# Setup and Introduction

**Maximisation Random Search (from Assignment 2)**

1. Choose $\mathbf{x} \in \{0,1\}^n$ where $\mathbf{x}$ has the least 1.
2. Produce $\mathbf{x}' \sim \text{Unif}(\{0,1\}^n)$ randomly.
3. Replace $\mathbf{x}$ by $\mathbf{x}'$ if $\mathsf{F}(\mathbf{x}') > \mathsf{F}(\mathbf{x})$.
4. Repeat Steps 2 and 3 forever.

Figure 1: Above is the pseudo-code bases on the provided Random Search in the Assignment 2

## OneMax Problem $\mathsf{F1}$

The OneMax problem is intened to maximise the number of ones inside a tuple of binary elements. Given $\mathbf{x} \in \{0,1\}^n$ thus we have that,

$$\arg\max_{\mathbf{x} \in \{0,1\}^n} \mathsf{F}(\mathbf{x}) \text{ where } \mathsf{F}(\mathbf{x}) = \sum_{i=1}^{n} x_i$$

### Problem

Prove that Random Search needs with probability

$$1 - e^{-\Omega(n)} \text{ at least a budget of } 2^{n/2}$$

fitness evaluations to reach an optimal search point for the function $\mathsf{F1}$.

*Proof.* For each of the element inside the tuple of $\mathbf{x}'$ is randomly chosen, which implies that each element can be a random variable of $X_i'$, thus

$$P(X_i' = 0) = P(X_i' = 1) = \frac{1}{2}$$

Suppose that the optimum is $\mathbf{x}^*$, we want to find probability of getting $\mathbf{x}^*$ with at least $2^{n/2}$ iterations when sampling $\mathbf{x}'$. For $n$ binary digits to be 1 the probability is $p = 1/2^n$ per iteration. Let $T$ be the number of iterations to reach $\mathbf{x}^*$ and $k = 2^{n/2}$, then

$$\begin{aligned} T \geq k &: \text{ at least } k \text{ iterations to reach } \mathbf{x}^* \\ T < k &: \text{ reach at least one } \mathbf{x}^* \text{ before } k \end{aligned}$$

Suppose we have each iteration $I_i$ to reach $\mathbf{x}^*$ with probability $p$, until $k-1$ iterations, then use the

Boole's inequality as

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq \sum_{i=1}^{k-1} P(I_i) \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq (k-1)p \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq \left(2^{n/2}-1\right)\frac{1}{2^n} \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq 2^{-n/2}-\frac{1}{2^n} \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq e^{-n/2\ln 2}-\frac{1}{2^n}
\end{aligned}
$$

Therefore we have that,

$$
\begin{aligned}
P(T<k) &\leq e^{-n/2\ln 2}-\frac{1}{2^n} \\
\Rightarrow 1-P(T<k) &\geq 1-e^{-n/2\ln 2}+\frac{1}{2^n} \\
\Rightarrow P(T\geq k) &\geq 1-e^{-n/2\ln 2}+\frac{1}{2^n}
\end{aligned}
$$

From the lecture slide we know that $a^n = a^{\Omega(n)}$, additionally since it is a bound we can change the sign from $\geq \rightarrow =$ and $1/2^n$ is very insignificant, thus

$$P(T\geq 2^{n/2}) = 1-e^{-\Omega(n)}$$

$\square$