MST209/J

Course Examination 2010

Mathematical Methods and Models

Friday 15 October 2010 **10.00 am – 1.00 pm**

Time allowed: 3 hours

Personal Identifier								
Examination No.								

You are **not** allowed to use a calculator in this examination.

There are THREE parts to this paper. In each part of the paper the questions are arranged in the order they appear in the course. There are 115 marks available, but scores greater than 100 will be rounded down to 100.

Part 1 consists of 15 questions each worth 2 marks. You are advised to spend no more than 1 hour on this part. Enter one option in each box provided on the question paper; use your answer book(s) for any rough work. Incorrect answers are not penalised. Cross out mistakes and write your answer next to the box provided.

Part 2 consists of 8 questions each worth 5 marks. You are advised to spend no more than $1\frac{1}{4}$ hours on this part.

Part 3 consists of 7 questions each worth 15 marks. Your best three marks will be added together to give a maximum of 45 marks.

In Parts 2 and 3: Write your answers in the answer book(s) provided. The marks allocated to each part of each question are given in square brackets in the margin. Unless you are directed otherwise in the question, you may use any formula or other information from the *Handbook* provided that you give a reference. Do **not** cross out any answers unless you have supplied a better alternative — everything not crossed out may receive credit.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used (as well as in the boxes above). **Failure to do so may mean that your work cannot be identified.** Use the paper fastener provided to fix together all your answer books, and the question paper, with your signed desk record on top.

PART 1

Each question in this part of the paper is worth 2 marks. Fill in the appropriate response in the box alongside the question.

Question 1

Given that

$$\ln(y+1) + \ln(y-1) - \ln 3 = -\ln(3x+9),$$

and that $x > -3$ and $y > 1$, select the option that gives y as a function of x.

Options Answer: $\square$

A $y = \dfrac{x+4}{x+3}$ **B** $y = \dfrac{x+3}{x+4}$

C $y = \sqrt{\dfrac{x+4}{x+3}}$ **D** $y = \sqrt{\dfrac{x+3}{x+4}}$

Question 2

This question concerns the differential equation

$$\frac{dy}{dx} = y(x+1) \quad (y > 0).$$

Which of the following options is correct?

Options Answer: $\square$

A The differential equation may be solved using the separable variables method but not the integrating factor method.

B The differential equation may be solved using integrating factor but not the the separable variables method method.

C The differential equation may be solved using either the separable variables method or the integrating factor method.

D The differential equation cannot be solved using either the separable variables method or the integrating factor method.

Question 3

If $\mathbf{a} = \mathbf{i} - 2\,\mathbf{j} + \,\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \,\mathbf{k}$ select the option that gives a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$.

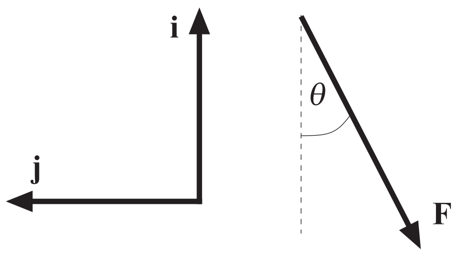
Options Answer: $\square$

A $\frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$ **B** $-\frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

C $\mathbf{i} + \mathbf{j} + \mathbf{k}$ **D** $2\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}$

Question 4

Consider the vector $\mathbf{F}$ and the axes shown below.



Select the option giving the $\mathbf{j}$-component of $\mathbf{F}$.

Options

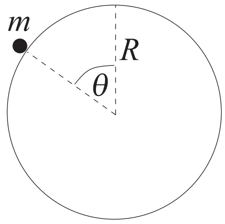
Answer:

A $|\mathbf{F}|\cos\theta$ **B** $|\mathbf{F}|\sin\theta$

C $-|\mathbf{F}|\cos\theta$ **D** $-|\mathbf{F}|\sin\theta$

Question 5

A particle of mass m slides down the outside of a smooth sphere of radius R. It starts from rest at the topmost point of the sphere, which is taken as the datum for potential energy. Select the option that gives the total energy of the particle when it makes an angle θ with the vertical, as shown.



Options

Answer:

A $mgR(1-\cos\theta)+\frac{1}{2}mR^2\dot{\theta}^2$ **B** $mgR(1-\cos\theta)-\frac{1}{2}mR^2\dot{\theta}^2$

C $mgR(\cos\theta-1)+\frac{1}{2}mR^2\dot{\theta}^2$ **D** $mgR(\cos\theta-1)-\frac{1}{2}mR^2\dot{\theta}^2$

Question 6

The eigenvalues of the matrix

$$\begin{bmatrix} -1 & 3 & 2 \\ -2 & 4 & 2 \\ -3 & 3 & a \end{bmatrix}$$

are 1, 2 and 4. Select the option that gives the value of a.

Options

Answer:

A 0 **B** 1 **C** 2 **D** 4

Question 7

Select the option that gives $\partial f/\partial\theta$ for the function $f(r,\theta,\phi) = \exp(r\sin\phi\cos\theta)$.

Options Answer:

A $\exp(-r\sin\phi\,\sin\theta)$ **B** $-\sin\theta\,\exp(r\cos\phi\,\cos\theta)$

C $-r\sin\phi\sin\theta\,\exp(r\sin\phi\,\cos\theta)$ **D** $r\cos\phi\cos\theta\,\exp(r\sin\phi\,\cos\theta)$

Question 8

Two populations x and y are modelled by the system of differential equations

$$\dot{x} = rx^2 - sxy$$
$$\dot{y} = 2qx - py^2,$$

where p, q, r and s are constants. Select the option that gives the Jacobian matrix for this system at the point (a, b).

Options Answer:

A $\begin{bmatrix} 2ra - sb & -sa \\ 2q & -2pb \end{bmatrix}$ **B** $\begin{bmatrix} ra^2 & -sab \\ 2qa & -pb^2 \end{bmatrix}$

C $\begin{bmatrix} -sa & 2ra - sb \\ -2pb & 2q \end{bmatrix}$ **D** $\begin{bmatrix} 2q & -2pb \\ 2ra - sb & -sa \end{bmatrix}$

Question 9

The Reynolds number of a fluid flow is a dimensionless constant Re defined as

$$\mathrm{Re} = \frac{\rho V L}{\mu},$$

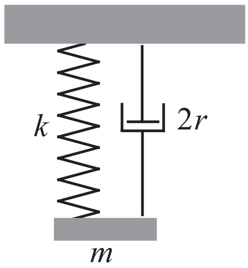
where ρ is the density of the fluid, V is its speed, L the distance through which the fluid has travelled, and μ is the dynamic viscosity of the fluid. Select the option that gives the dimensions of dynamic viscosity.

Options Answer:

A dimensionless **B** MLT **C** $\mathrm{M^{-1}LT}$ **D** $\mathrm{ML^{-1}T^{-1}}$

Question 10

A particle of mass m is connected to a support by a model spring of stiffness k and a model damper of damping constant $2r$. The mass hangs vertically below the point O on the support, and is displaced from equilibrium.



Select the option that guarantees that the particle will experience critical damping.

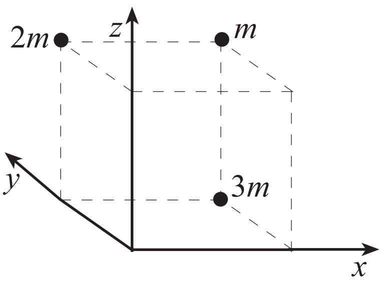
Options

A $4r^2 < mk$ **B** $4r^2 = mk$ **C** $r^2 < mk$ **D** $r^2 = mk$

Answer:

Question 11

Three particles are placed as if at three corners of a unit cube whose faces are aligned with the axes as shown.



The masses of the particles are as shown. Select the option that gives the centre of mass of the three particles.

Options

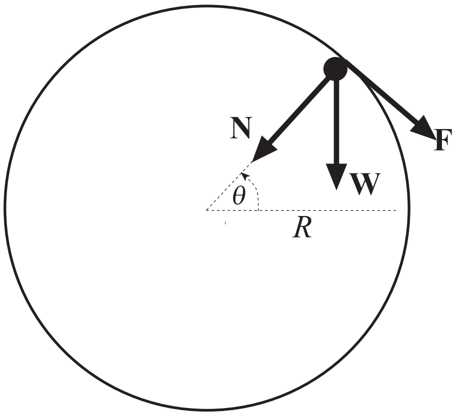
A $\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{2}\mathbf{k}$ **B** $\frac{2}{3}\mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$

C $\frac{1}{2}\mathbf{i} + \frac{2}{3}\mathbf{j} + \mathbf{k}$ **D** $\frac{1}{2}\mathbf{i} + \mathbf{j} + \frac{2}{3}\mathbf{k}$

Answer:

Question 12

A particle of mass m is moving on the inside of a circular cylinder of radius R. The particle is moving in an anti-clockwise direction in a vertical plane. The unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ take their usual meanings.



Which option gives the radial equation of motion of the bead?

Options

Answer:

A $mR\dot{\theta}^2 = -mg\sin\theta - |\mathbf{N}|$ **B** $mR\dot{\theta}^2 = -mg\sin\theta + |\mathbf{N}|$

C $mR\dot{\theta}^2 = mg\sin\theta - |\mathbf{N}|$ **D** $mR\dot{\theta}^2 = mg\sin\theta + |\mathbf{N}|$

Question 13

The method of separation of variables is to be applied to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\lambda^2}\frac{\partial u}{\partial t}.$$

Select the option that gives the resulting pair of ordinary differential equations.

Options

Answer:

A $X'' = kX$, $\dot{T} = k\lambda^2 T$ **B** $X' = kX$, $\ddot{T} = k\lambda^2 T$ **C** $\lambda^2 X'' = X$, $\dot{T} = T$ **D** $\lambda^2 X' = kX$, $\ddot{T} = kT$

Question 14

A scalar field is given in cylindrical polar coordinates by $h(\rho,\,\theta,\,z) = \rho\, z\cos\theta$. Which option gives **grad** h?

Options

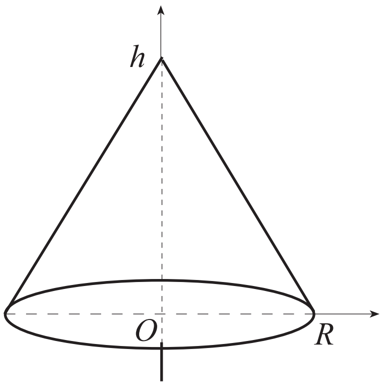
Answer:

A $z\cos\theta\,\mathbf{e}_\rho - z\sin\theta\,\mathbf{e}_\theta + \rho\cos\theta\,\mathbf{e}_z$ **B** $z\cos\theta\,\mathbf{e}_\rho + z\sin\theta\,\mathbf{e}_\theta + \rho\cos\theta\,\mathbf{e}_z$

C $z\cos\theta\,\mathbf{e}_\rho - \rho z\sin\theta\,\mathbf{e}_\theta - \rho\cos\theta\,\mathbf{e}_z$ **D** $z\cos\theta\,\mathbf{e}_\rho + \rho z\sin\theta\,\mathbf{e}_\theta - \rho\cos\theta\,\mathbf{e}_z$

Question 15

A cone of base radius R and height h is placed on the x-y-plane with the centre of its base at the origin. It is made of material whose density is given by cz, where z is height above the plane.



Select the option that gives the mass of the cone in cylindrical polar coordinates.

Options

Answer: $\square$

A $\displaystyle\int_0^h \int_{-\pi}^{\pi} \int_0^{R(h-z)/h} c\, z\, \rho\, d\rho\, d\theta\, dz$

B $\displaystyle\int_0^h \int_{-\pi}^{\pi} \int_0^{R(h-z)/h} c\, z\, \rho^2\, d\rho\, d\theta\, dz$

C $\displaystyle\int_0^h \int_{-\pi}^{\pi} \int_0^{R(h-z)/h} c\, z^2\, \rho\, d\rho\, d\theta\, dz$

D $\displaystyle\int_0^h \int_{-\pi}^{\pi} \int_0^{R(h-z)/h} c\, z^2\, \rho^2\, d\rho\, d\theta\, dz$

PART 2

Each question in this part of the paper is worth 5 marks.

Question 16

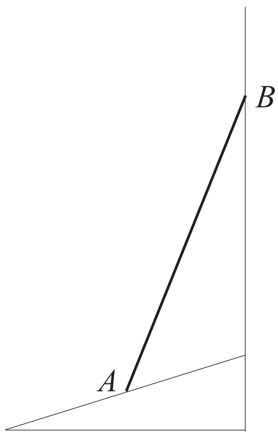
Solve the initial-value problem

$$\frac{dy}{dx} - x^3 y = e^{x^4/4}, \quad y(0) = 1,$$

expressing your answer in the form $y = f(x)$. [5]

Question 17

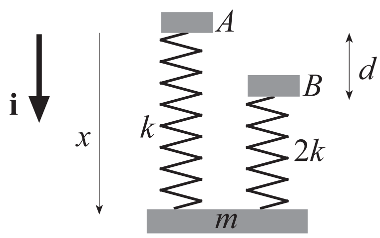
A ladder of mass m stands with its base at a point A on a rough plane inclined to the horizontal. Its other end rests against a smooth vertical wall.



Model the ladder as a model rod. Draw a force diagram showing all the forces acting on the ladder showing their direction. Briefly describe the nature of each force. [5]

Question 18

A particle of mass m is connected to two vertical model springs, each of natural length l_0. The left-hand spring has stiffness k and its other end is attached to a fixed point A. The right-hand spring has stiffness $2k$ and its other end is attached to a fixed point B, a vertical distance d below A. The particle is constrained to move in a vertical line, and there are no resistive forces.



(a) Draw a force diagram showing all the forces acting on the particle when it is a distance x from A. [*1*]

(b) Express each of these forces in vector form in terms of the given parameters. [*3*]

(c) Hence derive the equation of motion of the particle. [*1*]

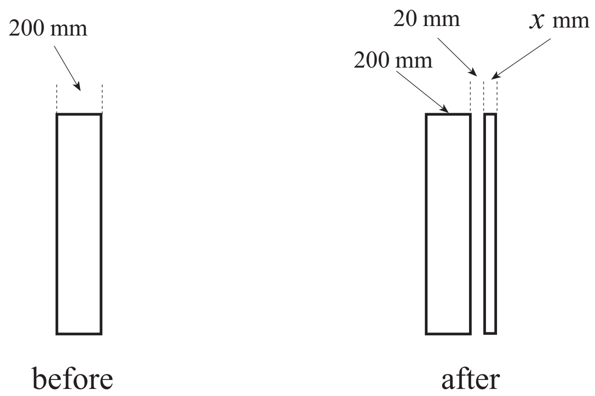
Question 19

Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}. \qquad [5]$$

Question 20

A solid brick wall consists of a layer of brick, 200 mm thick, and of thermal conductivity $0.5\,\mathrm{Wm^{-1}K^{-1}}$. It is proposed to dry-line the wall using plasterboard of thickness x mm and thermal conductivity $0.2\,\mathrm{Wm^{-1}K^{-1}}$, with an air gap of 20 mm.



The convective heat transfer coefficients for the inner and outer surfaces, respectively, are $10\,\mathrm{Wm^{-2}K^{-1}}$ and $100\,\mathrm{Wm^{-2}K^{-1}}$, and the combined heat transfer coefficient of this air gap is $10\,\mathrm{Wm^{-2}K^{-1}}$.

(a) Calculate the U-value of the wall before lining. *[2]*

(b) Calculate the U-value of the wall after lining. *[2]*

(c) Hence find the value of x if the rate of heat transfer is to be two thirds of its value prior to lining. *[1]*

Question 21

A ball of mass m moves with velocity $2\,\mathbf{i} - 2\,\mathbf{j}$. It collides with a second ball, of identical size and mass, that is initially at rest. After the collision the two balls have velocities $u_1\,\mathbf{i} + v_1\,\mathbf{j}$ and $u_2\,\mathbf{i} + v_2\,\mathbf{j}$, respectively.

(a) Find the energy of the system after the collision. *[1]*

(b) Show that if

$$(u_1\,\mathbf{i} + v_1\,\mathbf{j}) \cdot (u_2\,\mathbf{i} + v_2\,\mathbf{j}) = 0$$

then the collision is elastic. *[4]*

Question 22

A vector field is given in spherical polar coordinates by

$$\mathbf{V}(r, \theta, \phi) = \frac{1}{r}\,\mathbf{e}_\phi.$$

(a) Find div $\mathbf{V}$. *[2]*

(b) Find **curl** $\mathbf{V}$. *[3]*

Question 23

The Euler-Trapezoidal method is to be applied to the initial-value problem

$$\frac{dy}{dx} = x + y, \quad y(0) = 1,$$

with a step length of $h = 0.1$. Carry out the first step of the process in order to calculate $Y(0.1)$. *[5]*

PART 3

Each question in this part of the paper is worth 15 marks. All of your answers will be marked and the marks from your best three answers will be added together. A maximum of 45 marks can be obtained from this part.

Question 24

Consider the differential equation

$$4\frac{d^2y}{dt^2} + 25y = 20\cos(5t/2) + 40\sin(5t/2).$$

(a) Find the general solution to this equation. [*10*]

(b) Hence find the particular solution that satisfies $y(0) = 1$ and $\dot{y}(0) = 8$. [*5*]

Question 25

A curling stone of mass M is projected horizontally with speed v_0 in a straight line on an ice rink. During its subsequent motion the stone is subject to a force $\mathbf{F}_1$ from the ice, and a further resistive force $\mathbf{F}_2$ due to air resistance. The force due to the ice can be modelled as $\mathbf{F}_1 = -k_1\,\mathbf{v}$, and that due to air resistance by $\mathbf{F}_2 = -k_2|\mathbf{v}|\,\mathbf{v}$, where $\mathbf{v}$ is the velocity of the stone and k_1 and k_2 are positive constants. You may model the stone as a particle, and ignore all other forces and any rotational motion. Measure the distance the stone has travelled since its release by the coordinate x.

(a) Draw a force diagram showing all the forces acting on the particle. [*1*]

(b) Derive the equation of motion of the particle. [*2*]

(c) Determine the distance travelled, x, as a function of the speed v. [*7*]

(d) Show that the stone comes to rest after it has travelled a distance

$$x_{\max} = \frac{M}{k_2}\ln\left(\frac{k_1 + k_2 v_0}{k_1}\right).$$ [*4*]

(e) What happens to the stone after it comes to rest? [*1*]

Question 26

Consider the following inhomogeneous system of first-order differential equations

$$\begin{aligned}\dot{x} &= 2x + 3y\\ \dot{y} &= x + 4y - 26\sin t.\end{aligned}$$

(a) Express the system of equations in matrix form. [*1*]

(b) Find the eigenvalues of the matrix of coefficients and an eigenvector corresponding to each eigenvalue. Hence write down the complementary function for the system of equations. [*5*]

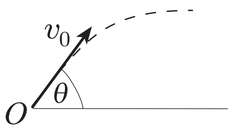
(c) Determine the particular integral to the inhomogeneous system and hence write down the general solution of the system given above. [*4*]

(d) If initially $x(0) = 1$ and $y(0) = 2$ find the particular solution at time t. [*3*]

(e) Describe the long term behaviour of this solution. [*2*]

Question 27

A particle of mass m is projected from a fixed point O with speed v_0 at an angle θ to the horizontal.



It moves through the air under the force due to gravity and a resistive force given by $-km\,\mathbf{v}$, where k is a positive constant and $\mathbf{v}$ is the particle's velocity at time t after launch. The object of this question is to find the coordinates of the particle as functions of time whilst it is in flight.

(a) Draw a force diagram for the particle. *[2]*

(b) Choose an appropriate coordinate system and express each force acting on the particle in terms of your system of unit vectors. *[2]*

(c) Show that the horizontal equation of motion of the particle may be written as

$$\ddot{x} = -k\dot{x},$$

where x denotes horizontal displacement from O at time t. *[1]*

(d) Find a corresponding differential equation for y. *[2]*

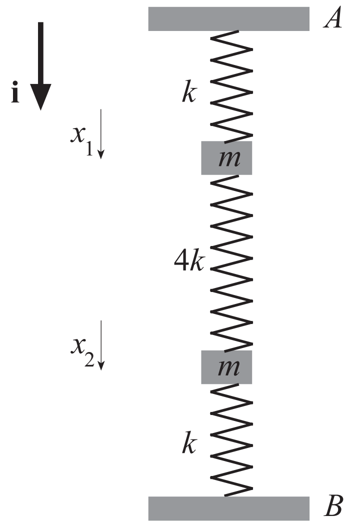
(e) Integrate these equations with respect to t and apply appropriate initial conditions. Hence show that the resulting equations may be expressed in the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -k & 0 \\ 0 & -k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} v_0 \cos\theta \\ v_0 \sin\theta - gt \end{bmatrix}.$$ *[3]*

(f) Find the general solution of the system in part (d) and hence determine the particular solution for x and y in terms of t. *[5]*

Question 28

Two particles, each of mass m, are attached to three springs, of stiffness k, $4k$ and k, respectively, as shown.



The free ends of two of the springs are attached to fixed points, A and B, and the particles are constrained to move in a vertical line. The displacements of the two particles from their equilibrium positions are denoted x_1, x_2, respectively, and the unit vector $\mathbf{i}$ points vertically as shown. You may ignore air resistance and any other frictional forces.

(a) Draw a force diagram showing the forces acting on each particle. [2]

(b) Express the changes in the spring forces (from their equilibrium values) acting on each particle in terms of the variables and parameters given above. [2]

(c) Write down the equation of motion of each particle, and hence show that the dynamic matrix of the system is

$$\begin{bmatrix} -\dfrac{5k}{m} & \dfrac{4k}{m} \\ \dfrac{4k}{m} & -\dfrac{5k}{m} \end{bmatrix}.$$ [3]

You are now given the additional information that $k = 8\,\mathrm{N\,m^{-1}}$ and $m = 2\,\mathrm{kg}$.

(d) Find the normal mode angular frequencies of the system. [3]

(e) Find an eigenvector corresponding to each angular frequency, and hence write down the displacement of each particle from its equilibrium position as a function of time. [3]

(f) The particles are released from rest with

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}.$$

Using the same axes, sketch the subsequent motion of the two particles. [2]

Question 29

This question is concerned with the periodic function defined by

$$f(t) = \begin{cases} 0 & -\pi \le t < 0 \\ 2t & 0 \le t < \pi/2 \\ \pi & \pi/2 \le t < \pi, \end{cases}$$

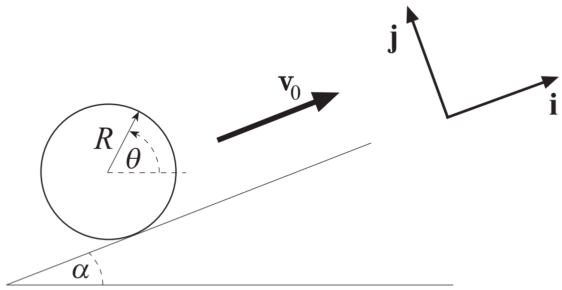
and $f(t + 2\pi) = f(t)$ for all t.

(a) Draw a sketch of the graph of f over two periods. State whether f is even, odd or neither. [*5*]

(b) Determine the coefficients A_0, A_n and B_n in the Fourier series expansion for $f(t)$. State all the terms up to and including $n = 4$. [*10*]

Question 30

In this question you may ignore air resistance. A uniform solid sphere of radius R and mass M is spinning with angular speed ω_0 about a horizontal diameter. The sphere is projected with speed v_0 up a rough plane inclined at an angle α to the horizontal.



The coefficient of sliding friction between the sphere and the plane is μ' and initially the sphere slides and rolls up the plane.

(a) Write down the moment of inertia I of the sphere about its diameter. [*1*]

(b) Draw a force diagram showing all the forces acting on the sphere during its ascent. [*2*]

(c) Take the distance the sphere has moved up the slope at time t after launch to be x. Find the equation of linear motion of the centre of mass in terms of μ', g and α. [*3*]

(d) Take the angle through which the sphere has rotated to be θ (measured anti-clockwise) at time t. Write down the equation of relative rotational motion about the centre of gravity. [*3*]

(e) Whilst the sphere is slipping, determine the velocity of the point of contact of the sphere with the inclined plane at time t. [*2*]

(f) Determine $\dot{x}$ and $\dot{\theta}$ as functions of time and hence show that the sphere stops slipping after a time

$$t = \frac{v_0 + R\omega_0}{g(\sin\alpha + \frac{7}{2}\mu'\cos\alpha)}.$$ [*4*]

[END OF QUESTION PAPER]