CS 236, Fall 2021
Midterm Exam

This exam is worth 90 points. You have 3.5 hours to complete and submit it. You are allowed to consult notes, books, the internet, and use a laptop. But no communication is allowed. Good luck!

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  - that they will do their share and take an active part in seeing to it that others as well as themselves uphold the spirit and letter of the Honor Code.
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Question	**Score**	**Question**	**Score**
1	/ 0	5	/ 10
2	/ 18	6	/ 10
3	/ 10	7	/ 12
4	/ 10	8	/ 20

Total score: / 90

Note: Partial credit will be given for partially correct answers. Zero points will be given to answers left blank.

1. **[0 points total] Stanford Honor Code**

- This exam is open-notes. This means you can reference notes, lectures slides, and other resources. If you use resources outside of notes and lecture slides, please cite your source.
- No form of collaboration is allowed.
- Please do not openly discuss anything about the contents of the exam (e.g. on Ed, Slack, in-person, etc.) with other people, both students and non-students, during the exam period and until exam grades have been released. This includes asking questions and receiving real-time assistance from Q&A answer sites such as Stack Overflow, Chegg, Yahoo Answers, etc.
- By taking this exam, you attest to following the Stanford Honor Code: I attest that I have not given or received aid in this examination, and that I have done my share and taken an active part in seeing to it that others as well as myself uphold the spirit and letter of the Honor Code.

2. **[18 points total] True/False**

For each of the statements below, state True or False. Explain your answer for full points.

(a) **[2 points]** Without any independence assumptions, the number of parameters for a tabular autoregressive model depends on the exact choice of variable ordering.

Answer: False. Tabular autoregressive models require the same number of parameters regardless of ordering if there are no independence assumptions.

(b) **[2 points]** Consider a discrete autoregressive model over greyscale images with pixel intensity values in $\{0, 1, \ldots, 255\}$. Using at most 256 forward passes, it is always possible to exactly compute the conditional distribution of *any* single missing pixel given the values for all the other pixels.

Answer: True. The conditional is an energy based model but computing the partition function is tractable since we are looking at a single pixel. Computation of this partition function requires marginalizing over all 256 possible choices for the missing pixel.

(c) **[2 points]** Given a latent variable model $p_\theta(x, z)$ and a fixed choice of observation $\bar{x}$, you can interpret the function $p_\theta(\bar{x}, z)$ as an unnormalized distribution with respect to z, and whose partition function is $p_\theta(\bar{x})$.

Answer: True. For fixed x, the function $p_\theta(\bar{x}, z)$ is an unnormalized distribution which, when normalized by $p_\theta(\bar{x})$, becomes $p_\theta(z \mid \bar{x})$.

(d) **[2 points]** Because the variational autoencoder objective contains a reconstruction term, an optimally-trained VAE will always make use of the latent space and thus have non-zero Kl-divergence to the prior: $D_{\mathrm{KL}}(q(z \mid x) \parallel p(z)) > 0$.

[Note: in this question, we define an optimally-trained VAE to be one with a tight ELBO (i.e., equality holds between the ELBO and the log-likelihood) and whose marginal distribution matches the data distribution.]

Answer: False. If the observation model $p_\theta(x \mid z)$ is sufficiently powerful or the data is sufficiently simple, the model can achieve an optimal ELBO (and likelihood) without ever using the latent space. This is called posterior collapse.

(e) **[2 points]** Any continuous autoregressive flow model $p_\theta(\mathbf{x}_{1:n}) = \prod_{i=1}^{n} p_\theta(\mathbf{x}_i \mid \mathbf{x}_{<i})$, where each $p_\theta(\mathbf{x}_i \mid \mathbf{x}_{<i})$ is a conditional probability density function, can be represented as a flow model with a uniform prior.

Answer: True. Since $p(\mathbf{x}_i \mid \mathbf{x}_{<i})$ is a probability density function, it has a conditional CDF $F(\mathbf{x}_i \mid \mathbf{x}_{<i})$. The conditional CDFs can be used as the invertible, differentiable function f that maps each $\mathbf{x}_i$ to the corresponding $\mathbf{z}_i$ drawn from a uniform distribution.

(f) **[2 points]** When training a GAN model with Minimax Loss, the gradient with respect to the generator parameters will be zero if we fix the discriminator so that it outputs a constant value for all inputs.

Answer: True. The GAN loss function will be constant when the discriminator is a constant function.

(g) **[2 points]** Let R be a rotation matrix (i.e., such that $R^T R = I$) and $X = f(Z)$ be a flow model where $Z \sim \mathcal{N}(0, I)$ is distributed as a Gaussian with unit covariance I. Then $g(Z) = f(RZ)$ is another flow model that achieves the same likelihood as f on any dataset.

Answer: True. R has unit determinant and the prior is rotationally invariant.

(h) **[2 points]** A normalizing flow model will map an observed random variable X to a lower dimensional latent variable Z.

Answer: False. X and Z need to have the same dimensionality.

(i) **[2 points]** Training an EBM always requires estimating its partition function.

Answer: False. Contrastive divergence, for example, does not require estimating the partition function. We can also use some f-divergence (eg. KL divergence) to train a EBM without estimating the partition function.

3. **[10 points total] Change of Variables**

(a) **[5 points]** You are dealing with a 32×32 greyscale image dataset whose pixel intensities are *real-valued* in the interval $[0, 255]$. A common pre-processing procedure is to scale your data by $1/127.5$ and then shifting it by -1, so that your data lies in the interval $[-1, 1]$, before training your Gaussian autoregressive model $p_\theta(\mathbf{x})$, where $\mathbf{x}$ has dimensionality 32×32. You do so and report a test set log-likelihood of

$$\frac{1}{N}\sum_{i=1}^{N} \ln p_\theta(\mathbf{x}^{(i)}) = 32.5, \tag{1}$$

where $\{\mathbf{x}^{(i)}\}_{i=1}^N$ is your test set and each $\mathbf{x}^{(i)}$ is a processed $[-1, 1]^{32\times32}$ image. However, Reviewer 2 requests that you report your model's test set log-likelihood in the original $[0, 255]^{32\times32}$ space for your report to be comparable with the literature. What is your test set log-likelihood in the original $[0, 255]^{32\times32}$ space? Report your value to the third significant digit in scientific notation. Explain how you got your answer for full credit.

Answer: You need to re-scale your model by adding 1 and then multiplying by 127.5. Since addition by 1 is a volume-preserving transformation, we only need to worry about the multiplication by 127.5.

This multiplication expands the volume of your space by $127.5^{32\times32}$, so your density d would correspondingly be divided by this expansion to yield $d_{\text{new}} = d/(127.5^{32\times32})$. Consequently, your new test set log-likelihood is

$$\begin{aligned}
\ln d_{\text{new}} &= \ln d - (32 \times 32) \cdot \ln 127.5 &(2)\\
&= 32.5 - (32 \times 32) \cdot \ln 127.5 &(3)\\
&\approx -4.93 \times 10^3. &(4)
\end{aligned}$$

Notice the value is significantly smaller. Shrinking the space increases the density, thus giving your previously reported value of 32.5 nats an unfair advantage over the literature.

(b) **[5 points]** Given a univariate Normal (i.e., Gaussian) random variable Y, its exponentiation $X = \exp(Y)$ is said to have a Log-Normal distribution. If Y is distributed according to $\mathcal{N}(\mu, \sigma^2)$, then we denote $X = \exp(Y)$ as being distributed according to $\text{LN}(\mu, \sigma^2)$. Using the definition of the Gaussian probability density function for p_Y and the change-of-variables formula that relates p_Y to p_X, prove that the probability density function for $X \sim \text{LN}(\mu, \sigma^2)$ is

$$p_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right). \tag{5}$$

Answer: By the change-of-variables formula, we know that

$$\begin{aligned}
p_X(x = \exp(y)) &= p_Y(y = \ln(x)) \cdot \left|\frac{\partial}{\partial x} \ln x\right| &(6)\\
&= \mathcal{N}(\ln x \mid \mu, \sigma^2) \cdot \frac{1}{x} &(7)\\
&= \frac{1}{x\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right). &(8)
\end{aligned}$$

4. **[10 points total] KL Divergence and Bounds**

In the VAE lectures, we have seen how addition/subtraction of a KL-divergence term can yield a bound (e.g. subtracting a KL term from the log-likelihood in a latent variable model yields the Evidence Lower Bound). We shall apply this same technique of adding/subtracting a KL term to answer the following questions.

(a) **[5 points]** Given a joint distribution $p(x, z)$, show that

$$\mathbb{E}_{p(x,z)}\left[\ln \frac{p(x, z)}{p(x)p(z)}\right] \leq \mathbb{E}_{p(z)} D_{\mathrm{KL}}(p(x \mid z) \parallel q(x)) \tag{9}$$

for any choice of q. Explicitly show the KL-divergence term you are adding/subtracting in your work.

Answer:

$$\begin{aligned}
\mathbb{E}_{p(x,z)}\left[\ln \frac{p(x, z)}{p(x)p(z)}\right] &\leq \mathbb{E}_{p(x,z)}\left[\ln \frac{p(x, z)}{p(x)p(z)}\right] + D_{\mathrm{KL}}(p(x) \parallel q(x)) && (10)\\
&= \mathbb{E}_{p(x,z)}\left[\ln \frac{p(x, z)}{p(x)p(z)}\right] + \mathbb{E}_{p(x)}\left[\ln \frac{p(x)}{q(x)}\right] && (11)\\
&= \mathbb{E}_{p(x,z)}\left[\ln \frac{p(x \mid z)}{p(x)} + \ln \frac{p(x)}{q(x)}\right] && (12)\\
&= \mathbb{E}_{p(x,z)}\left[\ln \frac{p(x \mid z)}{q(x)}\right] = \mathbb{E}_{p(z)}\mathbb{E}_{p(x|z)}\left[\ln \frac{p(x \mid z)}{q(x)}\right] && (13)\\
&= \mathbb{E}_{p(z)} D_{\mathrm{KL}}(p(x \mid z) \parallel q(x)). && (14)
\end{aligned}$$

(b) **[5 points]** Given a joint distribution $p(x, z)$, show that

$$\mathbb{E}_{p(x,z)}\left[\ln \frac{p(x, z)}{p(x)p(z)}\right] \geq -\mathbb{E}_{p(z)}\left[\ln p(z)\right] + \mathbb{E}_{p(z,x)}[\ln q(z \mid x)] \tag{15}$$

for any choice of q. Explicitly show the KL-divergence term you are adding/subtracting in your work.

Answer: Note: due to an typo in the initial question (we had written $q(x \mid z)$ instead of $q(z \mid x)$), we are throwing out this question. We provide the solution to the intended question below.

$$\begin{aligned}
\mathbb{E}_{p(x,z)}\left[\ln \frac{p(x, z)}{p(x)p(z)}\right] &= -\mathbb{E}_{p(z)}\left[\ln p(z)\right] + \mathbb{E}_{p(x,z)}\left[\ln \frac{p(x, z)}{p(x)}\right] && (16)\\
&= -\mathbb{E}_{p(z)}\left[\ln p(z)\right] + \mathbb{E}_{p(x,z)}\left[\ln p(z \mid x)\right] && (17)\\
&\geq -\mathbb{E}_{p(z)}\left[\ln p(z)\right] + \mathbb{E}_{p(x,z)}\left[\ln p(z \mid x)\right] - \mathbb{E}_{p(x)} D_{\mathrm{KL}}(p(z \mid x) \parallel q(z \mid x)) && (18)\\
&= -\mathbb{E}_{p(z)}\left[\ln p(z)\right] + \mathbb{E}_{p(x,z)}\left[\ln p(z \mid x)\right] - \mathbb{E}_{p(x)}\mathbb{E}_{p(z|x)}\left[\ln \frac{p(z \mid x)}{q(z \mid x)}\right] && (19)\\
&= -\mathbb{E}_{p(z)}\left[\ln p(z)\right] + \mathbb{E}_{p(z,x)}\left[\ln q(z \mid x)\right]. && (20)
\end{aligned}$$

5. **[10 points total] MCMC-Based Training of Latent Variable Models**

So far, we have seen how variational methods (i.e. ELBO maximization) can be used to train the latent variable model $p_\theta(x, z)$ where x is observed and z is latent. In this question, we shall consider a popular alternative called Markov Chain Monte Carlo (MCMC). For the purposes of this question, we shall simply treat MCMC as a black-box method that—with enough computation time—reliably allows us to sample from (but not compute!) the posterior $p_\theta(z \mid x)$. Fortunately, the ability to sample from the posterior (even if we cannot compute it) is sufficient for constructing an unbiased estimate of the gradient of the log-likelihood, thanks to the following identity,

$$\nabla_\theta \ln p_\theta(x) = \mathbb{E}_{p_\theta(z|x)} \nabla_\theta \ln p_\theta(x, z). \tag{21}$$

Prove this identity using the formula for the gradient of the logarithm function (log-derivative trick): $\nabla_\theta \ln p_\theta(x) = \frac{1}{p_\theta(x)} \cdot \nabla_\theta p_\theta(x)$.

Answer:

$$\begin{aligned}
\nabla_\theta \ln p_\theta(x) &= \frac{1}{p_\theta(x)} \nabla_\theta p_\theta(x) && (22)\\
&= \frac{1}{p_\theta(x)} \nabla_\theta \int p_\theta(x, z) \mathrm{d}z && (23)\\
&= \int \frac{1}{p_\theta(x)} \nabla_\theta p_\theta(x, z) \mathrm{d}z && (24)\\
&= \int \frac{p_\theta(z \mid x)}{p_\theta(x, z)} \nabla_\theta p_\theta(x, z) \mathrm{d}z && (25)\\
&= \int p_\theta(z \mid x) \nabla_\theta \ln p_\theta(x, z) \mathrm{d}z && (26)\\
&= \mathbb{E}_{p_\theta(z|x)} \nabla_\theta \ln p_\theta(x, z). && (27)
\end{aligned}$$

6. **[10 points total] Variational Perspective to Energy-Based Models**

   In this question, we will consider energy-based models from a variational perspective.

   (a) **[5 points]** Consider an EBM with an unnormalized distribution $\tilde{p}_\theta(x)$ and partition function $Z(\theta) = \int \tilde{p}_\theta(x)\mathrm{d}x$. Computing the log-partition function $\ln Z(\theta)$ is usually intractable. So we shall look to bounding this quantity instead. In particular, if we introduce a proposal distribution $q(x)$ that is easy to compute and sample from, we can construct the following lower bound for the log-partition function,

$$\ln Z(\theta) \geq \mathbb{E}_{q(x)}\left[\ln \frac{\tilde{p}_\theta(x)}{q(x)}\right]. \tag{28}$$

   Prove that this bound holds for any choice of q. It may help to notice the strong resemblance between this expression and the Evidence Lower Bound for a latent variable model.

   Answer: One approach is Jensen's Inequality.

$$\begin{aligned}
\ln Z(\theta) &= \ln \int \frac{q(x)}{q(x)} \cdot \tilde{p}_\theta(x)\mathrm{d}x && (29)\\
&= \ln \mathbb{E}_{q(x)}\left[\frac{\tilde{p}_\theta(x)}{q(x)}\right] && (30)\\
&\geq \mathbb{E}_{q(x)}\left[\ln \frac{\tilde{p}_\theta(x)}{q(x)}\right]. && (31)
\end{aligned}$$

   Another approach is to directly use the subtract-a-KL-trick.

$$\begin{aligned}
\ln Z(\theta) &= \ln \frac{\tilde{p}_\theta(x)}{p_\theta(x)} && (32)\\
&\geq \ln \frac{\tilde{p}_\theta(x)}{p_\theta(x)} - D_{\mathrm{KL}}(q(x) \parallel p_\theta(x)) && (33)\\
&= \ln \frac{\tilde{p}_\theta(x)}{p_\theta(x)} - \mathbb{E}_{q(x)}\left[\ln \frac{q(x)}{p_\theta(x)}\right] && (34)\\
&= \mathbb{E}_{q(x)}\left[\ln \frac{\tilde{p}_\theta(x)}{q(x)}\right]. && (35)
\end{aligned}$$

   (b) **[5 points]** Consider again the EBM with an unnormalized $\tilde{p}_\theta(x)$ and partition function $Z(\theta) = \int \tilde{p}_\theta(x)\mathrm{d}x$. Note that, when normalized, $p_\theta(x) = \tilde{p}_\theta(x)/Z(\theta)$. So far, we have learned from class that gradient-based optimization of the EBM's log-likelihood requires computing

$$\nabla_\theta \ln p_\theta(x) = \nabla_\theta \ln \tilde{p}_\theta(x) - \mathbb{E}_{p_\theta(x)} \nabla_\theta \ln \tilde{p}_\theta(x), \tag{36}$$

   We now take a variational perspective to this expression by introducing the variational family $\mathcal{Q}$, which we shall denote as the set of all possible distributions over x. Prove that

$$\nabla_\theta \ln p_\theta(x) = \nabla_\theta \ln \tilde{p}_\theta(x) - \mathbb{E}_{q^*(x)} \nabla_\theta \ln \tilde{p}_\theta(x), \tag{37}$$

   where

$$q^*(x) = \arg\max_{q \in \mathcal{Q}} \mathbb{E}_{q(x)}\left[\ln \frac{\tilde{p}_\theta(x)}{q(x)}\right]. \tag{38}$$

   You may make use of and do not have to prove Equation (36). [Hint: Prove that $q^* = p_\theta$.]

   Answer: The key insight here is that

$$\begin{aligned}
\ln Z(\theta) - \mathbb{E}_{q(x)}\left[\ln \frac{\tilde{p}_\theta(x)}{q(x)}\right] &= \ln \frac{\tilde{p}_\theta(x)}{p_\theta(x)} - \mathbb{E}_{q(x)}\left[\ln \frac{\tilde{p}_\theta(x)}{q(x)}\right] && (39)\\
&= D_{\mathrm{KL}}(q(x) \parallel p_\theta(x)). && (40)
\end{aligned}$$

Thus, since q is optimized for fixed θ,

$$q^*(x) = \arg\max_{q \in \mathcal{Q}} \mathbb{E}_{q(x)} \left[\ln \frac{\tilde{p}_\theta(x)}{q(x)} \right] \tag{41}$$

$$= \arg\min_{q \in \mathcal{Q}} D_{\mathrm{KL}}(q(x) \parallel p_\theta(x)). \tag{42}$$

Furthermore, since $\mathcal{Q}$ is the set of all distribution, the minimum KL-divergence of 0 is achieved, and uniquely so, by $q^*(x) = p_\theta(x)$. This equivalence therefore means that

$$\nabla_\theta \ln p_\theta(x) = \nabla_\theta \ln \tilde{p}_\theta(x) - \mathbb{E}_{p_\theta(x)} \nabla_\theta \ln \tilde{p}_\theta(x) \tag{43}$$

$$= \nabla_\theta \ln \tilde{p}_\theta(x) - \mathbb{E}_{q^*(x)} \nabla_\theta \ln \tilde{p}_\theta(x). \tag{44}$$

7. **[12 points total] Masked Autoencoder**

   In this question, we shall design a mask within a Masked Autoencoder. Our MADE model takes as input $\mathbf{x} \in \mathbb{R}^3$ and outputs predictions $\hat{\mathbf{x}}_i$ conditional on all preceding input dimensions $\mathbf{x}_{<i}$. We have provided the masks M_1, M_2, M_4, and M_5 in the figure below.

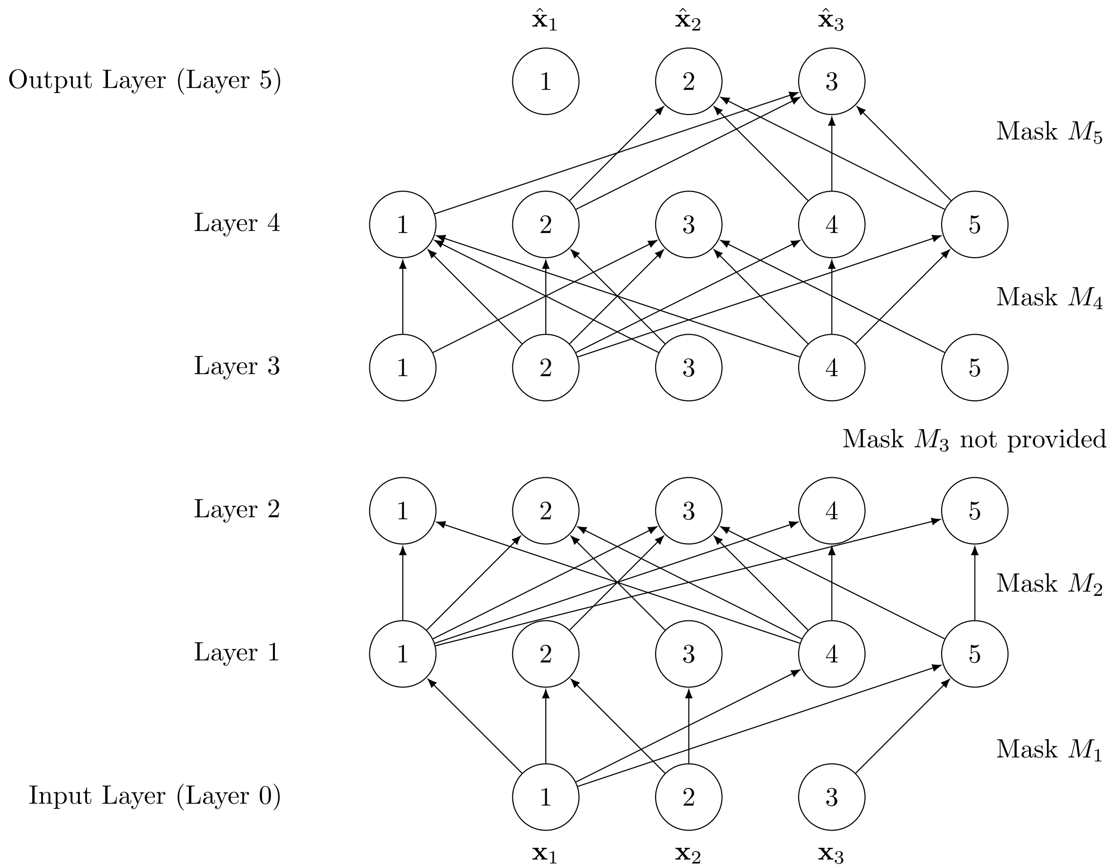

Figure 1: MADE Model with masks provided for M_1, M_2, M_4, and M_5.

The index for each neuron is provided in the figure. Each mask M_ℓ is a binary matrix, where the element $(M_\ell)_{ij} = 1$ if and only if the i^{th} neuron of layer $\ell - 1$ points to the j^{th} neuron of the subsequent layer ℓ, otherwise $(M_\ell)_{ij} = 0$.

We have not provided the mask M_3. Your objective is to design the densest possible binary mask $M_3 \in \{0,1\}^{5\times 5}$ that preserves the autoregressive property, $p(x) = \prod_{i=1}^{3} p(x_i \mid x_{<i})$, in our MADE model. In other words, we want M_3 to be a valid mask (i.e., preserving the autoregressive property) that has as many non-zero elements as possible. For your convenience, we are also providing the matrix multiplications for $M_1 \cdot M_2$ and $M_4 \cdot M_5$, which we denote as matrices A and B,

$$A = M_1 \cdot M_2 = \begin{pmatrix} 2 & 2 & 4 & 2 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \qquad B = M_4 \cdot M_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

Answer the following questions related to the determination of the densest valid mask for M_3.

(a) **[2 points]** What is the index of the largest-indexed Layer 0 neuron that has a path to the 2nd neuron of Layer 2? For full credit, describe how to get your answer using matrix A, and without relying on Figure 1.

Answer: Look for the last non-zero element in the 2nd column of matrix A. The answer is: 2. Since Figure 1 and the text explicitly uses 1-indexing for the neurons, we do not accept 0-indexing unless you explicitly state so.

(b) **[2 points]** What is the index of the smallest-indexed Layer 2 neuron that has a path from the 3rd neuron of Layer 0? For full credit, describe how to get your answer using matrix A, and without relying on Figure 1.

Answer: Look for the first non-zero element in the 3rd row of matrix A. The answer is: 3. Since Figure 1 and the text explicitly uses 1-indexing for the neurons, we do not accept 0-indexing unless you explicitly state so.

(c) **[8 points]** Based on our specific mask choices for M_1, M_2, M_4, M_5, express the densest valid mask for M_3 as an adjacency list, according to the following example format:

Layer 0 Neuron 1 points to Layer 1 Neurons: $1, 2, 4, 5$
Layer 0 Neuron 2 points to Layer 1 Neurons: $2, 3$
Layer 0 Neuron 3 points to Layer 1 Neurons: 5

For full credit, describe how to get your answer using the matrices A and B, and without relying on Figure 1.

Answer:

- Layer 2 Neuron 1 points to Layer 3 Neurons: $\{1, 2, 3, 4, 5\}$
- Layer 2 Neuron 2 points to Layer 3 Neurons: $\{1, 5\}$
- Layer 2 Neuron 3 points to Layer 3 Neurons: $\{5\}$
- Layer 2 Neuron 4 points to Layer 3 Neurons: $\{1, 2, 3, 4, 5\}$
- Layer 2 Neuron 5 points to Layer 3 Neurons: $\{5\}$

Note: We do not accept zero-indexing in this subquestion in any circumstance since our gradescope instructions and provided example makes the desired format explicitly clear.

Explanation:

First, for each neuron in Layer 2, we need to find the largest-indexed input neuron that has a path to it. To do so, inspect each column of A for the last non-zero element. This yields the vector

$$L_A = [1, 2, 3, 1, 3]. \tag{46}$$

Next, for each neuron in Layer 3, we need to find the smallest-indexed output neuron that has a path from it. To do so, inspect each row of B for the first non-zero element. This yields the vector

$$S_B = [3, 2, 2, 2, +\infty]. \tag{47}$$

Since none of the output neurons has a path from the 5th neuron of Layer 3, we can think of the smallest-indexed neuron as $+\infty$.

For M_3 to be valid, each output neuron can only connect to input neurons with strictly smaller indices. So we only make a connection between (Layer 2 neuron i) and (Layer 3 neuron j) if the largest input neuron pointing to (Layer 2 neuron i) is strictly smaller than the smallest output neuron that (Layer 3 neuron j) points to, i.e., $(L_A)_i < (S_B)_j$.

Note: We accept other solutions as long as you describe a concrete, polynomial-time procedure for finding M_3. For example: initializing M_3 as a zero-matrix and then iteratively checking what happens to each element if you set it to 1—keeping the element as 1 if and only if AM_3B remains strictly upper-triangular. Note that simply stating "choose densest possible M_3 such that AM_3B is upper triangular" does not count as a explanation since it does not explain how to actually construct such a matrix.

8. **[20 points total] GAN Loss and Weighted Jensen-Shannon Divergence**

This problem explores how the GAN objective function $\mathcal{L}$ relates to the Jensen-Shannon divergence. Given a distribution p that we wish to model, recall the GAN optimization problem

$$\min_q \max_D \mathcal{L}(q, D) = \min_q \max_D \mathbb{E}_{p(x)}\left[\ln D(x)\right] + \mathbb{E}_{q(x)}\left[\ln\left(1 - D(x)\right)\right], \tag{48}$$

where q is the generative model and D is the discriminator. In class, we showed that if the discriminator is optimized over all possible discriminative functions, the optimal discriminator $D^*(x)$ reduces the GAN objective to

$$\mathcal{L}(q, D^*) = 2 \cdot D_{\mathrm{JS}}(p \parallel q) - \ln(4). \tag{49}$$

where $D_{\mathrm{JS}}(p \parallel q)$ is the JS-divergence.

(a) **[10 points]** For any choice of weight $\pi \in (0, 1)$, we can define a π-weighted version of the Jensen-Shannon divergence as

$$D_{\mathrm{JS}_\pi}(p \parallel q) = \pi \cdot D_{\mathrm{KL}}\Big(p \,\Big\|\, \pi p + (1-\pi)q\Big) + (1-\pi) \cdot D_{\mathrm{KL}}\Big(q \,\Big\|\, \pi p + (1-\pi)q\Big). \tag{50}$$

For notational simplicity, we shall refer to the π-weighted JS-divergence simply as the weighted JS-divergence henceforth. A natural consideration is whether the weighted JS-divergence is an f-divergence. For the following choice of generator function f,

$$f(u) = \pi u \ln u - (\pi u + 1 - \pi)\ln\left(\pi u + 1 - \pi\right), \tag{51}$$

prove that

$$D_{\mathrm{JS}_\pi}(p \parallel q) = D_f(p \parallel q), \tag{52}$$

where $D_f(p \parallel q) = \mathbb{E}_{q(x)}[f(\frac{p(x)}{q(x)})]$ denotes the f-divergence.

Answer:

$$\begin{aligned}
D_{\mathrm{JS}_\pi}(p \parallel q) &= \pi \cdot D_{\mathrm{KL}}\Big(p \,\Big\|\, \pi p + (1-\pi)q\Big) + (1-\pi) \cdot D_{\mathrm{KL}}\Big(q \,\Big\|\, \pi p + (1-\pi)q\Big) \\
&= \pi \cdot \mathbb{E}_{p(x)} \ln \frac{p(x)}{\pi p(x) + (1-\pi)q(x)} + (1-\pi) \cdot \mathbb{E}_{q(x)} \ln \frac{q(x)}{\pi p(x) + (1-\pi)q(x)} \\
&= \pi \cdot \mathbb{E}_{q(x)} \frac{p(x)}{q(x)} \ln \frac{p(x)}{\pi p(x) + (1-\pi)q(x)} + (1-\pi) \cdot \mathbb{E}_{q(x)} \ln \frac{q(x)}{\pi p(x) + (1-\pi)q(x)} \\
&= \mathbb{E}_{q(x)} \pi u \ln \frac{p(x)}{\pi p(x) + (1-\pi)q(x)} + (1-\pi) \ln \frac{q(x)}{\pi p(x) + (1-\pi)q(x)}
\end{aligned}$$

where $u(x) = \frac{p(x)}{q(x)}$. Dividing numerator and denominator of the fraction inside the log by $q(x)$ and continuing, we have:

$$\begin{aligned}
&= \mathbb{E}_{q(x)} \pi u(x) \ln \frac{u(x)}{\pi u(x) + 1 - \pi} + (1-\pi) \ln \frac{1}{\pi u(x) + 1 - \pi} \\
&= \mathbb{E}_{q(x)} \pi u(x) \ln u(x) - (\pi u(x) + 1 - \pi) \ln\left(\pi u(x) + 1 - \pi\right) \\
&= \mathbb{E}_{q(x)} f(u(x)) \\
&= \mathbb{E}_{q(x)} f(\frac{p(x)}{q(x)}) \\
&= D_f(p \parallel q)
\end{aligned}$$

(b) **[10 points]** Since the weighted JS-divergence is an f-divergence, we can cast the weighted JS-divergence as a variational divergence problem instead. Recall that the Fenchel conjugate for any function f is

$$f^*(t) = \sup_{u \in \mathrm{dom}(f)} ut - f(u). \tag{53}$$

For the weighted JS-divergence, the Fenchel conjugate for its generator function is

$$f^*(t) = (1-\pi) \ln \left(\frac{1-\pi}{1 - \pi \cdot \exp\left(\frac{t}{\pi}\right)} \right), \tag{54}$$

where the domain of f^* is $t < -\pi \ln \pi$. Based on this and the equations from Question (8a), prove that

$$D_{\mathrm{JS}_\pi}(p \parallel q) \geq \mathbb{E}_{p(x)} [\ln D(x)] - \mathbb{E}_{q(x)} \left[(1-\pi) \ln \left(\frac{1-\pi}{1 - \pi D(x)^{\frac{1}{\pi}}} \right) \right], \tag{55}$$

for any choice of function $D : \mathcal{X} \to (0, (\frac{1}{\pi})^\pi)$. This gives us a GAN-like objective to approximately minimize the weighted JS-divergence.

Answer: From lecture, recall that

$$D_f(p \parallel q) \geq \mathbb{E}_{p(x)} T(x) - \mathbb{E}_{q(x)} f^*(T(x)), \tag{56}$$

for any choice of function $T : \mathcal{X} \to \mathrm{dom}(f^*)$. For our choice of f^*, note that the codomain of T is $\mathrm{dom}(f^*) = (-\infty, -\pi \ln \pi)$. Based on our previous results, we know that

$$\begin{aligned} D_{\mathrm{JS}_\pi}(p \parallel q) &= D_f(p \parallel q) \qquad (57) \\ &\geq \mathbb{E}_{p(x)} T(x) - \mathbb{E}_{q(x)} \left[(1-\pi) \ln \left(\frac{1-\pi}{1 - \pi \exp(\frac{1}{\pi} \cdot T(x))} \right) \right]. \qquad (58) \end{aligned}$$

We now simply reparameterize the function T as $T = \ln D$. Since $D = \exp(T)$, the codomain for D is thus $(0, \exp(-\pi \ln \pi)) = (0, (\frac{1}{\pi})^\pi)$. Replacing T with $\ln D$ thus shows that

$$\begin{aligned} D_{\mathrm{JS}_\pi}(p \parallel q) &= D_f(p \parallel q) \qquad (59) \\ &\geq \mathbb{E}_{p(x)} T(x) - \mathbb{E}_{q(x)} \left[(1-\pi) \ln \left(\frac{1-\pi}{1 - \pi \exp(\frac{1}{\pi} \cdot T(x))} \right) \right] \qquad (60) \\ &= \mathbb{E}_{p(x)} \ln D(x) - \mathbb{E}_{q(x)} \left[(1-\pi) \ln \left(\frac{1-\pi}{1 - \pi \exp(\frac{1}{\pi} \cdot \ln D(x))} \right) \right] \qquad (61) \\ &= \mathbb{E}_{p(x)} \ln D(x) - \mathbb{E}_{q(x)} \left[(1-\pi) \ln \left(\frac{1-\pi}{1 - \pi D(x)^{\frac{1}{\pi}}} \right) \right], \qquad (62) \end{aligned}$$

for any $D : \mathcal{X} \to (0, (\frac{1}{\pi})^\pi)$.