CS231A

Computer Vision:

From 3D Reconstruction to Recognition

Optimal Estimation

Perception as a Continuous Process

Perception as a Multi-Modal Experience

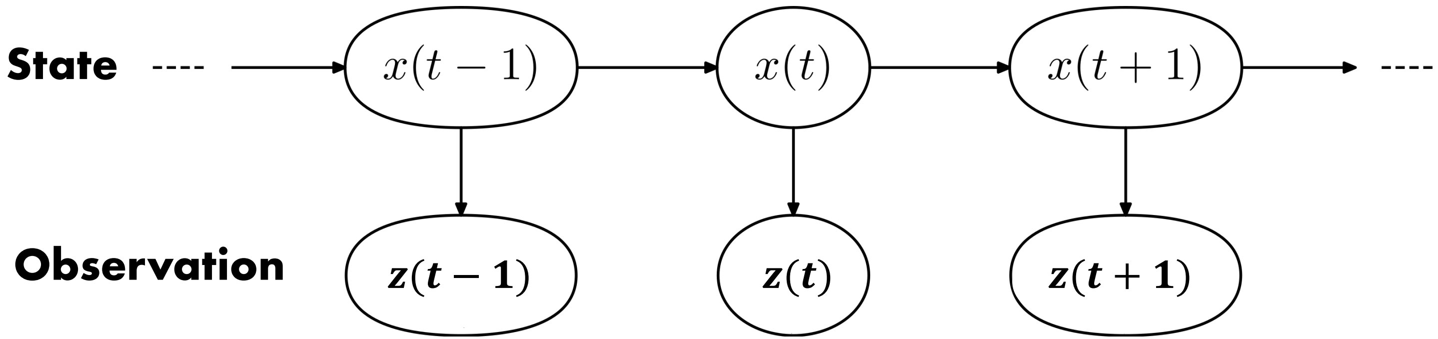
Perception as Inference

Recursive State Estimation

Mathematical Formalism to:

- continuously integrate measurements
- from different sensor sources
- to infer the state of a latent variable

What is a state? What is a representation?



Hidden Markov Model

Representations for Autonomous Driving

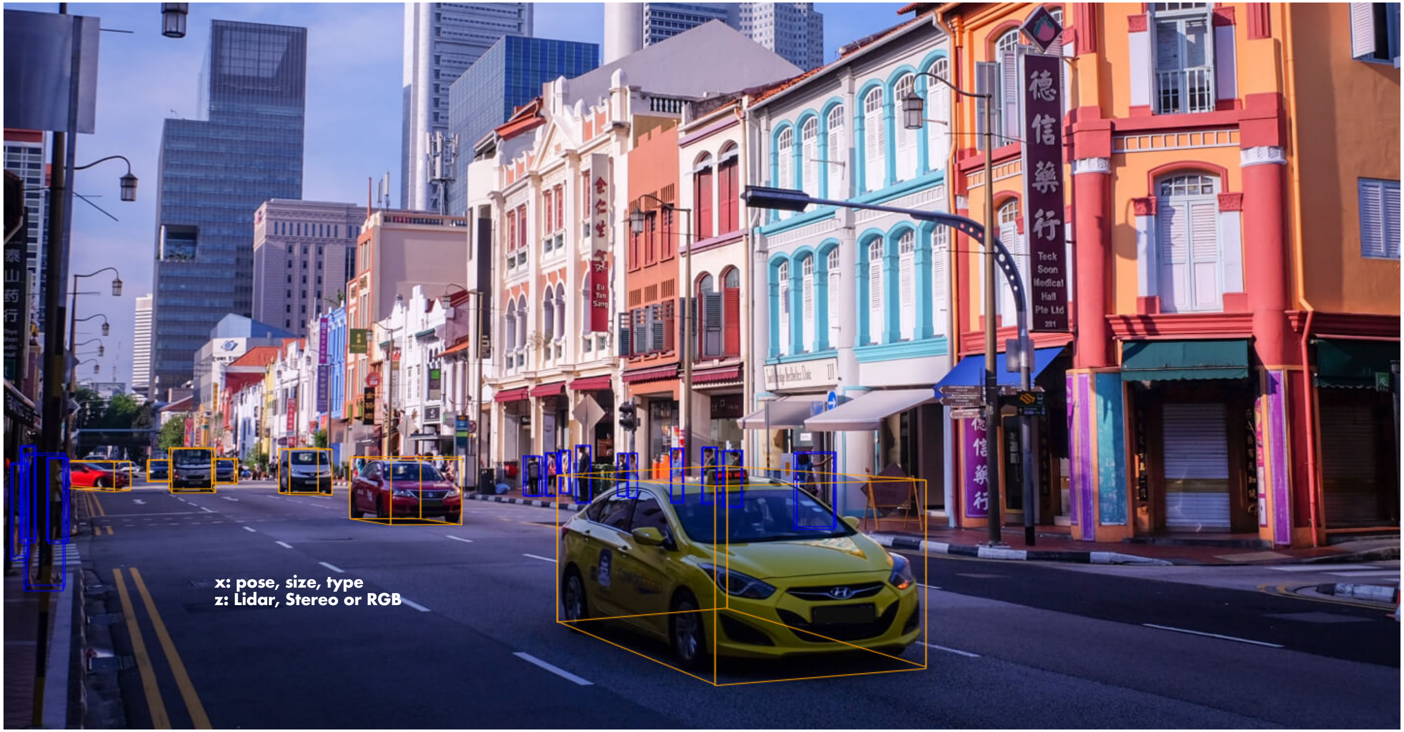
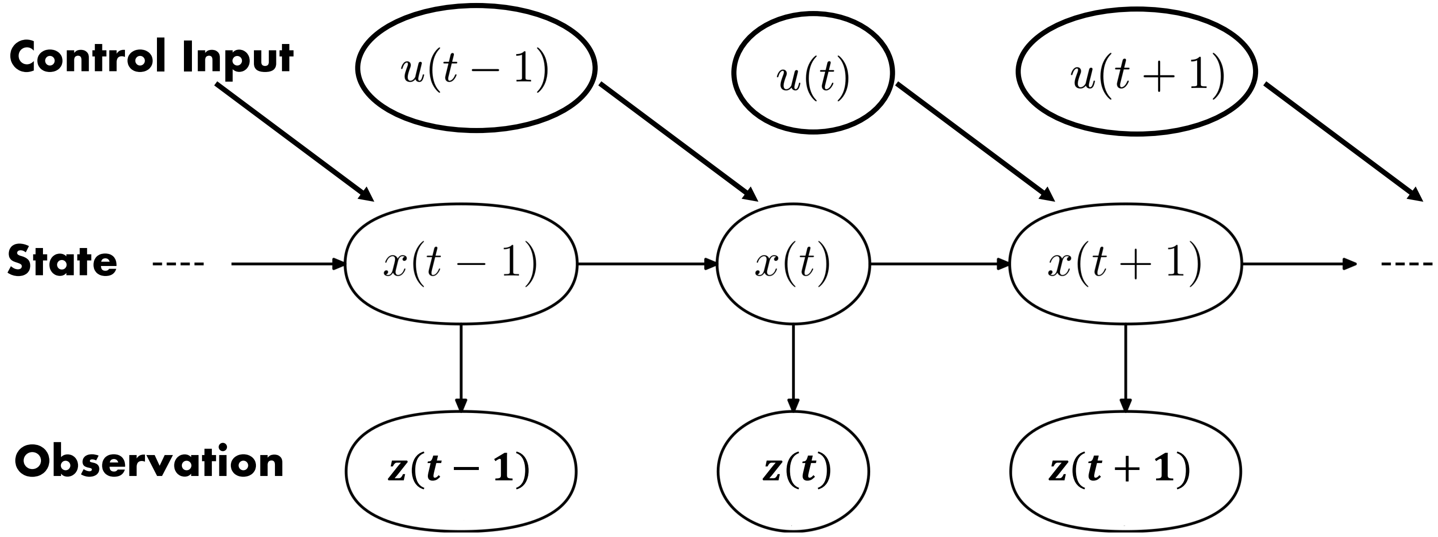


Image adapted from NuScenes by Motional. nuscenes.org

Representations for Manipulation

Manuel Wühtrich et al. “Probabilistic Object Tracking using a Depth Camera”, IROS 2013

Why do we care about state estimation in Robotics?

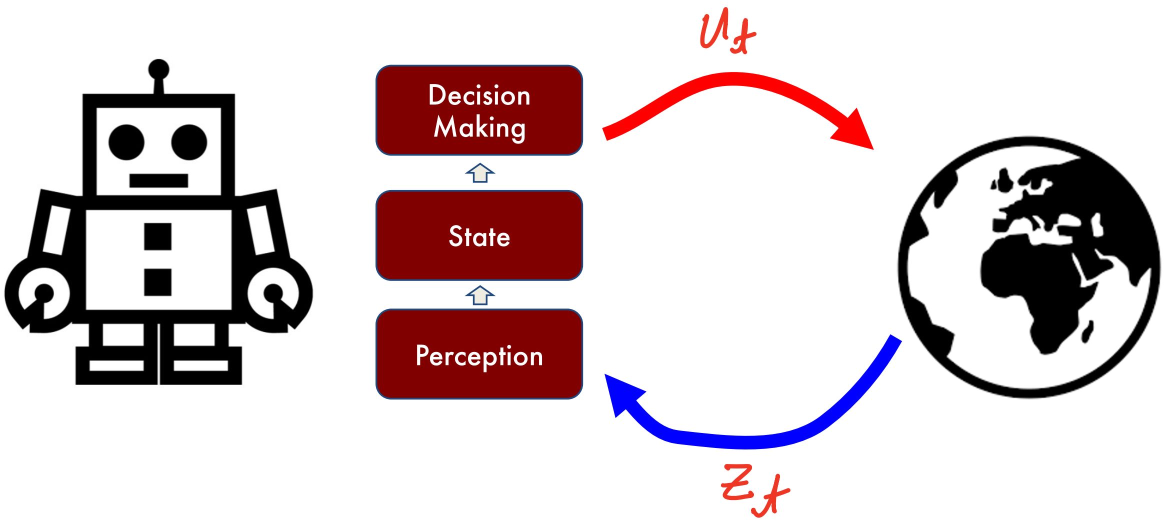


Partially Observable Markov Decision Process

Today

- Intro: Why state estimation?
- Bayes Filter
- Kalman Filter
- Extended Kalman Filter

The Agent and the Environment



Notation

Random variables

↓

$x \longrightarrow$ dynamic state

$x_t \longrightarrow$ state being optimized

$z \longrightarrow$ observation, dim K

$z_t \longrightarrow$ observation at time t

$u \longrightarrow$ actions, dim m

$u_t \longrightarrow$ action at time t

$p(x_t | z_{0:t}, u_{0:t})$

Markov assumption! $\longrightarrow$ makes the problem tractable

= State is complete

Probabilistic Generative Laws

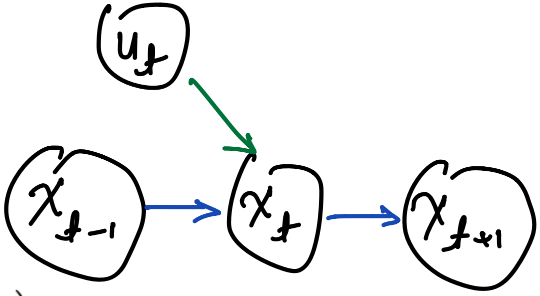
- Evolution of state and measurement governed by probabilistic laws
- x_t generated stochastically

State Transition Model

- Probability distribution conditioned on all previous states, measurements and controls

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})$$

- Assumption: State complete



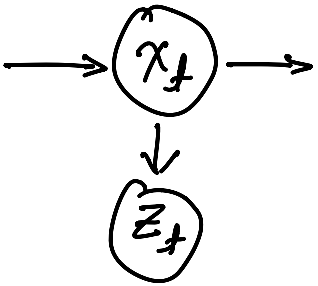
$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

Measurement Model

- Probability distribution conditioned on all previous states, measurements and controls

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t})$$

- Assumption: State complete



$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) \quad = \quad p(z_t \mid x_t)$$

z_t only depends on x_t

Belief Distribution

- Assigns probability to each possible hypothesis about what the true state may be
- Posterior distributions over state conditioned on all the data

$$bel(x_t) \quad = \quad p(x_t \mid z_{1:t}, u_{1:t})$$

- Before incorporating measurement = prediction

$$\overline{bel}(x_t) \quad = \quad p(x_t \mid z_{1:t-1}, u_{1:t})$$

The Bayes Filter

- Recursive filter for estimating x_t only from x_{t-1}, z_t and u_{t-1} and not from the ever-growing history $z_{1:t}, u_{1:t}$

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$)**:**
    for all x_t do
        $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$ ← Predict
        $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$ ← Update
    endfor
    return $bel(x_t)$

Simple example – Belief & Measurement Model

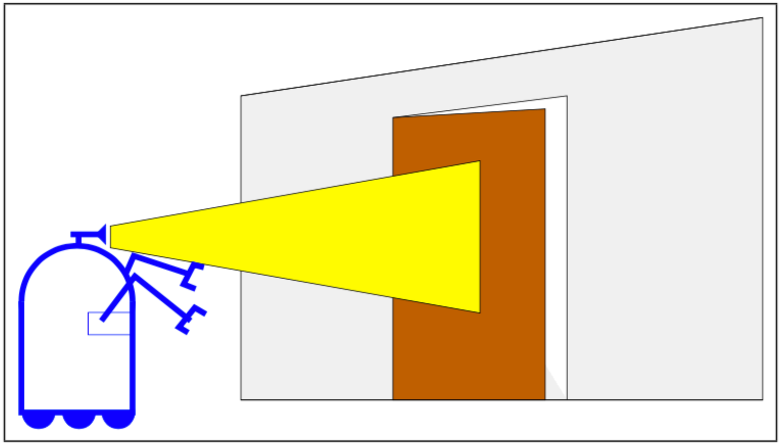


Figure 2.2 A mobile robot estimating the state of a door.

$$
\begin{aligned}
bel(X_0 = \mathbf{open}) &= 0.5 \\
bel(X_0 = \mathbf{closed}) &= 0.5
\end{aligned}
$$

$$
\begin{aligned}
p(Z_t = \mathbf{sense\_open} \mid X_t = \mathbf{is\_open}) &= 0.6 \\
p(Z_t = \mathbf{sense\_closed} \mid X_t = \mathbf{is\_open}) &= 0.4
\end{aligned}
$$

$$
\begin{aligned}
p(Z_t = \mathbf{sense\_open} \mid X_t = \mathbf{is\_closed}) &= 0.2 \\
p(Z_t = \mathbf{sense\_closed} \mid X_t = \mathbf{is\_closed}) &= 0.8
\end{aligned}
$$

Simple example – Transition Model

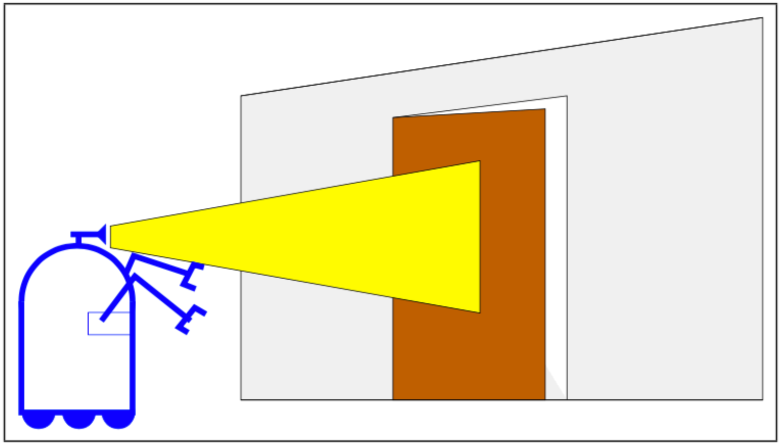


Figure 2.2 A mobile robot estimating the state of a door.

$$
\begin{aligned}
p(X_t = \textbf{is\_open} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_open}) &= 1 \\
p(X_t = \textbf{is\_closed} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_open}) &= 0 \\
p(X_t = \textbf{is\_open} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_closed}) &= 0.8 \\
p(X_t = \textbf{is\_closed} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_closed}) &= 0.2
\end{aligned}
$$

$$
\begin{aligned}
p(X_t = \textbf{is\_open} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_open}) &= 1 \\
p(X_t = \textbf{is\_closed} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_open}) &= 0 \\
p(X_t = \textbf{is\_open} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_closed}) &= 0 \\
p(X_t = \textbf{is\_closed} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_closed}) &= 1
\end{aligned}
$$

The Bayes Filter - Derivation

- Bayes Rule

Bayes Rule: $p(a|b)\,p(b) = p(b|a)\,p(a)$

$\Rightarrow p(a|b) = \dfrac{p(b|a)\,p(a)}{p(b)}$

$$p(x_t \mid z_{1:t}, u_{1:t}) = \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})} \longleftarrow \text{Normalization Constant}$$

$$= \eta\, \underbrace{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})}_{1}\, \underbrace{p(x_t \mid z_{1:t-1}, u_{1:t})}_{2}$$

The Bayes Filter - Derivation

- State is complete

$$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \quad = \quad p(z_t \mid x_t)$$

- Simplify

$$\begin{aligned} p(x_t \mid z_{1:t}, u_{1:t}) &= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})} \\ &= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \end{aligned}$$

The Bayes Filter - Derivation

measurement model

$$p(x_t \mid z_{1:t}, u_{1:t}) \quad = \quad \eta \, p(z_t \mid x_t) \, \underline{p(x_t \mid z_{1:t-1}, u_{1:t})}$$

$$bel(x_t) \quad = \quad \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$$

Still depends on entire history

1: **Algorithm Bayes_filter**($bel(x_{t-1}), u_t, z_t$)**:**
2: for all x_t do
3: $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \, bel(x_{t-1}) \, dx$
4: $bel(x_t) = \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$
5: endfor
6: return $bel(x_t)$

The Bayes Filter - Derivation

$$bel(x_t) \quad = \quad \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

- Total probability $\longrightarrow$ $p(a) = \int p(a \mid b)\, p(b)\, db$

$$\begin{aligned}\overline{bel}(x_t) &= p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, \underbrace{p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}}_{\text{Previous belief over } x}\end{aligned}$$

- State is complete

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

The Bayes Filter - Derivation

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

$$\begin{aligned}\overline{bel}(x_t) &= p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}\end{aligned}$$

$$\overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, u_t)\, \underline{p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}}$$

1: **Algorithm Bayes_filter**($bel(x_{t-1}), u_t, z_t$)**:**
2: for all x_t do Process model
3: $\overline{bel}(x_t) = \int \underline{p(x_t \mid u_t, x_{t-1})}\, \boxed{bel(x_{t-1})\, dx}$
4: $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
5: endfor Measurement
6: return $bel(x_t)$

$P(z_t \mid x_t)$ = Measurement model

Limitations

1. $p(x)$ is defined $\forall x$ – intractable
   - Discrete and small spaces
   - Continuous and/or large spaces – Moments, Finite # of samples
2. The integral term -> costly to compute

Re-Iterate Example

- Is door open or not?

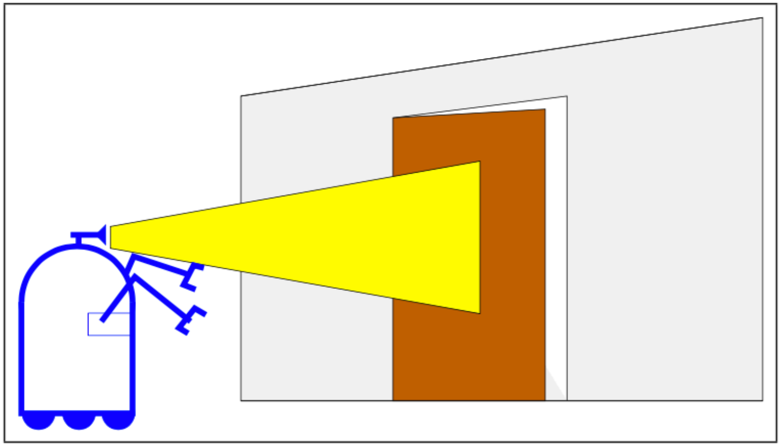
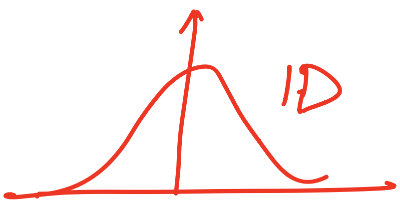


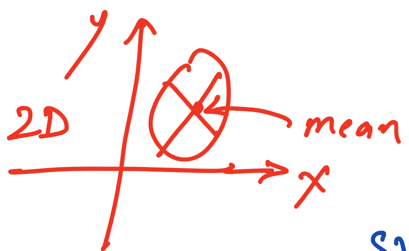
Figure 2.2 A mobile robot estimating the state of a door.

Gaussian Filters - Kalman Filter

$x \sim N(\mu, \Sigma)$

$$p(x) \quad = \quad \overbrace{\det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(x-\underbrace{\mu}_{\text{mean}})^T \underbrace{\Sigma^{-1}}_{\text{Covariance matrix}}(x-\mu)\right\}}^{\text{Multivariate Gaussian}}$$





SVD(Σ)

Eigenvectors = Axes of Ellipse

Eigenvalues = Length of Axes

Process Noise, $\varepsilon_t \sim \underbrace{N(0, R)}$
Gaussian Noise
Zero mean, Covariance R

Measurement Noise, $\delta_t \sim N(0, Q)$

Kalman Filter

- Gaussian Belief
- Linear Transition Model

Noise

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t \qquad x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{pmatrix} \qquad u_t = \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{m,t} \end{pmatrix}$$

- Linear Measurement Model

$$z_t = C_t x_t + \delta_t$$

noise

Noise is Gaussian and Additive

Kalman Filter

- Initial Belief $x_0 \sim N(\mu_0, \Sigma_0)$

$$bel(x_0) = p(x_0) = \det(2\pi\Sigma_0)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x_0-\mu_0)^T \Sigma_0^{-1}(x_0-\mu_0)\right\}$$

- Distribution over next state

Process Noise

$$p(x_t \mid u_t, x_{t-1}) \tag{3.4}$$
$$= \det(2\pi R_t)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x_t - \underline{A_t x_{t-1} - B_t u_t})^T R_t^{-1}(x_t - \underline{A_t x_{t-1} - B_t u_t})\right\}$$

Transition

- Likelihood of Measurement

Measurement Noise

Measurements

$$p(z_t \mid x_t) = \det(2\pi Q_t)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(z_t - C_t\, x_t)^T Q_t^{-1} (z_t - C_t\, x_t)\right\}$$

The Kalman Filter Algorithm

1: **Algorithm Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

Predict

2: $\bar{\mu}_t = A_t\, \mu_{t-1} + B_t\, u_t$

3: $\bar{\Sigma}_t = A_t\, \Sigma_{t-1}\, A_t^T + R_t$

$\boxed{\bar{x}_t \sim (\bar{\mu}_t, \bar{\Sigma}_t)}$

Update

4: $K_t = \bar{\Sigma}_t\, C_t^T (C_t\, \bar{\Sigma}_t\, C_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - \boxed{C_t\, \bar{\mu}_t})$

Prediction of measurement

6: $\Sigma_t = \underline{(I - K_t\, C_t)}\, \bar{\Sigma}_t$

7: return μ_t, Σ_t

uncertainty reduced

K_t = Kalman Gain $\approx \dfrac{R_t}{Q_t}$

if Q is large then K is small

R is large then K is large

1: **Algorithm Bayes_filter**$(bel(x_{t-1}), u_t, z_t)$**:**

2: for all x_t do

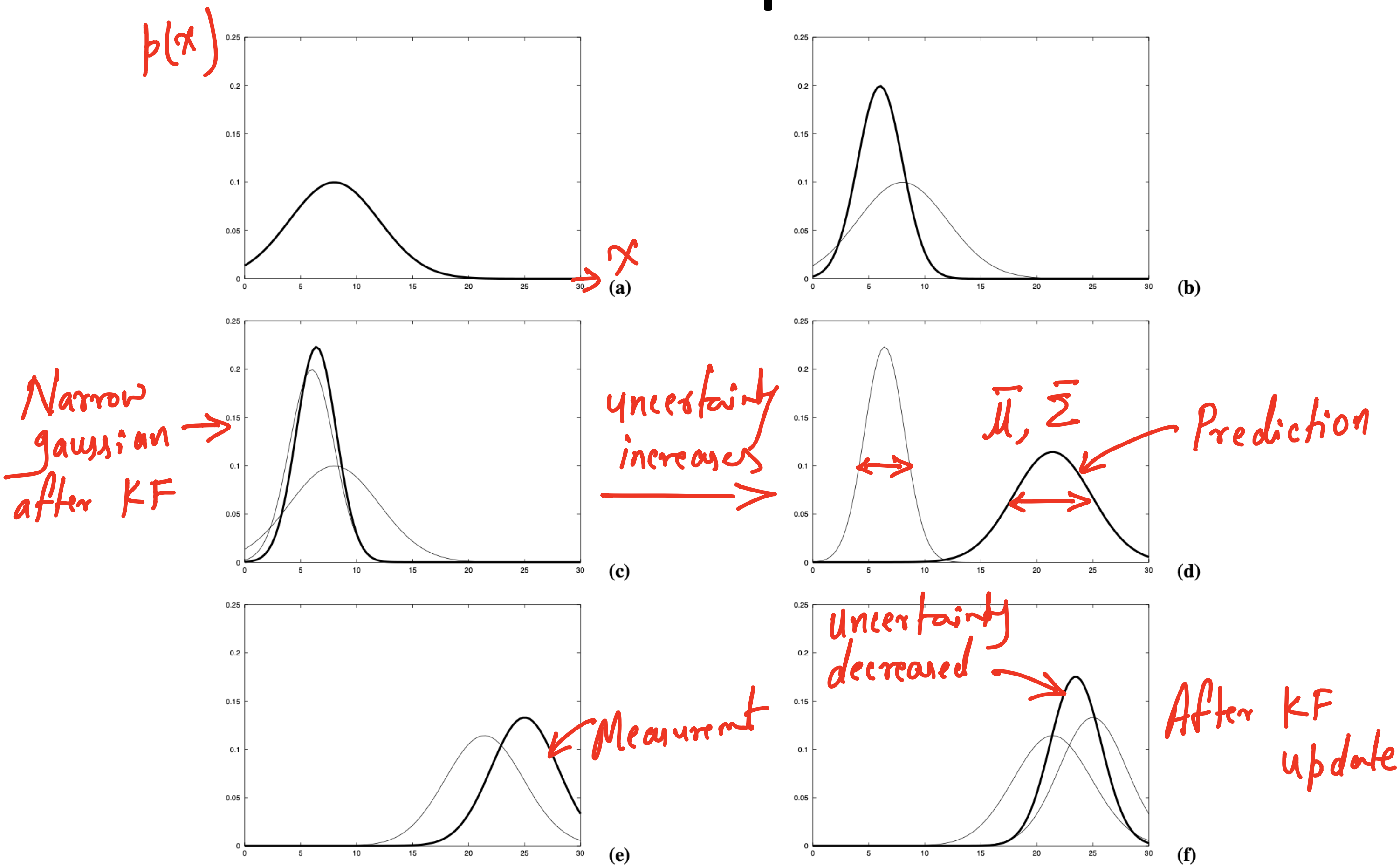
3: $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$

4: $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$

5: endfor

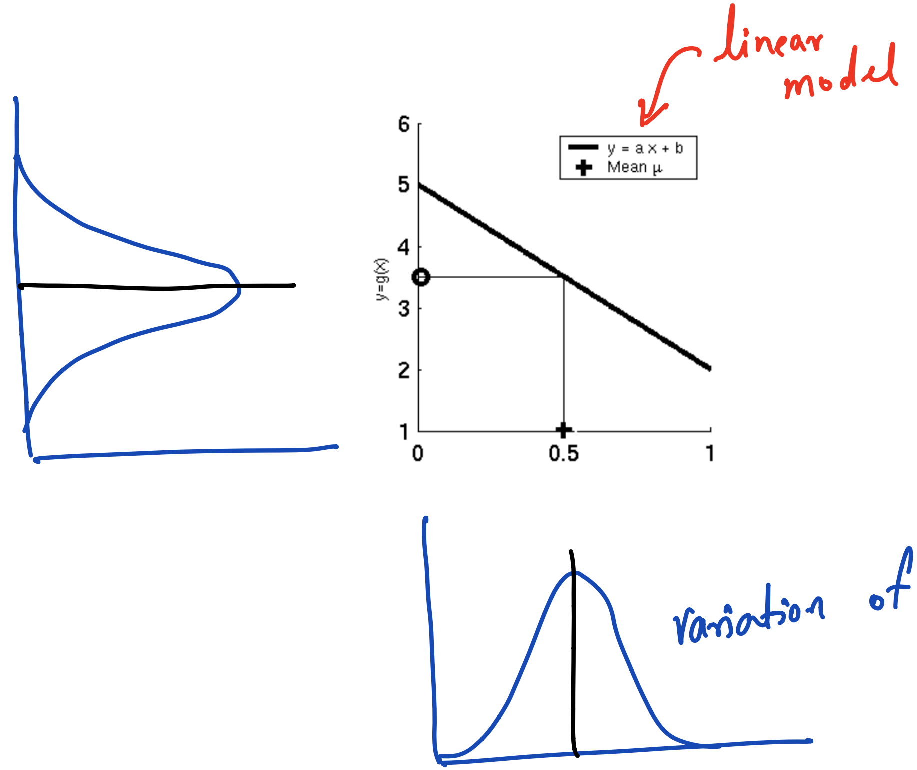
6: return $bel(x_t)$

Example

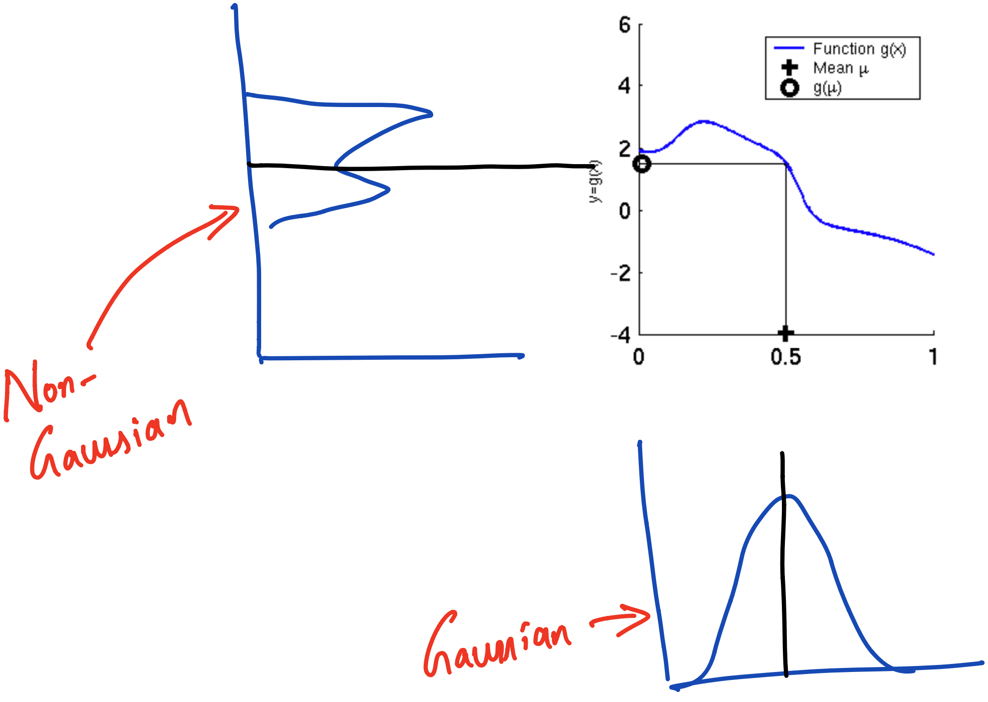


p(x)
x
(a)
(b)
Narrow gaussian after KF
(c)
uncertainty increases
$\bar{\mu}, \bar{\Sigma}$
Prediction
(d)
Measurement
(e)
uncertainty decreased
After KF update
(f)

Propagating a Gaussian through a Linear Model

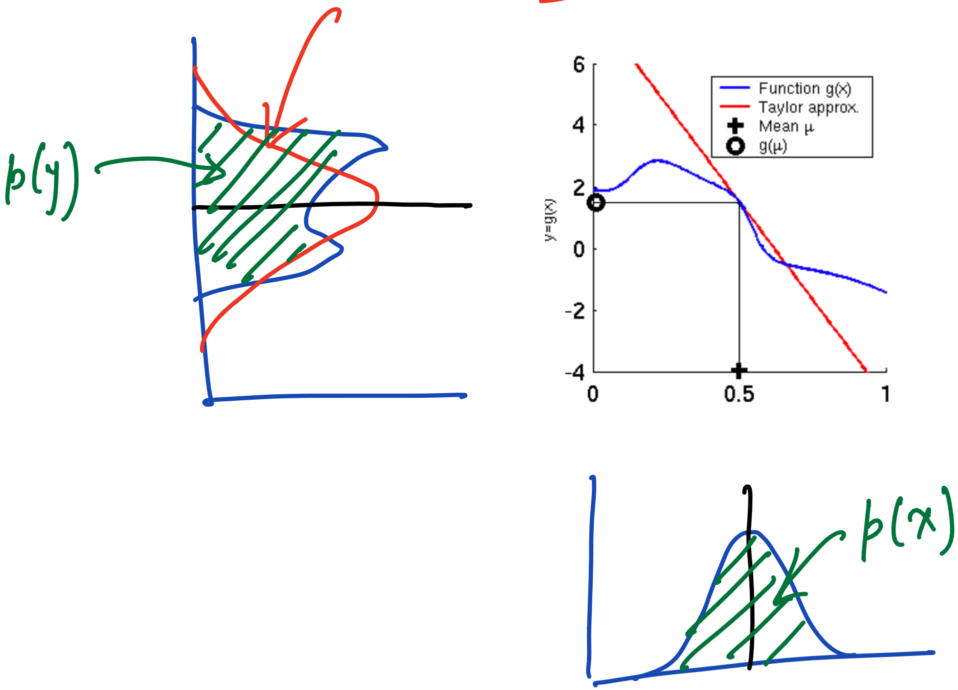


Propagating a Gaussian through a Non-Linear Model



Linearizing the Non-Linear Model

after linearizing the model



Extended Kalman filter - Process Model

$\varepsilon_t \sim \mathcal{N}(0, R)$
$\delta_t \sim \mathcal{N}(0, Q)$

$$
\begin{aligned}
x_t &= g(u_t, x_{t-1}) + \underline{\varepsilon_t} \\
z_t &= h(x_t) + \underline{\delta_t}\,.
\end{aligned}
$$

→ Process Model
→ Measurement Model

First order Taylor Expansion – linear approximation around value and slope

$$g'(u_t, x_{t-1}) := \frac{\partial g(u_t, x_{t-1})}{\partial x_{t-1}}$$

Gradient of non-linear function around x_{t-1}

Taylor Exp

$$
\begin{aligned}
g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + \underbrace{g'(u_t, \mu_{t-1})}_{=:\,G_t}\,(x_{t-1} - \mu_{t-1}) \\
&= g(u_t, \mu_{t-1}) + \underline{G_t\,(x_{t-1} - \mu_{t-1})}
\end{aligned}
$$

Jacobian

Extended Kalman filter - Process Model

$$
\begin{aligned}
g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + \underbrace{g'(u_t, \mu_{t-1})}_{=:\, G_t} (x_{t-1} - \mu_{t-1}) \\
&= g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})
\end{aligned}
$$

Same Eq. as in previous slide

Written as Gaussian:

linear

$A\mu_{t-1} - Bu_t$

$$
\begin{aligned}
&p(x_t \mid u_t, x_{t-1}) \\
&\approx \det(2\pi R_t)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2} [x_t - g(u_t, \mu_{t-1}) - G_t \, (x_{t-1} - \mu_{t-1})]^T \\
&\quad R_t^{-1} [x_t - g(u_t, \mu_{t-1}) - G_t \, (x_{t-1} - \mu_{t-1})]\}
\end{aligned}
$$

Extended Kalman Filter – Measurement Model

$$x_t = g(u_t, x_{t-1}) + \varepsilon_t$$
$$z_t = h(x_t) + \delta_t \, .$$

First order Taylor Expansion – linear approximation around value and slope

$$h(x_t) \approx h(\bar{\mu}_t) + \underbrace{h'(\bar{\mu}_t)}_{=:\, H_t} (x_t - \bar{\mu}_t)$$
$$= h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$

Jacobian

dimenss

$H_t \bar{\mu}_t$

Written as Gaussian:

$$p(z_t \mid x_t) = \det(2\pi Q_t)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2} \left[z_t - h(\bar{\mu}_t) - H_t \, (x_t - \bar{\mu}_t)\right]^T Q_t^{-1} \left[z_t - h(\bar{\mu}_t) - H_t \, (x_t - \bar{\mu}_t)\right]\right\}$$

The Extended Kalman Filter Algorithm

1: **Algorithm Extended_Kalman_filter(**$\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$**):**

Predict

2: $\bar{\mu}_t = \underline{g(u_t, \mu_{t-1})}$ – Nonlinear process model

3: $\bar{\Sigma}_t = \underline{G_t}\ \Sigma_{t-1}\ \underline{G_t^T} + R_t$

G_t, H_t Jacobians

Update

4: $K_t = \bar{\Sigma}_t\ \underline{H_t^T} (\underline{H_t}\ \bar{\Sigma}_t\ \underline{H_t^T} + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t (z_t - \underline{h(\bar{\mu}_t)})$ – Nonlinear measurement model

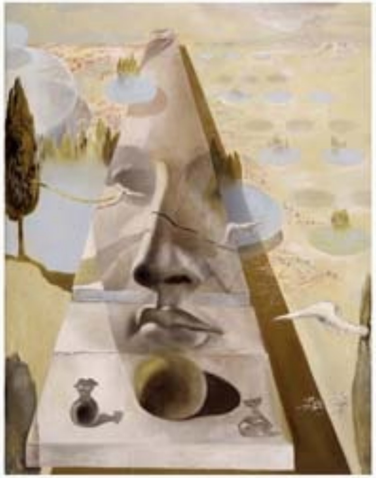
6: $\Sigma_t = (I - K_t\ \underline{H_t})\ \bar{\Sigma}_t$

7: return μ_t, Σ_t

	Kalman filter	EKF
state prediction (Line 2)	$A_t\ \mu_{t-1} + B_t\ u_t$	$g(u_t, \mu_{t-1})$
measurement prediction (Line 5)	$C_t\ \bar{\mu}_t$	$h(\bar{\mu}_t)$

CS231

Introduction to Computer Vision



Next lecture:

Optimal Estimation cont'