**CS 3333 Mathematical Foundations Spring 2011**

**Homework 6 Solutions**



Binomial coefficients: Pages 369-370: 3 (do this for [x+y]7), 6, 8, 12, 14, 22, 23 (give combinatorial argument), 28, 33 [use 12×8 grid; that is, find path from (0,0) to (11,7)].

Generalized permutations and combinations: Pages 379-382: 4, 10, 12, 14, 16, 17, 20, 24, 32, 37, 58, 66.  
  
Pages 388--390: 20, 21, 22, 23, 29, 33, 35, 36, 37.

Extra Problems

1. Show that C(2n, n) + C(2n, n-1) = C(2n+2, n+1)/2. [Hint: Apply Pascal's identity followed by the combinatorial identity in problem 23, page 369, to the expression on the left hand side.]
2. Use the above result to give an alternate proof for problem 25, page 369. [Hint: take the right hand side of the identity in this problem, expand it using Vandermonde's identity, and apply the result from Extra Problem 1.]

Maximum points: \*\* points



**Pages 369-370**

3. (1 point) .

6. (1 point) 14 = 330

8. (1 point) = 81,662,929,920

12. (1 point) 1 11 55 165 330 462 462 330 165 55 11 1

14. (4 points) Using the factorial formulae for computing binomial coefficients, we can see that  = . If k <= n/2, then < 1, so the “less than” signs are correct. Similarly, if k > n/2, then  > 1, so the “greater than” signs are correct. The middle equality is Corollary 2 in section 5.3, since  +  = n. The equalities at the ends are clear.

22. (3+1=4 points) a. Suppose that we have a set with n elements, and we wish to choose a subset A with k elements and another, disjoint, subset with r – k elements. The left-hand side gives us the number of ways to do this, namely the product of the number of ways to choose the r elements that are to go into one or the other of the subsets and the number of ways to choose which of these elements are to go into the first of the subsets. The right hand side gives us the number of ways to do this as well, namely the product of the number of ways to choose the first subset and the number of ways to choose the second subset from the elements that remain.

b. LHS:  = ∙ = 

RHS:  = ∙ = 

23. (3 points)

Let us use a different letter for k.

 is the same as .

Use 22 (by substituting r=α and k=1) to prove the latter identity.

Alternative combinatorial argument: Suppose that we have a set of n + 1 people, and we wish to choose k of them. Clearly there are  ways to do this. On the other hand, we can choose our set of k people by first choosing one person to be in the set (there are n + 1 choices), and then choosing k – 1 additional people to be in the set, from the n people remaining. This can be done in  ways. Therefore, apparently there are  ways to choose the set of k people. However, we have over counted: there are k ways that every such set can be chosen, since once we have the set, we realize that any of the k people could have chosen first. Thus we have over counted by a factor of k, and the real answer is 

Finally, we are asked to use this identity to give a recursive definition of the ’s. Note that this identity expresses  in terms of  for values of i and j less than n and k, respectively (namely i = n – 1 and j = k - 1). Thus the identity will be recursive part of the definition. We need the base cases to handle n = 0 or k = 0. Thus, our full definition is,

 = 

28. (3+1=4 points) a. To choose 2 people from a set of n men and m women, we can either choose 2 men:  ways to do so, or 2 women:  ways to do so, or one of each sex: n\*n ways to do so. Therefore the right hand side counts the number of ways to do this (by the sum rule). The left hand side counts the same thing, since we are simply choosing 2 people of 2n people.

b. 2 + n2 = n(n - 1) + n2 = 2n2 – n = n(2n – 1) = 2n(2n – 1)/2 = 

33. (2 points) m = 11, n = 7

a. Clearly, a path to the desired point must consist of 11 moves to the right and 7 moves up. Therefore each such path can be represented by a bit string consisting of 11 0’s and 7 1’s, with 0’s representing moves to the right and 1’s representing moves up. Note that the total length of the bit string is 11 + 7 = 18.

b. The number of bit strings of length 18 containing exactly 7 1’s is .

**Pages 379-382**

4. (1 point) There are 6 choices each of 7 times, so the answer is 67 = 279,936

10. (6 points)

a C (6+12-1, 12) = C (17, 12) = 6188

b. C (6+36-1, 36) = C (41, 36) = 749,398

c. If we first pick the two of each kind, then we have picked 2.6 = 12 croissants. This leaves one dozen left to pick without restriction, so the answer is the same as in part (a), namely C (6+12-1, 12) = C (17, 12) = 6188.

d. We first compute the number of ways to violate the restriction, by choosing at least three broccoli croissants. This can be done in C( 6+21-1,21) = C(26,21) = 65780 ways, since one we have picked the three broccoli croissants there are 21 left to pick without restriction. Since there are C (6+24-1, 24) = C (29,24) = 118755 ways to pick 24 croissants without any restriction, there must be 118755-65780 = 52,975 ways to choose two dozen croissants with no more than two broccoli.

e. Eight croissants are specified, so this problem is the same as choosing 24-6 = 16 croissants without restriction, which can be done in C( 6+16-1,16) = C(21,16) = 20,349 ways.

f. First let us include all the lower bound restrictions. If we choose the required 9 croissants, then there are 24 – 9 = 15 left to choose, and if there are no restriction on the broccoli croissants then there would be C (6+15-1, 15) = C (20, 15) = 15504 ways to make the selections. If in addition we were to violate the broccoli restriction by choosing at least four broccoli croissants, there would be C (6+11-1, 11) = C (16, 11) = 4368 choices. Therefore the number of ways to make the selection without violating the restriction is 15504 – 4368 = 11,136.

12. (1 point) There are 5 things to choose from, repetitions allowed, and we want to choose 20 things, order not important. Therefore by Theorem 2, the answer is C( 5+20-1,20) = C(24,20) = C(24,4) = 10,626.

14. (1 point) By theorem 2 the answer is C(4+17-1, 17) = C(20,17) = C(20,3) = 1140.

16. (3\*1+2=5 points) a. We require each xi >= 2. This uses up 12 of the 29 total required, so the problem is the same as finding the number of solutions to x1 + x2 + x3 + x4 + x5 + x6 = 17 with each xi a nonnegative integer. The number of solution is therefore C (6+17-1, 17) = C (22, 17) = 26,334

b. The restrictions use up 22 of the total, leaving a free total of 7. Therefore the answer is C (6+7-1, 7) = C (12, 7) = 792.

c. The number of solutions without restriction is C (6+29-1,29) = C(34,29) = 278256. The number of solution violating the restriction by having x1≥6 is C (6+23-1, 23) = C (28, 23) = 98280. Therefore the answer is 278256 – 98280 = 179,976.

d. The number of solutions with x2≥9(as required) but without the restriction on x1 is C (6+20-1, 20) = C (25, 20) = 53130. The number of solution violating the additional restriction by having x1≥8 is C (6 + 12 -1, 12) = C (17, 12) = 6188. Therefore the answer is 53130 – 6188 = 46, 942.

17. (1 point) From theorem 3 the answer is 10! / (2! . 3! .5!)=2520.

20. (1 point) We introduce the nonnegative slack variable x4, and our problem becomes the same as the problem of counting the number of nonnegative integer solutions x1+x2+x3+x4 = 11. By theorem 2 the answer is C (4+11 -1, 11) = C (14, 11) = C (14, 3) = 364.

24. (2 points) We assume that this problem leaves us free to pick which boxes get which numbers of balls. There are several ways to count this. Here is one. Line up the 15 objects in a row( 15! Ways to do that), and line up the five boxes in a row( 5! Ways to do that). Now put the first object into the first box, the next two into the second box, the next three into the third box, and so on. This over counts by a factor of 1!.2!.3!.4!.5!, since there are many ways to swap objects in the permutation without affecting the result. Therefore the answer is 15!5!/(1!.2!.3!.4!.5!) = 4,540,536,000.

32. (1 point) We can treat the 3 consecutive A’s as one letter. Thus we have 6 letters, of which 2 are the same (the two R’s), so by theorem 3 the answer is 6! /2! = 360.

37. (1 point) It follows directly from Theorem 3 that the answer is

7! / (2! . 2!. 3!) = 210.

58. (4 points) a. This is a straightforward application of the product rule: There are 7 choices for the first ball, 6 choices for the second ball, and so on, for an answer of 7.6.5.4.3 = 2520.

b. This is choosing 5 out of 7 balls that re put in boxes: C (7,5) = 21.

c. Since each box must have a ball and the balls are unlabeled, there is only one way to do this.

d. As noted in part (c), there is only one way to do this.

66. (1 point) The answer is C (3+100-1,100) = C (102,100) = C (102, 2) = 5151.

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20. (2 points) If the worm never gets sent to the same computer twice, then it will infect 100 computers on the first round of forwarding, 1002 = 10,000 other computers on the second round of forwarding, and so on. Therefore the maximum number of different computers this one computer can infect is 100 + 1002 + 1003 + 1004 + 1005 = 10,101,010,100. This figure of ten billion is probably comparable to the total number of computers in the world.

21. (6 points)

a) C(20, 12) = 125,970

b) The only choice is the choice of a variety, so the answer is 20.

c) C(12 + 20 -1, 12) = 141,120,525

d) Subtract the answer to (b) from that to (c), which asks form the number of ways this restriction can be violated. Therefore the answer if 141,120,505

e) C(20 + 6 – 1, 6) = C(25, 6) = 177,100.

f) C(12 + 20 -1, 12) - C(20 + 5 – 1, 5) = 141,120,525 – 42,504 = 141,078,021.

22. (3 points)

a) We want to solve n(n - 1) = 110. Simple algebra gives n = 11. (We ignore n = -10, since we need a positive integer for our answer).

b) We recall that 7! = 5040, so the answer is 7.

c) We need to solve the equation n(n - 1) (n - 2) (n - 3) = 12n(n – 1). Since we have n ≥ 4 in order for P(n, 4) to be defined, this equation reduces to (n - 2) (n - 3) = 12. Simple algebra gives n = 6.

23. (2 points)

a) n (n - 1) /2 = 45. Simple algebra gives n = 10.

b) n (n - 1) (n – 2)/6 = n (n - 1). Since P(n, 2) is not defined for n < 2, we know that neither n nor n - 1 is 0, so we can divide both sides by these factors, obtaining that n – 2 = 6. Simple algebra gives n = 8.

c) n = 7.

29. (1 points) Substitute x = 1 and y = 3 into the Binomial Theorem and we obtain exactly this identity.

33. (2 points) There are

such permutations.

35. (2 points) We assume that each student is to get one advisor, that there are no other restrictions, and that the students and advisors are to be considered distinct. Then there are 5 ways to assign each student, so by the product rule there are ways to assign all of them.

36. (2 points) This is equivalent to the number of nonnegative integer solutions to d’ + m’ + g’ = 3, where d’ = d – 3, m’ = m – 3, and g’ = g – 3. By Theorem 2 of Section 5.5, the answer is C(3+3-1, 3) = C(5, 3) = 10.

37. (2 points)

a) We let x1 = x1’ + 2, x2 = x2’ + 3, and x3 = x3’ + 4. Then the equivalent problem is the number of nonnegative integer solutions to the equation x1’ + x2’ + x3’ = 8. There are C(3 + 8 – 1, 8) = C(10, 8) = 45 of them.

b) The number of solutions with x3 > 5 is equivalent to the number of solutions to x1 + x2 + x3’ = 11, where x3 = x3’ + 6. There are C(3 + 11 – 1, 11) = 78 of these. Now we want to subtract the number of solutions for which also x1 ≥ 6. This is equivalent to the number of solutions to x1’ + x2 + x3’ = 5, where x1 = x1’ + 6. There are C(3 + 5 – 1, 5) = 21 of these. Therefore the answer to the problem is 78 – 21 = 57.

c) The number of solutions with x1 ≥ 4 is equivalent to the number of solutions to x1’ + x2 + x3’ = 7, where x1 = x1’ + 4, namely C(3 + 7 – 1, 7) = C(9, 7) = C(9, 2) = 36. The number of solutions with x2 ≥ 3 is equivalent to the number of solutions to x1 + x2’ + x3’ = 8, where x1 = x1’ + 3, namely C(3 + 8 – 1, 8) = C(10, 8) = 45. However, there are also solutions in which both restrictions are violated, namely the solutions to x1’ + x2’ + x3’ = 4. There are C(3 + 4 – 1, 4) = C(6, 4) = 15 of these. Therefore the number of solutions is 78 – (36 + 45 - 15) = 78 – 66 = 12 solutions of the given problem.

**Extra Problems**

1. (3 point) .

2. (3 point) According to Vandermonde’s Identity, the right hand side of the identity in this problem: .