CS 3333 Mathematical Foundations Spring '11

Recitation 9 Practiced on: 3/7 & 3/9 5:30 - 6:20 pm

Binomial Coefficients

Note: These problems are designed for practice during a 50 minute recitation.

a) **Easy** problems: expected to be solved in *5 min*.
b) **Medium** problems: expected to be solved in *30 min*.
c) **Hard** problems: expected to be solved in *15 min*.

During the recitation, you may discuss the problems with your peers and the TA. Please control your volume and don't annoy others. An electronic copy of these problems and solutions will be posted on the following URL: http://cs.utsa.edu/~btang/pages/teaching.html.

Solutions:

1. (Easy, 2 min) Find the expansion of $(x + y)^6$. (Textbook [KR] Page 369: 3)

   Answer: The coefficients are the binomial coefficients $\binom{6}{i}$, as i runs from 0 to 6, namely 1, 6, 15, 20, 15, 6, 1. Therefore $(x+y)^6 = \sum_{i=0}^{6} \binom{6}{i} x^{6-i} y^i = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

2. (Easy, 3 min) What is the coefficient of x^9 in $(2 - x)^{19}$? (Textbook [KR] Page 360: 7)

   Answer: By the Binomial Theorem the term involving x^9 in the expansion of $(2 + (-x))^{19}$ is $\binom{19}{9} 2^{10}(-x)^9$. Therefore the coefficient is $\binom{19}{9} 2^{10}(-1)^9 = -2^{10}\binom{19}{9} = -94{,}595{,}072$.

3. (Medium, 10min) What is the coefficient of x^{101} y^{99} in the expansion of $(2x - 3y)^{200}$? (Textbook [KR] Page 361: 9)

   Answer: Using the Binomial Theorem, we see that the term involving x^{101}in the expansion of $(2x + (-3y))^{200}$ is $\binom{200}{99}(2x)^{101}(-3y)^{99}$. Therefore the coefficient is $\binom{200}{99} 2^{101}(-3)^{99}$.

4. (Medium, 10 min) Show that $\binom{n}{k} \le 2^n$ for all positive integers n and all integers k with $0 \le k \le n$. (Textbook [KR] Page 361: 15)

   Answer: There are many ways to see why this is true. By Corollary 1 the sum of all the positive numbers $\binom{n}{k}$, as k runs from 0 to n, is 2^n, so certainly each one of them is no bigger than this sum. Another way to see this is to note that $\binom{n}{k}$ counts the number of subsets of an n-set having k

Bo Tang
Mail: CS Department, UT San Antonio, San Antonio TX 78249, USA
Phone: 210-458-5592 Email: btang[at]cs.utsa.edu Homepage: http://cs.utsa.edu/~btang/

elements and 2^n counts even more – the number of subsets of an n-set with no restriction as to size; so certainly the former is smaller than the latter.

5. (Medium, 10min) Prove Pascal's Identity, using the formula for $\binom{n}{r}$. (Textbook [KR] Page 361: 19)

   Answer: Using the formula (Theorem 2 in Section 5.3) we have:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!\,(n-(k-1))!} + \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!\,k + n!\,(n-k+1)}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)n!}{k!((n+1)-k)!} = \frac{(n+1)!}{k!((n+1)-k)!} = \binom{n+1}{k}.$$

6. (Hard, 15 min) Let n be a positive integer. Show that $\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}/2$. (Textbook [KR] Page 361: 25)

   Answer: We use Pascal's Identity twice (Theorem 2 of Section 5.4) and Corollary 1 of Section 5.3:

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = \frac{1}{2}\left(\binom{2n+1}{n+1} + \binom{2n+1}{n+1}\right)$$

$$= \tfrac{1}{2}\left(\binom{2n+1}{n+1} + \binom{2n+1}{n}\right) = \tfrac{1}{2}\binom{2n+2}{n+1}.$$

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