Lecture 5
Epipolar Geometry

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Lecture 5

Epipolar Geometry

- Why is stereo useful?
- Epipolar constraints
- Essential and fundamental matrix
- Estimating F
- Examples

Reading: [AZ] Chapter: 4 "Estimation – 2D perspective transformations
Chapter: 9 "Epipolar Geometry and the Fundamental Matrix Transformation"
Chapter: 11 "Computation of the Fundamental Matrix F"
[FP] Chapter: 7 "Stereopsis"
Chapter: 8 "Structure from Motion"

Recovering structure from a single view

From calibration rig	→ location/pose of the rig, K
From points and lines at infinity + orthogonal lines and planes	→ structure of the scene, K
Knowledge about scene (point correspondences, geometry of lines & planes, etc...	

Recovering structure from a single view

Why is it so difficult?

Intrinsic ambiguity of the mapping from 3D to image (2D)

Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)

Courtesy slide S. Lazebnik

Two eyes help!

Two eyes help!

$$P = l \times l'$$

[Eq. 1]

In practice the lines might not intersect becaus of noisy observation and noisy calibration estimation.

This is called triangulation

Triangulation

- Find P* that minimizes

$$d(p, M\ P^*) + d(p', M'P^*) \quad \text{[Eq. 2]}$$

Multi (stereo)-view geometry

- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.

- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.

- **Correspondence:** Given a point **p** in one image, how can I find the corresponding point **p′** in another one?

Epipolar geometry

- Epipolar Plane
- Baseline
- Epipolar Lines

- Epipoles e, e′

  = intersections of baseline with image planes
  = projections of the other camera center

Example of epipolar lines

Example: Parallel image planes

- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to u axis

Example: Parallel Image Planes

$\longrightarrow u$

Example: Forward translation

- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)

Epipolar Constraint

- Two views of the same object
- Given a point on left image, how can I find the corresponding point on right image?

Epipolar geometry

Epipolar Constraint

Epipolar Constraint

Cam1 to Cam2:
first rotation then translation: $[R \; T]$
Cam2 to Cam1:
first translation then rotation:

$$\begin{bmatrix} R^T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T & -R^T T \\ 0 & 1 \end{bmatrix}$$

$$M = K \begin{bmatrix} I & 0 \end{bmatrix}$$

$$M' = K' \begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$M\,P = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = p \quad \text{[Eq. 3]}$$

$$M'\,P = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = p' \quad \text{[Eq. 4]}$$

If camera is canonical, then $f=1$

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Epipolar Constraint

$$K_{canonical} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = K\begin{bmatrix} I & 0 \end{bmatrix}$$

K = K′ are known (canonical cameras)

$$M' = K'\begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

$$M = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{[Eq. 5]}$$

$$M' = \begin{bmatrix} R^T & -R^T T \end{bmatrix} \quad \text{[Eq. 6]}$$

Epipolar Constraint

p, p', and T lies in the same plane

$(T \times Rp') + \cancel{(T \times T)}^{\,0}$

p′ in first camera reference system is = $R\,p' + T$ → assuming canonical camera

$\underbrace{T \times ((R\,p') + T)}_{\text{Same plane}} = T \times (R\,p')$ is perpendicular to epipolar plane

$$\rightarrow p^T \cdot \left[T \times (R\,p')\right] = 0 \quad \text{[Eq. 7]}$$

Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

$a = [a_x \quad a_y \quad a_z]^T$

$b = [b_x \quad b_y \quad b_z]^T$

Epipolar Constraint

$$p^T \cdot [T \times (R\, p')] = 0 \rightarrow p^T \cdot \boxed{[T_\times] \cdot R}\, p' = 0$$

[Eq. 8] [Eq. 9]

E = Essential matrix

(Longuet-Higgins, 1981)

Epipolar Constraint

$$p^T \cdot E\, p' = 0$$

[Eq. 10]

- $l = E\,p'$ is the epipolar line associated with p'
- $l' = E^T p$ is the epipolar line associated with p
- $E\,e' = 0$ and $E^T e = 0$
- E is 3x3 matrix; 5 DOF → E is defined up to scale
- E is singular (rank two)

Epipolar Constraint

K no longer canonical

Cam 1 to Cam 2
$\rightarrow [R \;\; T]$ Rotation followed by translation

Inversing the process
Translation followed by Rotation.

Cam 2 to Cam 1
$R^T[I \;\; -T]$
$= [R^T \;\; -R^T T]$

Let p_c & p_c' be the projections of p (3D point) under the assumption that the camera is canonical

$$M = K[I \quad 0]$$

$$M' = K'\begin{bmatrix} R^T & -R^T T \end{bmatrix}$$

Projection of p in a canonical cam $\rightarrow$

$$p_c = K^{-1}\, p \quad \text{[Eq. 11]}$$

$$p_c' = K'^{-1}\, p' \quad \text{[Eq. 12]}$$

Epipolar Constraint

$$p_c = K^{-1}\, p$$

[Eq. 11]

$$p'_c = K'^{-1}\, p'$$

[Eq. 12]

[Eq.9]

$$p_c^T \cdot [T_\times] \cdot R\, p'_c = 0 \rightarrow (\mathrm{K}^{-1}\,\mathrm{p})^{\mathrm{T}} \cdot [\mathrm{T}_\times] \cdot \mathrm{R}\,\mathrm{K}'^{-1}\,\mathrm{p}' = 0$$

$$\mathrm{p}^{\mathrm{T}} \boxed{\mathrm{K}^{-\mathrm{T}} \cdot [\mathrm{T}_\times] \cdot \mathrm{R}\,\mathrm{K}'^{-1}}\,\mathrm{p}' = 0 \rightarrow \mathrm{p}^{\mathrm{T}} \boxed{\mathrm{F}}\,\mathrm{p}' = 0$$

[Eq. 13]

Epipolar Constraint

[Eq. 13]

$$\boxed{\mathrm{p}^{\mathrm{T}}\,\mathrm{F}\,\mathrm{p}' = 0}$$

$$F = K^{-T} \cdot [T_\times] \cdot R\, K'^{-1}$$

[Eq. 14]

F = Fundamental Matrix

(Faugeras and Luong, 1992)

Epipolar Constraint

$$p^T \cdot F\ p' = 0$$

- $l = F\,p'$ is the epipolar line associated with p'
- $l' = F^T p$ is the epipolar line associated with p
- $F\,e' = 0$ and $F^T e = 0$
- F is 3x3 matrix; 7 DOF → Proven mathematically
- F is singular (rank two)

We initially have 9 DOF because the fundamental matrix is composed of 9 parameters, which implies that we need 9 corresponding points to compute the fundamental matrix (F). But because of the following two reasons, we only need 7 corresponding points.

Reason 1

We lose 1 DOF because we are using homogeneous coordinates. This basically is a way to represent nD points as a vector form by adding an extra dimension. ie) A 2D point (0,2) can be represented as [0,2,1], in general [x,y,1]. There are useful properties when using homogeneous coordinates with 2D/3D transformation, but I'm going to assume you know that.

Now given the expression p and p' representing pixel coordinates:

p'=[u',v',1] and p=[u,v,1]
the fundamental matrix:

F = [f1,f2,f3]
[f4,f5,f6]
[f7,f8,f9]
and fundamental matrix equation:

(transposed p')Fp = 0
when we multiple this expression in algebra form, we get the following:

uu'f1 + vu'f2 + u'f3 + uv'f4 + vv'f5 + v'f6 + uf7 + vf8 + f9 = 0.
In a homogeneous system of linear equation form Af=0 (basically the factorization of the above formula), we get two components A and f.

A:
[uu',vu',u', uv',vv',v',u,v,1]
f (f is essentially the fundamental matrix in vector form):

[f1,f2'f3,f4,f5,f6,f7,f8,f9]
Now if we look at the components of vector A, we have 8 unknowns, but one known value 1 because of homogeneous coordinates, and therefore we only need 8 equations now.

Reason 2

det F = 0.
A determinant is a value that can be obtained from a square matrix.

Basically given some matrix A

A = [a,b,c]
[d,e,f]
[g,h,i]
The determinant can be computed using this formula:

det A = aei+bfg+cdh-ceg-bdi-afh
If we look at the determinant using the fundamental matrix, the algebra would look something like this:

F = [f1,f2,f3]
[f4,f5,f6]
[f7,f8,f9]

det F = (f1*f5*f8)+(f2*f6*f7)+(f3*f4*f8)-(f3*f5*f7)-(f2*f4*f9)-(f1*f6*f8)
Now we know the determinant of the fundamental matrix is zero:

det F = (f1*f5*f8)+(f2*f6*f7)+(f3*f4*f8)-(f3*f5*f7)-(f2*f4*f9)-(f1*f6*f8) = 0
So, if we work out only 7 of the 9 parameters of the fundamental matrix, we can work out the last parameter using the above determinant equation.
Therefore the fundamental matrix has 7DOF.

Why F is useful?

- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, we can compute the corresponding epipolar line in the second imag

Why F is useful?

- F captures information about the epipolar geometry of 2 views + camera parameters

- MORE IMPORTANTLY: F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching

Estimating F

The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)

$$p^T F p' = 0$$

Estimating F

[Eq. 13] $\mathrm{p}^{\mathrm{T}}\,\mathrm{F}\,\mathrm{p}' = 0$ ➡

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

➡

$$(uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

Estimating F

Estimating F

$$\left(u_i u'_i, u_i v'_i, u_i, v_i u'_i, v_i v'_i, v_i, u'_i, v_i', 1\right)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \qquad \text{[Eq. 14]}$$

Estimating F

$$\mathbf{W}\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix}}_{\mathbf{f}} = 0 \quad \text{[Eqs. 15]}$$

- Homogeneous system $\mathbf{W}\mathbf{f} = 0$
- Rank 8 $\longrightarrow$ A non-zero solution exists (unique)
- If N>8 $\longrightarrow$ Lsq. solution by SVD! $\longrightarrow$ $\hat{\mathrm{F}}$

$$\|\mathbf{f}\| = 1$$

$\hat{F}$ satisfies: $p^T \hat{F} p' = 0$

and estimated $\hat{F}$ may have full rank ($\det(\hat{F}) \neq 0$)

But remember: fundamental matrix is Rank2

Find F that minimizes $\left\| F - \hat{F} \right\| = 0$

Frobenius norm (*)

Subject to det(F)=0

SVD (again!) can be used to solve this problem

(*) Sq. root of the sum of squares of all entries

distance between matrices

Find F that minimizes $\left\| F - \hat{F} \right\| = 0$

Frobenius norm (*)

Subject to det(F)=0

$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

Where:

$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$

[HZ] pag 281, chapter 11, "Computation of F"

Data courtesy of R. Mohr and B. Boufama.

Mean errors:
10.0pixel
9.1pixel

Problems with the 8-Point Algorithm

$$\mathbf{W}\,\mathbf{f} = 0, \quad \|\mathbf{f}\| = 1 \xrightarrow[\text{by SVD}]{\text{Lsq solution}} \mathrm{F}$$

- Recall the structure of W:
  - do we see any potential (numerical) issue?

Problems with the 8-Point Algorithm

Could be order of 10^4

Could be order of 100s

$$\mathbf{Wf} = 0$$

$$\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

- Highly un-balanced (not well conditioned)
- Values of W must have similar magnitude
- This creates problems during the SVD decomposition

HZ pag 108

Normalization

IDEA: Transform image coordinates such that the matrix **W** becomes better conditioned (**pre-conditioning**)

For each image, apply a transformation T (translation and scaling) acting on image coordinates such that:

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels

Example of normalization

Coordinate system of the image before applying T

Coordinate system of the image after applying T

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels

Normalization

$$q_i = T\ p_i$$

$$q'_i = T'\ p'_i$$

The Normalized Eight-Point Algorithm

0. Compute T and T′ for image 1 and 2, respectively

1. Normalize coordinates in images 1 and 2:

$$q_i = T\ p_i \qquad q'_i = T'\ p'_i$$

2. Use the eight-point algorithm to compute $\hat{F}_q$ from the corresponding points q_i and q'_i .

1. Enforce the rank-2 constraint: $\rightarrow \mathrm{F_q}$ such that:

$$\begin{cases} \mathrm{q}^{\mathrm{T}}\ \mathrm{F_q}\ \mathrm{q}' = 0 \\ \det(\mathrm{F_q}) = 0 \end{cases}$$

2. De-normalize F_q: $F = T^T F_q\ T'$

Without normalization

Mean errors:
10.0pixel
9.1pixel

With normalization

Mean errors:
1.0pixel
0.9pixel

The Fundamental Matrix Song

Next lecture:

Stereo systems

Example: Parallel image planes

$K_1 = K_2$ = known

x parallel to O_1O_2

$\boxed{E = ?}$

Hint :

$\boldsymbol{R} = I \qquad \boldsymbol{T} = (T, 0, 0)$

Essential matrix for parallel images

$$\mathbf{E} = \left[\mathbf{T}_{\times}\right] \cdot \mathbf{R}$$

$$\mathbf{E} = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix} \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$$

[Eq. 20]

$$\mathbf{T} = [\ \mathrm{T}\ \ 0\ \ 0\ ]$$

$$\mathbf{R} = \mathbf{I}$$

Example: Parallel image planes

What are the directions of epipolar lines?

$$l = \mathrm{E}\, p' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -T \\ T\,v' \end{bmatrix}$$

horizontal!

Example: Parallel image planes

How are p and p′ related?

$$p^T \cdot E\, p' = 0$$

Example: Parallel image planes

How are p and p′ related?

$$\Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} u & v & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -T \\ Tv' \end{pmatrix} = 0 \Rightarrow Tv = Tv'$$

$$\Rightarrow v = v'$$

Example: Parallel image planes

Rectification: making two images "parallel"

Why it is useful?

- Epipolar constraint → v = v′
- New views can be synthesized by linear interpolation

Application: view morphing

S. M. Seitz and C. R. Dyer, *Proc. SIGGRAPH 96*, 1996, 21-30

Rectification

BAND-AID

BAND-AID

From its reflection!

Deep view morphing

D. Ji, J. Kwon, M. McFarland, S. Savarese, CVPR 2017

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