Handbook references are given as (p) for information only.

1. $\ln(4 - x^2)$ is defined only if $4 - x^2 > 0$ which gives $4 > x^2$ or $-2 < x < 2$ so the answer is C. (p11)

2. $2x + xy = (2 + y)x$ so the variables are separable and P is correct. The equation is also linear so q is correct. The answer is C. (p26)

3. Using formula on page 29 of Handbook, cross product is $(4+3)\mathbf{i} - (2+3)\mathbf{j} + (-1+2)\mathbf{k} = 7\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ so the answer is C.

4. The answer is D

5. Each string is extended by length l. PE in AB $= \frac{1}{2}kl^2$. PE in AC $= \frac{3}{2}kl^2$. PE in CB $= kl^2$. Total PE is $3kl^2$. The answer is C. (p35)

6. No solutions as the final line is inconsistent. (p37) The answer is C.

7. $\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} = -3\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. The eigenvalue is -3. (p41) The answer is D.

8. The answer is D. (p45)

9. $\mathrm{J}(x,y) = \begin{bmatrix} -qy & 2py - qx \\ 2rx & 2sy \end{bmatrix}$ so the answer is D. (p47)

10. $[F]$ = MLT^{-2} $[k] = [F]/[e]$ = MLT^{-2}/L so the answer is C. (p53)

11. Centre of mass $= \frac{1\times 2l_0 + 1\times 3l_0}{2+1+1} = \frac{5l_0}{4}$ so the answer is C. (p58)

12. The resultant force will be towards the centre so the answer is B. (p59)

13. The function is odd so the answer is D. (p60/61)

14. $\mathbf{r.r} = x^2 + y^2 + z^2$ so the answer is B. (p64)

15. Mass= $\int_B r\,dV = \int_B r \times r^2 \sin\theta\, d\varphi\, d\theta\, dr$ (p70) so the answer is B.

16. Separating the variables (p26) $\frac{2y}{1+y^2}\frac{dy}{dx} = \frac{1}{x^2}$

Integrating $\ln(1+y^2) = -\frac{1}{x} + C$

$y(1) = 0$ so $0 = -1 + C$ or $C = 1$

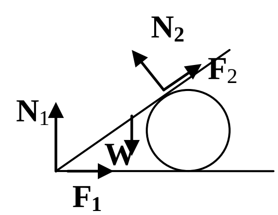
Giving $\ln(1+y^2) = 1 - \frac{1}{x} = (x-1)/x$

Taking exponentials $1 + y^2 = e^{(1-x)/x}$ $\}\, e^{\frac{x-1}{x}}$

$$y^2 = e^{\frac{-(1-x)}{x}} - 1 \quad y = \sqrt{e^{\frac{1-x}{x}} - 1}$$

$= \sqrt{e^{\frac{x-1}{x}} - 1}$?

17.



$\mathbf{F}_1$ is the frictional force between the floor and the rod.
$\mathbf{N}_1$ is the normal reaction between the floor and the rod.
$\mathbf{F}_2$ is the frictional force between the cylinder and the rod.
$\mathbf{N}_2$ is the normal reaction between the cylinder and the rod.
$\mathbf{W}$ is the weight of the rod.

18.a) $\xleftarrow{}$ $\mathbf{H}_1$ $\xrightarrow{}$ $\mathbf{H}_2$

b) $\mathbf{H}_1 = -k(x - l_0)\mathbf{i}$ (p34)

$\mathbf{H}_2 = -k(4l_0 - x - 2l_0)(-\mathbf{i})$

$= k(2l_0 - x)\mathbf{i}$

c) Equation of motion is $m\ddot{x}\mathbf{i} = \mathbf{H}_1 + \mathbf{H}_2$ (N2) (p33) Resolve in the **i**-direction

$$m\ddot{x} = -k(x - l_0) + k(2l_0 - x)$$
$$m\ddot{x} + 2kx = 3l_0k$$

19. a) $\begin{bmatrix}3 & 8\\3 & 1\end{bmatrix}\begin{bmatrix}1\\0.5\end{bmatrix} = \begin{bmatrix}7\\3.5\end{bmatrix} = 7\begin{bmatrix}1\\0.5\end{bmatrix}$

$$\begin{bmatrix}3 & 8\\3 & 1\end{bmatrix}\begin{bmatrix}1\\-0.75\end{bmatrix} = \begin{bmatrix}-3\\2.25\end{bmatrix} = -3\begin{bmatrix}1\\-0.75\end{bmatrix}$$

The method is direct iteration and converges to the eigenvector with the largest eigenvalue (which is 7) so this will be $[1 \quad 0.5]^T$. (p24)

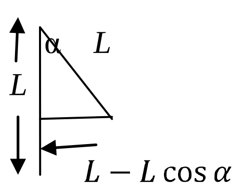
b) No as $\mathbf{e}_0 = 4\begin{bmatrix}1\\-0.75\end{bmatrix}$ so it is already an eigenvector.

20. a) Using the constant acceleration formulae with $a_0 = -g$ and $v_0 = u$, $v^2 - u^2 = -2gx$ (p33)
but $v = 0$ when $x = h$ so $u^2 = 2gh$

$$\text{or } u = \sqrt{2gh}$$

Alternatively you could use conservation of energy.(p35)
b) Maximum Range $= u^2/g$ (Handbook page 49) so maximum range is $2h$.

21. a) Let u be the speed of the bullet plus wood just after impact. Then by conservation of momentum
$mv_0 = (m+M)u$ or $u = \frac{mv_0}{m+M}$ (p58)
b) Taking the datum for the potential energy as the initial height of the block
KE at impact $\frac{=\frac{1}{2}(M+m)(m^2v_0^2)}{(m+M)^2} = \frac{m^2v_0^2}{2(m+M)}$ (p35)
PE at impact = 0
When they come to rest
KE = 0
PE= $(m+M)gL(1-\cos\alpha)$



By conservation of energy

$$\frac{m^2v_0^2}{2(m+M)} = (m+M)gL(1-\cos\alpha)$$

So $v_0^2 = 2(m+M)^2L(1-\cos\alpha)/m^2$
$\Rightarrow v_0 = \frac{m+M}{m}\sqrt{2gL(1-\cos\alpha)}$

22. a)

$$\text{div}\mathbf{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\theta}(V_\theta) + \frac{\partial}{\partial z}(V_z) \text{ (p67)}$$
$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^3\cos\theta) + \frac{1}{\rho}\frac{\partial}{\partial\theta}(-\sin\theta)$$
$$= 3\rho\cos\theta - \frac{1}{\rho}\cos\theta$$

b) $\mathbf{curl\ V} = \frac{1}{\rho}\begin{vmatrix}\mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z\\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ \rho^2\cos\theta & -\rho\sin\theta & 0\end{vmatrix}$ (p67)

$$= \frac{1}{\rho}\left(0\mathbf{e}_\rho + 0\mathbf{e}_\theta + (-\sin\theta + \rho^2\sin\theta)\mathbf{e}_z\right)$$

Curl V$= \frac{1}{\rho}(\rho^2 - 1)\sin\theta\,\mathbf{e}_z$

23. **Note** There is an error in the exam paper and Y_N for step size 0.01 should be 1.015. This should be obvious from the diagrams.
a) From the graph the likely order is second order as the graph of Y_N against h^2 is almost linear.
b) $p = 2$ (as 2nd order) (p73)
To achieve 6 dp accuracy we need

$$|Ch^p| \le 0.5 \times 10^{-6} \quad (1)$$

And $\quad Y_{N_1} - Y_{N_2} \cong C({h_1}^p - {h_2}^p)$
Giving $1.000 - 1.015 \cong C(0.02^2 - 0.01^2)$

$$-0.015 \cong C(0.0004 - 0.0001)$$
$$C \cong -\frac{0.015}{0.0003} = -50$$

So substituting in (1) $50h^2 \le 0.5 \times 10^{-6}$
$\Rightarrow h^2 \le 10^{-8} \ \Rightarrow h \le 10^{-4}$
The upper bound is 10^{-4}.

24. Auxiliary equation is $2\lambda^2 + 6\lambda + 5 = 0$ (p26)
So $\lambda = \frac{-6\pm\sqrt{36-40}}{4} = -\frac{3}{2} \mp \frac{1}{2}i$
Giving the complementary function as
$y_c = e^{-\frac{3}{2}t}\left(C\cos\left(\frac{t}{2}\right) + D\sin\left(\frac{t}{2}\right)\right)$

24. (cont) For particular integral try (p27)

$$y = p\cos\left(\frac{t}{2}\right) + q\sin\left(\frac{t}{2}\right)$$
$$\frac{dy}{dt} = -\frac{1}{2}p\sin\left(\frac{t}{2}\right) + \frac{1}{2}q\cos\left(\frac{t}{2}\right)$$
$$\frac{d^2y}{dt^2} = -\frac{1}{4}p\cos\left(\frac{t}{2}\right) - \frac{1}{4}q\sin\left(\frac{t}{2}\right)$$

Substituting into the equation gives

$$-\frac{1}{2}p\cos\left(\frac{t}{2}\right) - \frac{1}{2}q\sin\left(\frac{t}{2}\right) - 3p\sin\left(\frac{t}{2}\right)$$
$$+3q\cos\left(\frac{t}{2}\right) + 5p\cos\left(\frac{t}{2}\right) + 5q\sin\left(\frac{t}{2}\right)$$
$$= 39\sin\left(\frac{t}{2}\right)$$

Giving

$$\left(\frac{9}{2}p + 3q\right)\cos\left(\frac{t}{2}\right) + \left(\frac{9}{2}q - 3p\right)\sin\left(\frac{t}{2}\right)$$
$$= 39\sin\left(\frac{t}{2}\right)$$

Equating coefficients $\frac{9}{2}p + 3q = 0 \Rightarrow q = -\frac{3}{2}p$

and $\frac{9}{2}q - 3p = 39$ or $\frac{3}{2}q - p = 13$

which becomes $-\frac{9}{4}p - p = 13$

giving $p = -4$ and $q = 6$

so $y_p = -4\cos\left(\frac{t}{2}\right) + 6\sin\left(\frac{t}{2}\right)$

$y = y_c + y_p$ (p27)

$$y = e^{-\frac{3}{2}t}\left(C\cos\left(\frac{t}{2}\right) + D\sin\left(\frac{t}{2}\right)\right) - 4\cos\left(\frac{t}{2}\right)$$
$$+ 6\sin\left(\frac{t}{2}\right)$$

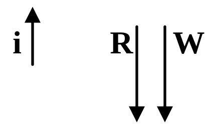
b) $\frac{dy}{dt} = -\frac{3}{2}e^{-\frac{3}{2}t}\left(C\cos\left(\frac{t}{2}\right) + D\sin\left(\frac{t}{2}\right)\right)$

$$+e^{-\frac{3}{2}t}\left(-\frac{1}{2}C\sin\left(\frac{t}{2}\right) + \frac{1}{2}D\cos\left(\frac{t}{2}\right)\right)$$
$$\overset{3}{-4}\cos\left(\frac{t}{2}\right) + \overset{2}{6}\sin\left(\frac{t}{2}\right)$$
$$y(0) = -1 \Rightarrow C - 4 = -1 \Rightarrow C = 3$$
$$\frac{dy(0)}{dx} = 1 \Rightarrow \frac{1}{2}D - \frac{3}{2}C + 3 = 1 \Rightarrow D = 5$$

The particular solution is

$$y = e^{-\frac{3}{2}t}(3\cos\left(\frac{t}{2}\right) + 5\sin\left(\frac{t}{2}\right)) - 4\cos\left(\frac{t}{2}\right)$$
$$+ 6\sin\left(\frac{t}{2}\right)$$

25. a)



R$= -kv^2\boldsymbol{i}$ **W**$= -mg$**i**

b) Equation of motion is m**a** = **R** + **W** (N2) (p33)

Resolving in the **i**-direction $ma = -kv^2 - mg$

Using $a = v\frac{dv}{dx}$ and dividing by m gives

$$v\frac{dv}{dx} = -\frac{k}{m}(v^2 + mg) = -\frac{g}{\lambda^2}(\lambda^2 + v^2)$$
$$\text{as } \lambda = \sqrt{\frac{mg}{k}}$$

c) Separating the variables (p26)

$$\frac{v}{\lambda^2 + v^2}\frac{dv}{dx} = -\frac{g}{\lambda^2}$$

Integrating $\frac{1}{2}\ln(\lambda^2 + v^2) = -\frac{g}{\lambda^2}x + C$

where C is the constant of integration.

When $x = 0$ $v = v_0$ so $C = \frac{1}{2}\ln(\lambda^2 + v_0^2)$ so

$$\frac{1}{2}\ln(\lambda^2 + v^2) - \frac{1}{2}\ln(\lambda^2 + v_0^2) = -\frac{g}{\lambda^2}x$$

or $\ln((\lambda^2 + v^2)/(\lambda^2 + v_0^2)) = -2gx/\lambda^2$ (1)

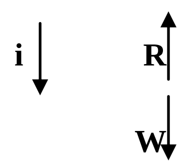
Taking exponentials gives $\frac{\lambda^2+v^2}{\lambda^2+v_0^2} = e^{-\frac{2gx}{\lambda^2}}$

So $v = \sqrt{(\lambda^2 + v_0^2)e^{-\frac{2gx}{\lambda^2}} - \lambda^2}$.

At greatest height $v = 0$ so from (1)

$x = -\frac{\lambda^2}{2g}\ln\left(\frac{\lambda^2}{\lambda^2+v_0^2}\right) = \frac{\lambda^2}{2g}\ln\left(\frac{\lambda^2+v_0^2}{\lambda^2}\right)$ (2)

d)



R$= -kv^2$i

W$= mg$i

The equation of motion is m**a** = **R** + **W**

Resolving in the **i**-direction gives

$ma = -kv^2 + mg$

e) At terminal speed $a = 0$ (Handbook page 34)

so $v_T^2 = \frac{mg}{k}$ or $v_T = \sqrt{\frac{mg}{k}} = \lambda$

f) The equation of motion becomes

$$v\frac{dv}{dx} = \frac{g}{\lambda^2}(\lambda^2 - v^2)$$

Separating the variables

$$\frac{v}{\lambda^2 - v^2}\frac{dv}{dx} = \frac{g}{\lambda^2}$$

Integrating $-\frac{1}{2}\ln(\lambda^2 - v^2) = \frac{g}{\lambda^2}x + D$

When $x = 0 \quad v = 0$ and so $D = -\frac{1}{2}\ln\lambda^2$

$x = -\frac{\lambda^2}{2g}\ln\left(\frac{\lambda^2 - v^2}{\lambda^2}\right)$ (3)

From (2) and using $v_0 = \lambda$ the particle will return to its original position when $x = \frac{\lambda^2}{2g}\ln\frac{\lambda^2+\lambda^2}{\lambda^2} = \frac{\lambda^2}{2g}\ln 2$

Substituting in (3) gives $\ln\left(\frac{\lambda^2 - v^2}{\lambda^2}\right) = -\ln 2$

Giving $\frac{\lambda^2 - v^2}{\lambda^2} = \frac{1}{2}$ or $v^2 = \frac{\lambda^2}{2} \Rightarrow v = \frac{\lambda}{\sqrt{2}} = \frac{v_T}{\sqrt{2}}$

26.a) $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-3 & 1\\ 3 & -5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}36t\\ 0\end{bmatrix}$

b) For eigenvalues $\begin{vmatrix}-3-\lambda & 1\\ 3 & -5-\lambda\end{vmatrix} = 0$

so $\lambda^2 + 8\lambda + 12 = 0$ which factorises to

$$(\lambda + 6)(\lambda + 2) = 0$$

So the eigenvalues are $\lambda = -6$ and $\lambda = -2$

When $\lambda = -6$ the eigenvectors are given by

$\begin{bmatrix}3 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ or $3x + y = 0$ so a possible eigenvector is $[1 \quad -3]^T$

When $\lambda = -2$ the eigenvectors are given by

$\begin{bmatrix}-1 & 1\\ 3 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ or $x - y = 0$ so a possible eigenvector is $[1 \quad 1]^T$.

The complementary function is

$$\begin{bmatrix}x\\ y\end{bmatrix} = C_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t} + C_2\begin{bmatrix}1\\ -3\end{bmatrix}e^{-6t}$$

c) For particular integral try $\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}at + b\\ ct + d\end{bmatrix}$

Substituting in the equations

$$a = -3(at + b) + ct + d + 36t$$
$$c = 3(at + b) - 5(ct + d)$$

Collecting terms

$$(3a - c - 36)t + a + 3b - d = 0$$
$$(5c - 3a)t + c - 3b + 5d = 0$$

Equating coefficients of t gives

$$3a - c = 36 \ (1) \text{ and } 5c - 3a = 0 \ (2)$$

Substituting (1) into (2) gives $3a - \frac{3a}{5} = 36$

$\Rightarrow a = 15 \Rightarrow c = 9$

Equating the constants gives

$a + 3b - d = 0$ or $3b - d = -15$ (3)

$c - 3b + 5d = 0$ or $3b - 5d = 9$ (4)

Subtracting (4) from (3) gives $4d = -24$ or $d = -6$ and substituting in (3) gives $b = -7$

Giving the particular solution

$$\begin{bmatrix}x_p\\ y_p\end{bmatrix} = \begin{bmatrix}15t - 7\\ 9t - 6\end{bmatrix}$$

And a general solution

$\begin{bmatrix}x\\ y\end{bmatrix} = C_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t} + C_2\begin{bmatrix}1\\ -3\end{bmatrix}e^{-6t} + \begin{bmatrix}15t - 7\\ 9t - 6\end{bmatrix}$

d) Using $x(0) = 3$ and $y(0) = 0$ gives

$$\begin{bmatrix}3\\ 0\end{bmatrix} = C_1\begin{bmatrix}1\\ 1\end{bmatrix} + C_2\begin{bmatrix}1\\ -3\end{bmatrix} + \begin{bmatrix}-7\\ -6\end{bmatrix}$$

Or $C_1 + C_2 = 10$ and $C_1 - 3C_2 = 6$

Subtracting $4C_2 = 4$ so $C_2 = 1$

Substituting in the first equation gives $C_1 = 9$

$\begin{bmatrix}x\\ y\end{bmatrix} = 9\begin{bmatrix}1\\ 1\end{bmatrix}e^{-2t} + \begin{bmatrix}1\\ -3\end{bmatrix}e^{-6t} + \begin{bmatrix}15t - 7\\ 9t - 6\end{bmatrix}$

e) In the long term the exponentials tend to zero and the population will grow linearly in the proportion 15:9.

27. Let θ_{in} be the temperature inside the pipe

θ_1 the temperature on the inside surface of the pipe

θ_2 the temperature on the outside surface of the pipe

θ_3 the temperature on the outside surface of the lagging

θ_{out} the temperature of the air around the pipe.

Using the formula given in the question for the pipe $q = -\kappa A\frac{d\theta}{dr}$, replacing A by $2\pi rL$ and rearranging we get

$$\frac{d\theta}{dr} = -\frac{q}{2\kappa L\pi r}$$

Integrating assuming q is constant

$$\theta = -\frac{q}{2\kappa L\pi}\ln r + C$$

$\theta = \theta_2$ when $r = R$ so $C = \theta_2 + \frac{q}{2\kappa L\pi}\ln R$

27. (cont)

Giving $\theta-\theta_2=\frac{q}{2\kappa L\pi}\ln\left(\frac{R}{r}\right)$

Now if we take $\theta=\theta_1$ at radius r, we have

$\theta_1-\theta_2=\frac{q}{2\kappa L\pi}\ln\left(\frac{R}{r}\right)$ (1)

Similarly for the lagging

$\theta_2-\theta_3=\frac{q}{2\kappa_{lag}L\pi}\ln\left(\frac{R+d}{R}\right)$ (2)

Using the given formula for the heat transfer at the surface of the pipe applied to the inside of the pipe with the above definitions of temperatures,

$\theta_{\text{in}}-\theta_1=\frac{q}{2\pi Lrh_{\text{in}}}$ (3)

And the outside of the pipe

$\theta_3-\theta_{\text{out}}=\frac{q}{2\pi L(R+d)h_{\text{out}}}$ (4)

Adding (3), (1), (2) and (4) we get

$$\theta_{\text{in}}-\theta_{\text{out}}=\frac{q}{2\pi L}\left(\frac{1}{rh_{\text{in}}}+\frac{1}{\kappa}\ln\left(\frac{R}{r}\right)+\frac{1}{\kappa_{lag}}\ln\left(\frac{R+d}{R}\right)+\frac{1}{(R+d)h_{\text{out}}}\right)$$

which rearranges to give the required result.

b) For maximum q we need to minimise

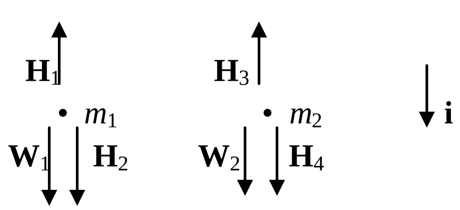
$U=\left(\frac{1}{rh_{\text{in}}}+\frac{1}{\kappa}\ln\left(\frac{R}{r}\right)+\frac{1}{\kappa_{lag}}\ln\left(\frac{R+d}{R}\right)+\frac{1}{(R+d)h_{\text{out}}}\right)$

with respect to d.

$$\frac{dU}{dd}=\frac{1}{\kappa_{\text{lag}}}\frac{1}{R+d}-\frac{1}{(R+d)^2h_{\text{out}}}=\frac{\left(\frac{R+d}{\kappa_{\text{lag}}}-\frac{1}{h_{\text{out}}}\right)}{(R+d)^2}$$

For a minimum and therefore q a maximum this needs to be zero so $d=\frac{\kappa_{\text{lag}}}{h_{\text{out}}}-R$ (which is valid because of the inequality given in the question.)

28. a)



b) Using the delta forces (p57)

$\Delta\text{H}_1=-k_1x_1\text{i}$

$\Delta\text{H}_2=-k_2(x_2-x_1)(-\text{i})=k_2(x_2-x_1)\text{i}$

$\Delta\text{H}_3=-\Delta\text{H}_2=-k_2(x_2-x_1)\text{i}$ (N3)

$\Delta\text{H}_4=-k_3(-x_2)(-\mathbf{i})=-k_3x_2\mathbf{i}$

Equation of motion for m_1 is $m_1\ddot{x}_1\mathbf{i}=\Delta\mathbf{H}_1+\Delta\mathbf{H}_2$

Resolving in the **i**-direction gives

$m_1\ddot{x}_1=-k_1x_1+k_2(x_2-x_1)$ (1)

Equation of motion for m_2 is $m_2\ddot{x}_2\mathbf{i}=\Delta\mathbf{H}_3+\Delta\mathbf{H}_4$

Resolving in the **i**-direction gives

$m_2\ddot{x}_2=-k_2(x_2-x_1)-k_3x_2$ (2)

c) (1) becomes $\ddot{x}_1=-\frac{k_1+k_2}{m_1}x_1+\frac{k_2}{m_2}x_2$

after collecting terms and dividing by m_1

Similarly (2) becomes $\ddot{x}_2=\frac{k_2}{m_2}x_1-\frac{k_2+k_3}{m_2}x_2$

Putting them in matrix form

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}=\begin{bmatrix}-\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1}\\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

which gives the required dynamic matrix. (p56)

d) The dynamic matrix becomes $\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$

For the eigenvalues $\begin{vmatrix}-2-\lambda & 1\\ 1 & -2-\lambda\end{vmatrix}=0$ (p41)

so $\lambda^2+4\lambda+3=0$

Factorising $(\lambda+3)(\lambda+1)=0$

The eigenvalues are $\lambda=-3$ and $\lambda=-1$

So the normal mode angular frequencies are 1 and $\sqrt{3}$

e) For eigenvectors for $\lambda=-3$

$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ so $u+v=0$

A typical eigenvector is $[1 \quad -1]^T$

For eigenvectors for $\lambda=-1$

$\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ so $u-v=0$

A typical eigenvector is $[1 \quad 1]^T$

28. e) The general solution is

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}(A\cos t+B\sin t)+\begin{bmatrix}1\\ -1\end{bmatrix}(C\cos\sqrt{3}\,t+D\sin\sqrt{3}t)$$

$$=\begin{bmatrix}1\\ 1\end{bmatrix}C_1\cos(t+\varphi_1)+\begin{bmatrix}1\\ -1\end{bmatrix}C_2\cos(\sqrt{3}t+\varphi_1)$$

(p57)

28. (cont)

f) As the vector $\begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = -3$, the particles will oscillate with angular frequency $\sqrt{3}$ out of phase.

29. a) $u(x,y) = X(x)Y(y)$

So $\frac{\partial^2 u}{\partial x^2} = X''(x)Y(y)$ and $\frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$

Substituting in the equation $X''Y + XY'' = 0$

Divide by XY and rearrange $\frac{X''}{X} = -\frac{Y''}{Y} = -\mu$ where μ is a constant.

So $X'' + \mu X = 0$ and $Y'' - \mu Y = 0$ (as required)

$u(x,0) = 0 \Rightarrow X(x)Y(0) = 0 \Rightarrow Y(0) = 0$

Similarly $u(x,a) = 0 \Rightarrow Y(a) = 0$

b) If $\mu > 0$ let $\mu = k^2$ where $k > 0$ so $Y'' - k^2 Y = 0$ which has solution $Y = Ae^{ky} + Be^{-ky}$

$Y(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$

$Y(a) = 0 \Rightarrow Ae^{ka} + Be^{-ka} = 0 \Rightarrow A\left(e^{ka} - e^{-ka}\right) = 0$

$k \neq 0$ so the bracket is not zero $\Rightarrow A = 0 \Rightarrow B = 0$ so only trivial solution.

If $\mu = 0$ $Y'' = 0$ and $Y = Cy + D$

$Y(0) = 0 \Rightarrow D = 0$ and $Y(a) = 0 \Rightarrow Ca = 0 \Rightarrow C = 0$

So only trivial solution.

If $\mu < 0$ let $\mu = -k^2$ where $k > 0$ so $Y'' + k^2 Y = 0$ which has solution $Y = E\cos ky + F\sin ky$

$Y(0) = 0 \Rightarrow E = 0$ and $Y(a) = 0 \Rightarrow F\sin ka = 0$

$F = 0$ gives the trivial solution so

$\sin ka = 0 \Rightarrow ka = r\pi \Rightarrow k = \frac{r\pi}{a} \Rightarrow \mu = -\frac{r^2\pi^2}{a^2}$

The solution is $Y = E\sin\frac{r\pi y}{a}$

c) $X'' - \mu X = 0 \Rightarrow X'' - \frac{r^2\pi^2}{a^2}X = 0$

$\Rightarrow \quad X = Ge^{\frac{r\pi x}{a}} + He^{-\frac{r\pi x}{a}}$

$u(x,y) \to 0$ as $x \to \infty$ so $X(x) \to 0$ as $x \to \infty$

$\Rightarrow \quad G = 0 \Rightarrow X = He^{-\frac{r\pi x}{a}}$.

For each r $u_r(x,y) = XY = B_r e^{-\frac{r\pi x}{a}} \sin\frac{r\pi y}{a}$ where $B_r = FH$.

Using the principle of superposition

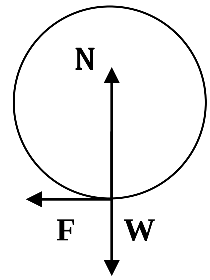
$$u(x,y) = \sum_{r=1}^{\infty} B_r e^{-\frac{r\pi x}{a}} \sin\frac{r\pi y}{a}$$

d) $u(0,y) = f(y) = \sum_{r=1}^{\infty} B_r \sin\frac{r\pi y}{a}$

Using the Fourier sine series of $f(y)$ (Handbook 61)

$B_r = \frac{2}{a}\int_0^a f(y)\sin\frac{r\pi y}{a}\,dy$.

30. a)



$\mathbf{N} = |\mathbf{N}|\mathbf{j}$

$\mathbf{F} = -|\mathbf{F}|\mathbf{i}$

$\mathbf{W} = -mg\mathbf{j}$

b) The equation of motion of the centre of mass is $M\ddot{\mathbf{x}} = \mathbf{N}+\mathbf{F}+\mathbf{W}$. Resolving in the **j**-direction

$|\mathbf{N}| - Mg = 0 \Rightarrow |\mathbf{N}| = Mg$

As we have kinetic friction $|\mathbf{F}| = \mu|\mathbf{N}| = \mu Mg$

Substituting in the equation of motion and cancelling M we get $\ddot{\mathbf{x}} = -\mu g\mathbf{i}$ (1)

c) As suggested we use the equation of relative rotational motion (Handbook 75) $I\ddot{\theta} = \Gamma_a^{rel}$ (2)

and $\boldsymbol{\Gamma}^{rel} = -R\mathbf{j} \times \mathbf{F} = -R\mathbf{j} \times -\mu Mg\mathbf{i} = -\mu RMg\mathbf{k}$

$I = \frac{2}{5}MR^2$ (Handbook 74)

Substituting in (2) $\frac{2}{5}MR^2\ddot{\theta} = -\mu RMg$ or $\ddot{\theta} = -\frac{5\mu g}{2R}$

d)Resolving (1) in the **i**-direction gives $\ddot{x} = -\mu g$

Integrating $\dot{x} = -\mu gt + C$ but $\dot{x} = v_0$ when $t = 0$

$\Rightarrow C = v_0 \Rightarrow \dot{x} = -\mu gt + v_0$ (3).

Integrating $\ddot{\theta} = -\frac{5\mu g}{2R}$ gives $\dot{\theta} = -\frac{5\mu g}{2R}t + D$ and

$\dot{\theta} = 0$ when $t = 0 \Rightarrow D = 0$

$\dot{x} + R\dot{\theta} = -\frac{7\mu gt}{2} + v_0$ and $\dot{x} + R\dot{\theta} = 0$ when $t = \frac{2v_0}{7\mu g}$

e) Integrating (3) using $x = 0$ when $t = 0$ gives

$x = -\frac{\mu gt^2}{2} + v_0 t$. At $t = \frac{2v_0}{7\mu g}$ $x = \frac{12v_0^2}{49\mu g}$.

At $t = \frac{2v_0}{7\mu g}$ $\dot{\theta} = -\frac{5\mu g}{2R}\frac{2v_0}{7\mu g} = -\frac{5v_0}{7R}$

Angular speed is $\frac{5v_0}{7R}$ as required and the distance the sphere slips is $\frac{12v_0^2}{49\mu g}$.