1 Consider $\varepsilon > 0$.

$$\text{Then } |a_n| < \varepsilon$$
$$\Leftrightarrow \left|\frac{(-1)^n}{3n^3 - 25}\right| < \varepsilon$$
$$\Leftrightarrow \frac{1}{3n^3 - 25} < \varepsilon \text{ (for } n \geq 3)$$
$$\Leftrightarrow 3n^3 - 25 > \tfrac{1}{\varepsilon} \quad \text{(since both sides are positive)}$$
$$\Leftrightarrow 3n^3 > 25 + \tfrac{1}{\varepsilon}$$
$$\Leftrightarrow n^3 > \tfrac{1}{3}(25 + \tfrac{1}{\varepsilon})$$
$$\Leftrightarrow n > \sqrt[3]{\tfrac{1}{3}(25 + \tfrac{1}{\varepsilon})}$$

Now choose $N = \left[\sqrt[3]{\tfrac{1}{3}\left(25 + \tfrac{1}{\varepsilon}\right)}\right]$. Then for all $n > N$, $|a_n| < \varepsilon$. The sequence is null.

2 (a) $a_n = \dfrac{2n^2 + 5n - 3}{2 + 3n - n^2} = \dfrac{2 + \frac{5}{n} - \frac{3}{n^2}}{\frac{2}{n^2} + \frac{3}{n} - 1} = \dfrac{2 + 5\left(\frac{1}{n}\right) - 3\left(\frac{1}{n^2}\right)}{2\left(\frac{1}{n^2}\right) + 3\left(\frac{1}{n}\right) - 1}$.

$\left\{\frac{1}{n}\right\}$ and $\left\{\frac{1}{n^2}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{2 + 5 \times 0 - 3 \times 0}{2 \times 0 + 3 \times 0 - 1} = -2$, as $n \to \infty$.

(b) $a_n = \dfrac{n^2 + 5(2^n)}{n^3 - 3(2^n)} = \dfrac{\frac{n^2}{2^n} + 5}{\frac{n^3}{2^n} - 3}$

$\left\{\frac{n^2}{2^n}\right\}$ and $\left\{\frac{n^3}{2^n}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{0 + 5}{0 - 3} = -\frac{5}{3}$, as $n \to \infty$.

(c) $a_n = \dfrac{1 + 2n - 3n^2}{2n^2 + n + 1} = \dfrac{\frac{1}{n^2} + 2\left(\frac{1}{n}\right) - 3}{2 + \frac{1}{n} + \frac{1}{n^2}}$

$\left\{\frac{1}{n}\right\}$ and $\left\{\frac{1}{n^2}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{0 + 2 \times 0 - 3}{2 + 0 + 0} = -\frac{3}{2}$, as $n \to \infty$.

(d) $a_n = \dfrac{n + 2(3^n) + 3(2^n)}{n^2 - 5(2^n) + 4(3^n)} = \dfrac{\frac{n}{3^n} + 2 + 3\left(\frac{2}{3}\right)^n}{\frac{n^2}{3^n} - 5\left(\frac{2}{3}\right)^n + 4}$.

$\left\{\frac{n}{3^n}\right\}$, $\left\{\left(\frac{2}{3}\right)^n\right\}$ and $\left\{\frac{n^2}{3^n}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{0 + 2 + 3 \times 0}{0 - 5 \times 0 + 4} = \frac{1}{2}$ as $n \to \infty$.

(e) $a_n = \dfrac{5n^3 - 3n + 6}{2n^3 + 4n - 1} = \dfrac{5 - 3\left(\frac{1}{n^2}\right) + 6\left(\frac{1}{n^3}\right)}{2 + 4\left(\frac{1}{n^2}\right) - \left(\frac{1}{n^3}\right)}$.

$\left\{\frac{1}{n^2}\right\}$ and $\left\{\frac{1}{n^3}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{5-3\times 0+6\times 0}{2+4\times 0-0} = \frac{5}{2}$ as $n \to \infty$.

(f) $$a_n = \frac{3^n + 3(n!)}{2^n + n^3 - 2(n!)} = \frac{\frac{3^n}{n!}+3}{\frac{2^n}{n!}+\frac{n^3}{n!}-2}.$$

$\left\{\frac{3^n}{n!}\right\}$, $\left\{\frac{2^n}{n!}\right\}$ and $\left\{\frac{n^3}{n!}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{0+3}{0+0-2} = -\frac{3}{2}$ as $n \to \infty$.

(g) $$a_n = \frac{n^2 - 2n! + 5}{n! - 2^n - 4n^3} = \frac{\frac{n^2}{n!} - 2 + 5\left(\frac{1}{n!}\right)}{1 - \left(\frac{2^n}{n!}\right) - 4\left(\frac{n^3}{n!}\right)}.$$

$\left\{\frac{n^2}{n!}\right\}$, $\left\{\frac{1}{n!}\right\}$, $\left\{\frac{2^n}{n!}\right\}$ and $\left\{\frac{n^3}{n!}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{0-2+5\times 0}{1-0-4\times 0} = -2$ as $n \to \infty$.

(h) $$a_n = \frac{2n^2 + n - 3}{8n^2 + 2n + 3} = \frac{2 + \frac{1}{n} - 3\left(\frac{1}{n^2}\right)}{8 + 2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n^2}\right)}.$$

$\left\{\frac{1}{n^2}\right\}$ and $\left\{\frac{1}{n^3}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{2+0-3\times 0}{8+2\times 0+3\times 0} = \frac{1}{4}$ as $n \to \infty$.

(i) $a_n = \dfrac{2n^3+5n-4}{6n^3+2n^2-3} = \dfrac{2+5\left(\frac{1}{n^2}\right)-4\left(\frac{1}{n^3}\right)}{6+2\left(\frac{1}{n}\right)-3\left(\frac{1}{n^3}\right)}$.

$\left\{\frac{1}{n}\right\}$, $\left\{\frac{1}{n^2}\right\}$ and $\left\{\frac{1}{n^3}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{2+5\times 0-4\times 0}{6+2\times 0-3\times 0} = \frac{1}{3}$ as $n\to\infty$.

(j) $a_n = \dfrac{2n^2+5n-3(n!)}{3^n-n!-2n^3} = \dfrac{2\left(\frac{n^2}{n!}\right)+5\left(\frac{n}{n!}\right)-3}{\frac{3^n}{n!}-1-2\left(\frac{n^3}{n!}\right)}$.

$\left\{\frac{n^2}{n!}\right\}$, $\left\{\frac{n}{n!}\right\}$, $\left\{\frac{3^n}{n!}\right\}$ and $\left\{\frac{n^3}{n!}\right\}$ are basic null sequences.

By the Combination Rules, $a_n \to \dfrac{2\times 0+5\times 0-3}{0-1-2\times 0} = 3$ as $n\to\infty$.

3 (a) $a_n = n^2 - \dfrac{4}{n} + 2^n > 0$ for $n>1$

$\dfrac{1}{a_n} = \dfrac{1}{n^2-\frac{4}{n}+2^n} < \dfrac{1}{n^2}$ for $n>1$

Because $0 \le \dfrac{1}{a_n} \le \dfrac{1}{n^2}$ and $\left\{\dfrac{1}{n^2}\right\}$ is basic null, $\left\{\dfrac{1}{a_n}\right\}$ is null by the Comparison Test.

These show that $a_n \to \infty$ by theReciprocal Rule.

(b) $a_n = 2-3(n!)+4n \Rightarrow -a_n = 3(n!)-2-4n \Rightarrow -a_n > 0$ for $n>2$

$\dfrac{1}{-a_n} = \dfrac{1}{3(n!)-2-4n} = \dfrac{\frac{1}{n!}}{3-2\left(\frac{1}{n!}\right)-4\left(\frac{n}{n!}\right)}$. $\left\{\frac{1}{n!}\right\}$ and $\left\{\frac{1}{(n-1)!}\right\}$ are null sequences.

By the Combination Rules, $\left\{-\dfrac{1}{a_n}\right\}$ is null.

By the Reciprocal Rule, $\{-a_n\}\to\infty$ and by definition, $\{a_n\}\to-\infty$.

4 (a) $a_n = \dfrac{1-(-1)^n}{1-2^{-n}}$. $\{(2^{-n})\} = \left\{\left(\frac{1}{2}\right)^n\right\}$ is a basic null sequence.

$(-1)^{2k} = 1$ and $(-1)^{2k+1} = -1$.

Using the Combination Rules,

$\lim\limits_{k\to\infty} a_{2k} = \dfrac{1-1}{1-0} = 0$ and $\lim\limits_{k\to\infty} a_{2k+1} = \dfrac{1+1}{1-0} = 2$.

The limits of these two subsequences are different and by the First Subsequence Rule $\{a_n\}$ diverges.

(b) $a_n = \dfrac{3n^2+(-1)^{n+1}n!}{2n+4(n!)} = \dfrac{3\left(\frac{n^2}{n!}\right)+(-1)^{n+1}}{2\left(\frac{n}{n!}\right)+4}$. $\left\{\frac{n^2}{n!}\right\}$ and $\left\{\frac{n}{n!}\right\}$ are basic null sequences.

$(-1)^{2k+1} = -1$ and $(-1)^{(2k+1)+1} = 1$.

By the Combination Rules,

$$\lim_{k\to\infty} a_{2k} = \frac{3\times 0+1}{2\times 0+4} = -\tfrac{1}{4} \text{ and } \lim_{k\to\infty} a_{2k+1} = \frac{3\times 0-1}{2\times 0+4} = \tfrac{1}{4}.$$

The subsequences have different limits and by the First Subsequence Rule $\{a_n\}$ diverges.

5 (a) $a_n = \dfrac{n!+2^n}{n^2+3(n!)+1} = \dfrac{1+\frac{2^n}{n!}}{\frac{n^2}{n!}+3+\frac{1}{n!}}$

$\left\{\frac{2^n}{n!}\right\}$, $\left\{\frac{n^2}{n!}\right\}$ and $\left\{\frac{1}{n!}\right\}$ are basic null sequences.

By theCombination Rules, $\lim\limits_{n\to\infty} a_n = \dfrac{1+0}{0+3+0} = \tfrac{1}{3}$.

(b) $a_n = \dfrac{n^2+4^n-4}{n^3+3^n-5} > 0$ and $\dfrac{1}{a_n} = \dfrac{n^3+3^n-5}{n^2+4^n-4} = \dfrac{\frac{n^3}{4^n}+\frac{3^n}{4^n}-5\left(\frac{1}{4^n}\right)}{\frac{n^2}{4^n}+1-4\left(\frac{1}{4^n}\right)}$.

$\left\{\frac{n^3}{4^n}\right\}$, $\left\{\left(\frac{3}{4}\right)^n\right\}$, $\left\{\frac{1}{4^n}\right\}$ and $\left\{\frac{n^2}{4^n}\right\}$ are basic null sequences.

By the Combination Rules, $\lim\limits_{n\to\infty}\left(\dfrac{1}{a_n}\right) = \dfrac{0+0-5\times 0}{0+1-4\times 0} = 0$.

By the Reciprocal Rule $\{a_n\}\to\infty$ and so diverges.

(c) $a_n = \dfrac{(-1)^n n^3}{4n^3+n+1} = \dfrac{(-1)^n}{4+\frac{1}{n^2}+\frac{1}{n^3}}$. $\left\{\frac{1}{n^2}\right\}$ and $\left\{\frac{1}{n^3}\right\}$ are basic null sequences.

$(-1)^{2k} = 1$ and $(-1)^{2k+1} = -1$.

By the Combination Rules,

$$\lim_{k\to\infty} a_{2k} = \frac{1}{4+0+0} = \tfrac{1}{4} \text{ and } \lim_{k\to\infty} a_{2k+1} = \frac{-1}{4+0+0} = -\tfrac{1}{4}.$$

By the First Subsequence Rule, $\{a_n\}$ diverges.

(d) $a_n = \dfrac{3^n+5n^2-3}{4^n+3n+1} = \dfrac{\frac{3^n}{4^n}+5\left(\frac{n^2}{4^n}\right)-3\left(\frac{1}{4^n}\right)}{1+3\left(\frac{n}{4^n}\right)+\frac{1}{4^n}}$. $\left\{\left(\frac{3}{4}\right)^n\right\}$, $\left\{\frac{n^2}{4^n}\right\}$, $\left\{\frac{n}{4^n}\right\}$ are all basic null sequences.

By the Combination Rules, $\lim\limits_{n\to\infty} a_n = \dfrac{0+5\times 0-3\times 0}{1+3\times 0+0} = 0$.

(e) $a_n = \dfrac{n(-1)^n+2}{3n+3} = \dfrac{(-1)^n+2\left(\frac{1}{n}\right)}{3+3\left(\frac{1}{n}\right)}$. $\left\{\frac{1}{n}\right\}$ is a basic null sequence.

$(-1)^{2k} = 1$ and $(-1)^{2k+1} = -1$.

By the Combination Rules,

$$\lim_{k\to\infty} a_{2k} = \frac{1+2\times 0}{3+3\times 0} = \tfrac{1}{3} \text{ and } \lim_{k\to\infty} a_{2k+1} = \frac{-1+2\times 0}{3+3\times 0} = -\tfrac{1}{3}.$$

By the First Subsequence Rule, $\{a_n\}$ diverges.

(f)

$a_n = \dfrac{n!+1}{1+n} > 0$ for all n. $\dfrac{1}{a_n} = \dfrac{1+n}{n!+1} = \dfrac{\frac{1}{n!}+\frac{n}{n!}}{1+\frac{1}{n!}}$. $\left\{\frac{1}{n!}\right\}$ and $\left\{\frac{n}{n!}\right\}$ are basic null sequences.

By the Combination Rules, $\lim\limits_{n\to\infty} \dfrac{1}{a_n} = \dfrac{0+0}{1+0} = 0$.

Now by the Reciprocal Rule, $a_n \to \infty$ as $n \to \infty$. The sequence diverges.