M337/K

Course Examination 2004

Complex Analysis

Friday 15 October 2004   2.30 pm – 5.30 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I (64% of the marks) you should attempt as many questions as you can.

In Part II (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Put all your used answer books and your question paper together with your signed desk record on top. Fix them all together using the fastener provided.

The use of calculators is **NOT** permitted in this examination.

Copyright © 2004 The Open University

PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

Let $\alpha = -2 + 2i$.

(a) Write down the values of

(i) $|\alpha|$,

(ii) $\operatorname{Arg}\alpha$. [2]

(b) Hence, or otherwise, evaluate each of the following, giving your answers in Cartesian form.

(i) $1/\alpha$

(ii) $\alpha^{1/3}$

(iii) $\operatorname{Log}\alpha$

(iv) $\operatorname{Log}(\alpha^3)$ [6]

Question 2

Let

$$A = \{z : 1 < |z + i| < 2\} \quad \text{and} \quad B = \{z : 0 \leq \operatorname{Arg} z \leq \frac{\pi}{2}\}.$$

(a) Make separate sketches of the sets A, B and $C = A - B$. [3]

(b) Write down which of the sets A, B and C, if any, is

(i) a region;

(ii) a simply-connected region;

(iii) neither open nor closed. [4]

(c) Using set notation, give an example of a set which is closed but not connected. [1]

Question 3

In this question Γ is the circle $\{z : |z| = 2\}$.

(a) (i) Write down the standard parametrization for Γ.

(ii) Evaluate

$$\int_\Gamma \overline{z}\, dz.$$ [3]

(b) Determine an upper estimate for the modulus of

$$\int_\Gamma \frac{\overline{z}^2 - 1}{z^2 - 1}\, dz.$$ [5]

Question 4

Evaluate the following integrals in which $C = \{z : |z - i| = 2\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C \frac{e^{i\pi z}}{z+1}\,dz$ [3]

(b) $\displaystyle\int_C \frac{e^{i\pi z}}{z+3}\,dz$ [2]

(c) $\displaystyle\int_C \frac{\sin(z - \frac{\pi}{2})}{z^3}\,dz$ [3]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{z^2+1}{z(z-\frac{1}{2})(z-2)}$$

at each of the poles of f. [3]

(b) Hence evaluate the integral

$$\int_0^{2\pi} \frac{\cos t}{5 - 4\cos t}\,dt.$$ [5]

Question 6

Let $f(z) = 2z^3 + 5z - 1$.

(a) (i) Show that f has three zeros lying inside the circle $C_1 = \{z : |z| = 2\}$.

(ii) Determine the number of zeros of f which lie inside the circle $C_2 = \{z : |z| = 1\}$. [7]

(b) Show that f has a zero inside C_2 which is real and positive. [1]

Question 7

Let $q(z) = i\bar{z}$ be a velocity function.

(a) Explain why q represents a model fluid flow on $\mathbb{C}$. [1]

(b) Determine a stream function for this flow. Hence find the equation of the streamline through the point $1 + i$, and sketch this streamline indicating the direction of flow. [5]

(c) Determine the flux of q across the path Γ, where

$$\Gamma : \gamma(t) = (1+i)t \quad (t \in [1, 2]).$$ [2]

Question 8

(a) Prove that the iteration sequence

$$z_{n+1} = z_n^2 + 6z_n + 5, \qquad n = 0, 1, 2, \ldots,$$

with $z_0 = -3$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 - 1, \qquad n = 0, 1, 2, \ldots,$$

with $w_0 = 0$. [3]

(b) Find the fixed points of $P_{-1}(z) = z^2 - 1$ and determine their nature. [3]

(c) Determine whether or not $\frac{1}{2} - i$ lies in the Mandelbrot set M. [2]

PART II

(i) You should attempt no more than **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Let f be the function

$$f(z) = \overline{z}(1+z).$$

(i) Write $f(x+iy)$ in the form $u(x,y) + iv(x,y)$, where u and v are real-valued functions.

(ii) Use the Cauchy-Riemann equations to show that f is differentiable at -1, but not analytic there.

(iii) Evaluate $f'(-1)$. [8]

(b) Let g be the function $g(z) = iz^3$.

(i) Show that g is conformal on $\mathbb{C} - \{0\}$.

(ii) Describe the effect of g on a small disc centred at 2.

(iii) Γ_1 and Γ_2 are the smooth paths meeting at 0 and 2 given by

$$\Gamma_1 : \gamma_1(t) = 2t \quad (t \in [0,1]),$$
$$\Gamma_2 : \gamma_2(t) = 1 + e^{it} \quad (t \in [0,\pi]).$$

Sketch these paths, clearly indicating their directions.

(iv) Using part (b)(ii), or otherwise, sketch the directions of $g(\Gamma_1)$ and $g(\Gamma_2)$ at $g(2)$.

(v) Show that g is not conformal at 0. [10]

Question 10

(a) Let $f(z) = \dfrac{2}{z(z-i)}$.

(i) Write down the singularities of f and determine their nature.

(ii) Determine the Laurent series about 0 for f on the set $\{z : 0 < |z| < 1\}$, giving the general term.

(iii) Determine the Laurent series about i for f on the set $\{z : |z-i| > 1\}$, giving the general term. [10]

(b) (i) Determine the Laurent series about 0 for the function g defined by $g(z) = z^2 \sin(1/z)$, giving the first three non-vanishing terms.

(ii) Classify the singularity of g at 0, justifying your answer.

(iii) Show that

$$\int_C z^2 \sin\left(\frac{1}{z}\right)\, dz = -\frac{\pi i}{3},$$

where $C = \{z : |z| = 1\}$.

(iv) Write down the value of

$$\int_C z^{2n} \sin\left(\frac{1}{z}\right)\, dz, \qquad \text{for } n = 1, 2, 3, \ldots,$$

where $C = \{z : |z| = 1\}$. [8]

Question 11

(a) Find the residues of the function

$$f(z) = \frac{\pi \cot \pi z}{9z^2 + 4}$$

at each of the points $0, \frac{2}{3}i, -\frac{2}{3}i$. [6]

(b) Hence determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{9n^2 + 4}.$$ [8]

(c) Use your result from part (b) to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{9n^2 + 4} = \frac{\pi}{6} \coth \frac{2\pi}{3}.$$ [4]

Question 12

(a) (i) Show that $\alpha = 1 + i$ and $\beta = 2(1 + i)$ are inverse points with respect to the circle $C = \{z : |z| = 2\}$.

(ii) Find the images of α and β under the Möbius transformation

$$g(z) = \frac{2}{z - (1 + i)},$$

and hence sketch the image of C under g.

(iii) Indicate $g(D)$ on your sketch, where $D = \{z : |z| < 2\}$. [8]

(b) Let $R = \{z : |z - 1| < 1,\ 0 < \operatorname{Arg}(z - 1) < \pi\}$,
$R_1 = \{z_1 : \operatorname{Re} z_1 > 0,\ \operatorname{Im} z_1 > 0\}$ and $S = \{w : \operatorname{Im} w > 0\}$.

(i) Sketch the regions R, R_1 and S.

(ii) Find a Möbius transformation f_1 which maps R to R_1, and use the conformality of f_1 to justify that it does indeed map R to R_1.

(iii) Write down a conformal mapping from R_1 to S and hence a conformal mapping f from R to S.

(iv) Explain why the function f is not conformal on $\overline{R}$ (the closure of R). [10]

[END OF QUESTION PAPER]