Question 1

a $\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{(1-i)(1-i)}{2}\right)^3 = \left(\frac{-2i}{2}\right)^3 = -i^3 = i$

b $\exp(2+i\frac{\pi}{6}) = \exp(2)(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \frac{\sqrt{3}e^2}{2} + i\frac{e^2}{2}$

c Log $\left(\frac{1+i\sqrt{3}}{2}\right)$ = Log $(\exp(i\frac{\pi}{3})) = i\frac{\pi}{3}$

d $\left(\frac{1+i\sqrt{3}}{2}\right)^{3-i}$ = exp(Log $(3-i)(\exp(i\frac{\pi}{3})) = \exp(\frac{\pi}{3} + i\pi)$ = $-\exp(\frac{\pi}{3})$

Question 3

Section a

Γ has a standard parametrisation of $\gamma(t)=(1-t)i+t, t\in[0,1]$ So Re z = t, Im z = 1-t

Therefore (Re z)(Im z) = $t-t^2$

With this parametrisation, $\frac{dz}{dt}=-i+1$

Since γ is a smooth path and (Re z)(Im z) is continuous along the path Γ , we have that ,

$\int_0^1 (t-t^2)(1-i)\,dt=(1-i)(\frac{1}{2}-\frac{1}{3})=\frac{1-i}{6}$

Question 3
Section b

As $f(z)=\frac{z^2-1}{\bar{z}^2+1}$ is continuous on the circle, we can use the Estimation Theorem

The length of the circle C is 4π

$|z^2-1|\le|z^2|+|1|=2^2+1=5$

Using the Backwards form of the Triangle Inequality, $|\bar{z}^2+1|\ge||\bar{z}^2|-|-1||=|2^2-1|=3$

Therefore $M=|\frac{z^2+1}{\bar{z}^2-1}|\le\frac{5}{3}$ for $\{z:|z|=2\}$

Therefore by the Estimation Theorem, an upper estimate for the modulus of the integral is

$ML=4\pi(\frac{5}{3})=\frac{20\pi}{3}$

Question 4

Let $\boldsymbol{R} = \{z : |z| < 3\}$

Section a

$\boldsymbol{R}$ is a simply connected region and $\frac{\cos z}{z-\pi}$ is analytic on $\boldsymbol{R}$.
C is a closed contour in $\boldsymbol{R}$

So by Cauchy's Theorem , $\int_C \frac{\cos z}{z-\pi} dz = 0$

Section b

$\boldsymbol{R}$ is a simply connected region and $f(z) = \cos z$ is analytic on $\boldsymbol{R}$, and $\frac{\pi}{3}$ is inside C and C is a simple closed contour in $\boldsymbol{R}$

So by Cauchy's Theorem , $\int_C \frac{\cos z}{z - \frac{\pi}{3}} dz = 2\pi i\, f(\frac{\pi}{3}) = 2\pi i (\frac{1}{2}) = \pi i$

Section c

$\boldsymbol{R}$ is a simply connected region and $f(z) = \cos z$ is analytic on $\boldsymbol{R}$, and $\frac{\pi}{2}$ is inside C and C is a simple closed contour in $\boldsymbol{R}$

$f'''(z) = \sin z$

So using Cauchy's n'th Derivative Formula , $\int_C \frac{\cos z}{(z - \frac{\pi}{2})^4} dz = \frac{2\pi i}{3!} \sin(\frac{\pi}{2}) = \frac{\pi}{3} i$

Question 5

Section a

$$f(z)=\frac{z+1}{z(z^2+4)}$$

$f(z)$ has simple poles at 0, 2i and -2i

Res (f,0) = the limit as $z\to 0,(z-0)f(z)=\frac{0+1}{0+4}=\frac{1}{4}$

Res (f,2i) = the limit as $z\to 2i,(z-2i)f(z)=\frac{2i+1}{2i(2i+2\mathrm{i})}=\frac{1+2\mathrm{i}}{-8}=-\frac{1+2i}{8}$

Res (f,-2i) = the limit as $z\to -2i,(z+2i)f(z)=\frac{-2i+1}{-2i(-2i-2i)}=\frac{2i-1}{8}$

Section b

Let $p(t)=t+1$ Let $q(t)=t(t^2+4)$

p and q are polynomial functions such that the degree of q exceeds that of p by at least 2 and the pole of p/q on the real axis is simple.

Therefore, $\int_{-\infty}^{\infty}(\frac{p(t)}{q(t)})dt=2\pi i S+\pi T$

where S is the sum of the residues of p/q at the poles in the upper half plane and
where T is the sum of the residues of p/q at the poles on the real axis.

As S = Res (p/q,2i) and T = Res (p/q,0) ,

$$\int_{-\infty}^{\infty}(\frac{p(t)}{q(t)})dt=-2\pi i(\frac{1+2i}{8})+\pi i(\frac{1}{4})=\frac{\pi}{2}$$

Question 6

$f(z)=z^7+3z^5-1$

The function is analytic on the simply connected region $\boldsymbol{R}=\mathbb{C}$ so Rouche's Theorem can be used.

Section a

Let $g_1(z)=z^7$

Using the Triangle Inequality, for $C_1=\{z:|z|=2\}$

$|f(z)-g_1(z)|=|3z^5-1|\leq|3z^5|+|-1|=96+1=97<2^7=|g_1(z)|$

Since C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouche's Theorem , f has the same number of zeros as g_1 inside the contour C_1 . Therefore f has 7 zeros inside C_1

Let $g_2(z)=3z^5$

Using the Triangle Inequality, for $C_2=\{z:|z|=1\}$

$|f(z)-g_2(z)|=|z^7-1|\leq|z^7|+|-1|=1+1=2<3(1^5)=|g_2(z)|$

Since C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouche's Theorem , f has the same number of zeros as g_2 inside the contour C_2 . Therefore f has 5 zeros inside C_2

Therefore , f has 7 – 5 = 2 solutions in the set $\{z:1\leq|z|<2\}$ Therefore we have to find if there are any solutions on the contour C_2

We have that , on the contour C_2 , $|z^7+3z^5-1|\geq|3z^5|-|z^7|-|-1|=3-1-1=1>0$

As $f(z)$ is non-zero on C_2 , then there are exactly 2 solutions of f(z) = 0 in the set $\{z:1<|z|<2\}$

Question 6

Section b

$f(z)$ is a polynomial with real coefficients. So if z is a solution of $f(z)=0$ then $\bar{z}$ will also be a solution.

We have that $f(0)=-1$ and $f(1)=3$ So there is at least one real solution of $f(z)=0$ in the interval $(0,1)$

We also have that $f'(z)=7z^6+15z^4$ If z is real, then $f'(z)>0$

So $f(z)$ is a strictly increasing function for real z, so there can only be one real solution of f(z) = 0.

Therefore there are 6 solutions of f(z) = 0 that do not lie on the real axis.

So there must be 3 solutions of f(z) = 0 in the upper half-plane, and the complex conjugates of these three solutions will also be solutions of f(z) = 0 and lie in the lower half-plane.

Q8

Part a

$z_{n+1}=2z_n^2-4z_n+2\,,\,z_0=1$

Multiplying by 2, we get that $2z_{n+1}=4z_n^2-8z_n+4$

Completing the square $2z_{n+1}=(2z_n-2)^2\Rightarrow 2z_{n+1}-2=(2z_n-2)^2-2$

So this is conjugate to the iteration sequence $w_{n+1}=w_n-2\,,\,n=0,1,2,3....$ with $w_{n+1}=2z_n-2$ and $w_0=2z_0-2=2-2=0$

Part b

What are the fixed points of $P_{-2}(z)=z^2-2$ and what are their nature?

We have that $P'_{-2}(z)=2z$

For a fixed point, $z=z^2-2$

So $z^2-z-2=0$ So the 2 fixed points are at $z=-1\,,\,z=2$

At the point $z=-1\,,|P'_{-2}(z)|=2(1)=2$ So this fixed point is a repelling point as the modulus is greater than 1

At the point $z=2\,,|P'_{-2}(z)|=2(2)=4$ So this fixed point is a repelling point as the modulus is greater than 1

Part c

Let $z=\frac{1}{2}+i$. Then $P^2(z)=z^2+z=-\frac{3}{4}+i+\frac{1}{2}+i=-\frac{1}{4}+2i$

But $|P^2(z)|=|-\frac{1}{4}+2i|=2+\frac{1}{16}>2$

Therefore, $\frac{1}{2}+i$ does not lie in the Mandelbrot set.

Question 9

Let $z=x+iy$ Then $f(z)=2e^{ix}-x+iy=2\cos x+2i\sin x-x+iy=(2\cos x-x)+i(2\sin x+y)$

Therefore , we can write $f(x+iy)$ in the form $u(x,y)+iv(x,y)$ where $u(x,y)=2\cos x-x$ and $v(x,y)=2\sin x+y$

So we have that $u_x=-2\sin x-1, u_y=0, v_x=2\cos x, v_y=1$

The converse of the Cauchy-Riemann Theorem says that f is not differentiable at the set of points G for which $u_x\neq v_y$ or for which $u_y\neq -v_x$

As the partial derivatives are continuous the Cauchy-Riemann Theorem says that f is differentiable at the set of points S for which $u_x=v_y$ and for which $u_y=v_x$

So $-2\sin x-1=1\Rightarrow \sin x=-1 x=\frac{3\pi}{2}+2n\pi$ where n is an integer.

We also have that $2\cos x=0\Rightarrow x=\frac{(2m+1)\pi}{2}$ where m is an integer.

As both conditions must be true , the set S of points at which f is differentiable is S = { z: Re ($z=\frac{3\pi}{2}+2n\pi, n\in\mathbb{Z}$ }

At those points , $u_x=1, u_y=0, v_x=0, v_y=1$, so f' is constant on S

Question 11

$$f(z)=\frac{z}{1-\cos z}$$

This is defined everywhere in the complex plane where $1-\cos z\neq 0 \Rightarrow z\neq\frac{(2n+1)\pi}{2}, n\in\mathbb{N}$

So the domain A of f is $\{\ z\in\mathbb{C}: z\neq\frac{(2n+1)\pi}{2}\ \},\quad n\in\mathbb{N}$

We need to derive the Laurent series about 0 for f

From Handbook page 35, the Taylor series of $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}....$

Thus, $1-\cos z=\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}....$ and $\frac{z}{1-\cos z}=\frac{z}{\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}....}=\frac{2}{z}(1-\frac{z^2}{12}+\frac{z^4}{720})^{-1}$

Using the given expansion on page 25 of $(1-z)^{-1}$ with $z=\frac{z^2}{12}-\frac{z^4}{720}$ and ignoring all powers of z beyond z^4, we get that $\frac{z}{1-\cos z}=\frac{2}{z}(1+\frac{z^2}{12}-\frac{z^4}{720}+(\frac{z^2}{12}-\frac{z^4}{720})^{2.}..)$

So $\frac{z}{1-\cos z}=\frac{2}{z}+\frac{z}{6}-\frac{z^3}{360}+\frac{z^3}{72.}...=\frac{2}{z}+\frac{1}{6}z+\frac{1}{120}z^3$

This will be valid if $z\neq 0$

We wish to evaluate the integral of $f(z)$ around the unit circle.

The residue of $f(z)$ at 0 is 2. As this is the only singularity inside the unit circle, we can use Cauchy's Theorem to evaluate the integral as $2\pi i\times 2=4\pi i$

Question 12

Section a

There is a formula for generating Moebius transformations given 3 points that map to 0,1 and infinity.

Using the formula :- $\hat{f}_1(z)=(\frac{z-(-1-i)}{z-(1+i)})(\frac{0-(-1-i)}{0-(1+i)})$

So $\hat{f}_1(z)=\frac{z+1+i}{z-1-i}(-1)=-(\frac{z+1+i}{z-1-i})$

Section b

If $z_1=x+iy$, S is the region below a dotted line with equation y = -x

We have that $w=-(\frac{z+1+i}{z-1-i})=-\frac{(x+1)+i(y+1)}{(x-1)+i(y-1)}$

So $|w|^2=\frac{(x+1)^2+(y+1)^2}{(x-1)^2+(y-1)^2}$ So $|w|^2=|\frac{x^2+y^2+2+2(x+y)}{x^2+y^2+2-2(x+y)}|=|\frac{1+\frac{2(x+y)}{x^2+y^2+2}}{1-\frac{2(x+y)}{x^2+y^2+2}}|$

Now, $x^2+y^2+2>0$, so $x+y<0\Leftrightarrow|\frac{x^2+y^2+2+2(x+y)}{x^2+y^2+2-2(x+y)}|<1$

So if and only if the real part of z_1 and the imaginary part of z_1 add up to less than zero, is the point z_1 transformed into a w where $|w|<1$ so $\hat{f}_1(S)=T$

Section iii.

Let the function g be defined by $g(z)=e^z$

Then $g(R)=S$ as S is the sector defined by $\{z_1:\frac{3\pi}{4}<Arg(z_1)<\frac{7\pi}{4}\}$

So the required conformal mapping from R onto T is $w=f(z)=-(\frac{e^z+1+i}{e^z-1-i})$

Question 12

Section iv

We have that $w=-(\frac{e^{z}+1+i}{e^{z}-1-i})$

So $w(e^{z}-1-i)=-(e^{z}+1+i)\Rightarrow we^{z}-w-wi+e^{z}+1+i=0$

Therefore , $(w+1)e^{z}=(w-1)+(w-1)i\Rightarrow e^{z}=\frac{(w-1)}{(w+1)}(1+i)$

So the inverse function has the rule $f^{-1}(z)=\log(\frac{(p-1)}{(p+1)}(1+i))$

The point p that f maps to 0 is $\log(-1-i)=\frac{1}{2}\log_{e}2+\frac{3\pi}{4}i$

Question – is p actually in R, as the imaginary part equals $\frac{3\pi}{4}$ and it should be less than that, not equal to that.