M337/C

Module Examination 2011
Complex Analysis

Tuesday 11 October 2011 **10.00 am – 1.00 pm**

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part 1 (64% of the marks) you should attempt as many questions as you can.

In Part 2 (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is **NOT** permitted in this examination.

PART 1

(i) You should attempt as many questions as you can in this part.
(ii) Each question in this part carries 8 marks.

Question 1

Determine each of the following complex numbers in Cartesian form, simplifying your answers as far as possible.

(a) $\left(\dfrac{1-i}{1+i}\right)^3$ [*2*]

(b) $\exp(2 + i\pi/6)$ [*2*]

(c) $\mathrm{Log}\left(\dfrac{1+i\sqrt{3}}{2}\right)$ [*2*]

(d) $\left(\dfrac{1+i\sqrt{3}}{2}\right)^{3-i}$ [*2*]

Question 2

Let A and B be the sets defined by:

$$A = \{z : -1 \le \mathrm{Re}\, z + \mathrm{Im}\, z \le 1\} \quad \text{and} \quad B = \{z : 2 < |z| < 3\}.$$

(a) Make separate sketches of the three sets A, B and $B - A$. [*3*]

(b) For each of the sets A, B and $B - A$, write down whether or not the set is:

(i) open;

(ii) connected;

(iii) a region;

(iv) bounded;

(v) compact. [*5*]

Question 3

(a) Evaluate

$$\int_\Gamma (\mathrm{Re}\, z)(\mathrm{Im}\, z)\, dz,$$

where Γ is the line segment from i to 1. [*3*]

(b) Determine an upper estimate for the modulus of

$$\int_C \frac{z^2 - 1}{\overline{z}^2 + 1}\, dz,$$

where C is the circle $\{z : |z| = 2\}$. [*5*]

Question 4

Evaluate the following integrals, in which $C = \{z : |z| = 2\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C \frac{\cos z}{z - \pi}\, dz$ [*2*]

(b) $\displaystyle\int_C \frac{\cos z}{z - \pi/3}\, dz$ [*3*]

(c) $\displaystyle\int_C \frac{\cos z}{(z - \pi/2)^4}\, dz$ [*3*]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{z+1}{z(z^2+4)}$$

at each of its poles. [*4*]

(b) Hence evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{t+1}{t(t^2+4)}\, dt.$$ [*4*]

Question 6

This question concerns the solutions of the equation

$$z^7 + 3z^5 - 1 = 0.$$

(a) Use Rouché's Theorem to show that there are exactly two solutions in the annulus $\{z : 1 < |z| < 2\}$. [*6*]

(b) Determine how many solutions lie in the upper half-plane $\{z : \operatorname{Im} z > 0\}$. [*2*]

Question 7

Let $q(z) = i/\bar{z}^2$ be a velocity function.

(a) Explain why q represents a model fluid flow on $\mathbb{C} - \{0\}$. [*1*]

(b) Determine a stream function for this flow. Hence find the equation of the streamline through the point 2, and sketch this streamline, indicating the direction of flow. [*5*]

(c) Evaluate the flux of q across the path Γ, where

$$\Gamma : \gamma(t) = 2t \quad (t \in [1, 2]).$$ [*2*]

Question 8

(a) Show that the iteration sequence

$$z_{n+1} = 2z_n^2 - 4z_n + 2, \qquad n = 0, 1, 2, \ldots,$$

with $z_0 = 1$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 - 2, \qquad n = 0, 1, 2, \ldots,$$

with $w_0 = 0$. *[3]*

(b) Find the fixed points of $P_{-2}(z) = z^2 - 2$ and determine their nature. *[3]*

(c) Determine whether or not $\frac{1}{2} + i$ lies in the Mandelbrot set M. *[2]*

PART 2

(i) You should attempt no more than **TWO** *questions in this part.*
(ii) Each question in this part carries 18 marks.

Question 9

(a) Let f be the function

$$f(z) = 2e^{i\,\mathrm{Re}\,z} - \overline{z}.$$

(i) Write $f(x+iy)$ in the form $u(x,y) + iv(x,y)$, where u and v are real-valued functions. [*1*]

(ii) Use the Cauchy–Riemann Theorem and its converse to determine the set of points S at which f is differentiable. [*6*]

(iii) Show that the derived function f' is constant on S. [*2*]

(b) Let g be the function $g(z) = z^2 + 2$.

(i) Show that g is conformal on $\mathbb{C} - \{0\}$. [*1*]

(ii) Describe the effect of g on a small disc centred at $2i$. [*2*]

(iii) Γ_1 and Γ_2 are the smooth paths given by

$$\Gamma_1 : \gamma_1(t) = t^2 - 1 + 2it \quad (t \in \mathbb{R}),$$
$$\Gamma_2 : \gamma_2(t) = (3t+2)i \quad (t \in \mathbb{R}).$$

Show that the two paths meet at $2i$, and find the angle from Γ_1 to Γ_2 at this point of intersection. [*2*]

(iv) Sketch the paths Γ_1 and Γ_2, clearly indicating their directions. [*2*]

(v) Using part (b)(ii), or otherwise, sketch the directions of $g(\Gamma_1)$ and $g(\Gamma_2)$ at $g(2i)$. [*2*]

Question 10

Let f be the function

$$f(z) = \frac{z}{1 - \cos z}.$$

(a) Write down the domain A of f. [*1*]

(b) Show that the Laurent series about 0 for f is

$$\frac{z}{1 - \cos z} = \frac{2}{z} + \frac{1}{6}z + \frac{1}{120}z^3 + \cdots, \qquad \text{for } 0 < |z| < 2\pi.$$

Hence evaluate the integral

$$\int_C f(z)\, dz,$$

where C is the unit circle $\{z : |z| = 1\}$. [*8*]

(c) Use the Uniqueness Theorem to show that f is the only analytic function with domain A such that

$$f(iy) = \frac{iy}{1 - \cosh y}, \qquad \text{for } y > 0.$$ [*4*]

(d) Show that f has singularities at points of the form $2k\pi$, $k \in \mathbb{Z}$, and classify these singularities. (*Hint*: $\cos z = \cos(z - 2k\pi)$ for $k \in \mathbb{Z}$.) [*5*]

Question 11

(a) Show that the functions

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n \qquad (|z| < 3)$$

and

$$g(z) = -\sum_{n=1}^{\infty} \left(\frac{3}{z}\right)^n \qquad (|z| > 3)$$

are indirect analytic continuations of each other. [*9*]

(b) Determine

$$\max\{|z^2 \exp(1 + z^2)| : |z| \leq 2\},$$

and find all points at which the maximum is attained, giving your answer in Cartesian form. [*9*]

Question 12

(a) Determine the extended Möbius transformation $\widehat{f_1}$ which maps

$$-1-i \text{ to } 0, \quad 0 \text{ to } 1 \quad \text{and} \quad 1+i \text{ to } \infty.$$ *[2]*

(b) Let

$$R = \{z : \tfrac{3}{4}\pi < \operatorname{Im} z < \tfrac{7}{4}\pi\},$$
$$S = \{z_1 : \operatorname{Re} z_1 + \operatorname{Im} z_1 < 0\},$$
$$T = \{w : |w| < 1\}.$$

(i) Sketch the regions R, S and T on separate diagrams. *[3]*

(ii) Explain why $\widehat{f_1}(S) = T$, where $\widehat{f_1}$ is the extended Möbius transformation from part (a). *[5]*

(iii) Hence determine a one-one conformal mapping f from R onto T. (*Hint*: you may find the exponential function useful.) *[3]*

(iv) Obtain a rule for the inverse function f^{-1}. *[3]*

(v) Hence find the point p in R that f maps to 0. State, with a reason, whether every conformal mapping from R to T maps p to 0? *[2]*

[END OF QUESTION PAPER]