Setup and Introduction

Maximisation Random Search (from Assignment 2)

1. Choose $\mathbf{x} \in \{0,1\}^n$ where $\mathbf{x}$ has the least 1.
2. Produce $\mathbf{x}' \sim \text{Unif}(\{0,1\}^n)$ randomly.
3. Replace $\mathbf{x}$ by $\mathbf{x}'$ if $\mathsf{F}(\mathbf{x}') > \mathsf{F}(\mathbf{x})$.
4. Repeat Steps 2 and 3 forever.

Figure 1: Above is the pseudo-code bases on the provided Random Search in the Assignment 2

OneMax Problem $\mathsf{F1}$

The OneMax problem is intened to maximise the number of ones inside a tuple of binary elements. Given $\mathbf{x} \in \{0,1\}^n$ thus we have that,

$$\arg\max_{\mathbf{x}\in\{0,1\}^n} \mathsf{F}(\mathbf{x}) \text{ where } \mathsf{F}(\mathbf{x}) = \sum_{i=1}^{n} x_i$$

Problem

Prove that Random Search needs with probability

$$1 - e^{-\Omega(n)} \text{ at least a budget of } 2^{n/2}$$

fitness evaluations to reach an optimal search point for the function $\mathsf{F1}$.

Proof. For each of the element inside the tuple of $\mathbf{x}'$ is randomly chosen, which implies that each element can be a random variable of X_i', thus

$$P(X_i' = 0) = P(X_i' = 1) = \frac{1}{2}$$

Suppose that the optimum is $\mathbf{x}^*$, we want to find probability of getting $\mathbf{x}^*$ with at least $2^{n/2}$ iterations when sampling $\mathbf{x}'$. For n binary digits to be 1 the probability is $p = 1/2^n$ per iteration. Let T be the number of iterations to reach $\mathbf{x}^*$ and $k = 2^{n/2}$, then

$$\begin{aligned} T \geq k &: \text{ at least } k \text{ iterations to reach } \mathbf{x}^* \\ T < k &: \text{ reach at least one } \mathbf{x}^* \text{ before } k \end{aligned}$$

Suppose we have each iteration I_i to reach $\mathbf{x}^*$ with probability p, until $k-1$ iterations, then use the

Boole's inequality as

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq \sum_{i=1}^{k-1} P(I_i) \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq (k-1)p \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq \left(2^{n/2}-1\right)\frac{1}{2^n} \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq 2^{-n/2} - \frac{1}{2^n} \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq e^{-n/2\ln 2} - \frac{1}{2^n}
\end{aligned}
$$

Therefore we have that,

$$
\begin{aligned}
P(T < k) &\leq e^{-n/2\ln 2} - \frac{1}{2^n} \\
\Rightarrow 1 - P(T < k) &\geq 1 - e^{-n/2\ln 2} + \frac{1}{2^n} \\
\Rightarrow P(T \geq k) &\geq 1 - e^{-n/2\ln 2} + \frac{1}{2^n} \\
\Rightarrow P(T \geq k) &> 1 - e^{-n/2\ln 2} \\
\Rightarrow P(T \geq k) &> 1 - e^{-(\ln 2/2)n}
\end{aligned}
$$

Recall that for lower bound,

$$\Omega(n) = \left\{g(n) : \exists c > 0 : \exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : g(n) \geq c \cdot n\right\}$$

Given that $g(n) = \frac{\ln 2}{2}n$, thus:

$$
\begin{aligned}
\frac{\ln 2}{2}n &\geq cn \\
\Rightarrow c &\leq \frac{\ln 2}{2}
\end{aligned}
$$

Therefore there exists $c \in (0, \ln 2/2]$, then we can say that

$$
\begin{aligned}
P(T \geq k) &= 1 - e^{-\Omega(n)} \\
P(T \geq 2^{n/2}) &= 1 - e^{-\Omega(n)}
\end{aligned}
$$

$\blacksquare$