MST209 2006

1.
$$xy=(2x+1)x+(2x+1)y$$
$$-(x+1)y=(2x+1)x$$
so **A.**

2. Linear equation so **B**.

3.
$$\mathbf{b}+\mathbf{c}=2\mathbf{j}+2\mathbf{k}$$
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{k}\times(2\mathbf{j}+2\mathbf{k})$$
so **A**.

4. **D**.

5. Left-hand spring is not stretched at all so has no energy stored. Right- hand spring is stretched by l_0 so **B**.

6. **B**.

7.
$$\begin{bmatrix}1 & 1 & -4\\ 1 & 3 & 1\\ -4 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}5\\0\\-5\end{bmatrix}=5\begin{bmatrix}1\\0\\-1\end{bmatrix}$$
so **D**.

8. **D**.

9.
$$U=\left(\frac{1}{10}+\frac{0.2}{0.6}+\frac{1}{20}\right)^{-1}$$
$$=\left(\frac{6+20+3}{60}\right)^{-1}$$
so **B**.

10. Energy has dimensions ML^2T^{-2} (Handbook Page 52) so if the rate of this has the dimensions of option **C**.

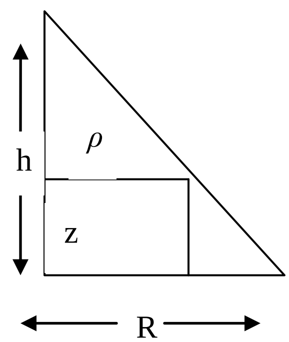
11. Using the formula from Handbook page 55 $\alpha=\dfrac{r}{2\sqrt{mk}}$ with r replaced by $3r$, m by $4m$ and k by $2k$ we get $\alpha=\dfrac{3r}{2\sqrt{8mk}}=\dfrac{3r}{4\sqrt{2mk}}$ so **D**.

12. Equation of motion is
$$M(-l\dot{\theta}^2\mathbf{e}_r+l\ddot{\theta}\mathbf{e}_\theta)$$
$$=-|\mathbf{T}|\mathbf{e}_r+Mg(\cos\theta\mathbf{e}_r-\sin\theta\mathbf{e}_\theta)$$
so **B**.

13. Equation becomes
$$X''T=XT' \text{ or } \frac{X''}{X}=\frac{T'}{T}=\alpha$$
so **B**.

14. $\mathbf{e}_\theta$ comp of grad$U=\dfrac{1}{\rho}\dfrac{\partial U}{\partial\theta}$ so **C**.

15.



By similar triangles
$\dfrac{h-z}{\rho}=\dfrac{h}{R}$ so $\rho=\dfrac{R}{h}(h-z)$. The limits for ρ are 0 to $R\left(1-\dfrac{z}{h}\right)$, limits for θ are 0 to 2π and limits for z are 0 to h. Using the formula from page 71 of Handbook the option is **D**.

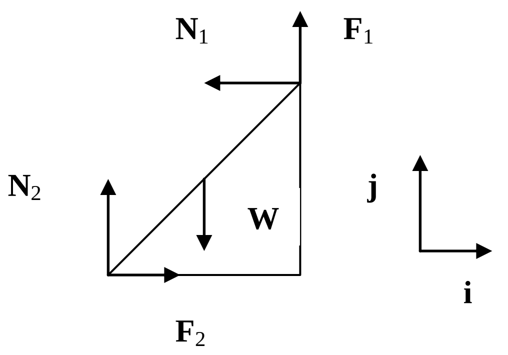
16. Separate the variables (as equation is not linear) $\dfrac{1}{1+y^2}\dfrac{dy}{dx}=\dfrac{1}{x^2}$.

Integrate $\displaystyle\int\frac{1}{1+y^2}\,dy=\int\frac{1}{x^2}dx$

$$\arctan y=-\frac{1}{x}+C$$
$y=0$ when $x=1$

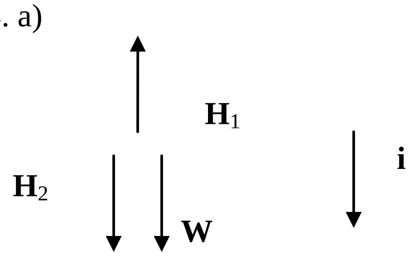
so $0=C-1$ giving $C=1$
$$y=\tan\left(1-\frac{1}{x}\right)\left(=\tan\left(\frac{x-1}{x}\right)\right)$$

17.



$$\mathbf{W} = -mg\mathbf{j} \quad \mathbf{N}_1 = -|\mathbf{N}_1|\mathbf{i} \quad \mathbf{F}_1 = \mu|\mathbf{N}_1|\mathbf{j}$$
$$\mathbf{N}_2 = |\mathbf{N}_2|\mathbf{j} \quad \mathbf{F}_2 = \mu|\mathbf{N}_2|\mathbf{i}$$

18. a)

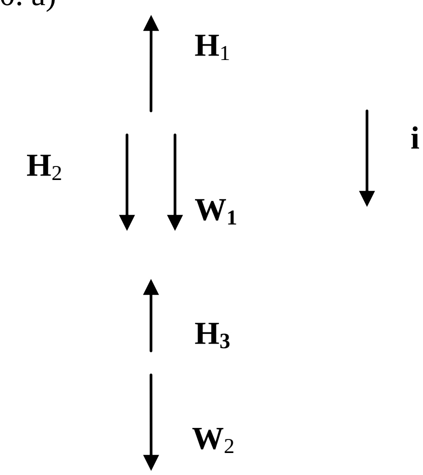


$$\mathbf{H}_1 = -k\left(x - l_0\right)\mathbf{i} \quad \mathbf{W} = mg\mathbf{i}$$
$$\mathbf{H}_2 = -k\left(4l_0 - x - 2l_0\right)\left(-\mathbf{i}\right) = k\left(2l_0 - x\right)\mathbf{i}$$

b) By Newton's second law
$$m\ddot{x}\mathbf{i} = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W}$$
Resolve in the **i**-direction
$$m\ddot{x} = -k\left(x - l_0\right) + k\left(2l_0 - x\right) + mg$$
$$m\ddot{x} + 2kx = 3kl_0 + mg$$

19. From Handbook page 49
$$y = x\tan\theta - \frac{x^2 g}{2u^2}\sec^2\theta$$
When $y = h$, $x = d$ and when $y = 0$, $x = 3d$ so
$$0 = 3d\tan\theta - \frac{9d^2 g}{2u^2}\sec^2\theta \quad (1)$$
$$h = d\tan\theta - \frac{d^2 g}{2u^2}\sec^2\theta \qquad (2)$$
$3\times(2)-(1)$ gives
$$3h = \frac{6d^2 g}{2u^2}\sec^2\theta \ \text{ or } \ u^2\cos^2\theta = \frac{d^2 g}{h}$$

20. a)



$$\Delta\mathbf{H}_1 = -3x\mathbf{i} \quad \Delta\mathbf{H}_2 = -3\left(y - x\right)\left(-\mathbf{i}\right) = 3\left(y - x\right)\mathbf{i}$$
$$\Delta\mathbf{H}_3 = -\Delta\mathbf{H}_2 = -3\left(y - x\right)\mathbf{i}$$
Equation of motion for particle 3/2 is
$$\frac{3}{2}\ddot{x}\mathbf{i} = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2$$
Resolving in the **i**-direction
$$\frac{3}{2}\ddot{x} = -3x + 3y - 3x \ \text{ or } \ddot{x} = -4x + 2y$$
Equation of motion for particle 1 is
$$\ddot{y}\mathbf{i} = \Delta\mathbf{H}_3$$
Resolving in the **i**-direction
$$\ddot{y} = 3x - 3y.$$
These two equations give
$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}-4 & 2\\ 3 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}.$$
b) For eigenvalues of the dynamic matrix
$$\begin{vmatrix}-4-\lambda & 2\\ 3 & -3-\lambda\end{vmatrix} = 0 \ \text{ so } \ \left(4+\lambda\right)\left(3+\lambda\right) - 6 = 0$$
$$\lambda^2 + 7\lambda + 6 = 0 \ \text{ or } \ \left(\lambda + 6\right)\left(\lambda + 1\right) = 0$$
$$\lambda = -6 \ \text{ or } \lambda = -1$$
$$\omega = \sqrt{-\lambda} \quad \text{so } \omega_1 = \sqrt{6} \text{ and } \omega_2 = 1$$

21. a) Linear momentum before $= mu\mathbf{i}$
Linear momentum after
$$= m\left(w\cos\beta\mathbf{i} + w\sin\beta\mathbf{j}\right) + m\left(v\cos\alpha\mathbf{i} - v\sin\alpha\mathbf{j}\right)$$
b) By conservation of linear momentum, these are equal so resolving in the **i**-direction gives
$$mu = m\left(w\cos\beta + v\cos\alpha\right) \quad (1)$$

21. cont.
and in the **j**-direction gives

$$0 = m\left(w\sin\beta - v\sin\alpha\right) \text{ or } w = \frac{v\sin\alpha}{\sin\beta}$$

Substituting into (1)

$$mu = m\left(\frac{v\sin\alpha\cos\beta}{\sin\beta} + v\cos\alpha\right)$$

$$u = \frac{v}{\sin\beta}\left(\sin\alpha\cos\beta + \sin\beta\cos\alpha\right)$$

$$= \frac{v}{\sin\beta}\left(\sin\left(\alpha+\beta\right)\right)$$

$$= \frac{v\sin\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}-\alpha\right)} = \frac{v}{\cos\alpha} = v\sec\alpha$$

using Handbook page 15.

22. a) From Handbook page 66

$$\mathbf{grad}f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi$$

$$= 2r\cos^2\theta\mathbf{e}_r - 2r\sin\theta\cos\theta\mathbf{e}_\theta$$

$$= 2r\cos^2\theta\mathbf{e}_r - r\sin 2\theta\mathbf{e}_\theta$$

b) From Handbook page 66, div**F**

$$= \frac{\partial F_r}{\partial r} + \frac{1}{r}\left(\frac{\partial F_\theta}{\partial\theta} + 2F_r\right) + \frac{1}{r\sin\theta}\left(\frac{\partial F_\phi}{\partial\phi} + F_\theta\cos\theta\right)$$

so div(**grad** *f*)
becomes

$$= 2\cos^2\theta + \frac{1}{r}\left(-2r\cos 2\theta + 4r\cos^2\theta\right)$$

$$+ \frac{1}{r\sin\theta}\left(-r\sin 2\theta\cos\theta\right)$$

$$= 2\cos^2\theta - 2\cos 2\theta + 4\cos^2\theta - 2\cos^2\theta$$

$$= 4\cos^2\theta - 2\left(\cos^2\theta - \sin^2\theta\right)$$

$$= 2\left(\cos^2\theta + \sin^2\theta\right) = 2$$

23. Euler's method is a first-order method and as shown by the graph Y_N has an approximate linear relationship with the step size.

$$Y_N \simeq y\left(b\right) + Ch$$

Taking the smallest two step sizes (as they are likely to be more accurate) we get

$$1 = Y_{20} \simeq y\left(b\right) + C(0.05)$$

$$1.0020 = Y_{100} \simeq y\left(b\right) + C(0.01)$$

Subtracting gives $.0020 = -C\left(0.04\right)$

Giving $C = \dfrac{-0.002}{0.04} = -\dfrac{1}{20} = -0.05$

From Handbook page 73 for 3 dp accuracy we have

$$\left|Ch\right| < 0.5\times10^{-3} \text{ or } 0.05h < 0.5\times10^{-3} \text{ or } h < 0.01$$

24. The auxiliary equation is

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \left(\lambda - 3\right)\left(\lambda - 2\right) = 0$$

$$\Rightarrow \quad \lambda = 3 \text{ or } \lambda = 2$$

Complementary function is $x_c = Ae^{3t} + Be^{2t}$

For particular integral try $x = a + bte^{2t}$
as e^{2t} is part of the complementary function (Handbook page 27)

$$\frac{dx}{dt} = 2bte^{2t} + be^{2t} \quad \frac{d^2x}{dt^2} = 4bte^{2t} + 4be^{2t}$$

Substituting into the equation gives

$$4bte^{2t} + 4be^{2t} - 10bte^{2t} - 5be^{2t} + 6a + 6bte^{2t} = 6 + e^{2t}$$

$$-be^{2t} + 6a = 6 + e^{2t}$$

Equating coefficients $b = -1$ $a = 1$
giving $x_P = 1 - te^{2t}$
The general solutions is

$$x = Ae^{3t} + Be^{2t} + 1 - te^{2t}$$

To fit the conditions

$$\frac{dx}{dt} = 3Ae^{3t} + 2Be^{2t} - e^{2t} - 2te^{2t}$$

$x = 1$ when $t = 0$ gives $1 = A + B + 1 \Rightarrow A = -B$

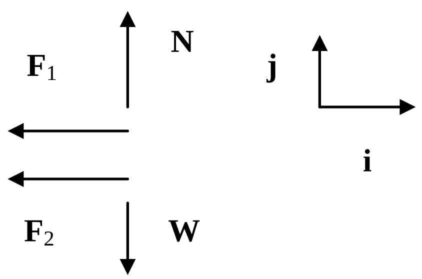
$\dot{x} = -5$ when $t = 0$ gives $-5 = 3A + 2B - 1$

$\Rightarrow 3A + 2B = -4$ or $-3B + 2B = -4$ using $A = -B$

so $B = 4$ and $A = -4$ giving

$$x = 4e^{2t} - 4e^{3t} + 1 - te^{2t}$$

25. (a)



$\mathbf{F}_1 = -k_1\mathbf{i}$ $\mathbf{F}_2 = -k_2 v\mathbf{i}$ $\mathbf{W} = Mg\mathbf{j}$ $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$

(b)By Newton's second law

$M\ddot{x}\mathbf{i} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{W} + \mathbf{N}$

Resolving in the **i**-direction

$M\ddot{x} = -k_1 - k_2 v = -(k_1 + k_2 v)$

(c) $M\dfrac{dv}{dt} = -(k_1 + k_2 v)$ as $\ddot{x} = \dfrac{dv}{dt}$

Separating the variables

$$\frac{1}{(k_1 + k_2 v)}\frac{dv}{dt} = -\frac{1}{M}$$

Integrating

$$\frac{1}{k_2}\ln(k_2 v + k_1) = -\frac{t}{M} + C$$

when $t = 0$ $v = v_0$ so $C = \dfrac{1}{k_2}\ln(k_2 v_0 + k_1)$

$$t = \frac{M}{k_2}\left(\ln(k_2 v_0 + k_1) - \ln(k_2 v + k_1)\right)$$

$$= \frac{M}{k_2}\ln\left(\frac{k_2 v_0 + k_1}{k_2 v + k_1}\right)$$

At rest $v = 0$ and

$$T = \frac{M}{k_2}\ln\left(\frac{k_2 v_0 + k_1}{k_1}\right) = \frac{M}{k_2}\ln\left(1 + \frac{k_2 v_0}{k_1}\right)$$

(d) Writing $\ddot{x}$ as $v\dfrac{dv}{dx}$ gives

$$Mv\frac{dv}{dx} = -(k_1 + k_2 v)$$

Separating the variables

$$\frac{v}{(k_1 + k_2 v)}\frac{dv}{dx} = -\frac{1}{M}$$

$$\left(\frac{1}{k_2} - \frac{k_1}{k_2(k_1 + k_2 v)}\right)\frac{dv}{dx} = -\frac{1}{M}$$

Integrating

$$\frac{v}{k_2} - \frac{k_1}{k_2{}^2}\ln(k_1 + k_2 v) = -\frac{x}{M} + C$$

$x = 0$ when $v = v_0$

$$C = \frac{v_0}{k_2} - \frac{k_1}{k_2{}^2}\ln(k_1 + k_2 v_0)$$

$$x = M\left(\frac{v_0 - v}{k_2} - \frac{k_1}{k_2{}^2}\ln\left(\frac{k_1 + k_2 v_0}{k_1 + k_2 v}\right)\right)$$

At $v = 0$

$$x = \frac{Mv_0}{k_2} - \frac{Mk_1}{k_2{}^2}\ln\left(\frac{k_1 + k_2 v_0}{k_1}\right)$$

$$= M\frac{v_0}{k_2} + \frac{Mk_1}{k_2{}^2}\ln\left(\frac{k_1}{k_1 + k_2 v_0}\right)$$

26. In matrix form the equations are

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}50\cos t\\ 0\end{bmatrix}$$

Solving the homogeneous equation

For eigenvalues

$$\begin{vmatrix}-1-\lambda & 3\\ 2 & -\lambda\end{vmatrix} = 0 \quad (1+\lambda)\lambda - 6 = 0 \quad \lambda^2 + \lambda - 6 = 0$$

$(\lambda + 3)(\lambda - 2) = 0 \Rightarrow \lambda = -3$ and $\lambda = 2$

For eigenvectors

For $\lambda = -3$ $\begin{bmatrix}2 & 3\\ 2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ or $2x + 3y = 0$

An eigenvector is $[3 \quad \text{-}2]^T$

For $\lambda = 2$ $\begin{bmatrix}\text{-}3 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ or $x - y = 0$

An eigenvector is $[1 \quad 1]^T$

$$\mathbf{x} = C\begin{bmatrix}1\\ 1\end{bmatrix}e^{2t} + D\begin{bmatrix}3\\ -2\end{bmatrix}e^{-3t}$$

For particular integral

$x = a\cos t + b\sin t \qquad y = c\cos t + d\sin t$

$\dot{x} = -a\sin t + b\cos t \quad \dot{y} = -c\sin t + d\cos t$

Substituting into the equations

$-a\sin t + b\cos t =$

$-a\cos t - b\sin t + 3c\cos t + 3d\sin t + 50\cos t$

$-c\sin t + d\cos t = 2a\cos t + 2b\sin t$

$(-a + b - 3d)\sin t + (b + a - 3c - 50)\cos t = 0$

$(-c - 2b)\sin t + (d - 2a)\cos t = 0$

Equating coefficients

$b - a - 3d = 0$ (1) $b + a - 3c = 50$ (2) $d = 2a$ (3)

$c = -2b$ (4) From (1) and (3) $b = 7a$ (5)

From (4) and (5) $c = -14a$

Substituting in (2) gives $50a = 50$ so $a = 1,\ \ b = 7,$

$c = -14$ and $d = 2$

so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -14 & 2 \end{bmatrix}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = C\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + D\begin{bmatrix} 3 \\ -2 \end{bmatrix}e^{-3t} + \begin{bmatrix} 1 & 7 \\ -14 & 2 \end{bmatrix}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

When $t = 0$, $x = 0$ and $y = 0$ so

$0 = C + 3D + 1 \Rightarrow C + 3D = -1$

$0 = C - 2D - 14 \Rightarrow C - 2D = 14$

Subtracting gives

$5D = -15$ or $D = -3$ so $C = 8$

Particular solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = 8\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} - 3\begin{bmatrix} 3 \\ -2 \end{bmatrix}e^{-3t} + \begin{bmatrix} 1 & 7 \\ -14 & 2 \end{bmatrix}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

or

$x = 8e^{2t} - 9e^{-3t} + \cos t + 7\sin t$

$y = 8e^{2t} + 6e^{-3t} - 14\cos t + 2\sin t$

27. (a) When there are no moles $x = 0$ $\dot{x} = 0$ and the equations reduce to $\dot{y} = y(B - y)$ which is the logistic equation. The population of worms will increase in number from their initial value but the increase will tail off as the population approaches the stable equilibrium population B.

(b) For equilibrium $\dot{x} = \dot{y} = 0$

$x(x + y - A) = 0$ (1)

$y(B + x - y) = 0$ (2)

From (1) $x = 0$ or $x + y = A$

Substitute $x = 0$ in (2) $y(B - y) = 0$

so $y = 0$ or $y = B$

Equilibrium points are (0, 0) and (0, B)

Substitute $x = A - y$ into (2)

$y(B + A - 2y) = 0$

so $y = 0$ or $y = \dfrac{A+B}{2}$

When $y = 0$ $x = A$ and when $y = \dfrac{A+B}{2}$

$x = A - \dfrac{A+B}{2} = \dfrac{A-B}{2}$

Equilibrium points $(A, 0)$ and $\left(\dfrac{A-B}{2}, \dfrac{A+B}{2}\right)$

so all the equilibrium points are

(0, 0) , (0, B), $(A, 0)$ and $\left(\dfrac{A-B}{2}, \dfrac{A+B}{2}\right)$.

(c)

$u = x^2 + xy - Ax \quad v = By + xy - y^2$

$$\mathbf{J}(x, y) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x + y - A & x \\ y & B + x - 2y \end{bmatrix}$$

(Handbook page 47)

$$\mathbf{J}\left(\frac{A-B}{2}, \frac{A+B}{2}\right) = \begin{bmatrix} \dfrac{A-B}{2} & \dfrac{A-B}{2} \\ \dfrac{A+B}{2} & -\dfrac{A+B}{2} \end{bmatrix}$$

For eigenvalues

$$\begin{vmatrix} \dfrac{A-B}{2} - \lambda & \dfrac{A-B}{2} \\ \dfrac{A+B}{2} & -\dfrac{A+B}{2} - \lambda \end{vmatrix} = 0$$

$$\left(\lambda - \frac{A-B}{2}\right)\left(\lambda + \frac{A+B}{2}\right) - \frac{(A+B)(A-B)}{4} = 0$$

$$\lambda^2 + B\lambda + \frac{-A^2 + B^2 - \left(A^2 - B^2\right)}{4} = 0$$

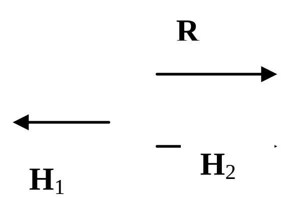
$$\lambda^2 + B\lambda + \frac{B^2 - A^2}{2} = 0$$

$$\lambda = \frac{-B \pm \sqrt{B^2 - 2\left(B^2 - A^2\right)}}{2}$$

$$\lambda = \frac{-B \pm \sqrt{2A^2 - B^2}}{2}$$

As $A > B$ $2A^2 - B^2 > 0$ and $\sqrt{2A^2 - B^2} > B$ the eigenvalues are real and distinct with different signs.
The equilibrium point is a saddle point. (Handbook page 47).

28. (a)



(b)

$\mathbf{H}_1 = -k\left(x - l_0\right)\mathbf{i}$

$\mathbf{H}_2 = -k\left(d - x - l_0\right)\left(-\mathbf{i}\right) = k\left(d - x - l_0\right)\mathbf{i}$

Length of dashpot is $l = y - x$ so

$\mathbf{R} = -r\left(\dot{y} - \dot{x}\right)\left(-\mathbf{i}\right) = r\left(\dot{y} - \dot{x}\right)\mathbf{i}$

Equation of motion is $m\ddot{x}\mathbf{i} = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{R}$

Resolve in the **i**-direction

$m\ddot{x} = -k\left(x - l_0\right) + k\left(d - x - l_0\right) + r\left(\dot{y} - \dot{x}\right)$

$m\ddot{x} + r\dot{x} + 2kx = kd + r\dot{y}$

(c) $\ddot{x} + 4\dot{x} + 2x = d + 4\cos t$

d only produces a constant solution so we need to solve $\ddot{x} + 4\dot{x} + 2x = 4\cos t$

Using Handbook page 56 the amplitude is given by $A = \dfrac{p}{\sqrt{\left(k - m\Omega^2\right)^2 + r^2\Omega^2}}$

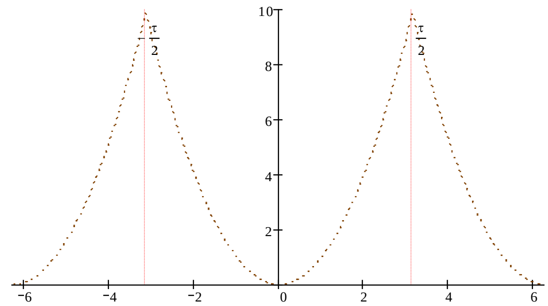
with $m = 1$ $r = 4$ $k = 2$ $P = 4$ and $\Omega = 1$, we get the amplitude

$$A = \frac{4}{\sqrt{\left(2-1\right)^2 + 16}} = \frac{4}{\sqrt{17}}$$

29. (a)

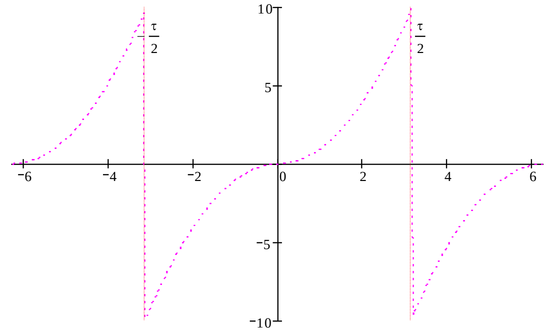
$f_{\text{even}}\left(x\right) = x^2 \quad -\pi < x < \pi$

$f_{\text{even}}\left(x + 2\pi\right) = f_{\text{even}}\left(x\right)$



$$f_{\text{odd}}\left(x\right) = \begin{cases} x^2 & 0 \le x < \pi \\ -x^2 & -\pi < x < 0 \end{cases}$$

$f_{\text{odd}}\left(x + 2\pi\right) = f_{\text{odd}}\left(x\right)$



(b) Using Handbook page 61

For the even function

$$A_0 = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$A_r = \frac{2}{\pi}\int_0^{\pi} x^2 \cos rx dx$$

$$= \frac{2}{\pi}\left[\frac{2rx\cos rx - 2\sin rx + r^2x^2\sin rx}{r^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\frac{2r\pi\cos r\pi}{r^3} = \frac{4\cos r\pi}{r^2}\left(= \frac{4(-1)^r}{r^2}\right)$$

$$F(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x + ...$$

For odd function

$$B_r = \frac{2}{\pi}\int_0^{\pi} x^2 \sin rx dx$$

$$= \frac{2}{\pi}\left[\frac{2rx\sin rx + 2\cos rx - r^2x^2\cos rx}{r^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi r^3}\left(2\cos r\pi - r^2\pi^2\cos r\pi - 2\right)$$

$$\left(= \frac{2}{\pi r^3}\left(2\left((-1)^r - 1\right) - r^2\pi^2(-1)^r\right)\right)$$

$$F(x) = \frac{2}{\pi}\left(\pi^2 - 4\right)\sin x - \pi\sin 2x$$

$$+ \frac{2}{27\pi}\left(9\pi^2 - 4\right)\sin 3x + ...$$

(c)

The odd approximation has discontinuities so the even approximation is better.

30. (a)

Moment of Inertia of assembly

= Moment of Inertia of the wheels plus Moment of Inertia of axle

$$= 2\left(\frac{1}{2}MR^2\right) + \frac{1}{2}mr^2 = MR^2 + \frac{1}{2}mr^2$$

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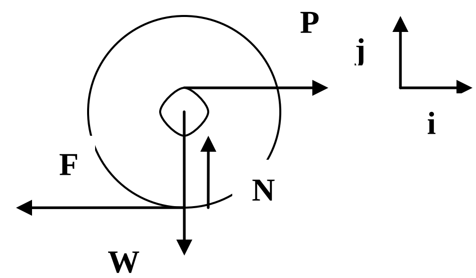
(b)



$\mathbf{W} = -(m+2M)g\mathbf{j}$ $\mathbf{P} = P\mathbf{i}$ $\mathbf{N} = |\mathbf{N}|\,\mathbf{j}$ $\mathbf{F} = -|\mathbf{F}|\,\mathbf{i}$

Equation of motion of the centre of mass

$(2M+m)\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{P} + \mathbf{N} + \mathbf{F}$

Resolving in the **i**-direction

$(2M+m)\ddot{x} = P - |\mathbf{F}|$ (1)

Equation of rotational motion about the centre of the axle is $I\ddot{\theta} = \Gamma_a$ where

$\mathbf{\Gamma} = r\mathbf{j}\times P\mathbf{i} + (-R)\,\mathbf{j}\times(-|\mathbf{F}|)\,\mathbf{i} = -(rP + R|\mathbf{F}|)\mathbf{k}$

so $\Gamma_a = -(rP + R|\mathbf{F}|)$ giving

$$\left(MR^2 + \frac{1}{2}mr^2\right)\ddot{\theta} = -(rP + R|\mathbf{F}|)$$

The rolling condition gives $x = -R\theta$ (as the angular velocity is $-\dot{\theta}\mathbf{k}$) giving $\ddot{x} = -R\ddot{\theta}$

so $\left(MR^2 + \frac{1}{2}mr^2\right)\frac{\ddot{x}}{R} = rP + R|\mathbf{F}|$

Substituting from (1)

$$\left(MR^2 + \frac{1}{2}mr^2\right)\frac{\ddot{x}}{R} = rP + R\left(P - (2M+m)\ddot{x}\right)$$

$$\left(MR^2 + \frac{1}{2}mr^2 + (2M+m)R^2\right)\ddot{x} = PR(r+R)$$

$$\ddot{x} = \frac{PR(r+R)}{3MR^2 + mR^2 + \frac{1}{2}mr^2}$$