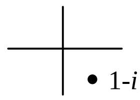
Based on tutorials of dr. Paul Reed and the OUSA revision weekend 1998, and my own
Typed by I.R. van de Stadt BSc. (Hons)

Part I

Question 1

(a) (i) $(1-i)^8$ [Put $1 - i$ in polar form]

$1-i = Re^{iq}$, with $R = \sqrt{1^2+1^2} = \sqrt{2} = 2^{1/2}$; $q = \tan^{-1}(-1/1) = -p/4$ or $3p/4$.



Since 1-*i* is in the 4th quadrant, we must have $q = -p/4$

So, $(1-i)^8 = \left(2^{1/2}e^{-ip/4}\right)^8 = 2^{8/2}e^{-i8p/4} = 2^4e^{-i2p} = 2^4 = 16$

[Remember $e^{iq} = \cos q + i\sin q$, so if $q = -2p$, we get $e^{-i2p} = \cos(-2p) + i\sin(-2p) = 1$]

(a) (ii) $(1+i)^8$ [Put $1 + i$ in polar form]

$1+i = \sqrt{2}e^{ip/4}$ [Note it is good to remember some of the common arguments like $\mathrm{Arg}(1+i) = \frac{p}{4}$ etc.]

So, $(1+i)^8 = \left(2^{1/2}e^{ip/4}\right)^i = 2^{i/2}e^{i\cdot ip/4} = e^{-p/4}2^{i/2}$

This is real

This needs to be simplified

Now, $2^{i/2} = e^{\frac{i}{2}\mathrm{Log}_e 2} = e^{i\log_e\sqrt{2}} = \cos\left(\log_e\sqrt{2}\right) + i\sin\left(\log_e\sqrt{2}\right)$

So the answer is $e^{-p/4}\left(\cos\left(\log_e\sqrt{2}\right) + i\sin\left(\log_e\sqrt{2}\right)\right)$

(b) $\tan(2i) = \dfrac{\sin(2i)}{\cos(2i)}$ [Now use the result that $\cos q = \frac{1}{2}\left(e^{iq} + e^{-iq}\right)$; $\sin q = \frac{1}{2i}\left(e^{iq} - e^{-iq}\right)$]

We get $\tan(2i) = \dfrac{e^{i2i} - e^{-i2i}}{i\left(e^{i2i} + e^{-i2i}\right)} = \dfrac{e^{-2} - e^{2}}{i\left(e^{-2} + e^{2}\right)}$

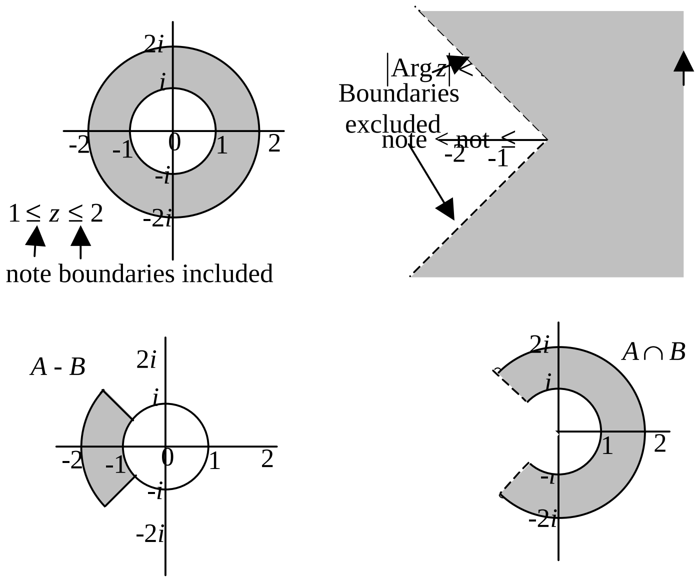
[Now take e^{-2} out as a factor in numerator and denominator]

We get $\tan(2i) = \dfrac{e^{-2} - e^{2}}{i\left(e^{-2} + e^{2}\right)} = \dfrac{e^{-2}\left(1 - e^4\right)}{ie^{-2}\left(1 + e^4\right)} = \dfrac{1 - e^4}{i\left(1 + e^4\right)} = \dfrac{-i\left(1 - e^4\right)}{1 + e^4} = i\dfrac{e^4 - 1}{e^4 + 1}$

[Here we have used the result that $\frac{1}{i} = -i$]

Question 2

(a)



(b) $A - B$ Not a region since not open
$A - B$ Compact since bounded and closed
$A \cap B$ Not a region since not open
$A \cap B$ Not compact since not closed

[If I am right then this is a tricky question since $A \cap B$ is not open and not closed. This is possible since open and closed are not opposites as you might think.]

Question 3

(a) The line joining 0 to 1 + *i* has equation $z = (1+i)t \quad 0 \le t \le 1$.

So, $\int_\Gamma \bar{z}^2 dz = \int_0^1 \left(\overline{(1+i)t}\right)^2 \frac{dz}{dt} dt = \int_0^1 (1-i)^2(1+i)t^2 dt = (1-i)^2(1+i)\left[\frac{1}{3}t^3\right]_0^1 = \frac{(1-i)^2(1+i)}{2} = \frac{2}{3}(1-i)$

[Note the equation of a line joining z_1 and z_2 is always given by $z_1 + t(z_2 - z_1) \quad 0 \le t \le 1$.]

(b) $\left|\int_\Gamma \exp(\bar{z}^2) dz\right| \le \max\left(\left|\exp(\bar{z}^2)\right|\right) \cdot L$

Since $\exp(\bar{z}^2)$ is continuous on Γ and where *L* is the length of Γ. Length of Γ is $\sqrt{2}$ (From Pythagoras). Also,

$\left|\exp(\bar{z}^2)\right| = \left|\exp((1-i)^2 t^2)\right| = \left|\exp(-2it^2)\right|$

and the maximum value of this is 1. [Note $\left|e^{iq}\right|$ is always 1 for q real.]

Finally we get $\left|\int_\Gamma \exp(\bar{z}^2) dz\right| \le 1 \cdot \sqrt{2} = \sqrt{2}$

[The problem with this question is that you might think you need to use the answer to part (a).
You DON'T.]

Question 4

(a) $f(z) = \frac{1}{z^2 - 1}$ [When asked to give the Laurent series about any point 'a' always rewrite $f(z)$ in terms of $z - a$ as shown below]

$f(z) = \frac{1}{z^2 - 1} = \frac{1}{(z-1)(z+1)} = \frac{1}{(z-1)((z-1)+2)}$ [Using $z + 1 = z - 1 + 2$]

Difference of two squares So here we have *z* - 1

Now since $|z-1| < 2$, we must rewrite $f(z)$ as below

$$f(z) = \frac{1}{(z-1)((z-1)+2)} = \frac{1}{2(z-1)\left(1 + \frac{(z-1)}{2}\right)} = \frac{1}{2(z-1)}\left(1 + \frac{(z-1)}{2}\right)^{-1}$$

[Now use the result that $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \quad$ for $|x| < 1$]

We get since $\left|\frac{z-1}{2}\right| < 1$

$$f(z) = \frac{1}{2(z-1)}\left(1 - \left(\frac{z-1}{2}\right) + \left(\frac{z-1}{2}\right)^2 + \cdots + (-1)^n\left(\frac{z-1}{2}\right)^n\right)$$

This is put in to give the alternating sign pattern.

$$\therefore\ f(z) = \frac{1}{2(z-1)} - \frac{1}{4} + \frac{1}{4}\left(\frac{z-1}{2}\right) + \cdots + \frac{1}{4}(-1)^n\left(\frac{z-1}{2}\right)^{n-1} +$$

(general term)

(b) [No need for much algebra here]

$$f(z) = \frac{1}{z^2 - 1} = \frac{1}{z^2\left(1 - \frac{1}{z^2}\right)} = \frac{1}{z^2}\left(1 - \frac{1}{z^2}\right)^{-1}$$

Since $|1/z| < 1$, we can expand using $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \quad$ for $|x| < 1$

We get

$$f(z) = \frac{1}{z^2}\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots + \frac{1}{z^{2n}} + \cdots\right) = \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \cdots + \frac{1}{z^{2(n+1)}} + \cdots$$

(general term)

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Question 5

(a) Simple poles when $z^3+1=0$ [Solve this equation]

$z^3=-1=e^{i(p+2pn)} \Rightarrow z=e^{i(p+2pn)/3}$

Now take n = 0, 1, 2 to give 3 roots:

n = 0, $z=e^{ip/3}$; **n = 1**, $z=e^{i(p+2p)/3}=e^{ip}$; **n = 2**, $z=e^{i(p+4p)/3}=e^{i5p/3}=e^{-ip/3}$

[To find residue, use the result on page 8 of Unit C1 = g/h rule HB1.2 p.28]

I don't know if it is strictly necessary to change to Principal Arg, but I have done

$$\text{Res}(f,a)=\left.\frac{1}{\frac{d}{dz}\left(z^3+1\right)}\right|_a=\left.\frac{1}{3z^2}\right|_a$$

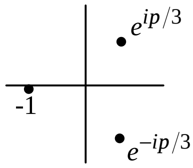
So Res at $a=e^{ip/3}$ is $\dfrac{1}{3\left(e^{ip/3}\right)^2}=\dfrac{1}{3e^{2ip/3}}=\dfrac{e^{-2ip/3}}{3}$

So Res at $a=e^{ip}$ is $\dfrac{1}{3\left(e^{ip}\right)^2}=\dfrac{1}{3e^{2ip}}=\frac{1}{3}$

So Res at $a=e^{-ip/3}$ is $\dfrac{1}{3\left(e^{-ip/3}\right)^2}=\dfrac{1}{3e^{-2ip/3}}=\dfrac{e^{2ip/3}}{3}$

No need to put answers in Cartesian form unless asked.

(b) $\displaystyle\int_{-\infty}^{\infty}\frac{1}{t^3+1}\,dt$ [Always draw a sketch showing poles]



the integrand $\dfrac{1}{t^3+1}$ satisfies the conditions for closing the contour in the upper half plane. Call this contour Γ.

[See Unit C1 for details page 25 onwards.]

We get

$$\int_{-\infty}^{\infty}\frac{1}{t^3+1}\,dt=\int_\Gamma\frac{1}{z^3+1}\,dz=pi\cdot\text{Res at -1}+2pi\cdot\text{Res at }e^{ip/3}$$

[note the factors of pi and $2pi$. pi is used when the pole is on the real axis. See page 27 of C1]

So we get

$$pi\cdot\tfrac{1}{3}+2pi\cdot e^{-2pi/3}=\frac{pi}{3}+\frac{2pi}{3}\left(\cos\left(\frac{2p}{3}\right)+i\sin\left(\frac{2p}{3}\right)\right)=\frac{pi}{3}+\frac{2pi}{3}\left(-\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)=\frac{-2pi\cdot i}{3}\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{3}p}{3}=\frac{p}{\sqrt{3}}$$

[To help remember if the pole is inside the contour we get a contribution of $2pi\times\text{Residue}$. If we have a simple pole on the contour the contribution is $pi\times\text{Residue}$]

Question 6

When $|z|=2$, the dominant term is $g_1(z)=z^6$. Both f and g_1 are analytic on (, which is a simply-connect region. $C_1=\{z:|z|=2\}$ is a simple-closed contour in (.

Now $\left|-3iz^4+1\right|\le\left|-48i+1\right|=\sqrt{48^2+1}<49<2^6=64=\left|g_1(z)\right|$ for $z\in C_1 \Rightarrow 6$ zeros in C_1.

When $|z|=1$, the dominant term is $g_2(z)=-3iz^4$. Then g_2 is analytic on (. $C_2=\{z:|z|=1\}$ is a simple-closed contour in (. Now $\left|z^6+1\right|\le 2<3=\left|g_2(2)\right|$ for $z\in C_2 \Rightarrow 4$ zeros in C_2.

And no zero's on the contours, since the inequalities are strict. So, there are 2 zeros in $\{z:1<|z|<2\}$.

Question 7

(a) $q(z) = \bar{z} + i$, so $q(z) = \overline{z - i}$ and $z - i$ is an analytic function. So q is the complex conjugate of an analytic function an do a model flow.

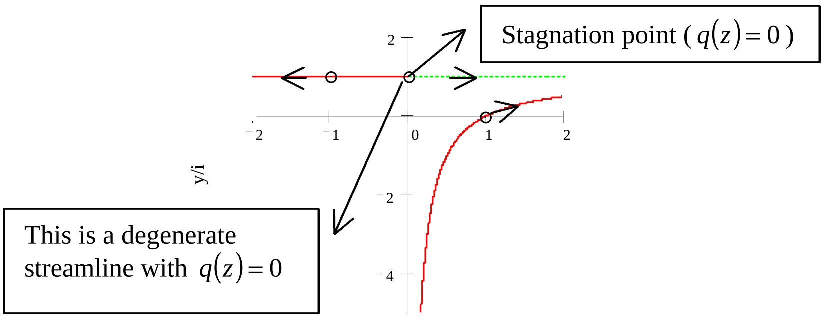
(b) $\Omega'(z) = \bar{q}(z) = z - i \Rightarrow \Omega(z) = \frac{1}{2}z^2 - iz$

Stream function is $\operatorname{Im}\Omega = \text{constant}$. So $\operatorname{Im}\left(\frac{x^2 - y^2 + 2ixy}{2} - ix + y\right) = xy - x = \text{constant} = k$

Point 1 $\quad x = 1, y = 0 \Rightarrow k = -1$. So $xy - x + 1 = 0 \Rightarrow y = \frac{x-1}{x} = 1 - \frac{1}{x}$ (since $x \neq 0$)

Point $-1 + i$ $\quad x = -1, y = 1 \Rightarrow -1 - (-1) = 0 = k$. So $xy - x = 0 \Rightarrow y - 1 = 0$ (since $x \neq 0$)

(c) $q(1) = 1 + i; \quad q(-1 + i) = -1 - i + i = -1$



[The definition of degenerate streamline is given on page 8 of D2. Essentially it is the set of points forming the stagnation point z_0 . That is the stagnation point itself $\{z_0\}$.]

Question 8

(a) [fixed points are solutions to the equation $f(z) = z$]

From $f(z) = z$ we have $z^2 + \frac{1}{4} = z \Leftrightarrow z^2 - z + \frac{1}{4} = 0$.

Using the formula for quadratic equations we have

$$z = \frac{1 \pm \sqrt{1 - 4 \times \frac{1}{4}}}{2} = \frac{1}{2}$$

So there is only one fixed point at $z = \frac{1}{2}$.

Now $f'(z) = 2z$

So $f'\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} = 1$

So the fixed point is indifferent. [see page 24 of unit D3]

(b) (i) This looks like it lies outside so try to show it is by using Corollary 1 on page 37 of D3.

We have $P_c(z) = z^2 - 1 + i \Rightarrow |P_c(0)| = |-1 + i| = \sqrt{2} < 2$

$P_c^2(0) = P_c(P_c(0)) = (-1 + i)^2 - 1 + i = -2i - 1 + i = -1 - i \Rightarrow \left|P_c^2(0)\right| = |-1 - i| = \sqrt{2} < 2$

$P_c^3(0) = P_c(P_c(P_c(0))) = P_c(-1 - i) = (-1 - i)^2 - 1 + i = 2i - 1 + i = -1 + 3i \Rightarrow$

$\left|P_c^2(0)\right| = |-1 + 3i| = \sqrt{1 + 9} = \sqrt{10} > 2$

Since $\left|P_c^3(0)\right| > 2$ c does not belong to the set M by Corollary 1 on page 37.

(b) (ii) $c = -\frac{1}{2} - \frac{1}{2}i$ looks like it does belong to M. Try to show c belongs to main cardiod (page 40 D3).

Look at

$$\left(8|c|^2 - \frac{3}{2}\right)^2 + 8\operatorname{Re} c = \left(8 \times \frac{1}{2} - \frac{3}{2}\right)^2 - 8 \times \frac{1}{2} = \left(\frac{5}{2}\right)^2 - 4 = \frac{25}{4} - 4 = \frac{9}{4} = 2\frac{1}{4} < 3$$

So $P_{-\frac{1}{2}-\frac{i}{2}}$ has an attracting fixed point (Theorem 2.4) and so $-\frac{1}{2} - \frac{1}{2}i \in M$ by theorem 4.3.

Part II

Question 9

(a) $f(z) = (x+iy)^2 + ax^2 + iby^2 = \underbrace{(a+1)x^2 - y^2}_{u(x,y)} + i\underbrace{\left(2xy + by^2\right)}_{v(x,y)}$

The Cauchy Riemann equations are satisfied in any region in which $f(z)$ is analytic.
The C.R. equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

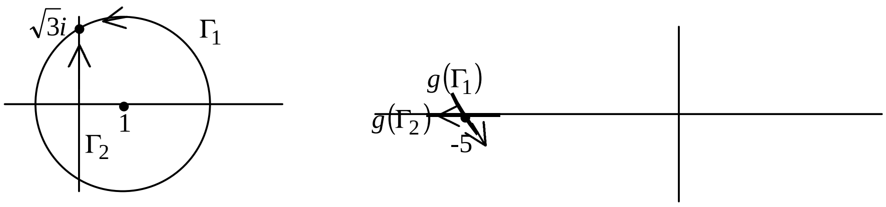
Now $u = (a+1)x^2 - y^2, \quad v = 2xy + by^2$.
We have

$$\frac{\partial u}{\partial x} = 2(a+1)x; \quad \frac{\partial v}{\partial y} = 2x + 2by; \quad \frac{\partial u}{\partial y} = -2y; \quad \frac{\partial v}{\partial x} = 2y$$

which are all continuous
So we arrive at the following C.R. equations
$2(a+1)x = 2x + 2by \Rightarrow a = 0$ and $b = 0, \forall x, y$
$-2y = -2y$ which is true for all y
So by C.R. Theorem and its converse *f* is analytic at (if and only if $a = b = 0$

(b) (i) Using HB A4.4.6 p. $g'(z) \neq 2z \neq 0 \Leftrightarrow z \neq 0$. So *g* is conformal on ($-\{0\}$
(or it's not conformal at 0 since the angle between the positive and negative real axis is not preserved.)

(b) (ii) The paths meet when $1 + 2e^{it_1} = it_2 \Leftrightarrow \cos t_1 = -\frac{1}{2}$ and $2\sin t_1 = t_2 \Rightarrow t_1 = \frac{2}{3}p, t_2 = \sqrt{3}$,
so they meet at $\sqrt{3}i$. Now $g(\sqrt{3}i) = \quad 5; g'(\sqrt{3}i) = 2\sqrt{3}i = 2\sqrt{3}e^{ip/2}$.
So maps a small disc round $\sqrt{3}i$ to a small disc round –5, rotated over $p/$
direction, and scaled by $2\sqrt{3}$



(a) e^x is entire, $\dfrac{1}{1-z}$ has a single singularity at 1, so it is analytic on ($-\{1\}$. So the composition
$f(z) = e^{\left(\frac{1}{1-z}\right)}$ is analytic on ($-\{1\}$ by the Composition Rule (HB 23.1 p. 19)
$f(z) = 1 - \dfrac{1}{z-1} + \dfrac{1}{2!(z-1)^2} - \dfrac{1}{3!(z-1)^3} + \cdots$, so the singularity is essential by HB 2.8(c) p.27.

(b) (i) 0, by Cauchy's theorem, HB 1.4 p.22, since *f* is analytic inside $R = \left\{z : |z| = \frac{3}{4}\right\}$ (a simply-connected region and C_1 is inside *R*.

(b) (ii) $\dfrac{f(z)}{(4z-1)^2} = \dfrac{f(z)}{16\left(z-\frac{1}{4}\right)^2}$. So by Cauchy's *n*th Derivative Formula (with *n* = 1) (HB p.22)

$$\frac{1}{16} f'\left(\tfrac{1}{4}\right) = \frac{1}{2pi}\int_{C_1} \frac{f(z)/16}{\left(z-\frac{1}{4}\right)^2}dz \Leftrightarrow \frac{1}{8}pif'\left(\tfrac{1}{4}\right) = \int_{C_1} \frac{f(z)/16}{\left(z-\frac{1}{4}\right)^2}dz = \frac{1}{8}\cdot\frac{16}{9}pie^{4/3} = \frac{2}{9}pie^{4/3}, \text{since } f'(z) = \frac{e^{\frac{1}{1-z}}}{(1-z)^2}$$

(b) (iii) $2pi$, since the singularity 1 is inside C_2. So by Cauchy's Residue Theorem HB p. 28 and HB 4.2 p.28

(b) (iv) $\operatorname{Res}\left(\frac{f}{z},0\right)= f(0)=e$ (By the Cover-up Rule) and Laurent series for $\frac{f}{z}$ about $z=1$ is

$$\frac{1}{1+w}\left(1-\frac{1}{1!\,w}+\frac{1}{2!\,w^2}-\frac{1}{3!\,w^3}\right)=\left(1-w+w^2+\cdots\right)\left(1-\frac{1}{1!\,w}+\frac{1}{2!\,w^2}-\frac{1}{3!\,w^3}+\cdots\right),\quad |w|<1,$$ where

$w=z-1$. So, $\operatorname{Res}\left(\frac{f}{z},1\right)=-\sum_{n=1}^{\infty}\frac{1}{n!}=-e+1$, and hence by Cauchy's Residue Theorem the contour integral equals $2pi(e-e+1)=2pi$

Question 11

(a) $f(z)=\frac{1}{9z^2+1}$ is analytic on ($-\left\{\pm\frac{1}{3}i\right\}$, so by HB 4.3 p.30 $\operatorname{Res}(f,0)=f(0)=1$

By the Cover-up Rule $\operatorname{Res}\left(f,\frac{1}{3}i\right)=\frac{p\operatorname{cosec}\left(\frac{1}{3}ip\right)}{9\left(\frac{1}{3}i+\frac{1}{3}i\right)}=\frac{p}{9\left(\frac{2}{3}i\right)\sinh\left(\frac{1}{3}ip\right)}=\frac{-p}{6\sinh\left(\frac{1}{3}p\right)}$

And $\operatorname{Res}\left(f,-\frac{1}{3}i\right)=\frac{p\operatorname{cosec}\left(-\frac{1}{3}ip\right)}{9\left(-\frac{1}{3}i-\frac{1}{3}i\right)}=\frac{p}{-9\left(\frac{2}{3}i\right)\sinh\left(-\frac{1}{3}ip\right)}=\frac{-p}{6\sinh\left(\frac{1}{3}p\right)}$

(b) $f(z)$ is even and is analytic on ($-\left\{\pm\frac{1}{3}i\right\}$

Also, if S_N is the square contour with vertices at $\left(N+\frac{1}{2}\right)(\pm1\pm i)$, then

$|\operatorname{cosec}pz|\le 1,\quad$ for $z\in S_N$, and $|z|\ge N+\frac{1}{2}$, so that by the Estimation Theorem,

$$\left|\int_{S_N} f(z)\,dz\right|\le\frac{p}{9\left(N+\frac{1}{2}\right)^2+1}\cdot 8\left(N+\frac{1}{2}\right)<\frac{p}{9\left(N+\frac{1}{2}\right)^2}\cdot 8\left(N+\frac{1}{2}\right)=\frac{8p}{9\left(N+\frac{1}{2}\right)}\to 0 \text{ as } N\to\infty$$

Hence, By HB C1 4.3 p.30

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{9n^2+1}=-\frac{1}{2}\left(1-\frac{2p}{6\sinh\left(\frac{1}{3}p\right)}\right)=\frac{p}{6\sinh\left(\frac{1}{3}p\right)}-\frac{1}{2}$$

(c) $f(z)=\frac{1}{k^2z^2+1}$ is analytic on ($-\left\{\pm\frac{1}{k}i\right\}$, so by HB 4.3 p.30 $\operatorname{Res}(f,0)=f(0)=1$

By the Cover-up Rule $\operatorname{Res}\left(f,\frac{1}{k}i\right)=\frac{p\operatorname{cosec}\left(\frac{1}{k}ip\right)}{k^2\left(\frac{1}{k}i+\frac{1}{k}i\right)}=\frac{-p}{2k\sinh\left(\frac{1}{k}p\right)}$

And $\operatorname{Res}\left(f,-\frac{1}{3}i\right)=\frac{p\operatorname{cosec}\left(-\frac{1}{k}ip\right)}{k^2\left(-\frac{1}{k}i-\frac{1}{k}i\right)}=\frac{-p}{2k\sinh\left(\frac{1}{k}p\right)}$

Now

$f(z)$ is even and is analytic on ($-\left\{\pm\frac{1}{k}i\right\}$

Also, if S_N is the square contour with vertices at $\left(N+\frac{1}{2}\right)(\pm1\pm i)$, then

$|\operatorname{cosec}pz|\le 1,\quad$ for $z\in S_N$, and $|z|\ge N+\frac{1}{2}$, so that by the Estimation Theorem,

$$\left|\int_{S_N} f_k(z)\,dz\right|\le\frac{p}{k^2\left(N+\frac{1}{2}\right)^2+1}\cdot 8\left(N+\frac{1}{2}\right)<\frac{p}{k^2\left(N+\frac{1}{2}\right)^2}\cdot 8\left(N+\frac{1}{2}\right)=\frac{8p}{k^2\left(N+\frac{1}{2}\right)}\to 0 \text{ as } N\to\infty$$

Hence, By HB C1 4.3 p.30

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{k^2n^2+1}=-\frac{1}{2}\left(1-\frac{2p}{2k\sinh\left(\frac{1}{k}p\right)}\right)=\frac{p}{2k\sinh\left(\frac{1}{k}p\right)}-\frac{1}{2}$$

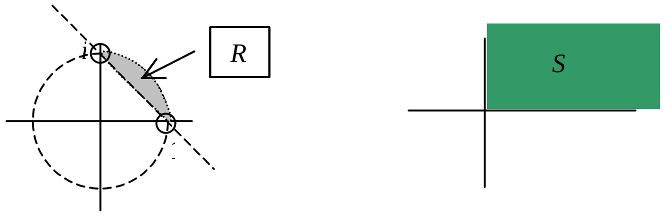
[It is probably not necessary to copy every line in the derivation]

Question 12

(a) This is the standard triple (HB 2.11 p.36) with $a = i, b = \frac{1}{2}(1+i), g = 1$. Hence

$$\hat{f}_1(z) = \frac{(z-i)\left(\frac{1}{2}+\frac{1}{2}i-1\right)}{(z-1)\left(\frac{1}{2}+\frac{1}{2}i-i\right)} = \frac{(z-i)\left(-\frac{1}{2}+\frac{1}{2}i\right)}{(z-1)\left(\frac{1}{2}-\frac{1}{2}i\right)} = \frac{(z-i)(-1+i)}{(z-1)(1-i)} = -\frac{(z-i)}{(z-1)}$$

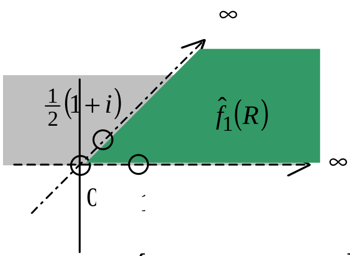
(b) (i)



(b) (ii) To determine the image of R use the three point trick:

$i \to 0, \frac{1}{2}(1+i) \to 1, \quad 1 \to \infty$, with the domain and image to the left

$-1 \to -\dfrac{-1-i}{2} = \frac{1}{2}(1+i), i \to 0, 1 \to \infty,$ with the domain and image to the right, so we get



So, $\hat{f}_1(R) = \left\{z_1 : 0 < \operatorname{Arg} z < \frac{1}{4}p\right\}$

(b) (iii) The quadratic doubles the argument, so $w = z_1^2$ maps $\hat{f}_1(R)$ to S. 1-1 and conformal, since $0 \notin \hat{f}_1(R)$.

Thus $\hat{f}(z) = \left(-\dfrac{z-i}{z+1}\right)^2$

(b) (iv) Since all components are 1-1 and conformal and $h = z^2$; $\hat{f} = \hat{h} \circ \hat{f}_1$, so that $\hat{f}^{-1} = \hat{f}_1^{1} \circ \hat{h}^{-1}$

Now by HB 2.6 p. 36

$\hat{f}_1^{-1}(w) = \dfrac{w+i}{w+i}$

So, $\hat{f}^{-1}(w) = \dfrac{\sqrt{w}+i}{\sqrt{w}+i}$, since in R we only had positive arguments, and hence the inverse of