M337/C

Third Level Course Examination 2003
Complex Analysis

Tuesday 14 October 2003 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I (64% of the marks) you should attempt as many questions as you can.

In Part II (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Put all your used answer books together with your signed desk record on top. Fix them all together using the fastener provided.

The use of calculators is **NOT** permitted in this examination.

PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

Give the Cartesian form of the following complex numbers, simplifying your answer as far as possible.

(a) $\dfrac{1}{(1-i)^4}$ [2]

(b) The square roots of i [2]

(c) $\operatorname{Log}(-2)$ [2]

(d) $(-2)^i$ [2]

Question 2

Let

$$A = \{z : 1 < |z - i| < 2\} \quad \text{and} \quad B = \{z : \pi/4 \le \operatorname{Arg} z \le 3\pi/4\}.$$

(a) Make separate sketches of the sets A, B and $C = A - B$. [3]

(b) Write down which of the sets A, B and C, if any, is

(i) a region;

(ii) a simply-connected region;

(iii) neither open nor closed. [4]

(c) Using set notation, give an example of a set which is closed but not bounded. [1]

Question 3

In this question Γ is the line segment from 1 to i.

(a) (i) Determine the standard parametrization for Γ.

(ii) Evaluate

$$\int_\Gamma \operatorname{Im} z \, dz.$$ [3]

(b) Determine an upper estimate for the modulus of

$$\int_\Gamma \frac{\cosh z}{4 + z^2} \, dz.$$ [5]

Question 4

Evaluate the following integrals, in which $C = \{z : |z| = 1\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C \frac{1}{z^3} \, dz$ [3]

(b) $\displaystyle\int_C \frac{\cos(z - \pi)}{z^3} \, dz$ [3]

(c) $\displaystyle\int_C \frac{\cos z}{(z - \pi)^3} \, dz$ [2]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{z^2+1}{z(2z+1)(z+2)} \qquad [3]$$

at all its poles.

(b) Hence evaluate the real integral

$$\int_0^{2\pi} \frac{\cos t}{5+4\cos t}\, dt. \qquad [5]$$

Question 6

(a) Evaluate the following expressions where Γ is the gamma function.

(i) $\Gamma(5)$

(ii) $\Gamma(5/2)$

(iii) $\dfrac{\Gamma(i+1)}{\Gamma(i-1)}$ [4]

(b) Prove that the series

$$\sum_{n=1}^{\infty} \frac{e^z}{n^2}$$

converges uniformly on the set $E = \{z : 0 \leq \operatorname{Re} z \leq 1\}$. [4]

Question 7

Let $q(z) = 1/\overline{z}$ be a velocity function on $\mathbb{C} - \{0\}$.

(a) Explain why q represents a model fluid flow on $\mathbb{C} - \{0\}$. [1]

(b) Determine a stream function for this flow. Hence find the equation of the streamline through the point $-1+i$, and sketch this streamline, indicating the direction of flow. [5]

(c) Evaluate the flux of q across the unit circle $\{z : |z| = 1\}$. [2]

Question 8

(a) Prove that the iteration sequence

$$z_{n+1} = z_n^2 + 4z_n + 3, \qquad n = 0, 1, 2, \ldots,$$

with $z_0 = -2$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + 1, \qquad n = 0, 1, 2, \ldots,$$

with $w_0 = 0$. [2]

(b) Find the fixed points of $P_1(z) = z^2 + 1$ and determine their nature. [3]

(c) By considering the sequence $\{P_1^n(0)\}$, show that 1 does not belong to the Mandelbrot set. Hence, using the Fatou-Julia Theorem, determine whether or not 0 is in the keep set K_1. [3]

PART II

(i) You should attempt no more than **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Show that the function $f(z) = \overline{z}$ is not differentiable at 0 by using the following methods:

(i) directly from the definition of the derivative as a limit;

(ii) by using the Cauchy-Riemann theorem. [*7*]

(b) Let g be the function $g(z) = 3iz^2 - 4$.

(i) Show that g is conformal on $\mathbb{C} - \{0\}$.

(ii) Describe the effect of g on a small disc centred at 2.

(iii) Γ_1 and Γ_2 are the smooth paths meeting at 0 and 2 given by

$$\Gamma_1 : \gamma_1(t) = 2t \quad (t \in [0, 1]),$$
$$\Gamma_2 : \gamma_2(t) = 1 + e^{it} \quad (t \in [0, \pi]).$$

Sketch these paths, clearly indicating their directions.

(iv) Using part (b)(ii), or otherwise, sketch the directions of $g(\Gamma_1)$ and $g(\Gamma_2)$ at $g(2)$.

(v) Show that g is not conformal at 0. [*11*]

Question 10

(a) Let f be the function

$$f(z) = \frac{\sin z}{z(z - i)^3}.$$

Write down the singularities of f and determine their nature. [*5*]

(b) (i) Write down the Laurent series about 0 for the function

$$g(z) = \exp(-1/z),$$

giving an expression for the general term of the series, and state its annulus of convergence.

(ii) Hence evaluate the integral

$$\int_C z^4 \exp(-1/z)\, dz,$$

where C is the unit circle $\{z : |z| = 1\}$. [*6*]

(c) Determine the first three non-zero terms of the Taylor series about 0 for the function

$$h(z) = \mathrm{Log}(\cosh z).$$

Hence determine the first three non-zero terms of the Taylor series about 0 for tanh. [*7*]

Question 11

(a) Let f be the function

$$f(z) = z^6 + 9z^2 + 101.$$

Show that

(i) f has no zeros in $\{z : |z| \leq 2\}$;

(ii) f has no zeros on the real axis;

(iii) f has exactly 3 zeros in $\{z : |z| > 2, \operatorname{Im} z > 0\}$;

(iv) f has exactly one zero in each of the 4 regions bounded by the real and imaginary axes, *given* that f has exactly two zeros on the imaginary axis. [*10*]

(b) Evaluate the improper real integral

$$\int_{-\infty}^{\infty} \frac{\sin t}{t(1+t^2)}\, dt.$$ [*8*]

Question 12

In this question g is the Möbius transformation given by

$$g(z) = \frac{z-1}{z+1}.$$

(a) (i) The points 1 and β are inverse points with respect to the extended imaginary axis. Write down the value of β, and the images of 1 and β under $\hat{g}$.

(ii) Deduce that the image of the extended imaginary axis under $\hat{g}$ is the unit circle $\{w : |w| = 1\}$.

(iii) Hence or otherwise show that the image of the region $\{z : \operatorname{Re} z > 0\}$ under g is $D = \{w : |w| < 1\}$. [*6*]

(b) (i) Sketch the regions

$$R = \{z : |z| < 1, \operatorname{Re} z < 0\} \quad \text{and} \quad R_1 = \{z_1 : \operatorname{Re} z_1 > 0, \operatorname{Im} z_1 > 0\}.$$

(ii) Determine a Möbius transformation f_1 which maps R onto R_1. (You must justify that f_1 maps the regions correctly.)

(iii) Using f_1 and g, defined above, and any intermediate conformal mapping needed, determine the rule of a conformal mapping f from R to $D = \{w : |w| < 1\}$. You do not need to simplify your answer or justify that it is conformal. [*10*]

(c) Write down an example of each of the following:

(i) a Möbius transformation which is conformal on $\mathbb{C}$;

(ii) a function which is conformal on $\mathbb{C}$ but is not a Möbius transformation. [*2*]

[END OF QUESTION PAPER]