Question 1

15/15

$$T(x,y) = 2x^2 - 2xy + 2x - 3y^2 + 7y$$

a) The **grad** T is

$$\begin{aligned} \mathbf{grad}\, T &= \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} \\ \implies &= (4x - 2y + 2)\mathbf{i} + (-2x - 6y + 7)\mathbf{j} \end{aligned}$$

so at point $(-1,1)$

$$\begin{aligned} \mathbf{grad}\, T(-1,1) &= (4(-1) - 2(1) + 2)\mathbf{i} + (-2(-1) - 6(1) + 7)\mathbf{j} \\ \implies &= (-4 - 2 + 2)\mathbf{i} + (2 - 6 + 7)\mathbf{j} \\ \implies &= -4\mathbf{i} + 3\mathbf{j} \end{aligned}$$

3/3

b) The derivative of a scalar field f in a direction specified by a unit vector $\hat{\mathbf{d}}$ is given by $\mathbf{grad}\, f \cdot \hat{\mathbf{d}}$. We are given that $\mathbf{d} = \mathbf{i} + \mathbf{j}$, so

$$\begin{aligned} \hat{\mathbf{d}} &= \frac{\mathbf{d}}{|\mathbf{d}|} \\ \implies &= \frac{\mathbf{i}+\mathbf{j}}{\sqrt{1^2 + 1^2}} \end{aligned}$$

Hence,

$$\hat{\mathbf{d}} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$$

So, the derivative of T at the point $(-1,1)$ is

$$\begin{aligned} \mathbf{grad}\, T(-1,1) \cdot \hat{\mathbf{d}} &= (-4\mathbf{i} + 3\mathbf{j}) \cdot \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} \\ \implies &= \left(\frac{-4}{\sqrt{2}} + \frac{3}{\sqrt{2}}\right) \\ &= \frac{-1}{\sqrt{2}} \end{aligned}$$

3/3

c) The maximum rate of rate of change in temperature at the point $(-1,1)$ is given by

$$\begin{aligned} |\mathbf{grad}\, T(-1,1)| &= |-3\mathbf{i} + 4\mathbf{j}| \\ \implies &= \sqrt{(-3)^2 + 4^2} \\ \implies &= 5 \end{aligned}$$

The direction is given by

$$\frac{\mathbf{grad}\, T(-1,1)}{|\mathbf{grad}\, T(-1,1)|} = \frac{-3\mathbf{i} + 4\mathbf{j}}{5}$$

2/2

d) Substituting $x = -1$ and $y = 1$ into

$$T(x,y) = 2x^2 - 2xy + 2x - 3y^2 + 7y$$

gives

$$t(-1,1) = 2(-1)^2 - 2(-1)(1) + 2(1) - 3(1) + 7(1)$$

Hence,

$$T(-1,1) = 6$$

Therefore, the point $(-1,1)$ is on the contour $T = 6$.

1/1

Now we need to find the equation of the tangent of the contour at $T = 6$ at $(-1,1)$.

We know that the $\mathbf{grad}\, T(-1,1) = -4\mathbf{i} + 3\mathbf{j}$, which is perpendicular to the tangent line to the contour at $(-1,1)$.

Note, the derivative of f does not change along the contour curve, now let $\mathbf{e}$, be a vector tangent to the contour curve at $(-1,1)$, then we have,

$$\begin{aligned} \mathbf{grad}\, T(-1,1)\cdot\mathbf{e} &= 0 \\ \implies (-4\mathbf{i}+3\mathbf{j})\cdot\mathbf{e} &= 0 \\ \implies (-4\mathbf{i}+3\mathbf{j})\cdot(e_1\mathbf{i}+e_2\mathbf{j}) &= 0 \\ \implies -4e_1+3e_2 &= 0 \end{aligned}$$

6/6

therefore, the vector $\mathbf{e} = 3\mathbf{i} + 4\mathbf{j}$ is parallel to the line forming the tangent.

The equation of the tangent is of the form $y = mx + c$. From $\mathbf{e}$ we know the gradient of the tangent is $4/3$, and as the point passes through $(-1,1)$ then $c = 7/3$. Hence, the equation of the tangent to the contour $T = 6$ at $(-1,1)$ is

$$y = \frac{4}{3}x + \frac{7}{3}$$

□

14/18

Question 2

$$h(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} + z$$

a) To convert this scalar field into cylindrical coordinates we can use equations (13) and (14) from Unit 15, i.e.

$$\rho = \sqrt{x^2+y^2}$$

and

$$z = z$$

hence

$$h(\rho,\phi,z) = \frac{1}{\sqrt{\rho^2+z^2}} + z$$

2/2

b) Using equation (24) from Unit 15

$$\mathbf{grad}\, h = \mathbf{e}_p\frac{\partial h}{\partial \rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial h}{\partial \phi} + \mathbf{e}_z\frac{\partial h}{\partial z}$$

Note that

$$\frac{\partial h}{\partial \phi} = 0$$

Now,

$$\frac{\partial h}{\partial \rho} = -\frac{\rho}{(\rho^2 + z^2)^{3/2}}$$

and,

$$\frac{\partial h}{\partial z} = 1 - \frac{z}{(\rho^2 + z^2)^{3/2}}$$

Hence, in cylindrical coordinates

$$\mathbf{grad}\, h = -\frac{\rho}{(\rho^2 + z^2)^{3/2}}\mathbf{e}_\rho + \left(1 - \frac{z}{(\rho^2 + z^2)^{3/2}}\right)\mathbf{e}_z$$

5/5

c) Using equations (28) and (29) from Unit 15 then in spherical coordinates the scalar field,

$$z = r\cos\theta$$

and

$$r = \sqrt{x^2 + y^2 + z^2}$$

hence

$$h(r, \theta, \phi) = \frac{1}{r} + r\cos\theta$$

2/2

d) The gradient function in spherical coordinates of a scalar field is given by equation (36) from Unit 15

$$\mathbf{grad}\, h = \mathbf{e}_r \frac{\partial h}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial h}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial h}{\partial \phi}$$

Note that

$$\frac{\partial h}{\partial \phi} = 0$$

Now,

$$\frac{\partial h}{\partial r} = -\frac{1}{r^2} + \cos\theta$$

and

$$\frac{\partial h}{\partial \theta} = -r\sin\theta$$

hence

$$\mathbf{grad}\, h = \left(\cos\theta - \frac{1}{r^2}\right)\mathbf{e}_r - \mathbf{e}_\theta \sin\theta$$

5/5

The $|\mathbf{grad}\, h|$ is the same as the magnitude of the e_r component, hence

$$|\mathbf{grad}\, h| = \left|\cos\theta - \frac{1}{r^2}\right|$$

0/4

$$|\text{grad}\, h| = \sqrt{\text{grad}\, h \cdot \text{grad}\, h} = \sqrt{\left(\cos\theta - \frac{1}{r^2}\right)^2 + \left(-\frac{1}{r}\right)^2} = \sqrt{1 - \frac{2\cos\theta}{r^2} + \frac{1}{r^4}}$$

□

$\frac{7}{8}$

Question 3

Evaluate the scalar line integral of the vector field

$$\mathbf{F}(x,y,z) = 2y\mathbf{i} - 2x\mathbf{j} + z^2\mathbf{k}$$

around the closed circular path C of radius 1 centred at the point $(1,0,2)$ defined by the parametric equations

$$x = \cos t + 1, \quad y = \sin t, \quad z = 2, \quad \text{for } 0 \le t \le 2\pi$$

Using equation (27) of Unit 16

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} dt$$

Substituting gives,

$$\int_0^{2\pi} (2y\mathbf{i} - 2x\mathbf{j} + z^2\mathbf{k}) \cdot \frac{d\mathbf{r}}{dt}\, dt$$
$$\implies \int_0^{2\pi} (2y\mathbf{i} - 2x\mathbf{j} + z^2\mathbf{k}) \cdot (-\mathbf{i}\sin t + \mathbf{j}\cos t)\, dt$$
$$\implies \int_0^{2\pi} (2\mathbf{i}\sin t - (2\cos t + 1)\mathbf{j} + 2^2\mathbf{k}) \cdot (-\mathbf{i}\sin t + \mathbf{j}\cos t)\, dt$$
$$\implies -2\int_0^{2\pi} \sin^2 t + (\cos^2 t - \cancel{\tfrac{1}{2}}\cos t)\, dt$$

just cos t

$\frac{5}{6}$

$$\implies -2\int_0^{2\pi} 1 - \cancel{\tfrac{1}{2}}\cos t)dt$$
$$\implies -2\left[t - \cancel{\tfrac{1}{2}}\sin t\right]_0^{2\pi}$$
$$= -4\pi$$

at sin 2π = sin 0 = 0 you do get correct answer but !!

As the curve is closed and,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \ne 0$$

then vector field is not conservative

$\frac{2}{2}$

□

Question 4

$$F(x,y,z) = (2xz^2 - 2yz)\mathbf{i} + (-2xz + 1)\mathbf{j} + (2x^2z - 2xy)\mathbf{k}$$

14/14

a) Using equation (14) from Unit 16

$$\mathbf{Curl\,F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

We have

$$F_1 = 2xz^2 - 2yz$$
$$F_2 = -2xz + 1$$
$$F_3 = 2x^2z - 2xy$$

so,

$$\frac{\partial F_1}{\partial y} = -2z\ ,\ \frac{\partial F_1}{\partial z} = 4xz - 2y$$
$$\frac{\partial F_2}{\partial x} = -2z\ ,\ \frac{\partial F_2}{\partial z} = -2x$$
$$\frac{\partial F_3}{\partial x} = -4xz - 2y\ ,\ \frac{\partial F_3}{\partial y} = -2x$$

5/5

Substituting gives

$$\mathbf{Curl}\quad \mathbf{F} = (-2x - (-2x))\,\mathbf{i} + (4xz - 2y - (-4xz - 2y))\,\mathbf{j} + (-2z - (-2z))\,\mathbf{k}$$
$$\implies = 0$$

As the $\mathbf{Curl\,F} = 0$ then $\mathbf{F}$ is conservative.

b) Using Procedure 1 on page 94, then

1. Using the parameterization of C from $(0,0,0)$ to the general point (a,b,c) by

$$\mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} \qquad (0 \le t \le 1)$$

So,

$$\frac{d\mathbf{r}}{dt} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

then

$$\int_C F\cdot d\mathbf{r} = \int_0^1 \big((2atc^2t^2 - 2btct)\mathbf{i} + (-2atct + 1)\mathbf{j} + (2a^2t^2ct - 2atbt)\mathbf{k}\big)\cdot\frac{d\mathbf{r}}{dt}dt$$
$$\implies = \int_0^1 \big((2a^2c^2t^3 - 2abct^2) + (-2abct^2 + b) + (2a^2c^2t^3 - 2abct^2)\big)\,dt$$
$$\implies = \int_0^1 \big(4a^2c^2t^3 - 6abct^2 + b\big)\,dt$$
$$\implies = 4a^2c^2\left[\frac{1}{4}t^4\right]_0^1 - 6abc\left[\frac{1}{3}t^3\right]_0^1 + b[t]_0^1$$
$$\implies = a^2c^2 - 2abc + b$$
$$\implies = U(0,0,0) - U(a,b,c)$$

7/7

Hence the potential function $U(x,y,z)$ is

$$U(x,y,z) = -x^2z^2 + 2xyz - y$$

c) From part (b) we know that

$$U(x,y,z) = -x^2z^2 + 2xyz - y$$

and from Eq (7) from Unit 15 that

$$\mathbf{grad}\ U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + + \frac{\partial U}{\partial z}\mathbf{k}$$

So, the $-\mathbf{grad}\ U$ is

$$-\mathbf{grad}\ U = (2xz^2 - 2yz)\mathbf{i} + (-2xy + 1)\mathbf{j} + (2x^2 + 2xy)\mathbf{k}$$

2/2

Hence

$$F = -\mathbf{grad}\ U$$

□

Question 5

3/3

$$\mathbf{F}(\rho, \phi, z) = \frac{z + \cos\phi}{\rho}\mathbf{e}_\rho + 2\rho\sin\phi\,\mathbf{e}_\phi + e^{-z}\mathbf{e}_z$$

From equation (2) of Unit 16 then

$$\text{div}\ \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

So from **F**

$$F_\rho = \frac{z + \cos\phi}{\rho}$$
$$F_\phi = 2\rho\sin\phi$$
$$F_z = e^{-z}$$

Now the partial derivatives are

$$\frac{\partial F_\rho}{\partial \rho} = -\frac{z + \cos\phi}{\rho^2}$$
$$\frac{\partial F_\phi}{\partial \phi} = 2\rho\cos\phi$$
$$\frac{\partial F_z}{\partial z} = -e^{-z}$$

Substituting gives

$$\text{div}\ \mathbf{F} = -\frac{z + \cos\phi}{\rho^2} + \frac{1}{\rho} \times \frac{z + \cos\phi}{\rho} + \frac{1}{\rho} \times 2\rho\cos\phi - e^{-z}$$

Hence

$$\text{div}\ \mathbf{F} = 2\cos\phi - e^{-z}$$

□

Question 6

5/5

The velocity field $\mathbf{v}$ is expressed in spherical coordinates as

$$\mathbf{v}(r,\theta,\phi) = r\sin\phi\sin(2\theta)\mathbf{e}_r + r\sin\phi\cos(2\theta)\mathbf{e}_\theta + r\cos\phi\cos\theta\mathbf{e}_\theta$$

From equation (17) of Unit 16, the curl of a vector field $F(r,\theta,\phi) = F_r e_r + F_\theta e_\theta + F_\phi e_\phi$ in spherical coordinates is

$$\begin{aligned}\nabla\times\mathbf{F} = &\left(\frac{1}{r}\frac{\partial F_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\phi} + \frac{\cot\theta}{r}F_\phi\right)\mathbf{e}_r\\ &+\left(-\frac{\partial F_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{1}{r}F_\phi\right)\mathbf{e}_\theta\\ &+\left(\frac{\partial F_\theta}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial\theta} + \frac{1}{r}F_\theta\right)\mathbf{e}_\phi\end{aligned}$$

Where

$$\begin{aligned}F_r &= r\sin\phi\sin(2\theta)\\ F_\theta &= r\sin\phi\cos(2\theta)\end{aligned}$$

and

$$F_\phi = r\cos\phi\cos\theta$$

So the partial derivatives are

$$\begin{aligned}\frac{\partial F_r}{\partial\theta} &= 2r\sin\phi\cos(2\theta) & \frac{\partial F_r}{\partial\phi} &= r\cos\phi\sin(2\theta)\\ \frac{\partial F_\theta}{\partial r} &= \sin\phi\cos(2\theta) & \frac{\partial F_\theta}{\partial\phi} &= r\cos\phi\cos(2\theta)\\ \frac{\partial F_\phi}{\partial r} &= \cos\phi\cos\theta & \frac{\partial F_\phi}{\partial\theta} &= -r\cos\phi\sin\theta\end{aligned}$$

Substituting gives

$$\begin{aligned}\nabla\times\mathbf{F} = &\left(-\frac{r\cos\phi\sin\theta}{r} - \frac{r\cos\phi\cos(2\theta)}{r\sin\theta} + \frac{\cot\theta}{r}r\cos\phi\cos\theta\right)\mathbf{e}_r\\ &+\left(-\cos\phi\cos\theta + \frac{r\cos\phi\sin(2\theta)}{r\sin\theta} - \frac{r\cos\phi\cos\theta}{r}\right)\mathbf{e}_\theta\\ &+\left(\sin\phi\cos(2\theta) - \frac{2r\sin\phi\cos(2\theta)}{r} + \frac{r\sin\phi\cos(2\theta)}{r}\right)\mathbf{e}_\phi\end{aligned}$$

Simplifying

$$\begin{aligned}&= \left(-\cos\phi\sin\theta - \frac{\cos\phi\cos(2\theta)}{\sin\theta} + \cot\theta\cos\phi\cos\theta\right)\mathbf{e}_r\\ &+\left(-\cos\phi\cos\theta + \frac{\cos\phi\sin(2\theta)}{\sin\theta} - \cos\phi\cos\theta\right)\mathbf{e}_\theta\\ &+(\sin\phi\cos(2\theta) - 2\sin\phi\cos(2\theta) + \sin\phi\cos(2\theta))\,\mathbf{e}_\phi\end{aligned}$$

Using the double angle formulae

$$
\begin{aligned}
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \text{ and, } \sin(2\theta) = 2\sin\theta\cos\theta \\
\implies &= \left(-\cos\phi\sin\theta - \frac{\cos\phi(\cos^2\theta - \sin^2)}{\sin\theta} + \frac{\cos\phi\cos^2\theta}{\sin\theta}\right)\mathbf{e}_r \\
&+ \left(-\cos\phi\cos\theta + \frac{2\cos\phi\sin\theta\cos\theta}{\sin\theta} - \cos\phi\cos\theta\right)\mathbf{e}_\theta \\
&+ (2\sin\phi\cos(2\theta) - 2\sin\phi\cos(2\theta)))\,\mathbf{e}_\phi \\
\\
\implies &= \left(-\cos\phi\sin\theta + \frac{\cos\phi\sin^2\theta}{\sin\theta}\right)\mathbf{e}_r \\
&+ (-\cos\phi\cos\theta + 2\cos\phi\ \cos\theta - \cos\phi\cos\theta)\,\mathbf{e}_\theta \\
&+ (2\sin\phi\cos(2\theta) - 2\sin\phi\cos(2\theta)))\,\mathbf{e}_\phi \\
\\
\implies &= (-\cos\phi\sin\theta + \cos\phi\sin\theta)\ \mathbf{e}_r \\
&+ (-2\cos\phi\cos\theta + 2\cos\phi\ \cos\theta)\,\mathbf{e}_\theta \\
&+ (2\sin\phi\cos(2\theta) - 2\sin\phi\cos(2\theta)))\,\mathbf{e}_\phi \\
\\
\implies &= \mathbf{0}
\end{aligned}
$$

Hence, because $\nabla \times \mathbf{v} = \mathbf{0}$ then $\mathbf{v}$ is conservative everywhere. □

Question 7

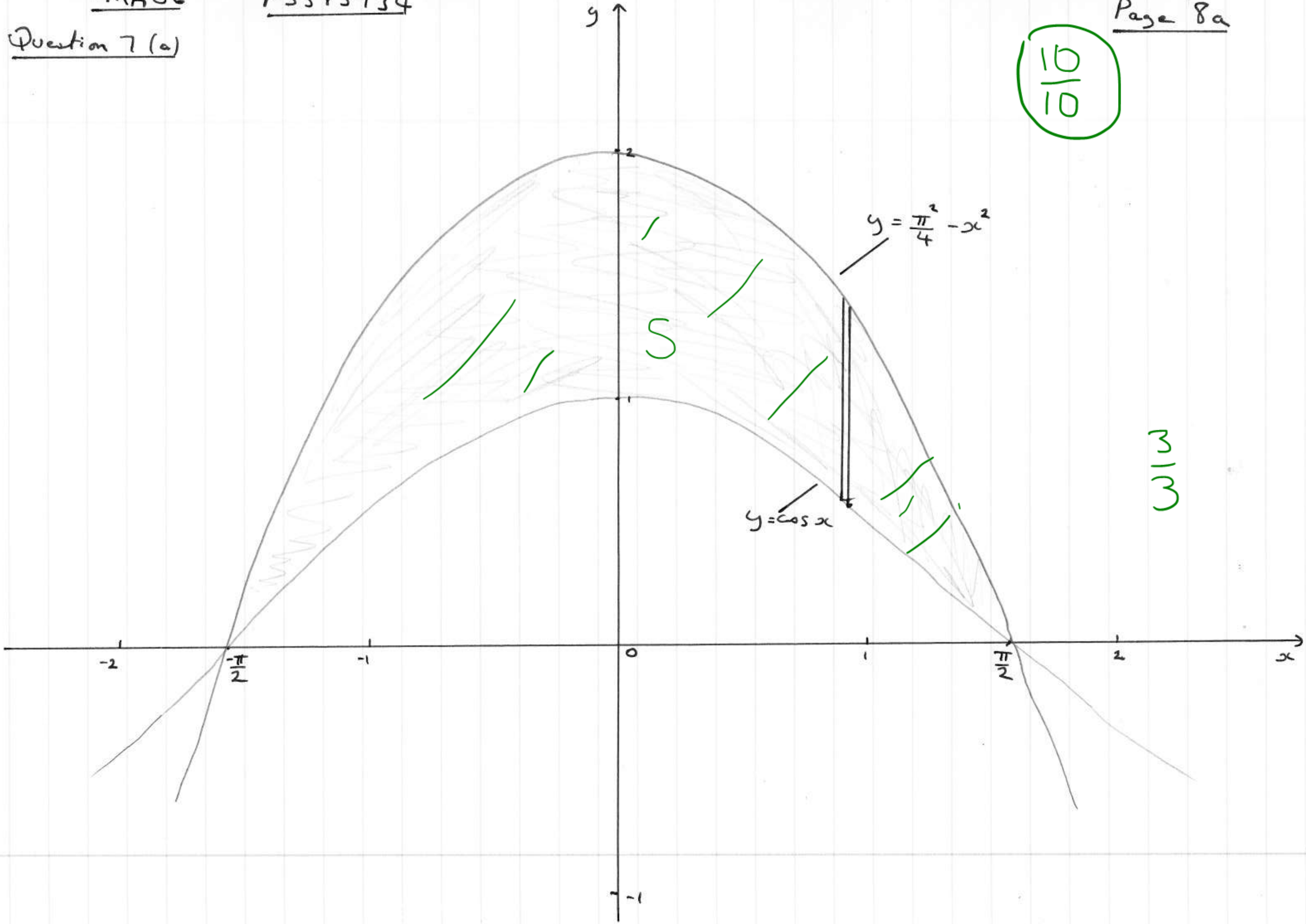
a) Please see the enclosed graph. Note, the y-limits are $y = \pi^2/4 - x^2$ and $y = \cos x$

TMA 06 P5585734

Page 8a

Question 7 (a)

10/10



3/3

b) Following procedure 1 of Unit 17 from Step 3.

Step 3. The x-limits are $x = \pi/2$ and $x = \pi/2$

Step 4.

3/3

$$\int_S f(x,y)dA = \int_{x=-\pi/2}^{x=\pi/2} \left(\int_{y=\cos x}^{y=\pi^2/4-x^2} 1dy \right) dx$$

$$\implies = \int_{x=-\pi/2}^{x=\pi/2} \Big[y \Big]_{y=\cos x}^{y=\pi^2/4-x^2} dx$$

$$\implies = \int_{x=-\pi/2}^{x=\pi/2} \frac{\pi^2}{4} - x^2 - \cos x \, dx$$

$$\implies = \left[\frac{\pi^2 x}{4} - \frac{1}{3}x^3 - \sin x \right]_{x=-\pi/2}^{x=\pi/2}$$

$$\implies = \left[\frac{\pi^3}{8} - \frac{\pi^3}{24} - 1 \right] - \left[-\frac{\pi^3}{8} - \frac{\pi^3}{24} - (-1) \right]$$

$$\implies = \left[\frac{3\pi^3}{24} - \frac{\pi^3}{24} - 1 \right] - \left[-\frac{3\pi^3}{24} - \frac{\pi^3}{24} - (-1) \right]$$

$$\implies = \left[\frac{\pi^3}{12} - 1 \right] - \left[-\frac{\pi^3}{12} + 1) \right]$$

$$\implies = \left[\frac{\pi^3}{6} - 2 \right]$$

$$\approx 3.17 \text{ to 2 d.p.}$$

4/4

exact answer please.

□

Question 8

8/9

The volume integral in cylindrical coordinates is given by equation (20) of Unit 17

$$M = \int_B f\, dV = \int_B f(\rho, \phi, z)\rho \, dz \, d\phi \, d\rho$$
$$= \int_{\rho=0}^{\rho=a} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=0}^{z=h} f \, \rho \, dz \right) d\phi \right) d\rho$$

The question says that the origin is in the centre of the cylinder so z limits are $-\frac{h}{2}$ to $\frac{h}{2}$

Where $f = Dz^2(1+\cos\phi)$ where D is a constant

$$\implies = \int_{\rho=0}^{\rho=a}\left(\int_{\phi=-\pi}^{\phi=\pi}\left(\int_{z=0}^{z=h} Dz^2(1+\cos\phi)\,\rho\,dz\right)d\phi\right)d\rho$$

$$\implies = \int_{\rho=0}^{\rho=a}\left(\int_{\phi=-\pi}^{\phi=\pi}\frac{1}{3}D\rho h^3(1+\cos\phi)\;d\phi\right)d\rho$$

$$\implies = \int_{\rho=0}^{\rho=a}\left[\frac{1}{3}D\rho h^3(\phi+\sin\phi)\right]_{\phi=-\pi}^{\phi=\pi}d\rho$$

$$\implies = \int_{\rho=0}^{\rho=a}\frac{2}{3}D\rho h^3\pi d\rho$$

$$\implies = \frac{1}{3}\left[D\rho^2h^3\pi\right]_{\rho=0}^{\rho=a}$$

(✓) (✓)

Hence, the mass of the cylinder is

$$\implies M = \frac{1}{3}Da^2h^3\pi$$

(✓)

will be $\frac{\pi}{12}Da^2h^3$

□

$\frac{13}{13}$

Question 9

a) The density of the shell is given as

$$\rho = A(x^2+y^2)z^2$$

Using equations (26)-(28) of Unit 15

$$\begin{array}{lll} x = r\sin\theta\cos\phi & \implies & x^2 = r^2\sin^2\theta\cos^2\phi \\ y = r\sin\theta\sin\phi & \implies & y^2 = r^2\sin^2\theta\sin^2\phi \\ z = r\cos\theta & \implies & z^2\cos^2\theta \end{array}$$

✓

Substituting gives

$$\rho = A\left(r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi\right)r^2\cos^2\theta$$

$$\implies = Ar^4\left(\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi\right)\cos^2\theta$$

$$\implies = Ar^4\left(\sin^2\theta\left(\cos^2\phi+\sin^2\phi\right)\right)\cos^2\theta$$

$$\implies = Ar^4\left(\sin^2\theta\right)\cos^2\theta$$

✓ ✓

hence as required

$$\rho = Ar^4\left(1-\cos^2\theta\right)\cos^2\theta$$

✓ $\frac{2}{2}$

b) Using equation (22) from Unit 17

$$M = \int_B f\,dv = \int_B f(r,\theta,\phi)r^2\sin\theta\,d\phi\,d\theta\,dr$$

✓

substituting gives

$$M = A\int_{r=a}^{r=2a}\left(\int_{\theta=0}^{\theta=\pi}\left(\int_{\phi=-\pi}^{\phi=\pi}\left(r^4(1-\cos^2\theta)\cos^2\theta\right)r^2\sin\theta\,d\phi\right)d\theta\right)dr$$

$$\implies = A\int_{r=a}^{r=2a}\left(\int_{\theta=0}^{\theta=\pi}\left[\phi\left(r^4(1-\cos^2\theta)\cos^2\theta\right)r^2\sin\theta\right]_{\phi=-\pi}^{\phi=\pi}d\theta\right)dr$$

$$\implies = 2\pi A\int_{r=a}^{r=2a}\left(r^6\int_{\theta=0}^{\theta=\pi}(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta\right)dr \qquad (1)$$

3/3

Using the substitution hint, let $u=\cos\theta$ then we can solve the indefinite integral

$$\int(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \qquad (2)$$

So differentiating u gives

$$du = -\sin\theta\,d\theta$$

substituting into (2) gives

$$-\int(1-u^2)u^2\,du$$

$$\implies \int u^4-u^2\,du$$

$$=\frac{1}{5}u^5-\frac{1}{3}u^3$$

As $u=\cos\theta$ then

$$\implies \int(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta = \frac{1}{5}\cos^5\theta-\frac{1}{3}\cos^3\theta$$

Using this result and the limits of θ then

$$\int_{\theta=0}^{\theta=\pi}(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta = \left[-\frac{1}{5}+\frac{1}{3}\right]-\left[\frac{1}{5}-\frac{1}{3}\right]$$

$$=\frac{4}{15}$$

Substituting this result into (1)

$$M = 2\pi A\int_{r=a}^{r=2a}\left(r^6\int_{\theta=0}^{\theta=\pi}(1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta\right)dr$$

$$\implies = 2\pi A\int_{r=a}^{r=2a}\frac{4}{15}r^6dr$$

$$\implies = \frac{8}{15}\pi A\left[\frac{1}{7}r^7\right]_{r=a}^{r=2a}$$

$$\implies = \frac{8}{105}\pi A\left[(2a)^7-a^7\right]$$

$$\implies = \frac{8}{105}\pi A\,[127a]$$

8/8

Hence, the mass of the shell is

$$M=\frac{1016}{105}\pi A a^7$$

□