Solutions for AA1

1

$$\frac{x^2-12}{x^2-5x+6}<2.$$

$$\Leftrightarrow \frac{x^2-12}{x^2-5x+6}-2<0$$

$$\Leftrightarrow \frac{x^2-12-2x^2+10x-12}{x^2-5x+6}<0$$

$$\Leftrightarrow \frac{-x^2+10x-24}{x^2-5x+6}<0$$

$$\Leftrightarrow \frac{x^2-10x+24}{x^2-5x+6}>0$$

$$\Leftrightarrow \frac{(x-6)(x-4)}{(x-3)(x-2)}>0$$

A sign table is used to find the solution.

x	$(-\infty,2)$	2	$(2,3)$	3	$(3,4)$	4	$(4,6)$	6	$(6,\infty)$
$(x-2)$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$(x-3)$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$(x-4)$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$(x-6)$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$\frac{(x-6)(x-4)}{(x-3)(x-2)}$	$+$	*	$-$	*	$+$	0	$-$	0	$+$

Hence the solution is $(-\infty,2)\cup(3,4)\cup(6,\infty)$

(*Students frequently construct the table leaving the numerator and denominator as quadratic expressions. While this is not wrong, it is much easier to construct the table with the linear factors.*)

2

Let $P(n)$ be the statement $3^n > 4n$.

$3^2=9$ while $4\times 2=8$. It follows that $P(2)$ is true.

Suppose $P(k)$ is true for some $k\geq 2$.

Then $3^k>4k$ and it follows that $3^{k+1}>12k$.

It is now sufficient to prove that $12k>4(k+1)$.

$12k>4(k+1)\Leftrightarrow 3k>k+1\Leftrightarrow 2k>1$ and this is true for $k\geq 2$.

Hence $3^{k+1}>12k>4(k+1)$.

By Mathematical Induction, $P(n)$ is true for all $n\geq 2$.

3

$$\left(1+\frac{1}{3n}\right)^n = 1+n\left(\frac{1}{3n}\right)+\frac{n(n-1)}{2}\left(\frac{1}{3n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{3n}\right)^3+\ldots+\left(\frac{1}{3n}\right)^n \quad \text{for } n \geq 2$$

$$\geq 1+\frac{1}{3}+\frac{n^2}{18n^2}-\frac{n}{18n^2} \quad \text{(since all omitted terms are positive)}$$

$$\geq 1+\frac{1}{3}+\frac{1}{18}-\frac{1}{18n}$$

$$=\frac{18+6+1}{18}-\frac{1}{18n}$$

$$=\frac{25}{18}-\frac{1}{18n}$$

and when $n=1$, $\left(1+\frac{1}{3n}\right)^n = 1+\frac{1}{3}=\frac{4}{3}=\frac{24}{18}$ and $\frac{25}{18}-\frac{1}{18n}=\frac{24}{18}$.

Hence for $n \geq 1$, $\left(1+\frac{1}{3n}\right)^n \geq \frac{25}{18}-\frac{1}{18n}$

4

$$E=\left\{5+\frac{3}{n^3} : n=1,2,\ldots\right\}$$

$5 \leq 5+\frac{3}{n^3}$ means that 5 is a lower bound of E.

Now consider any $M'>5$.

$$M'>5+\frac{3}{n^3}$$

$$\Leftrightarrow M'-5>\frac{3}{n^3}$$

$$\Leftrightarrow \frac{1}{M'-5}<n^3 \quad \text{since both sides are positive}$$

$$\Leftrightarrow \sqrt[3]{\frac{1}{M'-5}}<n$$

Since we can choose a sufficiently large n to make this true, such a value must mean that $M'>5+\frac{3}{n^3}$ and it follows that M' is not a lower bound.

Hence, 5 is the greatest lower bound of E.