1 (a) GA1 Let $\begin{pmatrix} a & 1-a \\ 0 & 1 \end{pmatrix} \in G$ and $(x, y) \in \mathbb{R}^2$.

Then $\begin{pmatrix} a & 1-a \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (ax + (1-a)y, y) \in \mathbb{R}^2$.

Hence GA1 is satisfied.

GA2 The identity of G is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \wedge (x, y)$

$$= (1x + 0y, y)$$
$$= (x, y) \text{ for every } (x, y) \in \mathbb{R}^2.$$

Hence GA2 is satisfied.

GA3 Let $\begin{pmatrix} a & 1-a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & 1-b \\ 0 & 1 \end{pmatrix} \in G$ and $(x, y) \in \mathbb{R}^2$.

$$\left(\begin{pmatrix} a & 1-a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 1-b \\ 0 & 1 \end{pmatrix} \right) \wedge (x, y) = \begin{pmatrix} ab & a(1-b) + 1 - a \\ 0 & 1 \end{pmatrix} \wedge (x, y)$$
$$= \begin{pmatrix} ab & 1-ab \\ 0 & 1 \end{pmatrix} \wedge (x, y)$$
$$= (abx + (1-ab)y, y)$$

$$\begin{pmatrix} a & 1-a \\ 0 & 1 \end{pmatrix} \wedge \left(\begin{pmatrix} b & 1-b \\ 0 & 1 \end{pmatrix} \wedge (x, y) \right) = \begin{pmatrix} a & 1-a \\ 0 & 1 \end{pmatrix} \wedge (bx + (1-b)y, y)$$
$$= (a(bx + (1-b)y) + (1-a)y, y)$$
$$= (abx + ay - aby + y - ay, y)$$
$$= (abx + (1-ab)y, y)$$

Since these are equal, GA3 is also satisfied.

GA1, GA2, GA3 being satisfied means that $\wedge$ is a group action.

(b) $$\mathrm{Orb}(1,0)=\left\{(x,y):(x,y)=\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}\wedge(1,0), a\in\mathbb{R}^*\right\}$$
$$=\{(x,y):(x,y)=(a+(1-a)0,0), a\in\mathbb{R}^*\}$$
$$=\{(a,0):a\in\mathbb{R}^*\}$$
$$\mathrm{Orb}(0,1)=\left\{(x,y):(x,y)=\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}\wedge(0,1), a\in\mathbb{R}^*\right\}$$
$$=\{(x,y):(x,y)=(a.0+(1-a)1,1), a\in\mathbb{R}^*\}$$
$$=\{(1-a,1):a\in\mathbb{R}^*\}$$
$$=\{(b,1):b\in\mathbb{R}, b\neq 1\}$$
$$\mathrm{Orb}(1,1)=\left\{(x,y):(x,y)=\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}\wedge(1,1), a\in\mathbb{R}^*\right\}$$
$$=\{(x,y):(x,y)=(a.1+(1-a)1,1), a\in\mathbb{R}^*\}$$
$$=\{(1,1)\}$$

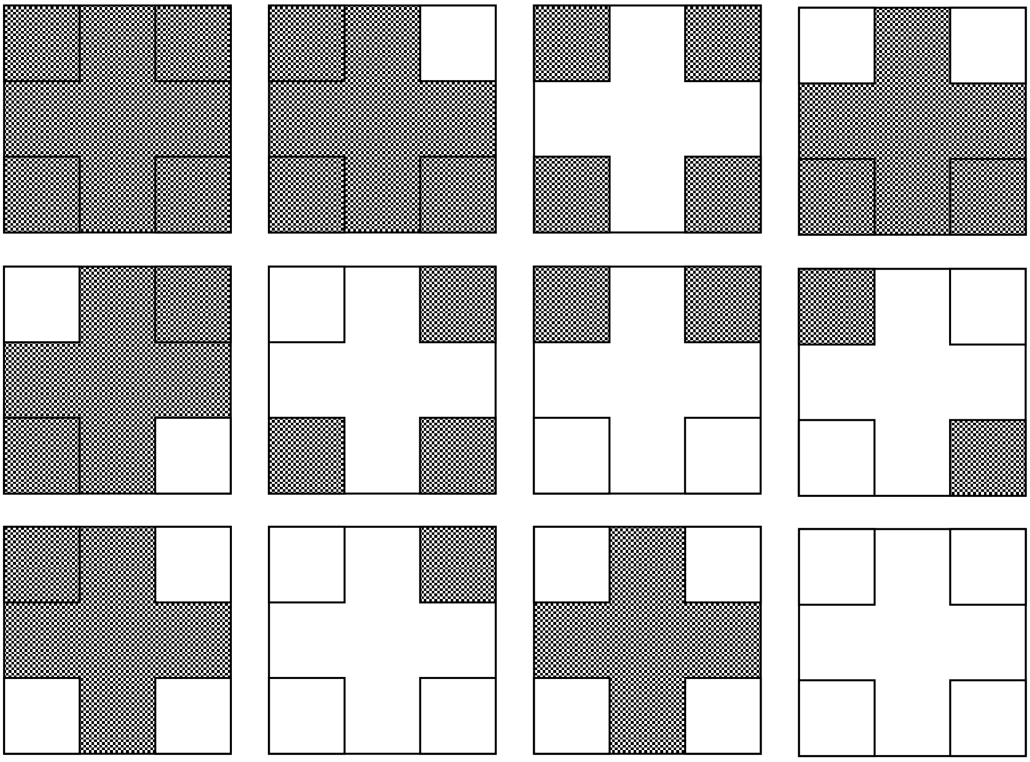
We deduce that the orbits of points of the form (a,a) contain just (a,a).
The orbits of points of the form (a,b), $a\neq b$ are the horizontal lines with equation $y=b$ but excluding the point (b,b).

(c) $$\mathrm{Stab}(1,0)=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}\wedge(1,0)=(1,0), a\in\mathbb{R}^*\right\}$$
$$=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:(a+(1-a).0,0)=(1,0), a\in\mathbb{R}^*\right\}$$
$$=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:(a,0)=(1,0), a\in\mathbb{R}^*\right\}=\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$$

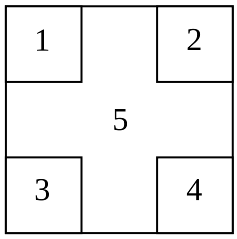
$$\mathrm{Stab}(0,1)=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}\wedge(0,1)=(0,1), a\in\mathbb{R}^*\right\}$$
$$=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:(a.0+(1-a).1,1)=(0,1), a\in\mathbb{R}^*\right\}$$
$$=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:(1-a,1)=(0,1), a\in\mathbb{R}^*\right\}=\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$$

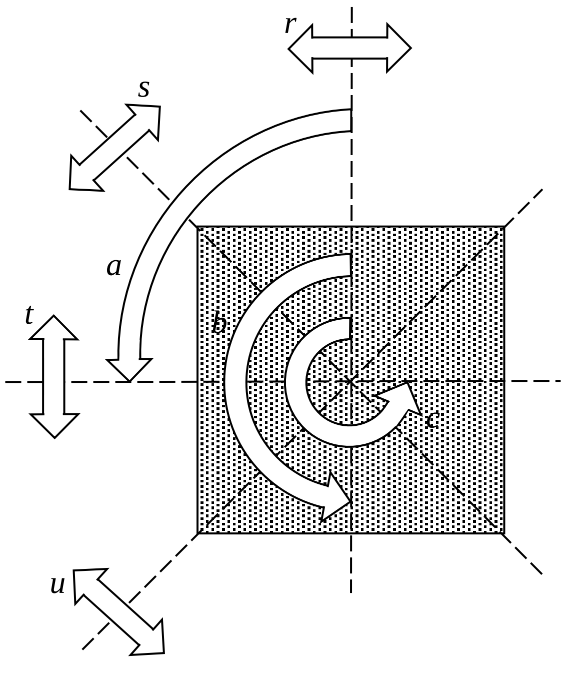
$$\mathrm{Stab}(1,1)=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}\wedge(1,1)=(1,1), a\in\mathbb{R}^*\right\}$$
$$=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:(a.1+(1-a).1,1)=(1,1), a\in\mathbb{R}^*\right\}$$
$$=\left\{\begin{pmatrix}a & 1-a\\ 0 & 1\end{pmatrix}:(1,1)=(1,1), a\in\mathbb{R}^*\right\}=G$$

2 (a)



(b) It is convenient to number the regions of the napkin.





element, g	$\lvert \mathrm{Fix}(g) \rvert$	comment
e	$2^5 = 32$	all five positions can be coloured independently
a	$2^2 = 4$	1, 2, 3,4 must be same
b	$2^3 = 8$	1,3 must be same; 2,4 must be same
c	$2^2 = 4$	1,2,3,4 must be same
r	$2^3 = 8$	1,2 must be same; 3,4 must be same
s	$2^4 = 16$	2,3 must be same
t	$2^3 = 8$	1,3 must be same; 2,4 must be same
u	$2^4 = 16$	1,4 must be same

The number of different napkins is the number of orbits, t, given by the Counting Theorem Handbook p80.

$$
\begin{aligned}
t &= \frac{1}{|G|}\sum_{g \in G} \lvert \mathrm{Fix}(g) \rvert \\
&= \frac{1}{8}(32+4+8+4+8+16+8+16) \\
&= \frac{1}{8}\times 96 \\
&= 12
\end{aligned}
$$