**CS 3333 Mathematical Foundations Fall 2011  
Homework 3 Solutions**

Matrices: Pages 254-256: 1 (use matrix A in problem 2a to solve this problem), 2b, 3 b & c, 6 (solve using matrix inverses), 8, 9, 10, 12a, 14, 15 (calculate a few powers of A and then generalize it to a formula), 16, 17, 18 (calculate both ways: first matrix \* second matrix; second matrix\*first matrix), 20, 21 (indicate the prior results used at each step), 22, 25  (give the number of multiplications required for each of the six approaches),   
27 (give the matrix equation-vector equation AX=B and calculate A-1 and A-1B).  
  
Page 260: 41, 42, 43, 44, 46, 48. Total points = 77.

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1. (2 point) Let A=.

a) What size is A? Answer: .

b) What is the third column of A? Answer: .

c) What is the second row of A? Answer: .

­d) What is the element of A in the (3, 2)th position? Answer: -2

e) What is At? Answer: .

2. (1 points)

b) A=, B = A + B =

3. (2 points) Find AB if

b) .

Answer: .

c) .

Answer: .

6. (3 points) Find a matrix A such that  A = 

Solution: Let B =  and C = . Then A = B-1C, if B is nonsingular.

Guass-Jordan method or Cofactors method gives B-1 = .

Gauss-Jordan elimination method gives the following intermediate steps. [Generated by a computer program; fractions appear as real numbers.]

Augmented Matrix

1.000 3.000 2.000 | 1.000 0.000 0.000

2.000 1.000 1.000 | 0.000 1.000 0.000

4.000 0.000 3.000 | 0.000 0.000 1.000

Iteration 1

1.000 3.000 2.000 | 1.000 0.000 0.000

0.000 -5.000 -3.000 | -2.000 1.000 0.000

0.000 -12.000 -5.000 | -4.000 0.000 1.000

Iteration 2

1.000 0.000 0.200 | -0.200 0.600 0.000

0.000 1.000 0.600 | 0.400 -0.200 -0.000

0.000 0.000 2.200 | 0.800 -2.400 1.000

Iteration 3

1.000 0.000 0.000 | -0.273 0.818 -0.091

0.000 1.000 0.000 | 0.182 0.455 -0.273

0.000 0.000 1.000 | 0.364 -1.091 0.455

∴ A =  ∙  = .

8. (2 points) Show that matrix addition is commutative; that is, show that if A and B are both n\*n matrices, then A + B = B + A.

Solution: Since the entries of A + B are aij + bij and the entries of B + A are bij + aij, that A + B = B + A follows from the commutativity of addition of real numbers.

9. (2 points) Show that matrix addition is associative; that is, show that if A,B and C are all m\*n matrices, then A+(B+C) = (A+B)+C.

Solution: A+ (B+C) = [aij+(bij+cij)] = [(aij+bij)+cij] = (A+B)+C

10. (3 points) Let A be a 3\*4 matrix, B be a 4\*5 matrix, and C be a 4\*4 matrix. Determine which of the following products are defined and find the size of those that are defined.

a) AB

Solution: This product is a 3\*5 matrix.

b) BA

Solution: This is not defined since the number of columns of B does not equal the number of rows of A.

c) AC

Solution: This product is a 3\*4 matrix.

d) CA

Solution: This is not defined since the number of columns of C does not equal the number of rows of A.

e) BC

Solution: This is not defined since the number of columns of B does not equal the number of rows of C.

f) CB

Solution: This product is a 4\*5 matrix.

12. (2 points) In this exercise we show that matrix multiplication is distributive over matrix addition.

a) Suppose that A and B are m \* k matrices and that C is a k \* n matrix. Show that (A + B)C = AC + BC.

Solution: We use the definition of matrix addition and multiplication. All summations here are from 1 to k..

( A + B )C = =  = AC + BC.

14. (3 points) The n\*n matrix A = [aij] is called a diagonal matrix if aij = 0 when i ≠ j. Show that the product of two n\*n diagonal matrices is again a diagonal matrix. Give a simple rule for determining this product.

Solution: Let A and B be two diagonal n\*n matrices. Let C = [cij] be the product AB. From the definition of matrix multiplication, cij = . Now all the terms aiq in this expression are 0 except for q = i, socij =aiibij. But bij = 0 unless i = j, so the only nonzero entries of C are the diagonal entries cii = aiibii.

15. (2 points) A2 = AA = , A3 = AA2 = , …, An = .

16. (1 point) Show that ( At )t = A.

Solution: The ( i, j )th entry of ( At )t  is the ( j, i )th entry of At , which is the ( i, j )th entry of A.

17. (1+2 = 3 points) Let A and B two n\*n matrices, Show that

a) (A+B)t = At + Bt

b) (AB)t = AtBt

Solution:

1. Let A = [aij] and B = [bij], Then A+B = [aij+bij]. We have (A+B)t = [aji+bji] = [aji]+[bji] = At+Bt.
2. The (i,j)th entry of (AB)t, which is the same as the (j,i)th entry of AB, is , which is a dot product of row j of A and column i of B.   
   On the other hand, the (i,j)th entry of BtAt , the dot product of row i of Bt and column j of At, is  since ith row of Bt is actually column i of B and jth column of At is actually row j of A.

18. (2 points) Show that  is the inverse of 

Solution: Multiplication of the two matrices gives I3. So they are inverses of each other.

20. (8 points) Let A = 

a) Find A-1.

Solution: Using Exercise 19, noting that |A| = ad – bc = -5, and A-1 = .

b) Find A3.

Solution: We multiply to obtain A2 =  and then A3 = 

c) Find (A-1)3

Solution: We multiply to obtain (A-1)2= and then (A-1)3=

d) Use your answers to (b) and (c) to show that (A-1)3  is the inverse of A3.

Solution: Applying the method of Exercise 19 for obtaining inverses to the answer in part (b), we obtain the answer in part (c). Therefore (A3)-1=(A-1)3

21.(3 points) Let A be an invertible matrix. Show that (An)-1 = (A-1)n whenever n is a positive integer.

Solution: An(A-1)n = A(A…(A(AA-1)A-1)…A-1)A-1 by the associative law. Because AA-1= I, working from the inside shows that An(A-1)n = I. Similarly (A-1)nAn = I. Therefore (An)-1 = (A-1)n.

22. (2 points) Let A be a matrix. Show that the matrix AAt is symmetric.

Solution: A matrix is symmetric if and only if it equals its transpose. So let us compute the transpose of AAt and see if we get this matrix back. Using Exercise 17b and Exercise 16, we have (AAt )t = ((At)t)At = AAt, as desired.

25. (6 points) What is the most efficient way to multiply the matrices A1, A2, A3, and A4 if the dimensions of these matrices are 10\*2, 2\*5, 5\*20 and 20\*3, respectively?

Solution: Use same procedure as 24. The most efficient way is A1 ((A2A3) A4).

(A1A2)(A3A4)

(A1A2)=10x5 dimension and 10\*2\*5=100 multiplications,

(A3A4) =5x3 dimension and 5\*20\*3=300 multiplications,

(A1A2)(A3A4) = 10x3 dimension and 10\*5\*3=150 multiplications.

Total= 550 multiplications.

((A1A2)A3)A4

(A1A2)=10x5 dimension and 10\*2\*5=100multiplications,

(A1A2)A3=10x20 dimension and 10\*5\*20=1000 multiplications,

((A1A2)A3)A4=10x3 dimension and 10\*20\*3=600 multiplications.

Total = 1700 multiplications.

(A1(A2A3))A4

(A2A3)=2x20 dimension and 2\*5\*20=200 multiplications

(A1(A2A3))=10x20 dimension and 10\*2\*20=400 multiplications,

(A1(A2A3))A4=10x3 dimension and 10\*20\*3=600 multiplications.

Total = 1200 multiplications.

A1(A2(A3A4))

(A3A4) =5x3 dimension and 5\*20\*3 = 300 multiplications,

A2(A3A4)=2x3 dimension and 2\*5\*3= 30 multiplications,

A1(A2(A3A4))= 10x3 dimension and 10\*2\*3=60 multiplications.

Total = 390 multiplications.

A1((A2A3)A4)

(A2A3)=2x20 dimension and 2\*5\*20=200 multiplications,

((A2A3)A4)=2x3 dimension and 2\*20\*3=120 multiplications,

A1((A2A3)A4) = 10x3 dimension and 10\*2\*3=60 multiplications.

Total = 380 multiplications.

27. (7 points) Use Exercise 18 and 26 to solve the system

7x1 - 8x2 + 5x3 = 5

-4x1 + 5x2 -3x3 = -3

x1 – x2 + x3 = 0

Solution: 











 is the solution vector.

41. (3 points) Find An if A is .

Answer: , , , , for .

42. (3 points) Show that if A = cI, where c is a real number and I is the n x n identity matrix, then AB = BA whenever B is an n x n matrix.

Answer: Since A is the matrix defined by and for , it is easy to see from the definition of multiplication that AB and BA are both the same as B except that every entry has been multiplied by c. Therefore these two matrices are equal.

43. (3 points) Show that if A is a matrix such that AB = BA whenever B is a matrix, then , where c is a real number and I is the identity matrix.   
***Answer:*** Suppose that , where a, b, c, and d are real numbers. Let . Because AB = BA, it follows that c = 0 and a = d. Let . Because AB = BA, it follows that b = 0. Hence, .

44. (3 points) From the definition of the matrix product, devise an algorithm for computing the product of two upper triangular matrices that ignores those products in the computation that are automatically equal to zero.

Answer: We just use Algorithm 1 in Section 3.8, where A and B are now upper triangular matrices, by replacing m by n in line 1, and having q iterate only from i to j, rather than from 1 to k.

46. (3 points) How many multiplications of entries are used by the algorithm found in Exercise 44 for multiplying two upper triangular matrices?

Answer: Looking at the nested loops, we see that the number of multiplications is given by the following expression: . This is simplified into: . As we know that . Doing the algebra, we obtain as the total number of multiplications.

48. (4 points) What is the best order to form the product ABCD if A, B, C, and D are matrices with dimensions , , , and , respectively? Assume that the number of multiplications of entries used to multiply a matrix and a matrix is pqr.

Answer: There are five ways to parenthesize the calculation ABCD, namely as (AB)(CD), as ((AB)C)D, as (A(BC))D, as **A((BC)D)**, and as A(B(CD)). We can compute the number of multiplications for each and compare. These numbers are (in thousands) 108, 117, 80, 44, and 81. Thus the most efficient method is the fourth, using 44,000 multiplications.