CS231A

Computer Vision:
From 3D Reconstruction to Recognition

Optimal Estimation

Perception as a Continuous Process

Perception as a Multi-Modal Experience

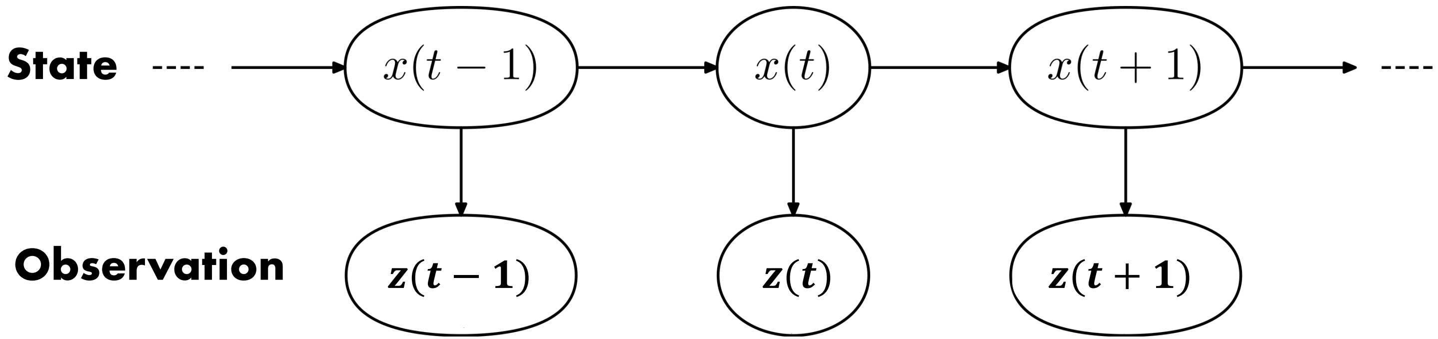
Perception as Inference

Recursive State Estimation

Mathematical Formalism to:

- continuously integrate measurements
- from different sensor sources
- to infer the state of a latent variable

What is a state? What is a representation?



Hidden Markov Model

Representations for Autonomous Driving

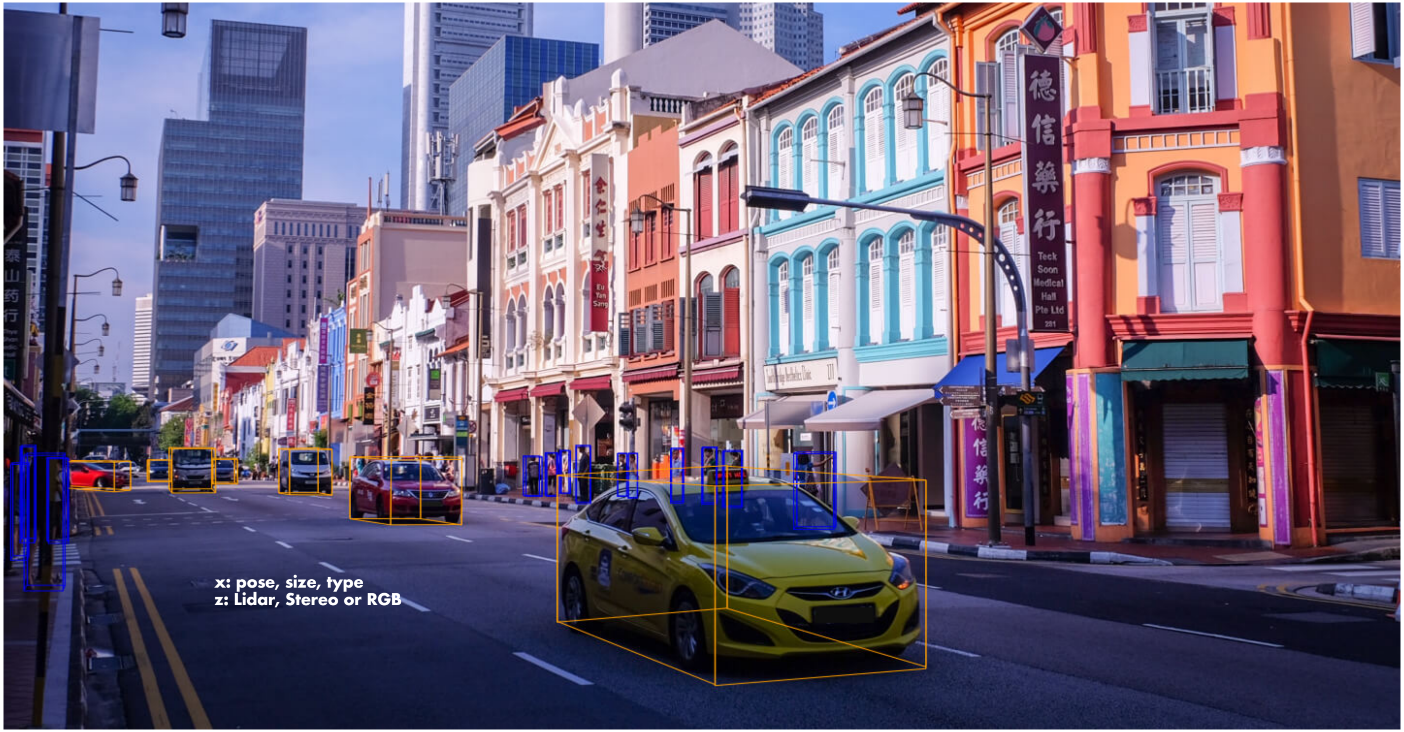
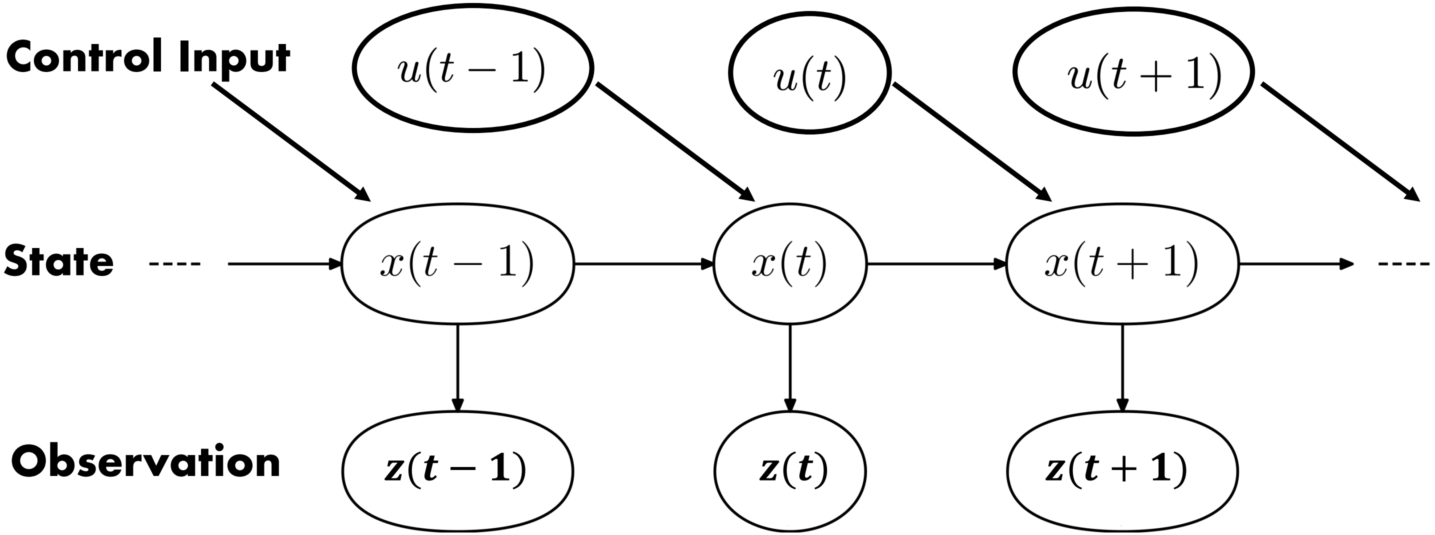


Image adapted from NuScenes by Motional. nuscenes.org

Representations for Manipulation

Manuel Wühtrich et al. "Probabilistic Object Tracking using a Depth Camera", IROS 2013

Why do we care about state estimation in Robotics?

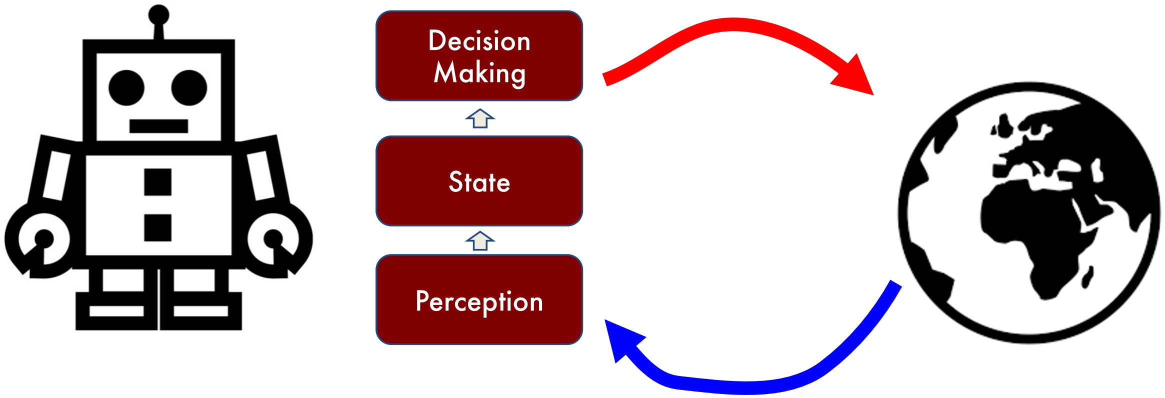


Partially Observable Markov Decision Process

Today

- Intro: Why state estimation?
- Bayes Filter
- Kalman Filter
- Extended Kalman Filter

The Agent and the Environment



Notation

x

x_t

z

z_t

u

u_t

$p(x_t | z_{0:t}, u_{0:t})$

Probabilistic Generative Laws

- Evolution of state and measurement governed by probabilistic laws
- x_t generated stochastically

State Transition Model

- Probability distribution conditioned on all previous states, measurements and controls

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})$$

- Assumption: State complete

$$p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

Measurement Model

- Probability distribution conditioned on all previous states, measurements and controls

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t})$$

- Assumption: State complete

$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) \quad = \quad p(z_t \mid x_t)$$

Belief Distribution

- Assigns probability to each possible hypothesis about what the true state may be
- Posterior distributions over state conditioned on all the data

$$bel(x_t) \quad = \quad p(x_t \mid z_{1:t}, u_{1:t})$$

- Before incorporating measurement = prediction

$$\overline{bel}(x_t) \quad = \quad p(x_t \mid z_{1:t-1}, u_{1:t})$$

The Bayes Filter

- Recursive filter for estimating x_t only from x_{t-1}, z_t and u_{t-1} and not from the ever-growing history $z_{1:t}, u_{1:t}$

1: **Algorithm Bayes_filter(**$bel(x_{t-1}), u_t, z_t$**):**
2: for all x_t do
3: $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$
4: $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
5: endfor
6: return $bel(x_t)$

Simple example – Belief & Measurement Model

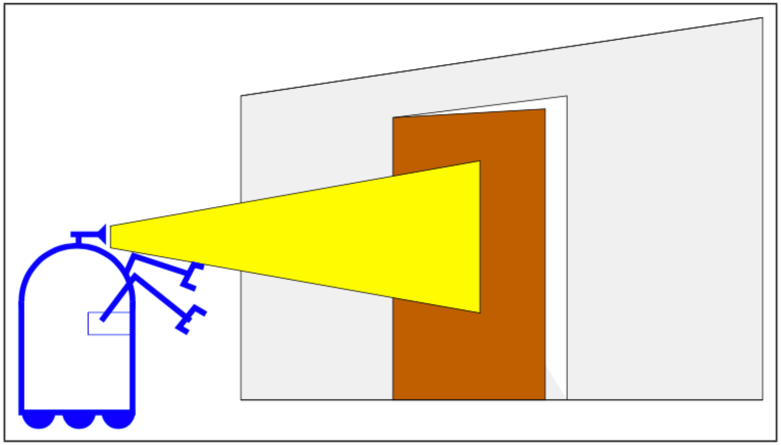


Figure 2.2 A mobile robot estimating the state of a door.

$$\begin{aligned} bel(X_0 = \textbf{open}) &= 0.5 \\ bel(X_0 = \textbf{closed}) &= 0.5 \end{aligned}$$

$$\begin{aligned} p(Z_t = \textbf{sense\_open} \mid X_t = \textbf{is\_open}) &= 0.6 \\ p(Z_t = \textbf{sense\_closed} \mid X_t = \textbf{is\_open}) &= 0.4 \end{aligned}$$

$$\begin{aligned} p(Z_t = \textbf{sense\_open} \mid X_t = \textbf{is\_closed}) &= 0.2 \\ p(Z_t = \textbf{sense\_closed} \mid X_t = \textbf{is\_closed}) &= 0.8 \end{aligned}$$

Simple example – Transition Model

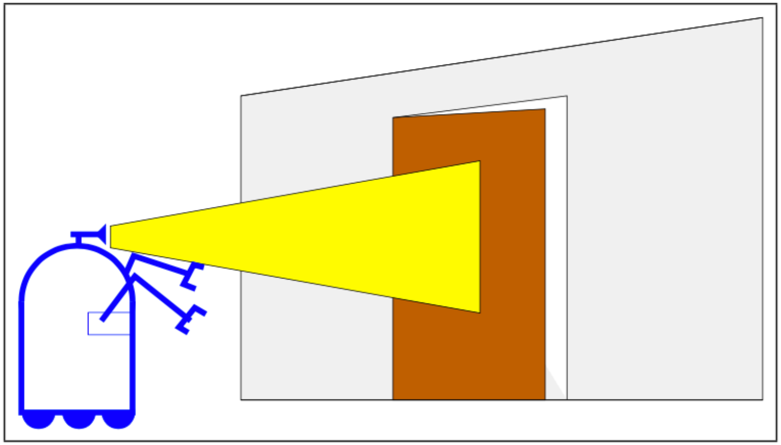


Figure 2.2 A mobile robot estimating the state of a door.

$$\begin{aligned} p(X_t = \textbf{is\_open} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_open}) &= 1 \\ p(X_t = \textbf{is\_closed} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_open}) &= 0 \\ p(X_t = \textbf{is\_open} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_closed}) &= 0.8 \\ p(X_t = \textbf{is\_closed} \mid U_t = \textbf{push}, X_{t\_1} = \textbf{is\_closed}) &= 0.2 \end{aligned}$$

$$\begin{aligned} p(X_t = \textbf{is\_open} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_open}) &= 1 \\ p(X_t = \textbf{is\_closed} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_open}) &= 0 \\ p(X_t = \textbf{is\_open} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_closed}) &= 0 \\ p(X_t = \textbf{is\_closed} \mid U_t = \textbf{do\_nothing}, X_{t\_1} = \textbf{is\_closed}) &= 1 \end{aligned}$$

The Bayes Filter - Derivation

- Bayes Rule

$$
\begin{aligned}
p(x_t \mid z_{1:t}, u_{1:t}) &= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})} \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})
\end{aligned}
$$

The Bayes Filter - Derivation

- State is complete

$$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \quad = \quad p(z_t \mid x_t)$$

- Simplify

$$\begin{aligned} p(x_t \mid z_{1:t}, u_{1:t}) &= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})} \\ &= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \end{aligned}$$

The Bayes Filter - Derivation

$$p(x_t \mid z_{1:t}, u_{1:t}) \quad = \quad \eta \, p(z_t \mid x_t) \, p(x_t \mid z_{1:t-1}, u_{1:t})$$

$$bel(x_t) \quad = \quad \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$$

Algorithm Bayes_filter$(bel(x_{t-1}), u_t, z_t)$**:**
  for all x_t do
    $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \, bel(x_{t-1}) \, dx$
    $bel(x_t) = \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$
  endfor
  return $bel(x_t)$

The Bayes Filter - Derivation

$$bel(x_t) \quad = \quad \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$$

- Total probability

$$\begin{aligned}
\overline{bel}(x_t) \quad &= \quad p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \quad \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \, p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, dx_{t-1}
\end{aligned}$$

- State is complete

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

The Bayes Filter - Derivation

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \quad = \quad p(x_t \mid x_{t-1}, u_t)$$

$$\begin{aligned}\overline{bel}(x_t) &= p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}\end{aligned}$$

$$\overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$$

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$)**:**
- for all x_t do
  - $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$
  - $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
- endfor
- return $bel(x_t)$

Limitations

1. $p(x)$ is defined $\forall x$ – intractable
   - Discrete and small spaces
   - Continuous and/or large spaces – Moments, Finite # of samples
2. The integral term -> costly to compute

Re-Iterate Example

- Is door open or not?

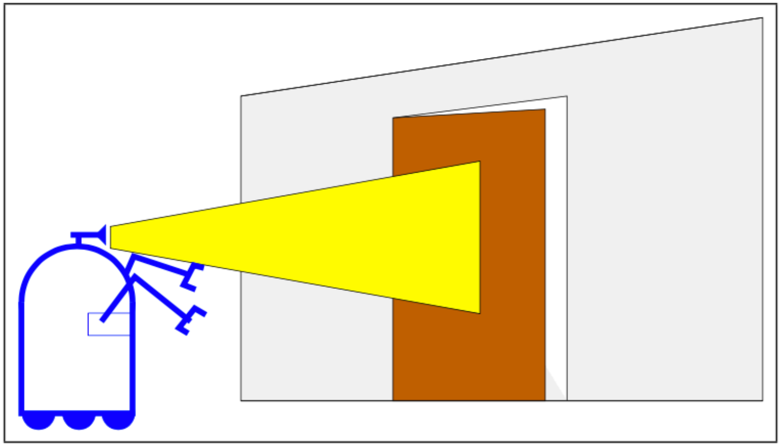


Figure 2.2 A mobile robot estimating the state of a door.

Gaussian Filters - Kalman Filter

$$p(x) \quad = \quad \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}$$

Kalman Filter

- Gaussian Belief
- Linear Transition Model

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t \qquad x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{pmatrix} \quad u_t = \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{m,t} \end{pmatrix}$$

- Linear Measurement Model

$$z_t = C_t x_t + \delta_t$$

Kalman Filter

- Initial Belief

$$bel(x_0) \quad = \quad p(x_0) \quad = \quad \det(2\pi\Sigma_0)^{-\frac{1}{2}} \; \exp\left\{-\tfrac{1}{2}\,(x_0-\mu_0)^T\Sigma_0^{-1}(x_0-\mu_0)\right\}$$

- Distribution over next state

$$p(x_t \mid u_t, x_{t-1}) \qquad (3.4)$$

$$= \quad \det(2\pi R_t)^{-\frac{1}{2}} \; \exp\left\{-\tfrac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1}(x_t - A_t x_{t-1} - B_t u_t)\right\}$$

- Likelihood of Measurement

$$p(z_t \mid x_t) \quad = \quad \det(2\pi Q_t)^{-\frac{1}{2}} \; \exp\left\{-\tfrac{1}{2}(z_t - C_t\, x_t)^T\, Q_t^{-1}\,(z_t - C_t\, x_t)\right\}$$

The Kalman Filter Algorithm

Algorithm Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

$\bar{\mu}_t = A_t \, \mu_{t-1} + B_t \, u_t$

$\bar{\Sigma}_t = A_t \, \Sigma_{t-1} \, A_t^T + R_t$

$K_t = \bar{\Sigma}_t \, C_t^T (C_t \, \bar{\Sigma}_t \, C_t^T + Q_t)^{-1}$

$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \, \bar{\mu}_t)$

$\Sigma_t = (I - K_t \, C_t) \, \bar{\Sigma}_t$

return μ_t, Σ_t

Algorithm Bayes_filter$(bel(x_{t-1}), u_t, z_t)$**:**

for all x_t do

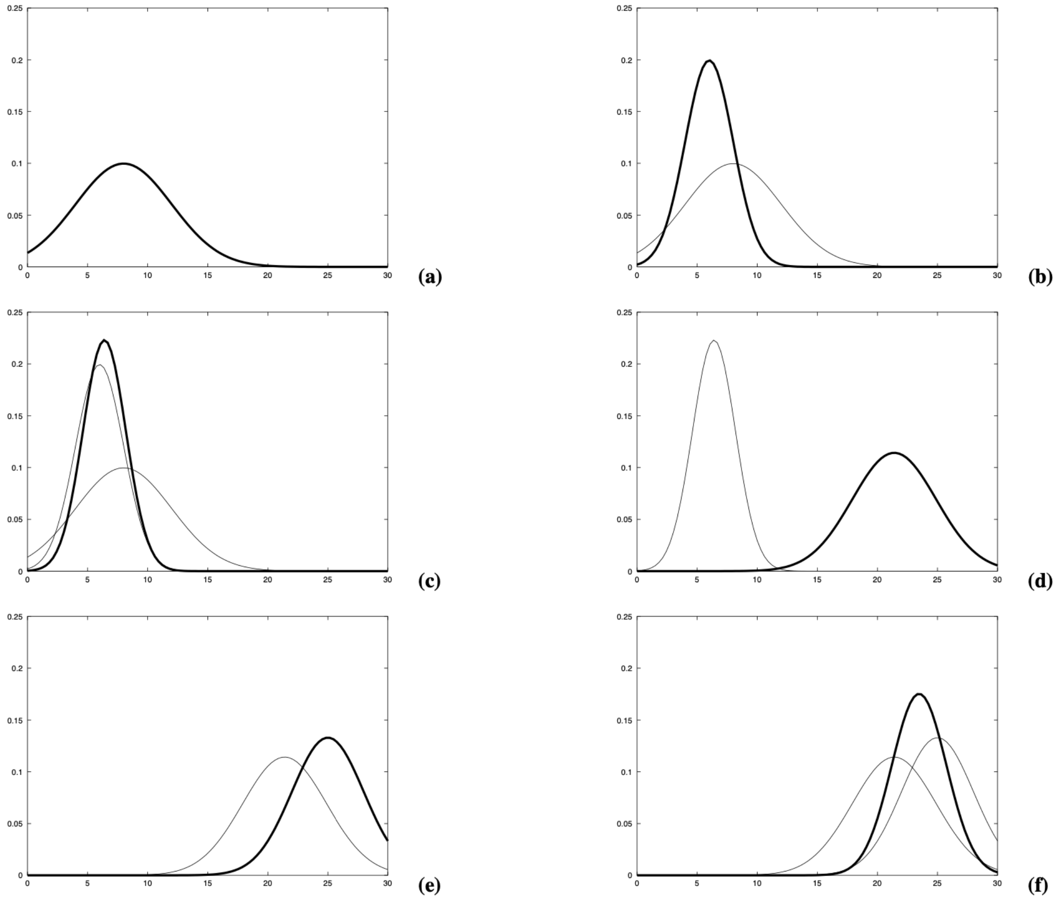
$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \, bel(x_{t-1}) \, dx$

$bel(x_t) = \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)$

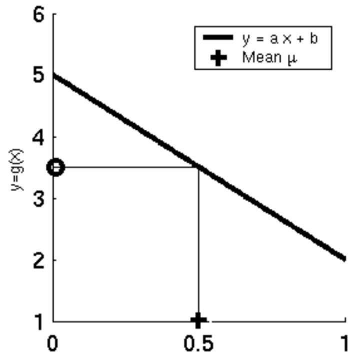
endfor

return $bel(x_t)$

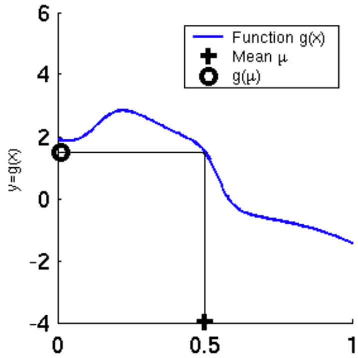
Example



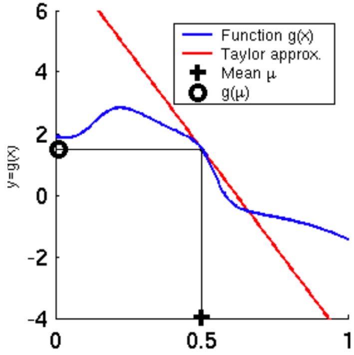
Propagating a Gaussian through a Linear Model



Propagating a Gaussian through a Non-Linear Model



Linearizing the Non-Linear Model



Extended Kalman filter

$$\boldsymbol{x}_t = f(\boldsymbol{x}_{t-1}, u_t) + \boldsymbol{w}_t$$

$$\boldsymbol{z}_t = h(\boldsymbol{x}_t) + \boldsymbol{v}_t$$

$$f(\boldsymbol{x}_{t-1}, u_t) \approx f(x_{t-1}, u_t) + f'(x_{t-1}, u_t)(\boldsymbol{x}_{t-1} - x_{t-1})$$

$$f'(x_{t-1}, u_t) = \left.\frac{\partial f(\boldsymbol{x}, u_t)}{\partial \boldsymbol{x}}\right|_{x_{t-1}} = F_t$$

$$h(\boldsymbol{x}_t) \approx h(x_t) + h'(x_t)(\boldsymbol{x}_t - x_t)$$

$$h'(x_t) = \left.\frac{\partial h(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{x_t} = H_t$$

Extended Kalman Filter

$$p(x_t|z_{t-1:1}, u_{t:1}, \boldsymbol{x}_0) = \mathcal{N}(\hat{x}_t, \hat{P}_t)$$

$$\hat{x}_t = f(x_{t-1}, u_t)$$

$$\hat{P}_t = F_t P_{t-1} F_t^T + Q_t$$

$$p(z_t|x_t) = \mathcal{N}(\hat{z}_t, \hat{S}_t)$$

$$\hat{z}_t = h(\hat{z}_t)$$

$$\hat{R}_t = H_t \hat{P}_t H_t^T$$

$$\hat{S}_t = \hat{R}_t + R_t$$

Extended Kalman Filter

$$p(x_t|z_{t:1}, u_{t:1}, \boldsymbol{x}_0) = \mathcal{N}(x_t, P_t)$$

$$x_t = \hat{x}_t + K_t(z_t - h(\hat{x}_t))$$

$$P_t = (I - K_t H_t)\hat{P}_t$$

$$K_t = \hat{P}_t H_t^T (H_t \hat{P}_t H_t^T + R_t)^{-1}$$

The Extended Kalman Filter Algorithm

1: **Algorithm Extended_Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):**

2: $\bar{\mu}_t = g(u_t, \mu_{t-1})$

3: $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$

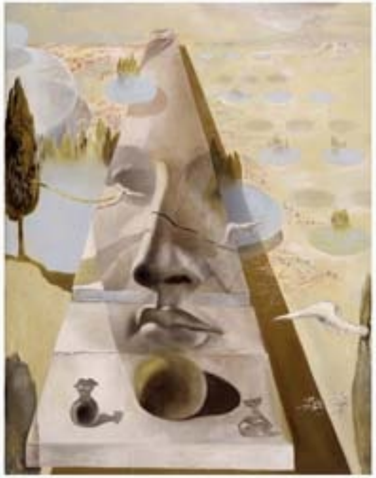
6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

7: return μ_t, Σ_t

	Kalman filter	EKF
state prediction (Line 2)	$A_t \, \mu_{t-1} + B_t \, u_t$	$g(u_t, \mu_{t-1})$
measurement prediction (Line 5)	$C_t \, \bar{\mu}_t$	$h(\bar{\mu}_t)$

CS231

Introduction to Computer Vision



Next lecture:

Optimal Estimation cont'