MST209/G

Module Examination 2013

Mathematical methods and models

Thursday 13 June 2013 **10.00 am–1.00 pm**

Time allowed: 3 hours

Personal Identifier								
Examination No.								

You are **not** allowed to use a calculator in this examination.

There are THREE parts to this paper. In each part of the paper the questions are arranged in the order they appear in the course. There are 115 marks available, but scores greater than 100 will be rounded down to 100.

Part 1 consists of 15 questions each worth 2 marks. You are advised to spend no more than 1 hour on this part. Enter one option in each box provided on the question paper; use your answer book(s) for any rough work. Incorrect answers are not penalised. Cross out mistakes and write your answer next to the box provided.

Part 2 consists of 8 questions each worth 5 marks. You are advised to spend no more than $1\frac{1}{4}$ hours on this part.

Part 3 consists of 7 questions each worth 15 marks. Your best three marks will be added together to give a maximum of 45 marks.

In Parts 2 and 3: Write your answers in the answer book(s) provided. The marks allocated to each part of each question are given in square brackets in the margin. Unless you are directed otherwise in the question, you may use any formula or other information from the *Handbook* provided that you give a reference. Do **not** cross out any answers unless you have supplied a better alternative — everything not crossed out may receive credit.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used (as well as in the boxes above). **Failure to do so may mean that your work cannot be identified.** Use the paper fastener provided to fix together all your answer books, and the question paper, with your signed desk record on top.

PART 1

Each question in this part of the paper is worth 2 marks. Fill in the appropriate response in the box alongside the question.

Question 1

For the implicit equation,

$$x^2y + x^3 - 2y^2 = 10,$$

which option gives the derivative?

Options Answer:

A $\dfrac{dy}{dx} = \dfrac{2xy + 3x^2}{4y - x^2}$ **B** $\dfrac{dy}{dx} = \dfrac{2xy + 3x^2}{x^2 - 4y}$

C $\dfrac{dy}{dx} = \dfrac{x^3}{2x - 4y}$ **D** $\dfrac{dy}{dx} = \dfrac{x^3}{4y - 2x}$

Question 2

This question concerns the differential equation

$$\frac{dy}{dx} + 4xy = x, \quad \left(y < \tfrac{1}{4}\right).$$

Which of the following options is correct?

Options Answer:

A The differential equation may be solved using the separation of variables method but not the integrating factor method.

B The differential equation may be solved using the integrating factor method but not the separation of variables method.

C The differential equation may be solved using either the integrating factor method or the separation of variables method.

D The differential equation cannot be solved using either the integrating factor method or the separation of variables method.

Question 3

Select the option that gives the angle between the two vectors

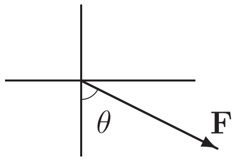
$$(\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}) \quad \text{and} \quad (7\,\mathbf{i} + 4\,\mathbf{j} - 4\,\mathbf{k}).$$

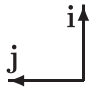
Options Answer:

A $\arccos(9)$ **B** $\arccos(-9)$ **C** $\arccos\left(\frac{1}{3}\right)$ **D** $\arccos\left(-\frac{1}{3}\right)$

Question 4

The diagram shows the force $\mathbf{F}$ and the directions of the unit vectors, $\mathbf{i}$ and $\mathbf{j}$.





Select the option that corresponds to the component of $\mathbf{F}$ in the $\mathbf{i}$-direction.

Options Answer:

A $|\mathbf{F}|\sin\theta$ **B** $-|\mathbf{F}|\sin\theta$ **C** $|\mathbf{F}|\cos\theta$ **D** $-|\mathbf{F}|\cos\theta$

Question 5

A particle P of mass m is a vertical distance x below O and moving horizontally with speed v. Choose the option that gives the total energy of the particle P at time t using O as the datum.

Options Answer:

A mgx **B** $-mgx$ **C** $mgx + \frac{1}{2}mv^2$ **D** $-mgx + \frac{1}{2}mv^2$

Question 6

The following stage has been reached in the solution by the Gaussian elimination method of four linear equations with four unknowns.

$$\begin{array}{r} \\ \mathbf{R}_2 - \mathbf{R}_1 \\ \mathbf{R}_3 - 2\mathbf{R}_1 \\ \mathbf{R}_4 + 3\mathbf{R}_1 \end{array} \left[\begin{array}{rrrr|r} 1 & 6 & 3 & 4 & 2 \\ 0 & 2 & -2 & 3 & 1 \\ 0 & -4 & 2 & -1 & 0 \\ 0 & 4 & 0 & 3 & -2 \end{array}\right] \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_{2a} \\ \mathbf{R}_{3a} \\ \mathbf{R}_{4a} \end{array}.$$

Choose the option that gives the next set of operations in the Gaussian elimination process.

Options Answer:

A $\mathbf{R}_{3b} = \mathbf{R}_{3a} + \mathbf{R}_{4a}$, $\mathbf{R}_{4b} = \mathbf{R}_{4a} + \mathbf{R}_{3a}$

B $\mathbf{R}_{3b} = \mathbf{R}_{3a} + 2\mathbf{R}_{2a}$, $\mathbf{R}_{4b} = \mathbf{R}_{4a} + \mathbf{R}_{3a}$

C $\mathbf{R}_{3b} = \mathbf{R}_{3a} + 2\mathbf{R}_{2a}$, $\mathbf{R}_{4b} = \mathbf{R}_{4a} - 2\mathbf{R}_{2a}$

D $\mathbf{R}_{3b} = \mathbf{R}_{3a} + \mathbf{R}_{4a}$, $\mathbf{R}_{4b} = \mathbf{R}_{4a} + 2\mathbf{R}_{2a}$

Question 7

The matrix $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ has an eigenvector $[1 \quad 1]^T$. What are the eigenvalues of $\mathbf{A}$?

Options Answer:

A $7, 2$ **B** $-7, -2$ **C** $-5, -4$ **D** $5, 4$

Question 8

At the point $(1,3)$, a function has partial derivatives

$$\frac{\partial f}{\partial x}(1,3) = 5 \quad \text{and} \quad \frac{\partial f}{\partial y}(1,3) = -1.$$

The maximum possible error in x is $\delta x = \pm 0.01$ and the maximum possible error in y is $\delta y = \pm 0.03$. Choose the option that gives an estimate of the maximum possible error in δf.

Options Answer:

A 0.02 **B** 0.04 **C** 0.06 **D** 0.08

Question 9

The acceleration of a particle is given by

$$\ddot{\mathbf{x}} = -6t\,\mathbf{i} + 4\,\mathbf{j}.$$

Initially the velocity is $\dot{\mathbf{x}}(0) = 2\,\mathbf{i} + 3\,\mathbf{j}$. Choose the option that gives the speed of the particle at time $t = 1$.

Options Answer:

A $-\mathbf{i} + 7\,\mathbf{j}$ **B** $\sqrt{50}$ **C** $2\,\mathbf{i} + 3\,\mathbf{j}$ **D** $\sqrt{13}$

Question 10

The dimensions of energy, E, are to be stated; select the option that gives the correct statement.

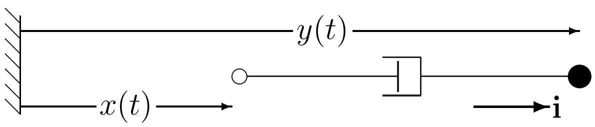
Options Answer:

A $[\mathrm{E}] = \mathrm{ML^2T^{-2}}$ **B** $\mathrm{E} = \mathrm{ML^2T^{-2}}$

C $[\mathrm{E}] = \mathrm{kg\,m^2s^{-2}}$ **D** $\mathrm{E} = \mathrm{kg\,m^2s^{-2}}$

Question 11

The left-hand end of a horizontal damper whose damping constant is r is a distance $x(t)$ from a fixed point to the left of the damper. The right-hand end of the damper is attached to a particle of mass m whose position is at a distance $y(t)$ from the fixed point. The unit vector $\mathbf{i}$ is horizontal and its direction is to the right.



Which option gives the damping force on the particle?

Options Answer:

A $-r\dot{x}\,\mathbf{i}$ **B** $-r\dot{y}\,\mathbf{i}$ **C** $-r(\dot{x} - \dot{y})\,\mathbf{i}$ **D** $-r(\dot{y} - \dot{x})\,\mathbf{i}$

Question 12

A particle of mass 4 kg is moving with speed $13\,\mathrm{m\,s^{-1}}$ in the direction $-5\,\mathbf{i} + 12\,\mathbf{j}$. Select the option that gives the momentum of the particle.

Options Answer: ☐

A 52 **B** 338 **C** $4(-5\,\mathbf{i} + 12\,\mathbf{j})$ **D** $52(-5\,\mathbf{i} + 12\,\mathbf{j})$

Question 13

Select the option that gives the even extension of the function

$$g(x) = 2 - x \quad 0 < x \leq 1.$$

Options Answer: ☐

A $g_{\text{even}}(x) = \begin{cases} 2 - x & -1 \leq x < 0 \\ 2 - x & 0 < x \leq 1 \end{cases}$

B $g_{\text{even}}(x) = \begin{cases} 2 + x & -1 \leq x < 0 \\ 2 - x & 0 < x \leq 1 \end{cases}$

C $g_{\text{even}}(x) = \begin{cases} -2 - x & -1 \leq x < 0 \\ 2 - x & 0 < x \leq 1 \end{cases}$

D $g_{\text{even}}(x) = \begin{cases} -2 + x & -1 \leq x < 0 \\ 2 - x & 0 < x \leq 1 \end{cases}$

Question 14

Choose the option that gives the gradient at $(1, -1)$ of the function

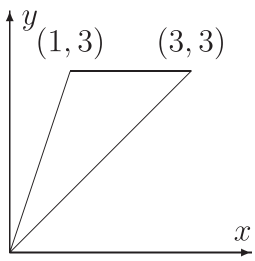
$$f(x, y) = x^2 - y.$$

Options Answer: ☐

A $-2\,\mathbf{i} + \mathbf{j}$ **B** $2\,\mathbf{i} - \mathbf{j}$ **C** $-2\,\mathbf{i} - \mathbf{j}$ **D** $+2\,\mathbf{i} + \mathbf{j}$

Question 15

The function $\sigma(x, y)$ is to be integrated over the area bounded by $y = 3$, $y = x$ and $y = 3x$.
Select the option that gives the correct integral.



Options Answer: ☐

A $\displaystyle\int_{y=0}^{y=3} \int_{x=0}^{x=3} \sigma(x, y)\, dx\, dy$ **B** $\displaystyle\int_{x=0}^{x=3} \int_{y=0}^{y=3} \sigma(x, y)\, dy\, dx$

C $\displaystyle\int_{y=0}^{y=3} \int_{x=\frac{1}{3}y}^{x=y} \sigma(x, y)\, dx\, dy$ **D** $\displaystyle\int_{x=0}^{x=3} \int_{y=x}^{y=3x} \sigma(x, y)\, dy\, dx$

PART 2

Each question in this part of the paper is worth 5 marks.

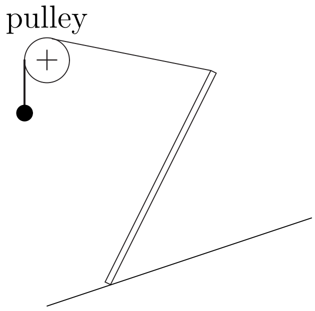
Question 16

Find the particular solution of the differential equation

$$2\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 6y = 0, \quad y(0) = 1, \frac{dy}{dt}(0) = -1. \qquad [5]$$

Question 17

One end of a plank rests on a rough slope. A cord is attached to the other end of the plank, and passes over a pulley with a mass hanging down on the other side of the pulley as shown in the diagram on the right. The system is in equilibrium. The plank can be taken as a model rod, the pulley as a model pulley, and the cord as a model string.



Draw a force diagram showing all the forces acting on the plank, and define each force identified. [5]

Question 18

Two fixed points are vertically one above the other and the vertical distance between them is $3l_0$. A particle of mass m is suspended by a model string from the upper fixed point, and a model spring with spring stiffness k and natural length l_0 is attached to the particle and the bottom fixed point. The length of the string is $L(< 2l_0)$. The particle is in equilibrium and is above the lower fixed point, so that the string is taut.

(a) Draw a force diagram showing all the forces acting on the particle. [*1*]

(b) Choose a suitable origin and the unit vector(s); express the forces as components of the unit vectors. [*2*]

(c) Find the magnitude of the tension in the string in terms of m, g, L, k and l_0. [*2*]

Question 19

Consider the pair of simultaneous non-linear differential equations:

$$\frac{dx}{dt} = x^2 - 3xy,$$
$$\frac{dy}{dt} = x - y^2 - 2.$$

(a) Confirm that $(3, 1)$ is an equilibrium point. [*1*]

(b) Classify this equilibrium point. [*4*]

Question 20

The matrix

$$\begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$$

has eigenvalues -1 and -4; the corresponding eigenvectors are $[1 \quad 1]^T$ and $[2 \quad -1]^T$.

The differential equation of motion for a system of two particles connected by springs is

$$\ddot{\mathbf{x}} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \mathbf{x}$$

where x_1 and x_2 are the displacements of the two particles from their respective equilibrium positions and $\mathbf{x} = [x_1 \; x_2]^T$.

(a) Write down the general solution for $\mathbf{x}$. *[2]*

(b) The system is at rest initially and $x_2(0) = 0.2$. What should be the displacement of the other particle for a single normal mode motion at the higher angular frequency? *[1]*

(c) The particles are given an initial displacement $\mathbf{x}(0) = [0.3 \quad 0]^T$ and released from rest. Determine an explicit vector expression for the displacement of each particle at time t. *[2]*

Question 21

The position vector, in plane polar unit vectors, of a particle of mass m, which is moving in a circle, is given by $\mathbf{r} = R\,\mathbf{e}_r$ where R is a constant. The particle has angular velocity $\boldsymbol{\omega} = \sin(\Omega t)\mathbf{k}$ where Ω is a constant.

(a) Find the velocity of the particle at time t. *[1]*

(b) Find the angular momentum of the particle about the origin. *[2]*

(c) Hence find the torque acting on the particle. *[2]*

Question 22

Consider the partial differential equation

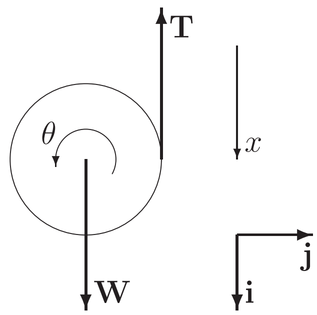
$$\frac{\partial^3 u}{\partial x^3} = \frac{1}{c^2}\left(\frac{\partial^2 u}{\partial t^2} + u\right).$$

Find the two ordinary differential equations that are created by the use of the separation of variables method. **Do not solve the differential equations.** *[5]*

Question 23

A model string is wrapped round the circumference of a circular disc of mass M, radius R and moment of inertia I. One end of the string is attached to a fixed point, and the disc is held below the fixed point so that the string is vertical and taut. The disc is released from this position and descends, rotating about its axis on its descent. Ignore air-resistance.

The forces acting on the disk are shown in the diagram as well as the direction of the unit vectors, with the unit vector **k** coming out of, and perpendicular to, the plane. x is the downward vertical displacement of the centre of the disc from some arbitrary origin, and θ is the anti-clockwise rotation of the disc around its axis from some arbitrary origin.



(a) Find the torque due to these forces about the centre of the disc, and hence derive a differential equation of rotational motion. *[3]*

(d) Derive a differential equation for the downward motion of the disc. *[1]*

(e) Hence, and using the relation $R\dot{\theta} = \dot{x}$, show that the component of downward acceleration is $\dfrac{MR^2}{MR^2 + I}g$. *[1]*

PART 3

Each question in this part of the paper is worth 15 marks. All of your answers will be marked and the marks from your best three answers will be added together. A maximum of 45 marks can be obtained from this part.

Question 24

(a) Solve the differential equation

$$\frac{dy}{dx} = (5\tan(x) + 6x^2)y^2, \quad y(0) = -\tfrac{1}{2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y < 0$$

expressing your answer in the form $y = f(x)$. [5]

(b) For the differential equation

$$\frac{dy}{dx} - 4xy^2 = x^3, \quad y(0) = 1,$$

use Euler's method with a step size of $h = 0.1$ to determine an approximate solution to $y(0.2)$ at $x = 0.2$. [4]

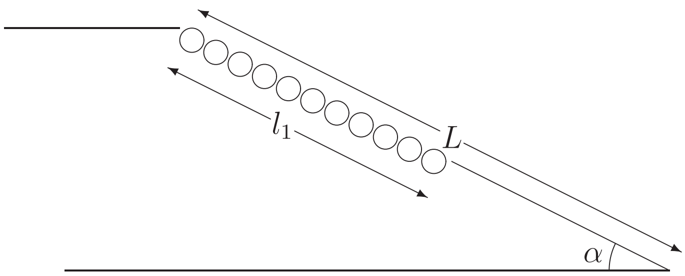
(c) Solve the differential equation

$$x\frac{dy}{dx} - 4y = x^2, \quad x > 0, \quad y(1) = \tfrac{1}{2},$$

expressing your answer in the form $y = f(x)$. [6]

Question 25

At an airport luggage handling system a straight ramp going from a higher level to a lower level is of length L, and at a constant angle of α to the horizontal. The upper part of the ramp is of length l_1 and has rollers, so that any luggage on this part of the ramp slides down freely. The lower part of the ramp is rough and the coefficient of sliding friction between the surface and any luggage is μ. Luggage arrives at the top of the ramp with negligible speed, and can be considered as a particle. The aim of the model is to find the length, l_1, of the rollers such that the luggage comes to rest at the bottom of the ramp.



(a) Defining any variables or symbols that you introduce, find the speed of the luggage just before it encounters the lower part of the ramp. [5]

(b) Defining any variables or symbols that you introduce, find the speed of the luggage just before it reaches the bottom of the ramp. [7]

(c) Find the value of l_1 such that the luggage comes to rest at the bottom of the ramp. [2]

(d) Consider the case where μ is very large; does your solution to part (c) seem reasonable? [1]

Question 26

(a) The matrix

$$\begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$$

has a repeated eigenvalue and only one eigenvector. Determine the value of the repeated eigenvalue and the corresponding eigenvector. *[4]*

Consider the simultaneous linear differential equations

$$\frac{dx}{dt} = 5x + 4y - 18$$
$$\frac{dy}{dt} = -x + y$$

which satisfy the initial conditions $x(0) = 3$, $y(0) = 1$.

(b) Express the pair of differential equations in matrix form. *[1]*

(c) Write down the solution to the associated homogeneous differential equations. *[3]*

(d) Find the general solution of the simultaneous differential equations. *[4]*

(e) Find the particular solution that satisfies the given initial conditions. *[3]*

Question 27

A particle of unit mass is initially on a track and following the path given parametrically by

$$x(t) = \tfrac{1}{12}t^4 - \tfrac{1}{2}t^2 + \tfrac{5}{3}t - \tfrac{1}{4}$$
$$y(t) = -\tfrac{1}{2}gt^2 - \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2 + gt + \tfrac{2}{3} - \tfrac{1}{2}g$$

where g is the magnitude of acceleration due to gravity. After some time the particle leaves the path and the time at which this happens will be investigated in part (c). The position vector of the particle is $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ where t represents time from an arbitrary origin. The unit vector $\mathbf{j}$ is vertically upwards. Air-resistance is negligible.

(a) Find the velocity and acceleration of the particle at time t whilst the particle is on the track. *[4]*

(b) Besides the weight acting on the particle, the only other force acting on the particle is the normal reaction from contact with the track. Find the normal reaction as a function of t whilst the particle is on the track. *[2]*

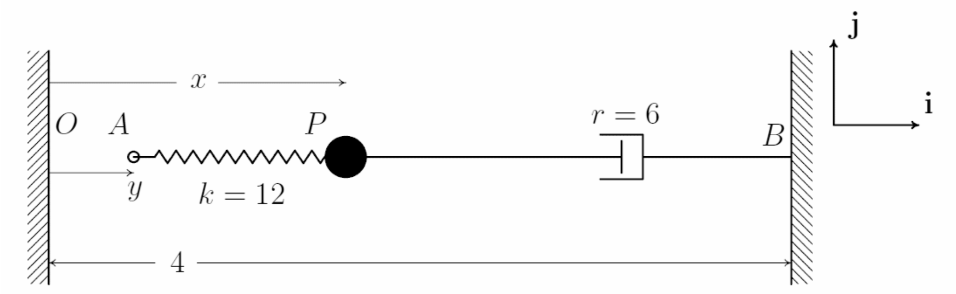
(c) Write down the condition that contact between the particle and the track is lost. Find the time at which this occurs. *[3]*

(d) After contact is lost with the track the particle behaves as a projectile.

Retaining the same coordinate axes, and the same origins for displacement and time, what are the initial conditions for the particle's motion as a projectile? Hence determine the position vector for the particle whilst it acts as a projectile. *[6]*

Question 28

A particle P of mass 3 kg moves in a straight line along a smooth horizontal track and is connected to damper, BP, and one model spring, AP, as shown in the diagram below. The left-hand end, A, of the spring AP is at a distance $y(t)$ from a fixed point O, and $x(t)$ is the distance of the particle from O. The right-hand end of the damper BP is attached to a fixed point on the right, which is a distance 4 m from O. Unit vectors $\mathbf{i}$ and $\mathbf{j}$ are defined horizontally and vertically as shown. The parameters for the damper and the spring are given in the table below.



Object	parameters
Damper BP	Damping coefficient, $6\,\mathrm{N\,s\,m^{-1}}$
Spring AP	Spring stiffness, $12\,\mathrm{N\,m^{-1}}$; natural length, 1 m

(a) Draw a force diagram to show all the forces acting on the particle. Define each force that you introduce. [2]

(b) At time t, derive in full each force in terms of the given unit vectors. [4]

(c) Determine a differential equation of motion and show that it may be expressed as

$$3\ddot{x} + 6\dot{x} + 12x = 12(1 + y).$$ [2]

(d) (i) Write down the natural angular frequency of this system. [1]

(ii) Find the damping ratio and hence state, with justification, whether this system is strongly or weakly damped. [1]

(e) Find the amplitude of the motion in the steady-state for forced periodic motion when $y = 1 + \frac{1}{2}\cos(\Omega t)$. [2]

(f) Find the value of the amplitude when $\Omega = 2$, $\Omega \to 0$ and $\Omega \to \infty$. Explain whether this system can exhibit resonance. Hence sketch the graph of the amplitude of the motion as a function of Ω. [3]

Question 29

Consider the vector field,

$$\mathbf{F}(x, y) = \left(x^2 + y^2\right)\mathbf{i} + 2y\,\mathbf{j}.$$

(a) A path is defined parameterically (where R is a positive constant):

$$x = R\cos(t), \quad y = R\sin(t) \quad (0 \le t \le 2\pi)$$

Show that the path is closed, and sketch the path, indicating the direction as t increases. *[2]*

(b) (i) Find the value of the scalar line integral along the path given in part (a). *[6]*

(ii) What can you deduce as to whether the vector field is conservative? *[1]*

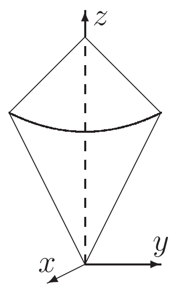
(c) Find $\mathbf{curl}\,\mathbf{F}$. *[4]*

(d) What can you deduce about the vector field? *[1]*

(e) Discuss whether there is a contradiction between your conclusions in parts (b)(ii) and (d)? *[1]*

Question 30

Consider the volume of a body whose surface is formed on the underside by the cone $z = 2\sqrt{x^2 + y^2}$ and bounded on top by the cone $z = 6 - \sqrt{x^2 + y^2}$ as illustrated in the diagram.



(a) Explain which coordinate system should be used in order to describe the shape easily, and transform the surfaces bounding the shape into this coordinate system (if necessary). *[2]*

(b) Find the equation of the curve of intersection of the two surfaces, and describe it. What is the value of the area bounded by the curve of intersection? *[3]*

(c) Set up the volume integral for the body, and calculate the volume. Compare your answer with that deduced from using the standard formula for the volume of a cone, i.e.
volume $= \frac{1}{3}$ (area of the base) $\times$ (perpendicular distance of the vertex). *[6]*

The density of the material is $\sigma(1 + \rho)$ where σ is a constant, and ρ is the distance from the z-axis.

(d) Set up the volume integral to find the mass of the body, M, and determine the value of the mass in terms of σ. *[3]*

(e) Set up the volume integral to find the moment of inertia of the body about the z-axis; **you are not required to solve the integral**. *[1]*

[END OF QUESTION PAPER]