Based on tutorials of dr. Paul Reed and the OUSA revision weekend 1998, and my own
Typed by I.R. van de Stadt BSc. (Hons)

Part I

Question 1

(a) $e^{1+ip/6} = e \cdot e^{ip/6} = e\left(\cos\frac{p}{6} + i\sin\frac{p}{6}\right) = \frac{1}{2}e\left(\sqrt{3}+i\right)$

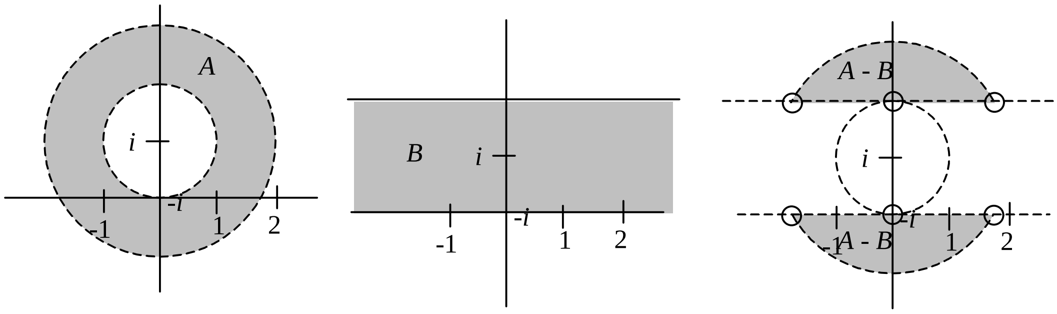
(b) $\dfrac{1}{(1-i)^4} = \dfrac{1}{\left(2^{1/2}e^{-ip/4}\right)^4} = \dfrac{1}{2^2 e^{-4ip/4}} = \frac{1}{4}e^{ip} = -\frac{1}{4}$

(c) $\operatorname{Log} i = \log_e|i| + i\operatorname{Arg} i = 0 + i\frac{1}{2}p = \frac{1}{2}ip$

(d) $i^{(i/p)} = \left(e^{ip/2}\right)^{i/p} = e^{(ip/2)\cdot(i/p)} = e^{-1/2} = \dfrac{1}{\sqrt{e}}$

Question 2

(a)



(b)

Set	(i) open	(ii) a region	(iii) closed	(iv) compact
A	True	True (basic region)	False	False
B	False	False (not open)	True	False (not bounded)
A-B	True	False (not connected	False	False

Question 3

(a) By HB 2.3 p.13 $g(t) = (1-t)\cdot 1 + it = 1 + (-1+i)t, \quad (t \in [0,1])$

(b) Let $z = 1 + (-1+i)t \Leftrightarrow \dfrac{dz}{dt} = i - 1$

Now, $\int_\Gamma \operatorname{Re} z\,dz = \int_0^1 (1-t)\dfrac{dz}{dt}dt = \int_0^1 (1-t)(i-1)dt = \left[(i-1)\left(t - \frac{1}{2}t^2\right)\right]_0^1 = \frac{1}{2}(-1+i)$

(c) *f* is continuous on Γ. Γ has length $\sqrt{2}$.

$|\cosh z| = \frac{1}{2}\left|e^z + e^{-z}\right| \le \frac{1}{2}\left(\left|e^z\right| + \left|e^{-z}\right|\right) = \frac{1}{2}\left(e^x + e^{-x}\right) \le \frac{1}{2}\left(e + e^{-1}\right), \quad (z \in \Gamma)$

$\left|4 + z^2\right| \ge \left|4 - |z|^2\right| \ge 3 \Rightarrow |f| \le \frac{1}{6}\left(e + e^{-1}\right)$

Therefore by the Estimation Theorem

$\left|\int_\Gamma \dfrac{\cosh(\operatorname{Re} z)}{4+z^2}dz\right| \le \frac{1}{6}\left(e+e^{-1}\right)\cdot\sqrt{2} = \frac{\sqrt{2}}{6}\left(e+e^{-1}\right)$

Question 4

(a) Cauchy's Residue Theorem:

Let $R = \{z : |z| < 1.1\}$ then *R* is a simply-connected region. *f* is analytic on *R* except for a pole at 0. *C* is a simple-closed contour in *R*, not passing through 0. So by Cauchy's Residue Theorem

$\int_\Gamma \dfrac{1}{z^3}dz = 2pi\operatorname{Res}\left(\dfrac{1}{z^3}, 0\right) = 0$

(b) Cauchy's *n*th Derivative Formula, with $n = 2$.
Let $R = \{z : |z| < 1.1\}$ then R is a simply-connected region. C is a simple-closed contour in R, $f(z) = \cos(z - p)$ is analytic on R. 0 is inside C and f is 2 times differentiable at 0.
So, by Cauchy's *n*th Derivative Formula, with $n = 2$

$$f''(0) = -\cos(z-p)\big|_{z=0} = -\cos p = 1 = \frac{1}{pi}\int_\Gamma \frac{\cos(z-p)}{z^3}dz \Rightarrow \int_\Gamma \frac{\cos(z-p)}{z^3}dz = pi$$

(c) Cauchy's Theorem:
Let $R = \{z : |z| < 1.1\}$ then R is a simply-connected region, then $\dfrac{\cos z}{(z-p)^3}$ is analytice on R. C is a closed contour in R. So by Cauchy's Theorem $\displaystyle\int_\Gamma \frac{\cos z}{(z-p)^3}dz = 0$

Question 5

(a) $f(z) = \dfrac{z^2+1}{2z\left(z+\frac{1}{2}\right)(z+2)}$. f has simple poles at 0, $-\frac{1}{2}$ and –2 (by HB 1.3 (b) p.27).

By the Cover-up Rule: $\operatorname{Res}(f, 0) = \dfrac{0^2+1}{2\left(0+\frac{1}{2}\right)(0+2)} = \frac{1}{2}$;

$$\operatorname{Res}\left(f, -\tfrac{1}{2}\right) = \frac{\left(-\frac{1}{2}\right)^2+1}{2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+2\right)} = \frac{5/4}{-3/2} = -\frac{5}{6};\ \operatorname{Res}(f, -2) = \frac{(-2)^2+1}{2(-2)\left(-2+\frac{1}{2}\right)} = \frac{5}{12/2} = \frac{5}{6}$$

(b) See HB 2.2 p.28. Let $z = e^{it} \Rightarrow \cos t = \frac{1}{2}\left(z + z^{-1}\right)$, $\dfrac{dt}{dz} = \dfrac{1}{iz}$, so that

$$\int_{|z|=1} \frac{\frac{1}{2}\left(z+z^{-1}\right)}{5+2\left(z+z^{-1}\right)} \cdot \frac{1}{iz}dz = \int_{|z|=1} \frac{\left(z+z^{-1}\right)}{10+4\left(z+z^{-1}\right)} \cdot \frac{1}{iz}dz = -i\int_{|z|=1} \frac{z^2+1}{2z\left(2z^2+5z+2\right)}dz =$$

$$\frac{-i}{2}\int_{|z|=1} \frac{z^2+1}{z(2z+1)(z+2)}dz = \frac{-i}{2} \cdot 2pi\left(\frac{1}{2} - \frac{5}{6}\right) = -\frac{1}{3}p$$

Question 6

(a) When $|z| = 2$, the dominant term is $g_1(z) = z^3$, then f and g_1 are analytic on the simply-connected region $R = \{z : |z| < 2.1\}$, $\{z : |z| = 2\}$ forms a simple-close contour in R and
$|z - 3| \le |z| + 3 \le 5 < 8 = |g_1(z)|, \quad (z \in \{z : |z| = 2\})$.
Hence by Rouché's Theorem f has 3 zero's inside $\{z : |z| < 2\}$
When $|z| = 1$, the dominant term is $g_2(z) = z$, then g_2 is also analytic on the simply-connected region R, $\{z : |z| = 1\}$ forms a simple-close contour in R and
$\left| {}^3 - z\right| = |z|^3 \quad | \,| \le \quad < 3 \quad |g_2(z)|, \quad (\ \in \{z \ \ |z| = 1\})$
Hence by Rouché's Theorem has no zero's in $\{z : |\ | \quad \}$.
So f has 3 zero's in $\{z \ \ 1 \le |z| < 2\}$

(b) has exactly 3 zeros by the Fundamental Theorem of Algebra, and since f reasing on R
$f(1) = -1$, $f(2) = 6$, so there is one real solution. The other two solutions form a conjugate pair, so there is exactly one solution on $\{z : \operatorname{Im} z > 0\}$.

Question 7

(a) q is continuous on $\mathbb{C}$ and $\overline{q}(z) = -iz$ is analytic on $\mathbb{C}$, so by HB 1.14 p.38 q represents a model fluid flow on $\mathbb{C}$.

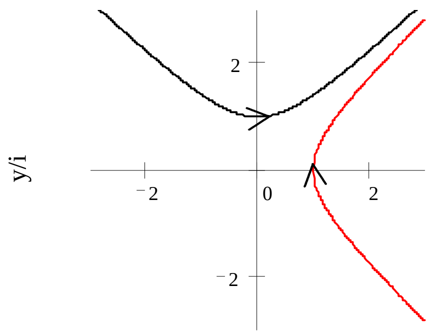
(b) $\Omega'(z) = \overline{q}(z) \Rightarrow \Omega(z) = -\frac{1}{2}iz^2$, The stream function is

$\operatorname{Im}\Omega(z) = \operatorname{Im}\left(-\frac{1}{2}iz^2\right) = \operatorname{Im}\left(-\frac{1}{2}i\left(x^2 - 2ixy - y^2\right)\right) = \frac{1}{2}\left(y^2 - x^2\right) = k$ where k is a constant in $\mathbb{R}$.

The streamline through 1 is $\frac{1}{2}\left(y^2 - x^2\right) = -\frac{1}{2}1^2 \Leftrightarrow y = \pm\sqrt{x^2 - 1}, \quad x > 0$

The streamline through i is $\frac{1}{2}\left(y^2 - x^2\right) = \frac{1}{2}1^2 \Leftrightarrow y = \sqrt{x^2 + 1}$ (The positive branch)

$q(1) = i; \quad q(i) = i \cdot (-i) = 1$. See for the streamlines the following graph:



(c) q is locally flux an circulation free, or $C_\Gamma = F_\Gamma = 0$ for any simple-closed curve Γ about 0. So 0 is neither a vortex not a source.

Question 8

(a) By HB 2.1 p. 41 $z_{n+1} = z_n^2 + 2z_n + 1 \Rightarrow a = 1, b = 2, c = 1$ is conjucate to

$w_{n+1} = w^2 + 1$, since $1 \cdot 1 + \frac{1}{2} \cdot 2 - \frac{1}{2} \cdot 2^2 = 1$

where $n = 0,1,2,\ldots$

The conjugating function is then $h(z) = z + 1$, so with $w_0 = z_0 + 1 = -1 + 1 = 0$, as required.

(b) We need to solve $z^2 + 1 = z \Leftrightarrow z^2 - z + 1 = 0 \Leftrightarrow z = \frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$.

Now $\left|P_1'\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i\right)\right| = 2\left|\frac{1}{2} + \frac{1}{2}\sqrt{3}i\right| = 2 > 1$, so both fixed points are repelling. (HB 1.5 p.41)

(c) According to HB 4.6 (c) p. 42 M meets the real axis in the interval $\left[-2, \frac{1}{4}\right]$, since $1 \notin \left[-2, \frac{1}{4}\right] \Rightarrow 1 \notin M$

So K_1 is not connected, and so by the Fatou-Julia Theorem $0 \notin K_1$, so $0 \in E_1$, that is

$P_1^n(0) \to \infty$ as $n \to \infty$, see HB 2.4 p.41.

Part II

Question 9

(a) $f(z) = \sin \bar{z} = \sin(x - iy)$ [Use double angle formula to expand $\sin(x - iy)$]

$\sin(x - iy) = \sin x \cos iy - \cos x \sin iy$ [Use $\cos iy = \cosh y$, $\sin iy = i \sinh y$]

$\therefore\ f(z) = \sin x \cosh y - i \cos x \sinh y = u + iv$ [Now we have to use Theorem 2.2 A4]

$$\frac{du}{dx} = \cos x \cosh y, \quad \frac{du}{dy} = \sin x \sinh y\ , \quad \frac{dv}{dx} = \sin x \sinh y, \quad \frac{dv}{dy} = -\cos x \cosh y$$

all derivatives exist and are continuous.

[The point is to see that the C-R equations are not generally satisfied *except* at certain points.]

We see that $\frac{du}{dx} \neq \frac{dv}{dy}$ and $\frac{dv}{dx} \neq -\frac{du}{dy}$, except when

$\cos x \cosh y = -\cos x \cosh y \Leftrightarrow \cos x = -\cos(x) \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

[Note that $\cosh y \neq 0$ this is why we can cancel it on both sides]

and

$\sin x \sinh y = -\sin x \sinh y \Rightarrow \sinh y = -\sinh y \Rightarrow \sinh y = 0 \Rightarrow y = 0$

since $\sin x \neq 0$, due to the restriction of the values of x from the first C-R equation.

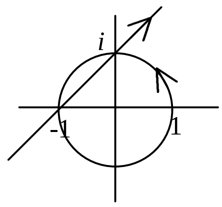
So f is differentiable at the points $z = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$

(b) (i) $g(z) = z^2 \Rightarrow g'(z) = 2z$, so g is conformal except when $2z = 0 \Rightarrow z = 0$, so g is conformal at i.

(b) (ii) [Be guided by what is being asked. Just find values of t which give value of i.]

$e^{it} = \cos t + i \sin t = i \Rightarrow t = \frac{\pi}{2} \Rightarrow g_1\left(\frac{\pi}{2}\right) = i$

$t - 1 + it = i \Rightarrow t = 1 \Rightarrow g_2(1) = i$



[e^{it} is always a circle and anything LINEAR in t is a straight line.]

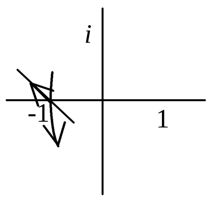
We have for g_2 : $x = t - 1, \quad y = t \Rightarrow x = y - 1 \Leftrightarrow y = x + 1$

[Straight line slope 45^{o} cutting y-axis at 1]

g_1 is the unit circle.

(b) (iii) We hve $g(z) = z^2 \Rightarrow g(i) = i^2 = -1$ and $|g'(i)| = |2i| = 2$ and $\operatorname{Arg}(g'(i)) = \frac{\pi}{2}$

1 So centre i mapped to –1
2 Lengths scaled by a factor 2
3 Rotated through $\frac{\pi}{2}$ anti-clockwise



Question 10

(a) Removable singularity at $z = 0$, because $\lim_{z\to 0} \frac{z \cdot \sin z}{z(z-2)^3} = \lim_{z\to 0} \frac{\sin z}{(z-2)^3} = \frac{\sin 0}{(0-2)^3} = 0$, by HB 3.1 (D) ⇔ (A) p.28, since *f* has a singularity at 0.

Pole of order 2 at $z = 2$, because $\lim_{z\to 2} \frac{(z-2)^3 \cdot \sin z}{z(z-2)^3} = \lim_{z\to 2} \frac{\sin z}{z} = \frac{\sin 2}{2} \neq 0$ and exists and *f* has a singularity at 2 by HB 3. (B) ⇔ (A) p.28. No other singularities.

(b) (i) [The question said write down s I think you should use the standard series and substitute $x = \frac{1}{z}$]

We have $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

So $\sin\frac{1}{z} = \frac{1}{z} - \frac{}{3!z^3} + \frac{1}{5!z^5} + \cdots + \frac{(-1)^n}{(2n+1)!z^{2n+1}}$, on $\mathbb{C} - \{0\}$

Annulus of convergence $\{z : 0 < |z| < r\}$ for any r.

(b) (ii) [Here we need the coefficient of the $\frac{1}{z}$ -term]

$\operatorname{Res}\left(z^6 \sin\frac{1}{z}\right) = -\frac{1}{7!}$

So using the Residue Theorem (or Cauchy's Theorem)

$\int_C z^6 \sin\frac{1}{z}\,dz = 2pi \operatorname{Res}\left(z^6 \sin\frac{1}{z}, 0\right) = -\frac{2pi}{7!}$

(c) [This is probably intended to be done using the Compoistion Rule, see HB p. 25 and 26]

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

$\operatorname{Log}(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

So,

$$\operatorname{Log}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \operatorname{Log}\left(1 - \left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)\right)$$

$$= -\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right) - \frac{1}{2}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)^2 - \frac{1}{3}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)^3 - \cdots$$

[Only need to keep terms up to z^6]

$$= -\frac{z^2}{2!} + z^4\left(\frac{1}{4!} - \frac{1}{2}\cdot\frac{1}{(2!)^2}\right) + z^6\left(-\frac{1}{6!} + \frac{1}{2}\cdot 2\cdot\frac{1}{2!4!} - \frac{1}{3}\cdot\frac{1}{(2!)^3}\right) - \cdots$$

$$= -\frac{1}{2}z^2 - \frac{1}{12}z^4 - \frac{1}{45}z^6 - \cdots$$

[I do not think I would simplify, but this may loos a mark, I do not know. But time is at a premium]

Now $\frac{d}{dz}\operatorname{Log}(\cos z) = -\frac{\sin z}{\cos z} = -\tan z$

So $\tan z = \frac{d}{dz}\left(\frac{1}{2}z^2 + \frac{1}{12}z^4 + \frac{1}{45}z^6 + \cdots\right) = \left(z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots\right)$

[I find getting all the numbers right difficult. I hate people who think mathematics is just hard long divisions. However this result equals the standard series for $\tan z$]

Question 11

(a) This is true since integrand is odd. Using HB 3.4 p. 25 we can show this

$f(-t) = \frac{-t}{(-t)^4 - 1} = \frac{-t}{t^4 - 1} = -f(t)$ and by HB 3.5 p.29 the required result follows.

(b) The integrand is even. So,

$$\int_0^\infty \frac{1}{t^4 - 1} dt = \tfrac{1}{2} \int_{-\infty}^\infty \frac{1}{t^4 - 1} dt$$

$t^4 - 1 = 0 \Rightarrow t = \pm 1$ or $t = \pm i$, so the integrand has 4 simple poles at those points, so the conditions of HB 3.8 p.29 are satisfied [Theorem 3.3 C1], and so

$\int_{-\infty}^\infty \frac{1}{2(t^4 - 1)} dt = 2piS + piT$ [drawing a sketch of the poles and the contour might help)

Using the f / h Rule gives

$$S = \operatorname{Res}(f, i) = \frac{1}{8 \cdot i^3} = -\frac{1}{8i} = \tfrac{i}{8}$$

$$T = \operatorname{Res}(f, 1) + \operatorname{Res}(f, -1) = \frac{1}{8 \cdot 1^3} + \frac{1}{8(-1)^3} = \tfrac{1}{8} - \tfrac{1}{8} = 0$$

So,

$$\tfrac{1}{2} \int_{-\infty}^\infty \frac{1}{t^4 - 1} dt = 2pi \cdot \tfrac{i}{8} = -\tfrac{1}{4} p$$

(c) The integrand satisfies the condition of HB 1.4 p. 33: the order of the denominato exceeds that of the numerator by more than 2 and the poles on the non-negative axis are simple (see part (b)). So,

$$\int_0^\infty \frac{t^{1/2}}{t^4 - 1} dt = -pe^{-pi/a} \operatorname{cosec}\left(\tfrac{p}{2}\right) S - p \cot\left(\tfrac{p}{2}\right) \cdot T = -pe^{-pi/a} \cdot S - p \cdot 0 \cdot T$$

$$S = \operatorname{Res}(f_1, -1) + \operatorname{Res}(f_1, i) + \operatorname{Res}(f_1, -i); \quad T = \operatorname{Res}(f_2, 1)$$

$$f_1 = \frac{1}{z^4 - 1} \exp\left(\tfrac{1}{2} \operatorname{Log}_{2p}(z)\right); \ f_2 = \frac{1}{z^4 - 1} \exp\left(\tfrac{1}{2} \operatorname{Log}(z)\right)$$

$$\operatorname{Res}(f_1, -1) = \left. \frac{\exp\left(\tfrac{1}{2} \operatorname{Log}_{2p}(z)\right)}{4z^3} \right|_{-1} = \frac{e^{0 + ip/2}}{-4} = -\tfrac{i}{4}$$

$$\operatorname{Res}(f_1, i) = \left. \frac{\exp\left(\tfrac{1}{2} \operatorname{Log}_{2p}(z)\right)}{4z^3} \right|_{i} = \frac{e^{0 + ip/4}}{-4i} = \frac{ie^{ip/4}}{4}$$

$$\operatorname{Res}(f_1, -i) = \left. \frac{\exp\left(\tfrac{1}{2} \operatorname{Log}_{2p}(z)\right)}{4z^3} \right|_{-i} = \frac{e^{0 + i3p/4}}{4(-i)^3} = -\frac{ie^{i3p/4}}{4}$$

The residue at 1 is not needed, since $\cot\left(\tfrac{p}{2}\right) = 0$. So the result is

$$\int_0^\infty \frac{t^{1/2}}{t^4 - 1} dt = -pe^{-pi/2} \cdot S = \frac{-pe^{-pi/2}}{4}\left(-i + ie^{ip/4} - ie^{i3p/4}\right) =$$

$$-p(-i)\tfrac{i}{4}\left(-1 + \cos\tfrac{p}{4} + i\sin\tfrac{p}{4} - \cos\tfrac{3p}{4} - i\sin\tfrac{3p}{4}\right) = \tfrac{p}{4}\left(-1 + 2\cos\tfrac{p}{4}\right) = -\tfrac{p}{4}\left(1 - \sqrt{2}\right)$$

Question 12

(a) This is the standard triple (HB 2.11 p.36) with $a = i, b = \infty, g = -i$. Hence

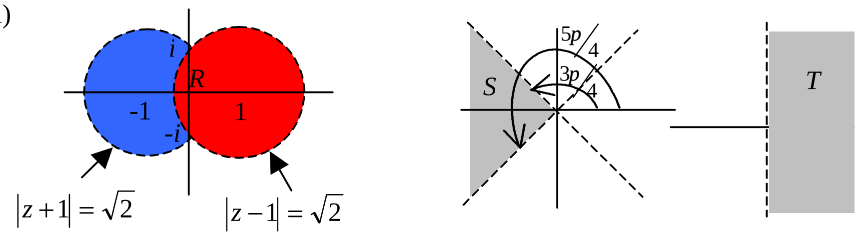
$$\hat{f}_1(z) = \frac{(z-i)(\infty+i)}{(z+i)(\infty-i)} = \frac{z-i}{z+i}$$

You can also argue as follows
We have i maps to 0, so numerator must contain $z-i$.
Also i goes to ∞ , so denominator must contaion $z+i$

So the transformation is $k\dfrac{z-i}{z+i}$.

Since ∞ goest to 1 this means $k=1$. The transformation is $\dfrac{z-i}{z+i}$

(b) (i)



(b) (ii) Generalised circles map to generalised circles. i is mapped ot 0 and $-i$ to ∞ , so the arcs of the circles between those points are mapped to half lines from zero. Now we can use the three point trick to find the area to which R is mapped to.

We can now apply the three point trick [*OUSA and Paul used* $\pm 1 \mp \sqrt{2}$ *, but that gets too messy for me and you don't have to restrict yourself to the boundaries in cases like this*]:
Points on the left circle

$-2-i \to \dfrac{-2-i-i}{-2-i+i} = \dfrac{-2-2i}{-2} = 1+i; \quad i \to 0; \quad -i \to \infty$, with the domain and image at the right side.

So the arc between i and $-i$ is mapped to the line $\left\{z : \operatorname{Arg}_{2p}(z) = \frac{5p}{4}\right\}$, with the image at the right.

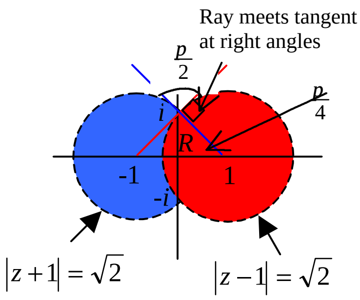
Points on the right circle

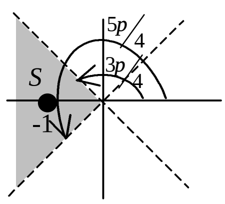
$2-i \to \dfrac{2-i-i}{2-i+i} = \dfrac{2-2i}{2} = 1-i; \quad i \to 0; \quad -i \to \infty$, with the domain and image at the left side.

So the arc between i and $-i$ is mapped to the line $\left\{z : \operatorname{Arg}_{2p}\, z = \frac{3p}{4}\right\}$, with the image at the left.

And so R is mapped to $\left\{z : \frac{3p}{4} < \operatorname{Arg}_{2p}\, z < \frac{5p}{4}\right\} = S$

[*Sketching the points and line segments might make it easier, I'm to lazy right know, drawing them on the computer*]
Here is another, using conformality.
Generalised circles map to generalised circles. i is mapped to 0 and $-i$ to ∞ , so the arcs of the circles between those points are mapped to half lines from zero to infinity.
Möbius transformations are conformal and one-one: angles between meeting lines are conserved. We know that the arcs of the right and left circles are mapped to half lines, and so will the parts of any generalised circle between i and $-i$!. Hence the imaginary axis between these points will also be mapped to a half line between zero and ∞. Now, the right circle and the left circle meet at i at and angle of $\frac{p}{2}$, and also the imaginary axis, cutting the angle in half. Now, ***i*** is mapped to 0 and the angles are preserved. A point on the imaginary axis $0 \to \dfrac{-i}{i} = -1$, so the imaginary axis between i and $-i$ is mapped to the negative real axis and the angles beteween the half lines are $\frac{p}{4}$, and so R is mapped to the area between the half lines, which is $\left\{z : \frac{3p}{4} < \operatorname{Arg}_{2p}\, z < \frac{5p}{4}\right\} = S$





(b) (iii) Here we need a map from *S* to *T* since we have on from *R* to *S* alraedy.
To get from *S* to *T* we need to double all the angles. We know squaring does this.
On *S* this is one-one (the area *S* in the sector has an angle less than p , and conformal since $0 \notin S$. Now *f* can be composed:

$$\hat{f}(z) = \left(\hat{f}_1(z)\right)^2 = \left(\frac{z-i}{z+i}\right)^2$$

(b) (iv) See HB 4.5 p.37, example 2

A map from the half plane *T* to the open unit disc is $\frac{w-1}{w+1}$. So a map from *R* to open disc would be

$$\frac{\left(\hat{f}_1(z)\right)^2 - 1}{\left(\hat{f}_1(z)\right)^2 + 1} = \frac{\left(\frac{z-i}{z+i}\right)^2 - 1}{\left(\frac{z-i}{z+i}\right)^2 + 1}$$