

Authors: Gihyuk Cho (gihyuk.cho@gmail.com) and Sashwat Tanay (sashwattanay@gmail.com)

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*This document gives some lengthy expressions related to the work presented in the following paper:*  
<https://arxiv.org/abs/2110.09608>

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### Local-in-time contributions (up to 4PN)

$$\begin{aligned} r &= ar * (1 - er * \text{Cos}[u]); \\ t &= \frac{1}{n} \left( u + \left( \frac{g4t}{c^4} + \frac{g6t}{c^6} + \frac{g8t}{c^8} \right) (-u + v) - et \text{Sin}[u] + \left( \frac{f4t}{c^4} + \frac{f6t}{c^6} + \frac{f8t}{c^8} \right) \text{Sin}[v] + \right. \\ &\quad \left. \left( \frac{i6t}{c^6} + \frac{i8t}{c^8} \right) \text{Sin}[2v] + \left( \frac{h6t}{c^6} + \frac{h8t}{c^8} \right) \text{Sin}[3v] + \frac{k8t \text{Sin}[4v]}{c^8} + \frac{j8t \text{Sin}[5v]}{c^8} \right); \\ \phi &= \frac{1}{2\pi} \Phi \left( v + \left( \frac{f4phi}{c^4} + \frac{f6phi}{c^6} + \frac{f8phi}{c^8} \right) \text{Sin}[2v] + \left( \frac{g4phi}{c^4} + \frac{g6phi}{c^6} + \frac{g8phi}{c^8} \right) \text{Sin}[3v] + \right. \\ &\quad \left. \left( \frac{i6phi}{c^6} + \frac{i8phi}{c^8} \right) \text{Sin}[4v] + \left( \frac{h6phi}{c^6} + \frac{h8phi}{c^8} \right) \text{Sin}[5v] + \right. \\ &\quad \left. \frac{k8phi \text{Sin}[6v]}{c^8} + \frac{j8phi \text{Sin}[7v]}{c^8} \right); \end{aligned}$$

ar =

$$\begin{aligned}
& -\frac{1}{2E} + \frac{-7+\eta}{4c^2} - \frac{34-22\eta+h^2E(1+10\eta+\eta^2)}{8c^4h^2} + \frac{1}{96c^6h^4} (h^2E(-864+(2212+3\pi^2)\eta-432\eta^2) - \\
& 24(134+(-281+5\pi^2)\eta+16\eta^2)+6h^4E^2(1-3\eta-2\eta^2+\eta^3)) + \\
& \frac{1}{921600c^8h^6} (6(-51187200-6(-18365728+415175\pi^2)\eta + \\
& (-34446144+1442225\pi^2)\eta^2+576000\eta^3)+4h^2E(-39600000+ \\
& (82674816-2641500\pi^2)\eta+(-45421472+3363525\pi^2)\eta^2+2520000\eta^3)+ \\
& h^4E^2(-4147200+6(-6345248+213375\pi^2)\eta+(19038208+4030875\pi^2)\eta^2+ \\
& 4262400\eta^3)-28800h^6E^3(1-4\eta+2\eta^2-11\eta^3+\eta^4)) \quad ;
\end{aligned}$$

$$\begin{aligned}
\text{er} = & \sqrt{1+2h^2E} + \frac{E(2(-6+\eta)+5h^2E(-3+\eta))}{2c^2\sqrt{1+2h^2E}} + \\
& \frac{E(-136+88\eta+12h^2E(-26+19\eta)+4h^4E^2(42-8\eta+\eta^2)+h^6E^3(415-290\eta+7\eta^2))}{8c^4h^2(1+2h^2E)^{3/2}} - \\
& \frac{1}{48c^6h^4(1+2h^2E)^{5/2}} E \\
& (48(134+(-281+5\pi^2)\eta+16\eta^2)+2h^2E(15900+(-37180+657\pi^2)\eta+2280\eta^2)+ \\
& 2h^6E^3(9276+(-44594+696\pi^2)\eta+3987\eta^2)+ \\
& 2h^4E^2(23508+(-68036+1185\pi^2)\eta+4692\eta^2)+3h^{10}E^5 \\
& (5679-6023\eta+997\eta^2+3\eta^3)+2h^8E^4(5070-(10915+12\pi^2)\eta+2070\eta^2+3\eta^3)) + \\
& (E(12(-51187200-6(-18365728+415175\pi^2)\eta+(-34446144+1442225\pi^2)\eta^2+ \\
& 576000\eta^3)+128h^{10}E^5(2118150-3(23006599+15525\pi^2)\eta+ \\
& (12346613+3218475\pi^2)\eta^2+708750\eta^3)+ \\
& 2h^2E(-2227852800-6(-799110304+19195875\pi^2)\eta+ \\
& (-1629119168+77202225\pi^2)\eta^2+32313600\eta^3)+8h^8E^4 \\
& (-644680800-6(-60020048+8999775\pi^2)\eta+95(-7436384+1218015\pi^2)\eta^2+ \\
& 44510400\eta^3)+8h^6E^3(-1667707200-6(-529705376+17959725\pi^2)\eta+ \\
& 5(-315341728+24751965\pi^2)\eta^2+52851600\eta^3)+ \\
& h^4E^2(-11815372800+(24952209216-665983350\pi^2)\eta+ \\
& (-9643245760+550612875\pi^2)\eta^2+237139200\eta^3)+ \\
& 3600h^{14}E^7(310331-440948\eta+144530\eta^2-4468\eta^3+11\eta^4)+ \\
& 16h^{12}E^6(51973200+6(-37081048+178575\pi^2)\eta+187(536584+21825\pi^2)\eta^2- \\
& 2289600\eta^3+1800\eta^4))) / (460800c^8h^6(1+2h^2E)^{7/2}) \quad ;
\end{aligned}$$

$$\begin{aligned}
\Phi = & 2\pi \left( 1 + \frac{3}{c^2h^2} - \frac{3(5(-7+2\eta)+2h^2E(-5+2\eta))}{4c^4h^4} + \right. \\
& \frac{1}{128c^6h^6} (96h^4E^2(5-5\eta+4\eta^2)+5(7392+(-8000+123\pi^2)\eta+336\eta^2)+ \\
& 2h^2E(10080+(-13952+123\pi^2)\eta+1440\eta^2)) - \\
& \left. \frac{1}{73728c^8h^8} (55296h^6E^3\eta^2(-5+4\eta)+7(-37065600+(63502592-1275315\pi^2)\eta+ \right.
\end{aligned}$$

$$\begin{aligned}
& 2400 (-6056 + 123 \pi^2) \eta^2 + 207360 \eta^3 + 12 h^4 E^2 \\
& (-1814400 + (5202688 - 106707 \pi^2) \eta + 240 (-12944 + 123 \pi^2) \eta^2 + 276480 \eta^3) + \\
& 12 h^2 E (-17297280 + (37556864 - 771585 \pi^2) \eta + \\
& 1920 (-7013 + 123 \pi^2) \eta^2 + 403200 \eta^3) \Bigg); \\
f4phi &= -\frac{(1 + 2 h^2 E) \eta (-1 + 3 \eta)}{8 h^4}; \\
f6phi &= \frac{1}{256 h^6} (256 + (-1076 + 49 \pi^2) \eta - 384 \eta^2 - 40 \eta^3 + \\
& 16 h^4 E^2 \eta (-11 - 40 \eta + 24 \eta^2) + 2 h^2 E (256 + (-1192 + 49 \pi^2) \eta - 336 \eta^2 + 80 \eta^3)) ; \\
f8phi &= \frac{1}{7372800 h^8} (177004800 + (-2446310192 + 72629250 \pi^2) \eta - \\
& 15 (-132716108 + 963535 \pi^2) \eta^2 - 86679000 \eta^3 + \\
& 1929600 \eta^4 - 7200 h^6 E^3 \eta (6844 - 13989 \eta - 1530 \eta^2 + 1888 \eta^3) + \\
& 4 h^4 E^2 (9273600 + 2 (-303923464 + 7907025 \pi^2) \eta + (567130588 + 8219475 \pi^2) \eta^2 - \\
& 26411400 \eta^3 + 1180800 \eta^4) + 4 h^2 E (84844800 + 10 (-149381636 + 4263405 \pi^2) \eta - \\
& 19 (-67975466 + 173325 \pi^2) \eta^2 - 54814500 \eta^3 + 2980800 \eta^4)) ; \\
g4phi &= -\frac{3 (1 + 2 h^2 E)^{3/2} \eta^2}{32 h^4}; \\
g6phi &= \frac{1}{768 h^6} \sqrt{1 + 2 h^2 E} \eta \\
& (220 + 3 \pi^2 + 96 \eta + 45 \eta^2 + 12 h^4 E^2 \eta (-9 + 26 \eta) + 2 h^2 E (220 + 3 \pi^2 + 312 \eta + 150 \eta^2)); \\
g8phi &= (44236800 + 2 (-5543017952 + 112713225 \pi^2) \eta - 3 (-5907471872 + 93782475 \pi^2) \\
& \eta^2 - 1527246000 \eta^3 + 50133600 \eta^4 - 172800 h^8 E^4 \eta^2 (36 - 95 \eta + 1226 \eta^2) + \\
& 4 h^4 E^2 (44236800 + 2 (-6842155424 + 127907475 \pi^2) \eta - \\
& 87 (-253902848 + 3210525 \pi^2) \eta^2 - 2262477600 \eta^3 + 39096000 \eta^4) + \\
& 2 h^2 E (88473600 + 10 (-2355674528 + 46488075 \pi^2) \eta + \\
& (37649997312 - 561808575 \pi^2) \eta^2 - 3436488000 \eta^3 + 103766400 \eta^4) - 24 h^6 E^3 \eta \\
& (-225 \pi^2 (24238 + 1697 \eta) + 64 (6320841 - 11422240 \eta + 1797975 \eta^2 + 142650 \eta^3))) \Bigg/ \\
& (176947200 h^8 \sqrt{1 + 2 h^2 E}) ; \\
i6phi &= \frac{(1 + 2 h^2 E)^2 \eta (5 + 28 \eta + 10 \eta^2)}{128 h^6}; \\
i8phi &= -\frac{1}{14745600 h^8} (1 + 2 h^2 E) \eta \\
& (75 \pi^2 (-23564 + 120829 \eta) + 28800 h^4 E^2 (440 - 1330 \eta + 700 \eta^2 + 173 \eta^3) - \\
& 32 (-5298382 + 12898185 \eta - 1825650 \eta^2 + 110475 \eta^3) + \\
& 2 h^2 E (75 \pi^2 (-23564 + 120829 \eta) - \\
& 32 (-5478382 + 12734835 \eta - 2195550 \eta^2 + 53550 \eta^3))) ; \\
h6phi &= \frac{5 (1 + 2 h^2 E)^{5/2} \eta^3}{256 h^6}; \\
h8phi &= -\frac{1}{6553600 h^8} (1 + 2 h^2 E)^{3/2} \eta (8000 h^4 E^2 \eta^2 (-39 + 86 \eta) +
\end{aligned}$$

$$\begin{aligned}
& 225 \pi^2 (-50 + 2577 \eta) + 16 (517 116 - 1 515 904 \eta + 435 125 \eta^2 - 30 650 \eta^3) + \\
& 2 h^2 E (225 \pi^2 (-50 + 2577 \eta) - 32 (-258 558 + 757 952 \eta - 237 625 \eta^2 + 7450 \eta^3)) ; \\
\text{k8phi} = & - \frac{(1 + 2 h^2 E)^3 \eta (1476 - 5755 \eta + 4154 \eta^2 + 150 \eta^3)}{24 576 h^8} ; \\
\text{j8phi} = & - \frac{35 (1 + 2 h^2 E)^{7/2} \eta^3 (-1 + 2 \eta)}{16 384 h^8} ;
\end{aligned}$$

$$\begin{aligned}
\text{et} = & \sqrt{1 + 2 h^2 E} + \frac{E (4 + h^2 E (17 - 7 \eta) - 4 \eta)}{2 c^2 \sqrt{1 + 2 h^2 E}} + \\
& \frac{1}{8 c^4 h^2 (1 + 2 h^2 E)^{3/2}} E (-68 + 44 \eta - 24 \sqrt{2} h \sqrt{-E} (-5 + 2 \eta) + \\
& 96 \sqrt{2} h^3 (-E)^{3/2} (-5 + 2 \eta) - 96 \sqrt{2} h^5 (-E)^{5/2} (-5 + 2 \eta) + \\
& 8 h^2 E (-17 + 16 \eta + 3 \eta^2) + 4 h^4 E^2 (86 + 5 \eta + 22 \eta^2) + h^6 E^3 (607 - 138 \eta + 79 \eta^2)) - \\
& \frac{1}{96 c^6 h^4 (1 + 2 h^2 E)^{5/2}} E (-2304 \sqrt{2} h^9 (-E)^{9/2} (45 - 31 \eta + 8 \eta^2) + \\
& 48 (134 + (-281 + 5 \pi^2) \eta + 16 \eta^2) - \sqrt{2} h \sqrt{-E} (10 080 + (-13 952 + 123 \pi^2) \eta + 1440 \eta^2) + \\
& 6 \sqrt{2} h^3 (-E)^{3/2} (11 700 + (-15 284 + 123 \pi^2) \eta + 1848 \eta^2) - \\
& 12 \sqrt{2} h^5 (-E)^{5/2} (15 480 + (-18 104 + 123 \pi^2) \eta + 2640 \eta^2) + \\
& 2 h^2 E (18 780 + (-38 140 + 597 \pi^2) \eta + 3012 \eta^2) + \\
& 8 \sqrt{2} h^7 (-E)^{7/2} (27 900 + (-26 876 + 123 \pi^2) \eta + 4968 \eta^2) + \\
& 8 h^4 E^2 (8949 + (-18 352 + 237 \pi^2) \eta + 2097 \eta^2 + 48 \eta^3) + \\
& 24 h^8 E^4 (-2843 - 342 \eta + 122 \eta^2 + 167 \eta^3) + 6 h^{10} E^5 (-10 801 + 1405 \eta - 771 \eta^2 + 407 \eta^3) + \\
& 4 h^6 E^3 (6348 + (-25 748 + 234 \pi^2) \eta + 4545 \eta^2 + 540 \eta^3)) + \\
& (E (-2 764 800 \sqrt{2} h^{13} (-E)^{13/2} (-1205 + 838 \eta - 351 \eta^2 + 82 \eta^3) - 15 \sqrt{2} h \sqrt{-E} \\
& (-17 297 280 + (37 556 864 - 771 585 \pi^2) \eta + 1920 (-7013 + 123 \pi^2) \eta^2 + 403 200 \eta^3) + \\
& 3 (-51 187 200 - 6 (-18 365 728 + 415 175 \pi^2) \eta + (-34 446 144 + 1 442 225 \pi^2) \eta^2 + \\
& 576 000 \eta^3) + 800 \sqrt{2} h^{11} (-E)^{11/2} (-13 076 640 + (16 578 880 - 153 939 \pi^2) \eta + \\
& 48 (-151 286 + 861 \pi^2) \eta^2 + 997 056 \eta^3) + 720 \sqrt{2} h^7 (-E)^{7/2} (-21 093 720 + \\
& (43 566 176 - 759 885 \pi^2) \eta + 10 (-1 773 292 + 22 755 \pi^2) \eta^2 + 1 144 560 \eta^3) + \\
& 10 \sqrt{2} h^3 (-E)^{3/2} (-227 612 160 + (498 316 928 - 9 962 295 \pi^2) \eta + \\
& 60 (-3 097 888 + 50 799 \pi^2) \eta^2 + 6 998 400 \eta^3) - 40 \sqrt{2} h^5 (-E)^{5/2} (-202 598 820 + \\
& (440 011 456 - 8 384 025 \pi^2) \eta + 30 (-5 708 246 + 84 993 \pi^2) \eta^2 + 8 297 640 \eta^3) + \\
& 8 h^4 E^2 (-486 446 400 - 75 (-13 992 592 + 308 667 \pi^2) \eta + \\
& 4 (-97 836 727 + 4 214 550 \pi^2) \eta^2 + 13 849 200 \eta^3) - 80 \sqrt{2} h^9 (-E)^{9/2} \\
& (-205 181 640 + (365 468 672 - 5 327 115 \pi^2) \eta + 60 (-2 590 870 + 25 707 \pi^2) \eta^2 + \\
& 14 072 400 \eta^3) + h^2 E (-1 223 596 800 - 210 (-12 684 256 + 286 605 \pi^2) \eta + \\
& (-903 434 368 + 38 453 775 \pi^2) \eta^2 + 21 744 000 \eta^3) + \\
& 14 400 h^{12} E^6 (34 110 + 113 675 \eta - 18 290 \eta^2 + 2727 \eta^3 + 2150 \eta^4) + \\
& 1800 h^{14} E^7 (768 955 - 11 012 \eta + 35 970 \eta^2 - 14 948 \eta^3 + 7963 \eta^4) + \\
& 24 h^6 E^3 (-270 783 600 - 120 (-4 574 036 + 92 785 \pi^2) \eta +
\end{aligned}$$

$$\begin{aligned}
& (-225\,289\,088 + 9\,672\,575\,\pi^2)\eta^2 + 12\,133\,200\,\eta^3 + 48\,000\,\eta^4) + \\
& 16\,h^8\,E^4(-399\,955\,500 - 60(-11\,628\,937 + 181\,890\,\pi^2)\eta + \\
& (-298\,818\,356 + 12\,160\,425\,\pi^2)\eta^2 + 25\,586\,100\,\eta^3 + 547\,200\,\eta^4) + 16\,h^{10}\,E^5 \\
& (-211\,721\,400 - 66(-5\,279\,848 + 36\,075\,\pi^2)\eta + (-123\,842\,812 + 3\,917\,775\,\pi^2)\eta^2 + \\
& 17\,124\,300\,\eta^3 + 1\,549\,800\,\eta^4)) / (230\,400\,c^8\,h^6(1 + 2\,h^2\,E)^{7/2}); \\
P = \frac{1}{n} = & \frac{1}{2\sqrt{2}(-E)^{3/2}} - \frac{-15 + \eta}{8\sqrt{2}\,c^2\sqrt{-E}} - \frac{3(32\sqrt{2}(-5 + 2\eta) + h\sqrt{-E}(35 + 30\eta + 3\eta^2))}{64\sqrt{2}\,c^4\,h} + \\
& \frac{1}{768\sqrt{2}\,c^6\,h^3} (1152\sqrt{2}\,h^2\,E(5 - 5\eta + 4\eta^2) + 4\sqrt{2} \\
& (10\,080 + (-13\,952 + 123\,\pi^2)\eta + 1440\eta^2) + 15\,h^3\sqrt{-E}\,E(21 - 105\eta + 15\eta^2 + 5\eta^3)) - \\
& \frac{1}{184\,320\sqrt{2}\,c^8\,h^5} (414\,720\sqrt{2}\,h^4\,E^2\eta^2(-5 + 4\eta) + 20\sqrt{2}\,h^2\,E \\
& (-1\,814\,400 + (5\,202\,688 - 106\,707\,\pi^2)\eta + 240(-12\,944 + 123\,\pi^2)\eta^2 + 276\,480\eta^3) + \\
& 6\sqrt{2}(-17\,297\,280 + (37\,556\,864 - 771\,585\,\pi^2)\eta + 1920(-7013 + 123\,\pi^2)\eta^2 + \\
& 403\,200\eta^3) + 315\,h^5(-E)^{5/2}(99 - 420\eta + 210\eta^2 + 60\eta^3 + 35\eta^4)); \\
g4t = & \frac{3\sqrt{2}\sqrt{-E}\,E(-5 + 2\eta)}{h}; \\
g6t = & \frac{(-E)^{3/2}(10\,080 + (-13\,952 + 123\,\pi^2)\eta + 1440\eta^2 + 72\,h^2\,E(95 - 55\eta + 18\eta^2))}{48\sqrt{2}\,h^3}; \\
g8t = & \frac{1}{46\,080\,h^5}\,E(829\,440\,h^3\,E^2(5 - 2\eta)^2 + 4320\sqrt{2}\,h^4(-E)^{5/2}(-3375 + 1600\eta - 755\eta^2 + 246\eta^3) + \\
& 10\sqrt{2}\,h^2\sqrt{-E}\,E(-3\,628\,800 + (7\,835\,008 - 128\,847\,\pi^2)\eta + \\
& 36(-98\,144 + 861\,\pi^2)\eta^2 + 293\,760\eta^3) + 3\sqrt{2}\sqrt{-E} \\
& (-17\,297\,280 + (37\,556\,864 - 771\,585\,\pi^2)\eta + 1920(-7013 + 123\,\pi^2)\eta^2 + 403\,200\eta^3)) ; \\
f4t = & \frac{\sqrt{-E}\,E\sqrt{1 + 2\,h^2\,E}\,\eta(4 + \eta)}{2\sqrt{2}\,h}; \\
f6t = & \frac{1}{48\,h^3\sqrt{2 + 4\,h^2\,E}}(-E)^{3/2}(1728 + (-4148 + 3\,\pi^2)\eta + 600\eta^2 + 33\eta^3 + \\
& 12\,h^4\,E^2\eta(-64 - 4\eta + 23\eta^2) + 2\,h^2\,E(1728 + (-4232 + 3\,\pi^2)\eta + 627\eta^2 + 105\eta^3)); \\
f8t = & (E(11\,059\,200\,h^3\,E^2\eta(-20 + 3\eta + 2\eta^2) + 44\,236\,800\,h^5\,E^3\eta(-20 + 3\eta + 2\eta^2) + 44\,236\,800 \\
& h^7\,E^4\eta(-20 + 3\eta + 2\eta^2) + 115\,200\sqrt{2}\,h^8(-E)^{9/2}\eta(4672 + 912\eta - 303\eta^2 + 902\eta^3) + \\
& 8\sqrt{2}\,h^6(-E)^{7/2}(331\,776\,000 + 1350(-919\,776 + 2377\,\pi^2)\eta + \\
& (568\,404\,992 + 2\,468\,925\,\pi^2)\eta^2 - 94\,248\,000\eta^3 - 16\,128\,000\eta^4) + \\
& \sqrt{2}\sqrt{-E}(-1\,712\,332\,800 + (8\,314\,359\,104 - 246\,319\,350\,\pi^2)\eta + \\
& (-4\,388\,287\,232 + 86\,487\,075\,\pi^2)\eta^2 + 184\,226\,400\eta^3 - 1\,944\,000\eta^4) + \\
& 2\sqrt{2}\,h^2(-E)^{3/2}(3\,607\,142\,400 + 2(-8\,729\,633\,504 + 247\,794\,225\,\pi^2)\eta + \\
& (9\,340\,505\,856 - 170\,534\,025\,\pi^2)\eta^2 - 471\,441\,600\eta^3 + 1\,152\,000\eta^4) + \\
& 4\sqrt{2}\,h^4(-E)^{5/2}(-2\,226\,585\,600 + (10\,348\,301\,504 - 252\,478\,050\,\pi^2)\eta + \\
& 9(-614\,377\,024 + 9\,064\,225\,\pi^2)\eta^2 + 383\,328\,000\eta^3 +
\end{aligned}$$

$$\begin{aligned}
& 10\,411\,200\,\eta^4)) / (7\,372\,800\,h^5(1+2h^2E)^{3/2}) \quad ; \\
i6t &= \frac{(-E)^{3/2}(1+2h^2E)\eta(23+12\eta+6\eta^2)}{8\sqrt{2}h^3}; \\
i8t &= \frac{1}{230\,400\sqrt{2}h^5}\sqrt{-E}E \\
& (-1\,857\,600 + (12\,167\,056 - 1\,072\,425\pi^2)\eta + (-43\,313\,932 + 3\,152\,775\pi^2)\eta^2 + \\
& 3\,709\,200\eta^3 - 126\,000\eta^4 + 1200h^4E^2\eta(-8904 + 12\,207\eta + 2356\eta^2 + 864\eta^3) + \\
& 2h^2E(-1\,857\,600 + (10\,986\,256 - 1\,072\,425\pi^2)\eta + \\
& (-38\,708\,632 + 3\,152\,775\pi^2)\eta^2 + 4\,891\,200\eta^3 + 176\,400\eta^4)) \quad ; \\
h6t &= \frac{13(-E)^{3/2}(1+2h^2E)^{3/2}\eta^3}{48\sqrt{2}h^3}; \\
h8t &= \frac{1}{3\,686\,400\sqrt{2}h^5}\sqrt{-E}E\sqrt{1+2h^2E}\eta(14\,400h^4E^2\eta^2(-839+526\eta) + \\
& 225\pi^2(10\,802+59\,407\eta) - 16(3\,594\,972 + 6\,795\,760\eta - 1\,441\,725\eta^2 + 56\,250\eta^3) + 2h^2E \\
& (225\pi^2(10\,802+59\,407\eta) + 32(-1\,797\,486 - 3\,397\,880\eta + 731\,775\eta^2 + 42\,750\eta^3))) ; \\
k8t &= \frac{\sqrt{-E}E(1+2h^2E)^2\eta(516-3303\eta+2444\eta^2+150\eta^3)}{1536\sqrt{2}h^5}; \\
j8t &= \frac{\sqrt{-E}E(1+2h^2E)^{5/2}\eta^3(-25+66\eta)}{1024\sqrt{2}h^5};
\end{aligned}$$

$$\begin{aligned}
\text{ephi} = & \sqrt{1 + 2 h^2 E} + \frac{E(-12 + h^2 E(-15 + \eta))}{2 c^2 \sqrt{1 + 2 h^2 E}} + \\
& \frac{1}{16 c^4 h^2 (1 + 2 h^2 E)^{3/2}} E(-408 + 232 \eta + 15 \eta^2 + 2 h^6 E^3 (415 - 90 \eta + 11 \eta^2) + \\
& 4 h^4 E^2 (-52 + 158 \eta + 21 \eta^2) + 2 h^2 E(-584 + 408 \eta + 33 \eta^2)) - \\
& (E(3(27776 + (-65436 + 1325 \pi^2) \eta + 3440 \eta^2 - 70 \eta^3) + \\
& 12 h^2 E(35128 + 2(-47540 + 953 \pi^2) \eta + 5949 \eta^2 - 60 \eta^3) + \\
& 24 h^{10} E^5 (5679 - 2075 \eta + 369 \eta^2 + 59 \eta^3) + 4 h^4 E^2 (157176 + 2(-278452 + 5475 \pi^2) \eta + \\
& 44769 \eta^2 + 189 \eta^3) + 16 h^8 E^4 (-3306 + (-8642 + 3 \pi^2) \eta + 4803 \eta^2 + 369 \eta^3) + \\
& 8 h^6 E^3 (21624 + 4(-47500 + 873 \pi^2) \eta + 23901 \eta^2 + 663 \eta^3))) / \\
& (384 c^6 h^4 (1 + 2 h^2 E)^{5/2}) + (E(-33533337600 + (103173396416 - 2712160050 \pi^2) \eta + \\
& (-45489430272 + 860961825 \pi^2) \eta^2 + 1000096200 \eta^3 + 2300400 \eta^4 + \\
& 115200 h^{14} E^7 (310331 - 159012 \eta + 38922 \eta^2 + 4798 \eta^3 + 1343 \eta^4) - \\
& 16 h^2 E(15270912000 + 2(-25312361504 + 694658625 \pi^2) \eta + \\
& (24809416576 - 454368375 \pi^2) \eta^2 - 626903775 \eta^3 + 168750 \eta^4) + \\
& 256 h^{10} E^5 (-16754400 - 180(-2496472 + 377705 \pi^2) \eta + \\
& 4(-570897209 + 5834250 \pi^2) \eta^2 + 206180325 \eta^3 + 2477250 \eta^4) + \\
& 16 h^8 E^4 (-16220217600 + (109003797056 - 4575024150 \pi^2) \eta + \\
& (-115039428704 + 1631484675 \pi^2) \eta^2 + 5955298200 \eta^3 + 2790000 \eta^4) - \\
& 64 h^6 E^3 (11258236800 + 2(-25202037344 + 826097325 \pi^2) \eta + \\
& (35493836272 - 575573625 \pi^2) \eta^2 - 1331836875 \eta^3 + 3198150 \eta^4) + \\
& 64 h^{12} E^6 (-329961600 - 150(5596096 + 14961 \pi^2) \eta - \\
& (253128992 + 2828925 \pi^2) \eta^2 + 174699000 \eta^3 + 10076400 \eta^4) - \\
& 4 h^4 E^2 (161974886400 + 6(-100059588896 + 2931093675 \pi^2) \eta - \\
& 83(-4094885888 + 71525475 \pi^2) \eta^2 - 10193754600 \eta^3 + \\
& 21070800 \eta^4))) / (14745600 c^8 h^6 (1 + 2 h^2 E)^{7/2}) ; \\
v = & 2 \text{ArcTan}\left[\text{Sqrt}\left[\frac{1 + \text{ephi}}{1 - \text{ephi}}\right] \text{Tan}\left[\frac{u}{2}\right]\right];
\end{aligned}$$

(\*u in terms of  $\hat{u}$ ; x = post-Newtonian parameter as defined in the paper\*)

$$\begin{aligned}
u = \hat{u} + & \frac{x^2 \left( 24(-5 + 2 \eta) \text{ArcTan}\left[\frac{(-1 + \sqrt{1 - \text{et}}) \text{Sin}[\hat{u}]}{\text{et} + (-1 + \sqrt{1 - \text{et}}) \text{Cos}[\hat{u}]}\right] (-1 + \text{et} \text{Cos}[\hat{u}]) + \sqrt{1 - \text{et}} \text{et} \eta (4 + \eta) \text{Sin}[\hat{u}] \right)}{8 \sqrt{1 - \text{et}} (-1 + \text{et} \text{Cos}[\hat{u}])^2} - \\
& \frac{1}{192 (1 - \text{et})^{3/2} (-1 + \text{et} \text{Cos}[\hat{u}])} x^3 \\
& \left( (2880 + (-10880 + 123 \pi^2) \eta + 960 \eta^2 + 96 \text{et}^2 (30 - 29 \eta + 11 \eta^2)) \hat{u} - \right. \\
& \left. (2880 + (-10880 + 123 \pi^2) \eta + 960 \eta^2 + 96 \text{et}^2 (30 - 29 \eta + 11 \eta^2)) \right)
\end{aligned}$$

$$\begin{aligned}
& \left( -2 \operatorname{ArcTan} \left[ \frac{(-1 + \sqrt{1 - \text{et}}) \operatorname{Sin}[\hat{u}]}{\text{et} + (-1 + \sqrt{1 - \text{et}}) \operatorname{Cos}[\hat{u}]} \right] + \hat{u} \right) + \frac{1}{(-1 + \text{et} \operatorname{Cos}[\hat{u}])^3} \sqrt{1 - \text{et}} \text{et} \\
& (7488 + (-7652 + 156 \text{et}^2 + 3 \pi^2) \eta - 16 (-82 + 7 \text{et}^2) \eta^2 + (-8 - 44 \text{et}^2 + 52 \text{et}^4) \eta^3 - \\
& 2 \text{et} (7488 + (-7598 + 102 \text{et}^2 + 3 \pi^2) \eta - 40 (-31 + \text{et}^2) \eta^2 + 56 (-1 + \text{et}^2) \eta^3) \operatorname{Cos}[\hat{u}] + \\
& \text{et}^2 (7488 + (-7544 + 48 \text{et}^2 + 3 \pi^2) \eta + 16 (73 + 2 \text{et}^2) \eta^2 + 52 (-1 + \text{et}^2) \eta^3) \operatorname{Cos}[\hat{u}]^2) \\
& \operatorname{Sin}[\hat{u}] \Big) + \frac{1}{14\,745\,600 (\sqrt{1 - \text{et}} - \sqrt{1 - \text{et}} \text{et} \operatorname{Cos}[\hat{u}])^6} x^4 \\
& \left( 1280 \sqrt{1 - \text{et}} (1\,460\,160 + (-7\,928\,128 + 334\,605 \pi^2) \eta - 720 (-7464 + 205 \pi^2) \eta^2 - \right. \\
& 76\,800 \eta^3) \operatorname{ArcTan} \left[ \frac{(-1 + \sqrt{1 - \text{et}}) \operatorname{Sin}[\hat{u}]}{\text{et} + (-1 + \sqrt{1 - \text{et}}) \operatorname{Cos}[\hat{u}]} \right] (1 - \text{et} \operatorname{Cos}[\hat{u}])^5 - \\
& 614\,400 \text{et}^4 (72 (5 - 2 \eta)^2 + \sqrt{1 - \text{et}} (-360 + 111 \eta - 403 \eta^2 + 286 \eta^3)) \\
& \operatorname{ArcTan} \left[ \frac{(-1 + \sqrt{1 - \text{et}}) \operatorname{Sin}[\hat{u}]}{\text{et} + (-1 + \sqrt{1 - \text{et}}) \operatorname{Cos}[\hat{u}]} \right] (1 - \text{et} \operatorname{Cos}[\hat{u}])^5 - \\
& 22\,118\,400 (5 - 2 \eta)^2 \operatorname{ArcTan} \left[ \frac{(-1 + \sqrt{1 - \text{et}}) \operatorname{Sin}[\hat{u}]}{\text{et} + (-1 + \sqrt{1 - \text{et}}) \operatorname{Cos}[\hat{u}]} \right] (-1 + \text{et} \operatorname{Cos}[\hat{u}])^5 - \\
& 22\,118\,400 \text{et}^2 (5 - 2 \eta)^2 \operatorname{ArcTan} \left[ \frac{(-1 + \sqrt{1 - \text{et}}) \operatorname{Sin}[\hat{u}]}{\text{et} + (-1 + \sqrt{1 - \text{et}}) \operatorname{Cos}[\hat{u}]} \right] (-1 + \text{et} \operatorname{Cos}[\hat{u}])^5 + \\
& 1600 \sqrt{1 - \text{et}} \text{et}^2 (-2\,056\,320 + (10\,105\,600 - 159\,843 \pi^2) \eta + 96 (-63\,776 + 615 \pi^2) \eta^2 + \\
& 568\,320 \eta^3) \operatorname{ArcTan} \left[ \frac{(-1 + \sqrt{1 - \text{et}}) \operatorname{Sin}[\hat{u}]}{\text{et} + (-1 + \sqrt{1 - \text{et}}) \operatorname{Cos}[\hat{u}]} \right] (-1 + \text{et} \operatorname{Cos}[\hat{u}])^5 + 32 (1 - \text{et}) \text{et}^2 \\
& (46\,094\,400 + (-85\,176\,256 - 1\,728\,000 \sqrt{1 - \text{et}} + 1\,134\,000 \text{et}^2 + 1\,101\,225 \pi^2) \eta + \\
& (70\,060\,432 + 259\,200 \sqrt{1 - \text{et}} - 4\,407\,300 \text{et}^2 - 3\,159\,975 \pi^2) \eta^2 - \\
& 1200 (-144 (-35 + \sqrt{1 - \text{et}}) + 913 \text{et}^2) \eta^3 - 1800 (2 + 257 \text{et}^2) \eta^4) \\
& (\text{et} - \operatorname{Cos}[\hat{u}]) (1 - \text{et} \operatorname{Cos}[\hat{u}])^3 \operatorname{Sin}[\hat{u}] - 800 \sqrt{1 - \text{et}} \text{et}^5 \eta \\
& (1152 (-20 + \sqrt{1 - \text{et}}) - 384 (-9 + 4 \sqrt{1 - \text{et}}) \eta + (2304 - 18\,337 \sqrt{1 - \text{et}}) \eta^2 - \\
& 6286 \sqrt{1 - \text{et}} \eta^3) (-1 + \text{et} \operatorname{Cos}[\hat{u}])^4 \operatorname{Sin}[\hat{u}] + 32 \sqrt{1 - \text{et}} \text{et} \\
& (-57\,600 (450 + 1727 \sqrt{1 - \text{et}}) - 32 (-720\,000 + \sqrt{1 - \text{et}} (-10\,972\,271 + 273\,525 \pi^2)) \\
& \eta + 15 (-299\,520 + \sqrt{1 - \text{et}} (-11\,651\,168 + 205\,725 \pi^2)) \eta^2 + \\
& 200 (-1152 + 40\,529 \sqrt{1 - \text{et}}) \eta^3 + 2800 \sqrt{1 - \text{et}} \eta^4) \\
& (-1 + \text{et} \operatorname{Cos}[\hat{u}])^4 \operatorname{Sin}[\hat{u}] + \sqrt{1 - \text{et}} \text{et}^3 (-33\,177\,600 (-25 + 52 \sqrt{1 - \text{et}}) - \\
& 150 (5\,038\,080 + \sqrt{1 - \text{et}} (-18\,978\,272 + 22\,161 \pi^2)) \eta + \\
& (146\,534\,400 - \sqrt{1 - \text{et}} (1\,667\,201\,792 + 2\,123\,325 \pi^2)) \eta^2 +
\end{aligned}$$



$$\begin{aligned}
& 8000 \left( 1152 + 37069 \sqrt{1-et} \right) \eta^3 + 80000 \sqrt{1-et} \eta^4 \left( -1 + et \cos[\hat{u}] \right)^4 \sin[\hat{u}] + \\
& et^3 \eta \left( 70421952 - 2430450 \pi^2 - 5 \left( -25524992 + 2673315 \pi^2 \right) \eta - \right. \\
& \quad \left. 1200 \left( 14972 + 603 et^2 \right) \eta^2 - 2400 \left( 1532 + 685 et^2 \right) \eta^3 \right) \\
& \left( \sqrt{1-et} - \sqrt{1-et} et \cos[\hat{u}] \right)^2 \left( 1 - 4 et^2 + 6 et \cos[\hat{u}] + (-4 + et^2) \cos[\hat{u}]^2 \right) \sin[\hat{u}] + \\
& 9600 (1-et) et^4 \eta \left( 516 - 3303 \eta - 52 \eta^2 + 774 \eta^3 \right) (1-et \cos[\hat{u}]) \\
& \left( et - 2 et^3 + (-1 + 4 et^2) \cos[\hat{u}] + et (-4 + et^2) \cos[\hat{u}]^2 - (-2 + et^2) \cos[\hat{u}]^3 \right) \sin[\hat{u}] - \\
& 3600 (1-et) et^5 \left( 25 - 66 \eta \right) \eta^3 \left( 1 - 12 et^2 + 16 et^4 - 20 et (-1 + 2 et^2) \cos[\hat{u}] - \right. \\
& \quad \left. 6 (2 - 9 et^2 + 2 et^4) \cos[\hat{u}]^2 + 20 et (-2 + et^2) \cos[\hat{u}]^3 + (16 - 12 et^2 + et^4) \cos[\hat{u}]^4 \right) \\
& \sin[\hat{u}] - \left( 230400 (1-et)^2 (1-et \cos[\hat{u}]) \left( 24 (-1 + \sqrt{1-et}) (-5 + 2 \eta) \right. \right. \\
& \quad \left. \left. (-1 + et \cos[\hat{u}])^2 (-1 + \sqrt{1-et} + et \cos[\hat{u}]) + \sqrt{1-et} et \eta (4 + \eta) (et - \cos[\hat{u}]) \right. \right. \\
& \quad \left. \left. \left( (-1 + \sqrt{1-et})^2 + et^2 + 2 (-1 + \sqrt{1-et}) et \cos[\hat{u}] \right) \right) \left( 24 (-5 + 2 \eta) \text{ArcTan} \left[ \right. \right. \right. \\
& \quad \left. \left. \left. \frac{(-1 + \sqrt{1-et}) \sin[\hat{u}]}{et + (-1 + \sqrt{1-et}) \cos[\hat{u}]} \right] (-1 + et \cos[\hat{u}] + \sqrt{1-et} et \eta (4 + \eta) \sin[\hat{u}]) \right) \right) \right) / \\
& \left( (-1 + \sqrt{1-et})^2 + et^2 + 2 (-1 + \sqrt{1-et}) et \cos[\hat{u}] \right) - 115200 (1-et)^2 et \\
& (1-et \cos[\hat{u}]) \sin[\hat{u}] \left( 24 (-5 + 2 \eta) \text{ArcTan} \left[ \frac{(-1 + \sqrt{1-et}) \sin[\hat{u}]}{et + (-1 + \sqrt{1-et}) \cos[\hat{u}]} \right] \right. \\
& \quad \left. \left. (-1 + et \cos[\hat{u}] + \sqrt{1-et} et \eta (4 + \eta) \sin[\hat{u}])^2 \right) \right);
\end{aligned}$$

Nonlocal-in-time 4PN contributions

n\_tail =

$$\begin{aligned}
 & x^{3/2} \left( 2 \left( (1 - et^2) \left( 1 - \frac{56374811 et^2}{24380301} + \frac{103729937 et^4}{57112735} - \frac{105413189 et^6}{194334558} + \frac{49804512 et^8}{1158420851} + \right. \right. \right. \\
 & \quad \left. \left. \frac{4447985 et^{10}}{4076572203} \right) \left( \frac{263415291 et}{819998} - \frac{3032926060 et^3}{3359177} - \frac{974131935 et^5}{4216762} + \right. \right. \\
 & \quad \left. \left. \frac{2317329404 et^7}{2491579} - \frac{3829962720 et^9}{13601521} - \frac{40653054 et^{11}}{124087307} + \frac{199674118 et^{13}}{328008227} \right) + \right. \\
 & \quad \left. (-1 + et^2) \left( -\frac{112749622 et}{24380301} + \frac{414919748 et^3}{57112735} - \frac{105413189 et^5}{32389093} + \right. \right. \\
 & \quad \left. \left. \frac{398436096 et^7}{1158420851} + \frac{44479850 et^9}{4076572203} \right) \right. \\
 & \quad \left( \frac{286746937}{12927762} + \frac{263415291 et^2}{1639996} - \frac{758231515 et^4}{3359177} - \frac{324710645 et^6}{8433524} + \right. \\
 & \quad \left. \frac{579332351 et^8}{4983158} - \frac{382996272 et^{10}}{13601521} - \frac{6775509 et^{12}}{248174614} + \frac{14262437 et^{14}}{328008227} \right) + \\
 & \quad 7 et \left( 1 - \frac{56374811 et^2}{24380301} + \frac{103729937 et^4}{57112735} - \frac{105413189 et^6}{194334558} + \right. \\
 & \quad \left. \frac{49804512 et^8}{1158420851} + \frac{4447985 et^{10}}{4076572203} \right) \left( \frac{286746937}{12927762} + \frac{263415291 et^2}{1639996} - \right. \\
 & \quad \left. \frac{758231515 et^4}{3359177} - \frac{324710645 et^6}{8433524} + \frac{579332351 et^8}{4983158} - \right. \\
 & \quad \left. \frac{382996272 et^{10}}{13601521} - \frac{6775509 et^{12}}{248174614} + \frac{14262437 et^{14}}{328008227} \right) \eta x^4 \Big/ \\
 & \quad \left( 5 et (-1 + et^2)^4 \left( 1 - \frac{56374811 et^2}{24380301} + \frac{103729937 et^4}{57112735} - \frac{105413189 et^6}{194334558} + \right. \right. \\
 & \quad \left. \left. \frac{49804512 et^8}{1158420851} + \frac{4447985 et^{10}}{4076572203} \right)^2 \right) - \frac{1}{90 et^2 (-1 + et^2)^4} \\
 & \quad \left( 1209 et^6 + 8 et^2 (3637 - 2917 \sqrt{1 - et^2}) + 4 et^4 (5982 - 2803 \sqrt{1 - et^2}) - \right. \\
 & \quad \left. 1152 (-1 + \sqrt{1 - et^2}) \right) \eta x^4 + \\
 & \quad \frac{2 (1256 + 1608 et^2 + 111 et^4) \eta x^4 \operatorname{Log}\left[\frac{4 e^{\operatorname{EulerGamma}} (1 - et^2) (1 - \sqrt{1 - et^2}) \sqrt{x}}{et^2}\right]}{15 (-1 + et^2)^4} \Big)
 \end{aligned}$$

(\*The tail part of n is given here in terms of x and et, rather than E and h as has been done above for the local in time contributions\*)

In[ ]:=