

Setup and Introduction

Maximisation Random Search (from Assignment 2)

1. Choose $\mathbf{x} \in \{0, 1\}^n$ where \mathbf{x} has the least 1.
2. Produce $\mathbf{x}' \sim \text{Unif}(\{0, 1\}^n)$ randomly.
3. Replace \mathbf{x} by \mathbf{x}' if $F(\mathbf{x}') > F(\mathbf{x})$.
4. Repeat Steps 2 and 3 forever.

Figure 1: Above is the pseudo-code bases on the provided Random Search in the Assignment 2

OneMax Problem F1

The OneMax problem is intened to maximise the number of ones inside a tuple of binary elements. Given $\mathbf{x} \in \{0, 1\}^n$ thus we have that,

$$\arg \max_{\mathbf{x} \in \{0, 1\}^n} F(\mathbf{x}) \text{ where } F(\mathbf{x}) = \sum_{i=1}^n x_i$$

Problem

Prove that Random Search needs with probability

$$1 - e^{-\Omega(n)} \text{ at least a budget of } 2^{n/2}$$

fitness evaluations to reach an optimal search point for the function F1.

Proof. For each of the element inside the tuple of \mathbf{x}' is randomly chosen, which implies that each element can be a random variable of X'_i , thus

$$P(X'_i = 0) = P(X'_i = 1) = \frac{1}{2}$$

Suppose that the optimum is \mathbf{x}^* , we want to find probability of getting \mathbf{x}^* with at least $2^{n/2}$ iterations when sampling \mathbf{x}' . For n binary digits to be 1 the probability is $p = 1/2^n$ per iteration. Let T be the number of iterations to reach \mathbf{x}^* and $k = 2^{n/2}$, then

$$\begin{aligned} T \geq k &: \text{ at least } k \text{ iterations to reach } \mathbf{x}^* \\ T < k &: \text{ reach at least one } \mathbf{x}^* \text{ before } k \end{aligned}$$

Suppose we have each iteration I_i to reach \mathbf{x}^* with probability p , until $k - 1$ iterations, then use the

Boole's inequality as

$$\begin{aligned}
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq \sum_{i=1}^{k-1} P(I_i) \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq (k-1)p \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq (2^{n/2} - 1) \frac{1}{2^n} \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq 2^{-n/2} - \frac{1}{2^n} \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq e^{-n/2 \ln 2} - \frac{1}{2^n}
\end{aligned}$$

Therefore we have that,

$$\begin{aligned}
P(T < k) &\leq e^{-n/2 \ln 2} - \frac{1}{2^n} \\
\Rightarrow 1 - P(T < k) &\leq 1 - e^{-n/2 \ln 2} + \frac{1}{2^n} \\
\Rightarrow P(T \geq k) &\leq 1 - e^{-n/2 \ln 2} + \frac{1}{2^n}
\end{aligned}$$

From the lecture slide we know that $a^n = a^{\Omega(n)}$, additionally since it is a bound we can change the sign from $\geq \rightarrow =$ and $1/2^n$ is very insignificant, thus

$$P(T \geq 2^{n/2}) = 1 - e^{-\Omega(n)}$$

□