CS 3333 Mathematical Foundations Spring '11

Recitation 5 Practiced on: 2/7 & 2/9 5:30 - 6:20 pm

Matrices

Note: These problems are designed for practice during a 50 minute recitation.

a) **Easy** problems: expected to be solved in *5 min*.
b) **Medium** problems: expected to be solved in *30 min*.
c) **Hard** problems: expected to be solved in *15 min*.

During the recitation, you may discuss the problems with your peers and the TA. Please control your volume and don't annoy others. An electronic copy of these problems and solutions will be posted on the following URL: http://cs.utsa.edu/~btang/pages/teaching.html.

Solutions:

1. (Easy, 2 min) Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$. (Textbook [KR] Page 254: 1d & e)

   a) What is the element of A in the (3, 2)th position?

      Answer: 1.

   b) What is A^t?

      Answer: The transpose of A is a 4×3 matrix: $A^t = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$.

2. (Easy, 3 min) Find A + B, where (Textbook [KR] Page 254: 2a)

   $A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$

   Answer: $A + B = \begin{bmatrix} 0 & 3 & 9 \\ 1 & 4 & -1 \\ 2 & -5 & -3 \end{bmatrix}$.

3. (Medium, 5 min)If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$, find AB.(Textbook [KR] Page 255: 3a)

   Answer: $AB = \begin{bmatrix} 2 \cdot 0 + 1 \cdot 1 & 2 \cdot 4 + 1 \cdot 3 \\ 3 \cdot 0 + 2 \cdot 1 & 3 \cdot 4 + 2 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 11 \\ 2 & 18 \end{bmatrix}$.

4. (Medium, 10 min) Show that if A is a 2×2 matrix such that AB = BA whenever B is a 2×2 matrix, then $A = cI$, where c is a real number and I is the 2×2 identity matrix.

   A_n $n \times n$ matrix is called **upper triangular** if $a_{ij} = 0$ whenever $i > j$. (Textbook [KR] Page 260: 43)

   Answer: Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, and d are real numbers. Let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Because

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AB = BA, it follows that c = 0 and a = d. Let $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Because AB = BA, it follows that b = 0. Hence, $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI$.

5. (Medium, 15 min) What is the most efficient way to multiply the matrices A_1, A_2, A_3, and A_4 if the dimensions of these matrices are $10 \times 2, 2 \times 5, 5 \times 20$, and 20×3, respectively? (Textbook [KR] Page 256: 25)

   Answer: $A_1((A_2A_3)A_4)$.

   Method 1: If we compute the product as $((A_1A_2)A_3)A_4$, then it will take $10 \cdot 2 \cdot 5$ multiplications for the first product, then $10 \cdot 5 \cdot 20$ for the second and then $10 \cdot 20 \cdot 3$ for the second. This is a total of 1700 multiplications.

   Method 2: If we compute the product as $((A_1A_2)(A_3A_4))$, then it will take $10 \cdot 2 \cdot 5$ multiplications for the first product, then $5 \cdot 20 \cdot 3$ for the second, and then $10 \cdot 5 \cdot 3$ for the second. This is a total of 550 multiplications.

   Method 3: If we compute the product as $A_1((A_2A_3)A_4)$, then it will take $2 \cdot 5 \cdot 20$ multiplications for the first product, then $2 \cdot 20 \cdot 3$ for the second, and then $10 \cdot 2 \cdot 3$ for the second. This is a total of 380 multiplications.

6. (Hard, 15 min) Let A be the 2×2 matrix: $\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that if $ad - bc \neq 0$, then: $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$. (Textbook [KR] Page 256: 19)

   Answer: As we have to do is form the products AA^{-1} and $A^{-1}A$, using the purported A^{-1}, and see that both of them are the 2×2 identity matrix. It is easy to see that the upper left and lower right entries in each case are $(ad - bc)/(ad - bc) = 1$, and the upper right and lower left entries are all 0.

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