons.
- What is the probability that the flight is delayed?

Independent events

Independent events

- Two events A and B are said to be independent events if

$$
\begin{aligned}
& P(A \mid B) = P(A) \\
\text{or} \quad & P(B \mid A) = P(B) \\
\text{or} \quad & P(A \cap B) = P(A)\,P(B)
\end{aligned}
$$

- The interpretation of two events being independent is that knowledge about one event does not affect the probability of the other event.

Games of chance

- In the roll of a fair die, consider the events

$$\text{even} = \{2, 4, 6\} \ \text{ and } \ \text{high-score} = \{4, 5, 6\}$$

  - Are the events "even" and "high-score" independent?

Games of chance

- In the roll of a fair die, consider the events

$$\text{low-score} = \{1, 2, 3\} \;\; \text{and} \;\; \text{high-score} = \{4, 5, 6\}$$

    - Are the events "low-score" and "high-score" independent?

Independence and mutually exclusive events

- Two events are said to be independent if the occurrence of one event does not affect the occurrence of the other
- Two events are said to be mutually exclusive if both events cannot happen simultaneously
- Two mutually exclusive events could be either dependent or independent

Independence and mutually exclusive events

- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ be two events of an experiment rolling a fair die
  - A and B are mutually exclusive but not independent (Why?)
- Let $R = \{\text{drawing a red card}\}$ and $A = \{\text{drawing an Ace}\}$ are two events of an experiment of drawing a card from a pack of cards
  - R and A are independent, but they are not mutually exclusive

Posterior Probabilities

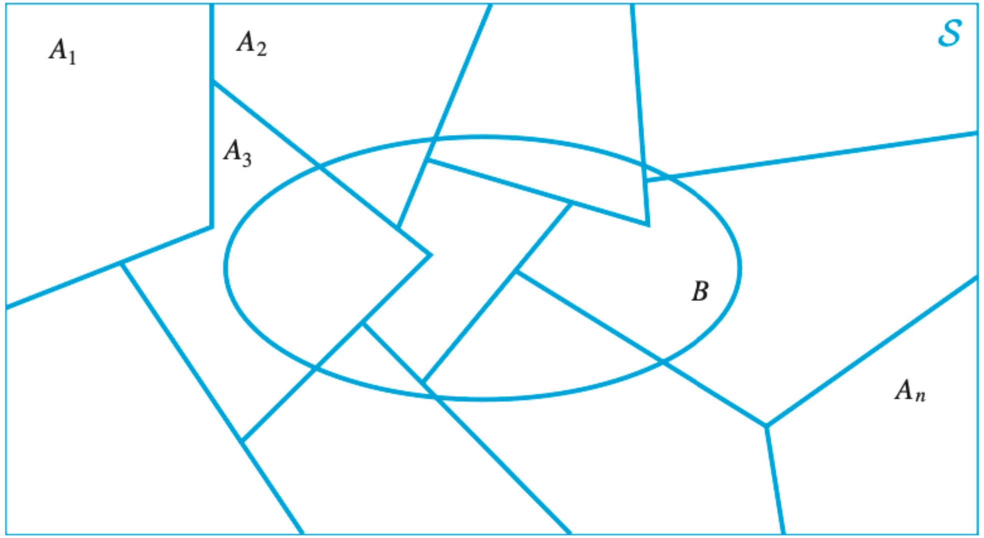
Law of total probability

- Let $A_1, \ldots, A_n$ be a partition of the sample space S so that A_i's are mutually exclusive

$$S = A_1 \cup A_2 \cup \cdots \cup A_n$$

- Consider an event B such that

$$B = (A_1 \cap B) \cup \cdots \cup (A_n \cap B)$$



Law of total probability

- Consider an event B such that

$$B = (A_1 \cap B) \cup \cdots \cup (A_n \cap B)$$

- We can express the probability of B in terms of $P(A_i)$ and $P(B \mid A_i)$

$$\begin{aligned} P(B) &= P(A_1 \cap B) + \cdots + P(A_n \cap B) \\ &= P(A_1)P(B \mid A_1) + \cdots + P(A_n)P(B \mid A_n) \end{aligned}$$

  - This expression is known as the *law of total probability*

Example

- A company sells a certain type of car that it assembles in one of four possible plants: I, II, III, and IV
- The probabilities of a purchased car being from each of the four plants:
  - $P(I) = 0.20$, $P(II) = 0.24$, $P(III) = 0.25$), and $P(IV) = 0.31$.
- Each new car sold carries a one-year warranty with corresponding claim probabilities
  - $P(\text{claim} \mid I) = .05$, $P(\text{claim} \mid II) = .11$, $P(\text{claim} \mid III) = .03$, and $P(\text{claim} \mid IV) = .08$
- What is $P(\text{claim})$, the probability of a claim being made on a car warranty?

Bayes' theorem

- If $A_1, \ldots, A_n$ is a partition of a sample space

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{\sum_{i=1}^{n} P(A_i)P(B \mid A_i)}$$

  - $P(A_i) \rightarrow$ prior probability
  - $P(A_i \mid B) \rightarrow$ posterior probability (updated probability)

Example . . .

- When a customer buys a car, the (prior) probabilities of it having been assembled in a particular plant are

$$P(I) = 0.20, P(II) = 0.24, P(III) = 0.25, P(IV) = 0.31$$

- Calculate the posterior probabilities

$$P(I \,|\, \text{claim}) = ? \quad P(II \,|\, \text{claim}) = ?$$
$$P(III \,|\, \text{claim}) = ? \quad P(VI \,|\, \text{claim}) = ?$$

Exercise 161

- It is known that 1% of the population suffers from a particular disease.
- A blood test has a 97% chance of identifying the disease in diseased individuals, but also has a 6% chance of falsely indicating that a healthy person has the disease.
    - What is the probability that a person will have a positive blood test?
    - If your blood test is positive, what is the chance that you have the disease?
    - If your blood test is negative, what is the chance that you do not have the disease?

Exercise 1.6.2

- Bag A contains 3 red balls and 7 blue balls. Bag B contains 8 red balls and 4 blue balls. Bag C contains 5 red balls and 11 blue balls.
- A bag is chosen at random, with each bag being equally likely to be chosen, and then a ball is chosen at random from that bag. Calculate the probabilities:
    - (i) A red ball is chosen, (ii) a blue ball is chosen, and (iii) A red ball from bag B is chosen
    - (iv) A red ball is chosen, what is the probability that it comes from bag A?
    - (v) A blue ball is chosen, what is the probability that it comes from bag B?

Supplementary problems

1.10.6

- Two cards are drawn from a pack of cards.
- Is it more likely that two hearts will be drawn when the drawing is with replacement or without replacement?

Supplementary problems

1.10.7

- Two fair dice are thrown.
- A is the event that the sum of the scores is no larger than four, and B is the event that the two scores are identical.
- Calculate the probabilities:

$$A \cap B,\ A \cup B,\ A' \cup B$$

Supplementary problems

1.10.8: **Two fair dice are thrown, calculate**

- $P(\text{the first die is 5} \mid \text{sum of scores is 8})$
- $P(\text{either die is 5} \mid \text{sum of scores is 8})$
- $P(\text{sum of scores is 8} \mid \text{either die is 5})$

Supplementary problems

1.10.10 **Which is more likely:**

- obtaining at least one head in two tosses of a fair coin, or
- at least two heads in four tosses, of a fair coin?

Thank You