

# Setup and Introduction

## Maximisation Random Search (from Assignment 2)

1. Choose  $\mathbf{x} \in \{0, 1\}^n$  where  $\mathbf{x}$  has the least 1.
2. Produce  $\mathbf{x}' \sim \text{Unif}(\{0, 1\}^n)$  randomly.
3. Replace  $\mathbf{x}$  by  $\mathbf{x}'$  if  $F(\mathbf{x}') > F(\mathbf{x})$ .
4. Repeat Steps 2 and 3 forever.

Figure 1: Above is the pseudo-code bases on the provided Random Search in the Assignment 2

## OneMax Problem F1

The OneMax problem is intened to maximise the number of ones inside a tuple of binary elements. Given  $\mathbf{x} \in \{0, 1\}^n$  thus we have that,

$$\arg \max_{\mathbf{x} \in \{0, 1\}^n} F(\mathbf{x}) \text{ where } F(\mathbf{x}) = \sum_{i=1}^n x_i$$

### Problem

Prove that Random Search needs with probability

$$1 - e^{-\Omega(n)} \text{ at least a budget of } 2^{n/2}$$

fitness evaluations to reach an optimal search point for the function F1.

*Proof.* For each of the element inside the tuple of  $\mathbf{x}'$  is randomly chosen, which implies that each element can be a random variable of  $X'_i$ , thus

$$P(X'_i = 0) = P(X'_i = 1) = \frac{1}{2}$$

Suppose that the optimum is  $\mathbf{x}^*$ , we want to find probability of getting  $\mathbf{x}^*$  with at least  $2^{n/2}$  iterations when sampling  $\mathbf{x}'$ . For  $n$  binary digits to be 1 the probability is  $p = 1/2^n$  per iteration. Let  $T$  be the number of iterations to reach  $\mathbf{x}^*$  and  $k = 2^{n/2}$ , then

$$\begin{aligned} T \geq k &: \text{ at least } k \text{ iterations to reach } \mathbf{x}^* \\ T < k &: \text{ reach at least one } \mathbf{x}^* \text{ before } k \end{aligned}$$

Suppose we have each iteration  $I_i$  to reach  $\mathbf{x}^*$  with probability  $p$ , until  $k - 1$  iterations, then use the

Boole's inequality as

$$\begin{aligned}
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq \sum_{i=1}^{k-1} P(I_i) \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq (k-1)p \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq (2^{n/2} - 1) \frac{1}{2^n} \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq 2^{-n/2} - \frac{1}{2^n} \\
P\left(\bigcup_{i=1}^{k-1} I_i\right) &\leq e^{-n/2 \ln 2} - \frac{1}{2^n}
\end{aligned}$$

Therefore we have that,

$$\begin{aligned}
P(T < k) &\leq e^{-n/2 \ln 2} - \frac{1}{2^n} \\
\Rightarrow 1 - P(T < k) &\geq 1 - e^{-n/2 \ln 2} + \frac{1}{2^n} \\
\Rightarrow P(T \geq k) &\geq 1 - e^{-n/2 \ln 2} + \frac{1}{2^n} \\
\Rightarrow P(T \geq k) &> 1 - e^{-n/2 \ln 2} \\
\Rightarrow P(T \geq k) &> 1 - e^{-(\ln 2/2)n}
\end{aligned}$$

Recall that for lower bound,

$$\Omega(n) = \{g(n) : \exists c > 0 : \exists n_0 \in \mathbb{N}^+ : \forall n \geq n_0 : g(n) \geq c \cdot n\}$$

Given that  $g(n) = \frac{\ln 2}{2}n$ , thus:

$$\begin{aligned}
\frac{\ln 2}{2}n &\geq cn \\
\Rightarrow c &\leq \frac{\ln 2}{2}
\end{aligned}$$

Therefore there exists  $c \in (0, \ln 2/2]$ , then we can say that

$$\begin{aligned}
P(T \geq k) &= 1 - e^{-\Omega(n)} \\
P(T \geq 2^{n/2}) &= 1 - e^{-\Omega(n)}
\end{aligned}$$

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