## MST209 2005

1. $\frac{x}{y}+1=2x-1$ so $\frac{x}{y}=2x-2$
   cross multiplying gives **B**.

2. **B**

3. $\mathbf{b}-\mathbf{c}=\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ so $\mathbf{a}.(\mathbf{b}-\mathbf{c})=1-2+2=1$
   **C**

4. **C**

5. Integrating
$$\mathbf{v}=\left(\frac{t^2}{2}-\frac{\sin 2t}{2}\right)\mathbf{i} \text{ and } \mathbf{r}=\left(\frac{t^3}{6}+\frac{\cos 2t}{4}+C\right)\mathbf{i}$$
$\mathbf{r}=\mathbf{0}$ when $t=0$ so $0=\frac{1}{4}+C\Rightarrow C=-\frac{1}{4}$
   **D**

6. Right hand spring is not extended **A**

7. Block is below datum level so minus.
Vertical distance is $d\sin\theta$ so **D**

8.
$$\begin{bmatrix}2&1&3\\4&0&2\\-4&1&-1\end{bmatrix}\begin{bmatrix}1\\2\\-1\end{bmatrix}=\begin{bmatrix}1\\2\\-1\end{bmatrix} \text{ so } \mathbf{B}$$

9. Using the composite rule **D.**

10.
$\mathbf{v}=\mathbf{j}+2t\mathbf{k}$. At $t=1$ $\mathbf{v}=\mathbf{j}+2\mathbf{k}$
$|\mathbf{v}|=\sqrt{1+4}$ so **B**

11. $\alpha=\frac{r}{2\sqrt{m(2k)}}>1$ so $r>2\sqrt{2mk}$ so **D**

12. Momentum is $m(u\mathbf{i}+v\mathbf{j})+2m(U\mathbf{i}+V\mathbf{j})$ **B**

13. $f(-x)=\cos\left(e^x\right)\neq\pm f(x)$ so **C**

14. $\text{div}\mathbf{F}=2xy^3z+2xyz^2+2x^3yz$ so **B**

15. **C**

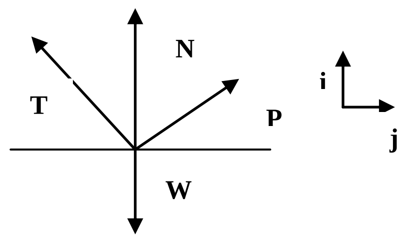
16. Using page 26 of Handbook
$$p(x)=\exp\left(\int\frac{2}{x}dx\right)=\exp(2\ln x)=x^2$$
Equation becomes $\frac{d}{dx}\left(x^2y\right)=2x^3$

Integrating $x^2y=\frac{x^4}{2}+C$

When $x=1$ $y=0$ so $C=-\frac{1}{2}$

$$y=\frac{x^2}{2}-\frac{1}{2x^2}$$

17. (a)

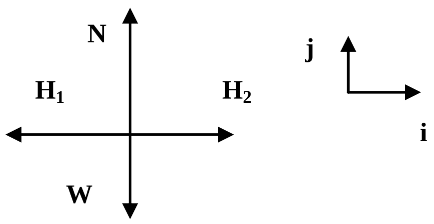



**T** = tension **N** = normal reaction
**W** = weight

(b)
$\mathbf{W}=-mg\mathbf{j}$ $\mathbf{N}=|\mathbf{N}|\mathbf{j}$
$$\mathbf{P}=|\mathbf{P}|\left(\cos\frac{\pi}{6}\mathbf{i}+\sin\frac{\pi}{6}\mathbf{j}\right)=|\mathbf{P}|\left(\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)$$
$$\mathbf{T}=|\mathbf{T}|\left(-\cos\frac{\pi}{4}\mathbf{i}+\sin\frac{\pi}{4}\mathbf{j}\right)=\frac{|\mathbf{T}|}{\sqrt{\mathbf{2}}}(-\mathbf{i}+\mathbf{j})$$
(c)
For equilibrium $\mathbf{W}+\mathbf{N}+\mathbf{P}+\mathbf{T}=\mathbf{0}$
Resolving in the **i**-direction
$$-\frac{|\mathbf{T}|}{\sqrt{2}}+\frac{\sqrt{3}|\mathbf{P}|}{2}=0 \text{ so } |\mathbf{T}|=\sqrt{\frac{3}{2}}|\mathbf{P}|$$
Resolving in the **j**-direction
$$-mg+|\mathbf{N}|+\frac{|\mathbf{P}|}{2}+\frac{|\mathbf{T}|}{\sqrt{2}}=0$$
$$|\mathbf{N}|=mg-\frac{|\mathbf{P}|}{2}-\frac{|\mathbf{T}|}{\sqrt{2}}=mg-\frac{|\mathbf{P}|}{2}-\frac{\sqrt{3}|\mathbf{P}|}{2}$$

To remain in contact
$$|\mathbf{N}|>0 \text{ so } m>|\mathbf{P}|\frac{\left(1+\sqrt{3}\right)}{2g}$$

18.




$\mathbf{N} + \mathbf{W} = \mathbf{0}$ as no vertical motion

$\mathbf{H_1} = -k\left(x - l_0\right)\mathbf{i}$

$\mathbf{H_2} = -2k\left(3l_0 - x - \frac{l_0}{2}\right)(-\mathbf{i}) = 2k\left(\frac{5l_0}{2} - x\right)\mathbf{i}$

Newton's second law gives

$m\ddot{x}\mathbf{i} = \mathbf{H_1} + \mathbf{H_2}$

Resolve in **i**-direction

$m\ddot{x} = -k\left(x - l_0\right) + k\left(5l_0 - 2x\right)$

$m\ddot{x} + 3kx = 6kl_0$

19. a)

From Handbook page 49

$y = x\tan\theta - \frac{x^2 g}{2u^2}\left(1 + \tan^2\theta\right)$

When $x = d,\, y = h$ so

$h = d\tan\theta - \frac{d^2 g}{2u^2}\left(1 + \tan^2\theta\right) \quad (1)$

When $x = 2d,\, y = h$ so

$h = 2d\tan\theta - \frac{4d^2 g}{2u^2}\left(1 + \tan^2\theta\right) \quad (2)$

$(1)\times 4$ gives

$4h = 4d\tan\theta - \frac{4d^2 g}{2u^2}\left(1 + \tan^2\theta\right) \quad (3)$

$(3)-(2)$ gives

$3h = 2d\tan\theta$ or $\tan\theta = \frac{3h}{2d}$

b) Substitute into (1)

$h = \frac{3h}{2} - \frac{d^2 g}{2u^2}\left(1 + \frac{9h^2}{4d^2}\right)$

$\frac{g}{2u^2}\left(d^2 + \frac{9h^2}{4}\right) = \frac{h}{2}$

$u^2 = \frac{g}{4h}\left(4d^2 + 9h^2\right)$

20. a) From Handbook page 51

$q = UA\left(\theta_{\text{in}} - \theta_{\text{out}}\right)$ and $U = \left(\frac{1}{h_{\text{in}}} + \frac{b}{k} + \frac{1}{\text{h}_{\text{out}}}\right)^{-1}$

Here $h_{\text{in}} = 10,\ \text{h}_{\text{out}} = 100,\ b = 0.01,\ k = 1$

so $U = \left(\frac{1}{10} + 0.01 + \frac{1}{100}\right)^{-1} = \left(\frac{12}{100}\right)^{-1} = \frac{100}{12}$

$\theta_{\text{in}} = 15,\quad \theta_{\text{out}} = 0,\quad A = 2$

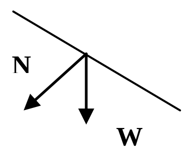
so $q = \frac{100}{12}\times 2\times 15 = 250$ W

b) From Handbook page 51

$q = h_{\text{out}}A(\theta_s - \theta_{\text{out}})$ where $\theta_s =$ Temp at outer surface

so $250 = 100\times 2\theta_s$ so $\theta_s = 1.25^\circ C$

21. a)




b) $\mathbf{N} = -|\mathbf{N}|\mathbf{e}_r \quad \mathbf{W} = mg\left(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta\right)$

c) By Newton's second law

$m\ddot{\mathbf{r}} = \mathbf{N} + \mathbf{W}$

Using page 59 of Handbook

$-mR\dot{\theta}^2\mathbf{e}_r + mR\ddot{\theta}\mathbf{e}_\theta = -|\mathbf{N}|\mathbf{e}_r + mg\left(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta\right)$

Resolve in the $\mathbf{e}_r$-direction

$-mR\dot{\theta}^2 = mg\cos\theta - |\mathbf{N}|$

Resolve in the $\mathbf{e}_\theta$-direction

$mR\ddot{\theta} = -mg\sin\theta$

22. a)

Substitute $u = XT$ into equation giving $X''T = XT''$

Divide by $XT$ giving $\frac{X''}{X} = \frac{T''}{T} = k$ a constant

as RHS is a function of $t$ only and LHS is a function of $x$ only.

So $X'' - kX = 0$.

b)

$u(0,t) = u(\pi,0) = 0 \;\Rightarrow\; X(0) = X(\pi) = 0$

If $k > 0$ i.e $k = \lambda^2$ the equation becomes $X'' - \lambda^2 X = 0$

The solution of which is $X = Ae^{\lambda x} + Be^{\lambda x}$

Fitting the boundary equations

$0 = A + B$ and $0 = Ae^{\lambda\pi} + Be^{-\lambda\pi} \;\Rightarrow\; A = B = 0$

22 b) (cont)
If $k=0$ the equation becomes
$X''=0$. The solution of which is $X=Ax+B$
Fitting the boundary conditions
$0=B$ and $0=A\pi \Rightarrow A=B=0$.
If $k<0$ i.e. $k=-\omega^2$ the equation becomes
$X''+\omega^2X=0$ which has solution
$X=A\cos\omega x+B\sin\omega x$. Fitting the boundary conditions $0=A$ and $0=B\sin\omega\pi$
For non trivial solutions
$B\neq 0 \Rightarrow \sin\omega\pi=0 \quad \omega=n$ where $n=1,2,3\ldots$
so $X=B\sin nx$ where $k=-n^2$ and $n=1,2,3\ldots$

23. a) M of I of child = 0 so M of I of roundabout together with child = $I$ making the angular momentum = $I\omega$
b) New M of I of child = $mR^2$.
Total M of I = $I+mR^2$
Let new angular speed be $\Omega$.
As there are no external forces by conservation of angular momentum

$$\left(I+mR^2\right)\Omega=I\omega \Rightarrow \Omega=\frac{I\omega}{I+mR^2}$$

c) KE at beginning = $\frac{1}{2}I\omega^2$
K E at end = KE of child + K E of roundabout $=\frac{1}{2}mR^2\Omega^2+\frac{1}{2}I\Omega^2$
Substituting for $\Omega$ gives K E at end

$$=\frac{1}{2}\left(mR^2+I\right)\frac{I\omega^2}{\left(I+mR^2\right)^2}=\frac{I\omega^2}{2\left(I+mR^2\right)}$$

$$\text{Change in K E}=\frac{I\omega^2}{2\left(I+mR^2\right)}-\frac{1}{2}I\omega^2$$

$$=-\frac{I\omega^2}{2}\left(\frac{mR^2}{I+mR^2}\right)$$

24.
From page 26 of Handbook auxiliary equation is

$$\lambda^2+2\lambda+5=0 \text{ so } \lambda=\frac{-2\pm\sqrt{4-4\times5}}{2}=-1\pm 2i$$

and complementary function is
$x=e^{-t}\left(A\cos 2t+B\sin 2t\right)$

For particular integral (page 27 Handbook) try
$x=a+b\cos 2t+c\sin 2t$

$$\frac{dx}{dt}=-2b\sin 2t+2c\cos 2t$$

$$\frac{d^2x}{dt^2}=-4b\cos 2t-4c\sin 2t$$

Substituting into the equation gives

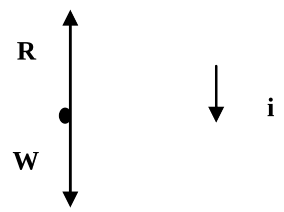
$$-4b\cos 2t-4c\sin 2t-4b\sin 2t+4c\cos 2t$$
$$+5a+5b\cos 2t+5c\sin 2t=20+17\cos 2t$$
$$5a+\left(4c+b\right)\cos 2t+\left(c-4b\right)\sin 2t=20+17\cos 2t$$

Equating coefficients
$5a=20 \Rightarrow a=4$
$b+4c=17$ and $c-4b=0 \Rightarrow$ 17b=17
so $b=1$ and $c=4$
$x_p=4+\cos 2t+4\sin 2t$
Adding the two solutions together
$x=e^{-t}\left(A\cos 2t+B\sin 2t\right)+4+\cos 2t+4\sin 2t$
Fitting the initial conditions

$$\frac{dx}{dt}=e^{-t}\left(-2A\sin 2t+2B\cos 2t\right)$$
$$-e^{-t}\left(A\cos 2t+B\sin 2t\right)-2\sin 2t+8\cos 2t$$

$x=5$ when $t=0$ gives $5=A+4+1$ so $A=0$
$\frac{dx}{dt}=0$ when $t=0$ gives $0=2B-A+8$ so $B=-4$
$x=-4e^{-t}\sin 2t+4+\cos 2t+4\sin 2t$

25. a) Choose the co-ordinate axis vertically downwards as motion is downwards




**R** is the air resistance force and **W** is the weight so $\mathbf{R}=-c_2D^2v^2\mathbf{i}$ and $\mathbf{W}=mg\mathbf{i}$.
By Newton's second law $m\mathbf{a}=\mathbf{R}+\mathbf{W}$
Resolving in the **i**-direction
$ma=-c_2D^2v^2+mg$
b) For terminal speed $a=0$
so $mg-c_2D^2v_T=0$ giving $v_T=\sqrt{\frac{mg}{c_2D^2}}$

25. b) (cont)

Using this and $a = v\dfrac{dv}{dt}$

(Handbook page 32) the equation of motion

becomes $mv\dfrac{dv}{dt} = mg - \dfrac{mgv^2}{{v_T}^2}$

Cancelling $m$ and tidying up gives

$$v\frac{dv}{dx} = -\frac{g}{{v_T}^2}\left(v^2 - {v_T}^2\right)$$

c)

Separating the variables gives

$$\int \frac{vdv}{v^2 - {v_T}^2} = -\int \frac{g}{{v_T}^2}dx \text{ as } v_0 > v_T \quad v > v_T$$

Integrating $\dfrac{1}{2}\ln\left(v^2 - {v_T}^2\right) = -\dfrac{g}{{v_T}^2}x + C$

$v = v_0$ when $x = 0$ so $C = \dfrac{1}{2}\ln\left({v_0}^2 - {v_T}^2\right)$

so $\dfrac{1}{2}\ln\left({v_0}^2 - {v_T}^2\right) - \dfrac{1}{2}\ln\left(v^2 - {v_T}^2\right) = \dfrac{gx}{{v_T}^2}$

or $x = \dfrac{{v_T}^2}{2g}\ln\left(\dfrac{\left({v_0}^2 - {v_T}^2\right)}{\left(v^2 - {v_T}^2\right)}\right)$

26.

In matrix form the equations are

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 6e^{-t} \end{bmatrix}$$

For eigenvalues

$$\begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - 3\lambda + 2 = 0$$

giving $(\lambda - 1)(\lambda - 2) = 0$

so $\lambda = 1$ or $\lambda = 2$

For eigenvectors using $\lambda = 1$

$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $v = 2u$ an eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For eigenvectors using $\lambda = 2$

$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $v = u$ an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Using page 43 of Handbook
the complementary function is

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^t + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$$

For particular integral try $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}e^{-t}$

Substituting into the equation

$$-\begin{bmatrix} a \\ b \end{bmatrix}e^{-t} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}e^{-t} + \begin{bmatrix} 0 \\ 6 \end{bmatrix}e^{-t}$$

so $-a = 3a - b$ giving $b = 4a$

and $-b = 2a + 6$ giving $6a = -6$

$\Rightarrow\ a = -1,\ b = -4$

The particular integral is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}e^{-t}$

General solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^t + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} - \begin{bmatrix} 1 \\ 4 \end{bmatrix}e^{-t}$$

Using the initial conditions

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ so

$C_1 + C_2 = 1$ and $2C_1 + C_2 = 4$ giving $C_1 = 3$ and $C_2 = -2$

so $x = 3e^t - 2e^{2t} - e^{-t}\quad y = 6e^t - 2e^{2t} - 4e^{-t}$

27. a)

As $x = 0$ the equation for $y$ becomes

$\dot{y} = y(B - y)$ which is the logistic equation.

The starting point is between the equilibrium points and so $y(B - y)$ is positive and the population will grow tailing off as it approaches the equilibrium point $y = B$.

b)

For equilibrium $\dot{y} = \dot{x} = 0$ so that

$x(y - A) = 0 \Rightarrow x = 0$ or $y = A$

Also $y(B - x - y) = 0$ so if $x = 0$ then $y = 0$ or $y = B$

If $y = A,$ then $B - x - A = 0$ or $x = B - A$

The equilibrium points $(0,0)$, $(0,B)$ and $(B - A, A)$

c)

$u = xy - Ax$ so $\dfrac{\partial u}{\partial x} = y - A$ and $\dfrac{\partial u}{\partial y} = x$

$v = By - xy - y^2$ so $\dfrac{\partial v}{\partial x} = -y$ and $\dfrac{\partial v}{\partial y} = B - x - 2y$

$$\mathbf{J}(B - A, A) = \begin{bmatrix} 0 & B - A \\ -A & -A \end{bmatrix}$$

27 c) i)

$B=\dfrac{6A}{5}$ makes $\mathbf{J}=\begin{bmatrix}0 & \dfrac{A}{5}\\ -A & -A\end{bmatrix}$

For eigenvalues

$\begin{vmatrix}-\lambda & \dfrac{A}{5}\\ -A & -\lambda-A\end{vmatrix}=0$ so $\lambda^2+A\lambda+\dfrac{A^2}{5}=0$

$$\lambda=\frac{-A\pm\sqrt{A^2-\frac{4}{5}A^2}}{2}=-\frac{A}{2}\left(1\pm\frac{1}{\sqrt{5}}\right)$$

Eigenvalues are both real and negative so the point is a stable sink. (Handbook page 47)

ii)

$B=2A$ makes $\mathbf{J}=\begin{bmatrix}0 & A\\ -A & -A\end{bmatrix}$

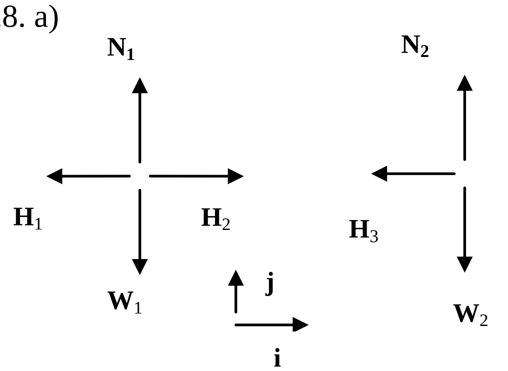
For eigenvalues

$\begin{vmatrix}-\lambda & A\\ -A & -\lambda-A\end{vmatrix}=0$ so $\lambda^2+A\lambda+A^2=0$

$$\lambda=\frac{-A\pm\sqrt{A^2-4A^2}}{2}=-\frac{A}{2}\left(1\pm\sqrt{3}i\right)$$

Eigenvalues are complex with negative real part so the point is a stable spiral sink.

28. a)

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b)

$\Delta\mathbf{H}_1=-6x\mathbf{i}$ $\Delta\mathbf{H}_2=-2(y-x)(-\mathbf{i})=2(y-x)\mathbf{i}$

$\Delta\mathbf{H}_3=-\Delta\mathbf{H}_2=-2(y-x)\mathbf{i}$

c)

$\mathbf{W}+\mathbf{N}=\mathbf{0}$ for each particle as the motion is horizontal. For the first particle

Newton's second law gives $4\ddot{x}\mathbf{i}=\Delta\mathbf{H}_1+\Delta\mathbf{H}_2$

Resolving in the $\mathbf{i}$ –direction

$4\ddot{x}=-6x+2y-2x$ or $\ddot{x}=-2x+\dfrac{y}{2}$

For the second particle $\ddot{y}\mathbf{i}=\Delta\mathbf{H}_3$

Resolving in the $\mathbf{i}$ –direction $\ddot{y}=-2y+2x$

In matrix form

$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix}=\begin{bmatrix}-2 & \frac{1}{2}\\ 2 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$ as required.

d)

For eigenvalues

$\begin{vmatrix}-2-\lambda & \frac{1}{2}\\ 2 & -2-\lambda\end{vmatrix}=0$ so $(2+\lambda)^2-1=0$

or $\lambda^2+4\lambda+3=0$ $(\lambda+3)(\lambda+1)=0$

so eigenvalues are $-3$ and $-1$

Normal mode angular frequencies are

$\omega=\sqrt{3}$ and $\omega=1$..

e)

For $\lambda=-3$ $\begin{bmatrix}1 & \frac{1}{2}\\ 2 & 1\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ so $v=-2u$

An eigenvector is $\begin{bmatrix}1 & -2\end{bmatrix}^T$

For $\omega=\sqrt{3}$ the components of an eigenvector have different signs so the motion is phase-opposed.

For $\lambda=-1$ $\begin{bmatrix}-1 & \frac{1}{2}\\ 2 & -1\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ so $v=2u$

An eigenvector is $\begin{bmatrix}1 & 2\end{bmatrix}^T$

For $\omega=1$. the components of an eigenvector have the same sign so the motion is in-phase.

29. a)

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \alpha xyz-\alpha x & x^2z-2y & x^2y\end{vmatrix}$$

$$=\mathbf{i}\left(x^2-x^2\right)-\mathbf{j}(2xy-\alpha xy)+\mathbf{k}(2xz-\alpha xz)$$

$$=(\alpha-2)xy\mathbf{j}+(2-\alpha)xz\mathbf{k}$$

b)

$\mathrm{curl}\mathbf{F}=\mathbf{0}$ when $\mathbf{F}$ is conservative so $\alpha=2$.

c)

Choosing a straight line segment from $O$ to the point $(a,b,c)$ we have

$$\mathbf{r}(t)=at\mathbf{i}+bt\mathbf{j}+ct\mathbf{k}\ \ (0\le t\le 1).$$

29 d)

Assuming $C$ is the path above along the path

$$\mathbf{F} = 2at\left(bct^2 - 1\right)\mathbf{i} + \left(a^2ct^3 - 2bt\right)\mathbf{j} + a^2bt^3\mathbf{k}$$

$$\frac{d\mathbf{r}}{dt} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

so $\mathbf{F}.\dfrac{d\mathbf{r}}{dt} = 2a^2bct^3 - 2a^2t + a^2bct^3 - 2b^2t + a^2bct^3$

$$= 4a^2bct^3 - 2\left(a^2 + b^2\right)t$$

$$\int_C \mathbf{F}.d\mathbf{r} = \int_0^1 \left(4a^2bct^3 - 2\left(a^2 + b^2\right)t\right)dt$$

$$= \left[a^2bct^4 - \left(a^2 + b^2\right)t^2\right]_0^1$$

$$= a^2bc - \left(a^2 + b^2\right)$$

e)

If $\mathbf{F} = -\operatorname{grad} U$ then

$U\left(a,b,c\right) = -\int_C \mathbf{F}.d\mathbf{r} = a^2 + b^2 - a^2bc$ .

As $a$, $b$ and $c$ are arbitrary they can be replaced by $x$, $y$ and $z$ respectively giving

$U\left(x,y,z\right) = x^2 + y^2 - x^2yz$

$$-\operatorname{grad} U = -\begin{pmatrix} \dfrac{\partial}{\partial x}\left(x^2 + y^2 - x^2yz\right)\mathbf{i} + \\ \dfrac{\partial}{\partial y}\left(x^2 + y^2 - x^2yz\right)\mathbf{j} \\ + \dfrac{\partial}{\partial z}\left(x^2 + y^2 - x^2yz\right)\mathbf{k} \end{pmatrix}$$

$$= -\left(2x - 2xyz\right)\mathbf{i} - \left(2y - x^2z\right)\mathbf{j} + x^2y\mathbf{k}$$

$$= \mathbf{F}$$

30. a)

In cylindrical polars the equation of the sphere is $\rho^2 + z^2 = R^2$ so the volume is given by using Handbook page 70

$$V = \int_{\frac{R}{2}}^{R}\int_0^{\sqrt{R^2-z^2}}\int_{-\pi}^{\pi} \rho\, d\theta\, d\rho\, dz = 2\pi\int_{\frac{R}{2}}^{R}\int_0^{\sqrt{R^2-z^2}} \rho\, d\rho\, dz$$

$$= \pi\int_{\frac{R}{2}}^{R}\left[\rho^2\right]_0^{\sqrt{R^2-z^2}} dz = \pi\int_{\frac{R}{2}}^{R} R^2 - z^2 dz$$

$$= \pi\left[R^2z - \frac{z^3}{3}\right]_{\frac{R}{2}}^{R} = \pi\left(\frac{2R^3}{3} - \left(\frac{R^3}{2} - \frac{R^3}{24}\right)\right)$$

$$= \frac{\pi R^3}{24}\left(16 - 12 + 1\right) = \frac{5\pi R^3}{24}$$

b)

$$I = \int_B z\,dV = \int_{\frac{R}{2}}^{R}\int_0^{\sqrt{R^2-z^2}}\int_{-\pi}^{\pi} z\rho\, d\theta\, d\rho\, dz$$

$$= 2\pi\int_{\frac{R}{2}}^{R}\int_0^{\sqrt{R^2-z^2}} z\rho\, d\rho\, dz = \pi\int_{\frac{R}{2}}^{R}\left[z\rho^2\right]_0^{\sqrt{R^2-z^2}} dz$$

$$= \pi\int_{\frac{R}{2}}^{R} zR^2 - z^3 dz = \pi\left[\frac{z^2R^2}{2} - \frac{z^4}{4}\right]_{\frac{R}{2}}^{R}$$

$$= \pi\left(\frac{R^4}{4} - \left(\frac{R^4}{8} - \frac{R^4}{64}\right)\right) = \frac{\pi R^4}{64}\left(16 - 8 - 1\right)$$

$$= \frac{9\pi R^4}{64}$$

c)

The centre of mass lies on the axis of symmetry i.e. the $z$-axis. Using Handbook page 71 $M = \int_B \rho_B dV$ and

$$z_G = \frac{\int_B z\rho_B dV}{\int_B \rho_B dV} = \frac{\int_B z\,dV}{\int_B dV} = \frac{I}{V}$$ as $\rho_B$ is constant.

So $z_G = \dfrac{\frac{9\pi R^4}{64}}{\frac{5\pi R^3}{24}} = \dfrac{27R}{40}$

The centre of mass of the object is

$\left(0, 0, \dfrac{27R}{40}\right)$.