Question 1

a $\left(\frac{1-i}{1+i}\right)^3=\left(\frac{(1-i)(1-i)}{2}\right)^3=\left(\frac{-2i}{2}\right)^3=-i^3=i$

b $\exp(2+i\frac{\pi}{6})=\exp(2)(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})=\frac{\sqrt{3}e^2}{2}+i\frac{e^2}{2}$

c Log $\left(\frac{1+i\sqrt{3}}{2}\right)$ = Log $(\exp(i\frac{\pi}{3}))=i\frac{\pi}{3}$

d $\left(\frac{1+i\sqrt{3}}{2}\right)^{3-i}$ = exp(Log $(3-i)(\exp(i\frac{\pi}{3}))=\exp(\frac{\pi}{3}+i\pi)$ = $-\exp(\frac{\pi}{3})$

Question 3

Section a

Γ has a standard parametrisation of $\gamma(t)=(1-t)i+t, t\in[0,1]$ So Re z = t, Im z = 1-t

Therefore (Re z)(Im z) = $t-t^2$

With this parametrisation, $\frac{dz}{dt}=-i+1$

Since γ is a smooth path and (Re z)(Im z) is continuous along the path Γ , we have that ,

$\int_0^1 (t-t^2)(1-i)\,dt=(1-i)(\frac{1}{2}-\frac{1}{3})=\frac{1-i}{6}$

Question 3
Section b

As $f(z)=\frac{z^2-1}{\bar{z}^2+1}$ is continuous on the circle, we can use the Estimation Theorem

The length of the circle C is 4π

$|z^2-1|\leq|z^2|+|1|=2^2+1=5$

Using the Backwards form of the Triangle Inequality, $|\bar{z}^2+1|\geq||\bar{z}^2|-|-1||=|2^2-1|=3$

Therefore $M=|\frac{z^2+1}{\bar{z}^2-1}|\leq\frac{5}{3}$ for $\{z:|z|=2\}$

Therefore by the Estimation Theorem, an upper estimate for the modulus of the integral is

$ML=4\pi(\frac{5}{3})=\frac{20\pi}{3}$

Question 4

Let $\boldsymbol{R}=\{z:|z|<3\}$

Section a

$\boldsymbol{R}$ is a simply connected region and $\dfrac{\cos z}{z-\pi}$ is analytic on $\boldsymbol{R}$.
C is a closed contour in $\boldsymbol{R}$

So by Cauchy's Theorem , $\int_C \dfrac{\cos z}{z-\pi}dz=0$

Section b

$\boldsymbol{R}$ is a simply connected region and $f(z)=\cos z$ is analytic on $\boldsymbol{R}$, and $\dfrac{\pi}{3}$ is inside C and C is a simple closed contour in $\boldsymbol{R}$

So by Cauchy's Theorem , $\int_C \dfrac{\cos z}{z-\frac{\pi}{3}}dz=2\pi i\, f(\frac{\pi}{3})=2\pi i(\frac{1}{2})=\pi i$

Section c

$\boldsymbol{R}$ is a simply connected region and $f(z)=\cos z$ is analytic on $\boldsymbol{R}$, and $\dfrac{\pi}{2}$ is inside C and C is a simple closed contour in $\boldsymbol{R}$

$f'''(z)=\sin z$

So using Cauchy's n'th Derivative Formula , $\int_C \dfrac{\cos z}{(z-\frac{\pi}{2})^4}dz=\dfrac{2\pi i}{3!}\sin(\frac{\pi}{2})=\frac{\pi}{3}i$

Question 5

Section a

$$f(z)=\frac{z+1}{z(z^2+4)}$$

$f(z)$ has simple poles at 0, 2i and -2i

Res (f,0) = the limit as $z\to 0,(z-0)f(z)=\frac{0+1}{0+4}=\frac{1}{4}$

Res (f,2i) = the limit as $z\to 2i,(z-2i)f(z)=\frac{2i+1}{2i(2i+2\mathrm{i})}=\frac{1+2\mathrm{i}}{-8}=-\frac{1+2i}{8}$

Res (f,-2i) = the limit as $z\to -2i,(z+2i)f(z)=\frac{-2i+1}{-2i(-2i-2i)}=\frac{2i-1}{8}$

Section b

Let $p(t)=t+1$ Let $q(t)=t(t^2+4)$

p and q are polynomial functions such that the degree of q exceeds that of p by at least 2 and the pole of p/q on the real axis is simple.

Therefore, $\int_{-\infty}^{\infty}\left(\frac{p(t)}{q(t)}\right)dt=2\pi i S+\pi T$

where S is the sum of the residues of p/q at the poles in the upper half plane and
where T is the sum of the residues of p/q at the poles on the real axis.

As S = Res (p/q,2i) and T = Res (p/q,0) ,

$$\int_{-\infty}^{\infty}\left(\frac{p(t)}{q(t)}\right)dt=-2\pi i\left(\frac{1+2i}{8}\right)+\pi i\left(\frac{1}{4}\right)=\frac{\pi}{2}$$

Question 6

$f(z)=z^7+3z^5-1$

The function is analytic on the simply connected region $\boldsymbol{R}=\mathbb{C}$ so Rouche's Theorem can be used.

Section a

Let $g_1(z)=z^7$

Using the Triangle Inequality, for $C_1=\{z:|z|=2\}$

$$|f(z)-g_1(z)|=|3z^5-1|\leq|3z^5|+|-1|=96+1=97<2^7=|g_1(z)|$$

Since C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouche's Theorem , f has the same number of zeros as g_1 inside the contour C_1 . Therefore f has 7 zeros inside C_1

Let $g_2(z)=3z^5$

Using the Triangle Inequality, for $C_2=\{z:|z|=1\}$

$$|f(z)-g_2(z)|=|z^7-1|\leq|z^7|+|-1|=1+1=2<3(1^5)=|g_2(z)|$$

Since C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouche's Theorem , f has the same number of zeros as g_2 inside the contour C_2 . Therefore f has 5 zeros inside C_2

Therefore , f has 7 – 5 = 2 solutions in the set $\{z:1\leq|z|<2\}$ Therefore we have to find if there are any solutions on the contour C_2

We have that , on the contour C_2 , $|z^7+3z^5-1|\geq|3z^5|-|z^7|-|-1|=3-1-1=1>0$

As $f(z)$ is non-zero on C_2 , then there are exactly 2 solutions of f(z) = 0 in the set $\{z:1<|z|<2\}$

Question 6

Section b

$f(z)$ is a polynomial with real coefficients. So if z is a solution of $f(z)=0$ then $\bar{z}$ will also be a solution.

We have that $f(0)=-1$ and $f(1)=3$ So there is at least one real solution of $f(z)=0$ in the interval $(0,1)$

We also have that $f'(z)=7z^6+15z^4$ If z is real, then $f'(z)>0$

So $f(z)$ is a strictly increasing function for real z, so there can only be one real solution of f(z) = 0.

Therefore there are 6 solutions of f(z) = 0 that do not lie on the real axis.

So there must be 3 solutions of f(z) = 0 in the upper half-plane, and the complex conjugates of these three solutions will also be solutions of f(z) = 0 and lie in the lower half-plane.