Handbook references are given as (p) for information only.

1. Combining the logarithms using the rules on (p11)

$$\ln\left(\frac{(y+1)(y-1)}{3}\right)=\ln\left(\frac{1}{3x+9}\right)$$

Exponentiating each side and using the difference of two squares gives $\frac{y^2-1}{3}=\frac{1}{3x+9}$ or $y^2-1=\frac{1}{x+3}$ which gives

$y^2=\frac{1}{x+3}+1$ or $y^2=\frac{x+4}{x+3}$ so the answer is C.

2. The equation can be written as $\frac{1}{y}\frac{dy}{dx}=x+1$ so we can solve it by separation of variables. The equation can be written as $\frac{dy}{dx}-(x+1)y=0$ so we can solve it by the integrating factor method. (p26). The answer is C.

3. $\mathbf{a}\times\mathbf{b}$ is perpendicular to **a** and **b** (p29) and evaluating this by Sarrus's Rule or formula

$$\begin{array}{ccccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\
1 & -2 & 1 & 1 & -2 \\
1 & 0 & -1 & 1 & 0
\end{array}$$

$$-2\mathbf{k} \quad 0\mathbf{i} \quad -\mathbf{j} \qquad 2\mathbf{i} \quad \mathbf{j} \quad 0\mathbf{k}$$

So $\mathbf{a}\times\mathbf{b}=2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$. Making it into a unit vector we divide by the magnitude $\sqrt{4+4+4}=2\sqrt{3}$ (p25) giving the answer B as the vector will be perpendicular to **a** and **b** regardless of the sign.

4. The angle between **F** and the **j** direction is $\frac{\pi}{2}+\theta$ and so the **j-**component is $|\mathbf{F}|\cos\left(\frac{\pi}{2}+\theta\right)=-|\mathbf{F}|\sin\theta$ (p31) so the answer is D.

5. Kinetic energy is $\frac{1}{2}mv^2=\frac{1}{2}m\left(R\dot{\theta}\right)^2$ (p35/59) and potential energy is $-mg(R-R\cos\theta)$ - minus as the distance is measured down from the top but the distance above the centre is $R\cos\theta$. The total energy is the sum of these and so the answer is C.

6. The sum of the eigenvalues is the trace of the matrix (p41) so $7=3+a$ giving the answer D.

7. Using the composite rule (p19) and remembering that θ is the variable (p45) the answer is C.

8. $u=rx^2-sxy \quad v=2qx-py^2$

So $\mathbf{J}(x,y)=\begin{bmatrix}2rx-sy & -sx\\ 2q & -2py\end{bmatrix}$ (p47)

Substituting $x=a,\ y=b$ gives the answer A.

9. $[\rho]=\mathrm{ML}^{-3} \quad [V]=\mathrm{LT}^{-1} \quad [L]=\mathrm{L}$ (p53)

As Re is dimensionless $1=((\mathrm{ML}^{-3})(\mathrm{LT}^{-1})(\mathrm{L}))/[\mu]$

$[\mu]=\mathrm{ML}^{-1}\mathrm{T}^{-1}$ so the answer is D.

10. In this case $\alpha=\frac{2r}{2\sqrt{mk}}$ as the damping constant is $2r$ not r. For critical damping $\alpha=1$ (p55) so the answer is D.

11. Using the position vectors of the appropriate corners, the centre of mass is given by

$$\mathbf{r}_G=\frac{3m(\mathbf{i}+\mathbf{j})+m(\mathbf{i}+\mathbf{j}+\mathbf{k})+2m(\mathbf{j}+\mathbf{k})}{6m}=\frac{4\mathbf{i}+6\mathbf{j}+3\mathbf{k}}{6}$$

So the answer is B. (p58)

12. The equation of motion is given by $m\ddot{\mathbf{r}}=\mathbf{W}+\mathbf{N}+\mathbf{F}$ where $\mathbf{W}=-mg\sin\theta\,\mathbf{e}_r-mg\cos\theta\,\mathbf{e}_\theta$

and $\mathbf{N}=-|\mathrm{N}|\mathbf{e}_r$ (Negative signs as $\mathbf{e}_r$ points away from the centre). Resolving the equation of motion in the $\mathbf{e}_r$ direction gives $-mR\dot{\theta}^2=-mg\sin\theta-|\mathbf{N}|$ so the answer is D. (p59)

13. If $u=X(x)T(t)$ then the equation becomes $X''T=\frac{1}{\lambda^2}XT'$ which divided by $X\,T$ gives

$\frac{X''}{X}=\frac{T'}{\lambda^2T}=k$. This separates to give option A. (p63 but here the separation constant is k)

14. $\frac{\partial h}{\partial\rho}=z\cos\theta \quad \frac{\partial h}{\partial\theta}=-\rho z\sin\theta$ and $\frac{\partial h}{\partial z}=\rho\cos\theta$

Using the formula from p65 the answer is A.

15. Using the volume integral in cylindrical polars (p70) with $f = cz$ we get option A. (The limits are the same in all cases)

16. The integrating factor is

$$p(x) = \exp \int -x^3 dx = \exp\left(-\frac{x^4}{4}\right)$$

Multiply the equation by $p(x)$

$$\frac{d}{dx}\left(e^{-\frac{x^4}{4}} y\right) = 1$$

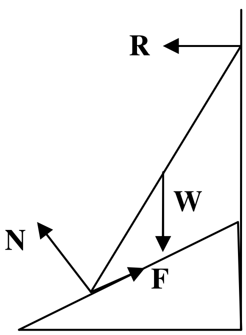
Integrate $\quad e^{-\frac{x^4}{4}} y = x + C$

Rearranging $y = (x + C)e^{\frac{x^4}{4}}$

Using the initial conditions $y(0) = 1 = C$

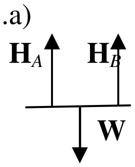
So $y = (x + 1)e^{\frac{x^4}{4}}$

17.



N = normal reaction of the plane on the ladder
F = frictional force of the plane on the ladder
W = weight of the ladder
R = Normal reaction of the wall on the ladder

18.a)



$\mathbf{H}_A$ = force in spring A
$\mathbf{H}_B$ = force in spring B
W = weight of particle

b) Using Hooke's law (p 34)
$\mathbf{H}_A = -k(x - l_0)\mathbf{i}$
$\mathbf{H}_B = -2k(x - d - l_0)\mathbf{i}$
(as the length of the spring is $x - d$)
$\mathbf{W} = mg\mathbf{i}$
c) The equation of motion is

$$m\ddot{x}\mathbf{i} = \mathbf{H}_A + \mathbf{H}_B + \mathbf{W}$$

Resolving in the **i**-direction

$$\begin{aligned} m\ddot{x} &= -k(x - l_0) - 2k(x - d - l_0) + mg \\ &= -3kx + 3kl_0 + 2kd + mg \end{aligned}$$

Or $\quad m\ddot{x} + 3kx = 3kl_0 + 2kd + mg$

19. Using section 32 on p39

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) & \begin{array}{l} R_1 \\ R_{2a} = R_2 + R_1 \\ R_3 \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{array}\right) & \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} = R_3 - \frac{1}{4}R_{2a} \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{2} \end{array}\right) & \begin{array}{l} R_1 \\ R_{2b} = \frac{R_{2a}}{4} \\ R_{3b} = \frac{R_{3a}}{2} \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{2} \end{array}\right) & \begin{array}{l} R_{1a} = R_1 - 2R_{2b} \\ R_{2b} \\ R_{3b} \end{array} \end{array}$$

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{2} \end{pmatrix}$$

20.

a) From p 51 $U = \left(\frac{1}{h_{in}} + \frac{b}{k} + \frac{1}{h_{out}}\right)^{-1}$

So $U = \left(\frac{1}{10} + \frac{0.2}{0.5} + \frac{1}{100}\right)^{-1}$

$$= \left(\frac{10 + 40 + 1}{100}\right)^{-1} = \left(\frac{51}{100}\right)^{-1} = \frac{100}{51}$$

b) Using the same method the U-value with the lining is

$$\begin{aligned} u_L &= \left(\frac{1}{10} + \frac{0.2}{0.5} + \frac{1}{10} + \frac{0.00x}{0.2} + \frac{1}{100}\right)^{-1} \\ &= \left(\frac{61}{100} + \frac{x}{200}\right)^{-1} \end{aligned}$$

c) Let q be the original heat transfer rate and q_L be the heat transfer rate with the lining. Then $q_L = \frac{2}{3}q$ or $U_L = \frac{2}{3}U$ Inverting gives

$$\frac{61}{100} + \frac{x}{200} = \frac{3}{2}\left(\frac{51}{100}\right)$$

$$\frac{x}{200} = \frac{153}{200} - \frac{61}{100} = \frac{31}{200}$$

So $x = 31\text{mm}$

21.

a) Kinetic energy after collision is $\frac{1}{2}m(u_1^2+v_1^2)+\frac{1}{2}m\,(u_2^2+v_2^2)=\frac{1}{2}m(u_1^2+u_2^2+v_1^2+v_2^2)$

b) $(u_1\mathbf{i}+v_1\mathbf{j}).(\mathrm{u}_2\mathbf{i}+v_2\mathbf{j})=u_1u_2+v_1v_2=0$ (1)
(p29 for evaluating the dot product)
By conservation of momentum (p58)

$$m(u_1\mathbf{i}+v_2\mathbf{j})+m(u_2\mathbf{i}+v_2\mathbf{j})=m(2\mathbf{i}-2\mathbf{j})$$

Cancelling *m* and equating coefficients gives

$$u_1+u_2=2 \text{ and } v_1+v_2=-2$$

So $(u_1+u_2)^2=u_1^2+u_2^2+2u_1u_2=2^2$
Giving $u_1^2+u_2^2=4-2u_1u_2$
Similarly $v_1^2+v_2^2=4-2v_1v_2$
Using these energy after collision $=\frac{1}{2}m(u_1^2+u_2^2+v_1^2+v_2^2)=\frac{1}{2}m\big(8-2(u_1u_2+v_1v_2)\big)=4m$ using (1)
Energy before collision $=\frac{1}{2}m(2^2+2^2)=4m$
As there is no loss of energy the collision is elastic.(p58)

22.

a) Using $V_r=V_\theta=0$ and replacing *F* by *V* in the formula on p66 we get
div $\mathbf{V}=\frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}=0$ as $V_\phi=\frac{1}{r}$.
b) Using the formula for **curl** in spherical polars (p67)

$$\textbf{curl V}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\boldsymbol{e}_r & r\mathbf{e}_\theta & r\sin\theta\,\mathbf{e}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ 0 & 0 & \sin\theta\end{vmatrix}$$

$$=\frac{1}{r^2\sin\theta}(\cos\theta\,\mathbf{e}_r)=\frac{1}{r^2}\cot\theta\,\mathbf{e}_r$$

23. $y(0)=1,\ h=0.1,\ f(x_i,y_i)=x_i+y_i$

$$Y_0=1,\qquad x_0=0$$

Using the procedure on page 72 of the Handbook

$$F_{1,0}=x_0+Y_0=1$$
$$x_1=x_0+h=0.1$$
$$Y_E=Y_0+hF_{1,0}=1+0.1*1=1.1$$
$$F_{2,0}=f(x_i,Y_E)=1.2$$
$$Y_1=Y_0+\frac{1}{2}h\big(F_{1,0}+F_{2,0}\big)=1+\frac{1}{2}0.1(1+1.2)$$
$$=1+\frac{1}{2}0.1(2.2)=1.11$$

24.a) Auxiliary equation is $4\lambda^2+25=0$

$$\text{or }\lambda^2=-\frac{25}{4}\text{ so }\lambda=\pm\frac{5}{2}i$$

The complementary function is

$$y_c=C\cos\left(\frac{5t}{2}\right)+D\sin\left(\frac{5t}{2}\right)$$

For the particular integral as the right hand side is of the form of the complementary function we neeed to multiply the obvious trial solution by *t*.
Try $y=t\left(p\cos\left(\frac{5t}{2}\right)+q\sin\left(\frac{5t}{2}\right)\right)$
So $\frac{dy}{dt}=p\cos\left(\frac{5t}{2}\right)+q\sin\left(\frac{5t}{2}\right)+t\left(-\frac{5p}{2}\sin\left(\frac{5t}{2}\right)+\frac{5q}{2}\cos\left(\frac{5t}{2}\right)\right)$
And `$\frac{d^2y}{dt^2}=-\frac{5p}{2}\sin\left(\frac{5t}{2}\right)+\frac{5q}{2}\cos\left(\frac{5t}{2}\right)-\frac{5p}{2}\sin\left(\frac{5t}{2}\right)+\frac{5q}{2}\cos\left(\frac{5t}{2}\right)+t\left(-\frac{25p}{4}\cos\left(\frac{5t}{2}\right)-\frac{25q}{4}\sin\left(\frac{5t}{2}\right)\right)$

$=-5p\sin\left(\frac{5t}{2}\right)+5q\cos\left(\frac{5t}{2}\right)-\frac{25t}{4}\left(p\cos\left(\frac{5t}{2}\right)+q\sin\left(\frac{5t}{2}\right)\right)$

Substituting in the original equation
Gives
$4\left(-5p\sin\left(\frac{5t}{2}\right)+5q\cos\left(\frac{5t}{2}\right)\right)-25t\left(p\cos\left(\frac{5t}{2}\right)+q\sin\left(\frac{5t}{2}\right)\right)+25t\left(p\cos\left(\frac{5t}{2}\right)+q\sin\left(\frac{5t}{2}\right)\right)$
$=20\cos\left(\frac{5t}{2}\right)+40\sin\left(\frac{5t}{2}\right)$
So $-20p\sin\left(\frac{5t}{2}\right)+20q\cos\left(\frac{5t}{2}\right)=20\cos\left(\frac{5t}{2}\right)+40\sin\left(\frac{5t}{2}\right)$
Equating coefficients $-20p=40$ and $20q=20$
So $p=-2$ and $q=1$
The particular integral is

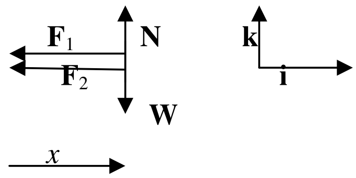
$$y_p=t\left(\sin\left(\frac{5t}{2}\right)-2\cos\left(\frac{5t}{2}\right)\right)$$

The general solution is $y=y_c+y_p$
So $y=\ C\cos\left(\frac{5t}{2}\right)+D\sin\left(\frac{5t}{2}\right)+\ t\left(\sin\left(\frac{5t}{2}\right)-2\cos\left(\frac{5t}{2}\right)\right)$

b)
$\frac{dy}{dt}=-\frac{5C}{2}\sin\left(\frac{5t}{2}\right)+\frac{5D}{2}\cos\left(\frac{5t}{2}\right)+\sin\left(\frac{5t}{2}\right)-2\cos\left(\frac{5t}{2}\right)+t\left(\frac{5}{2}\cos\left(\frac{5t}{2}\right)+5\sin\left(\frac{5t}{2}\right)\right)$
When $t=0$ $\frac{dy}{dt}=8$
So $8=\frac{5D}{2}-2$ so $D=\frac{20}{5}=4$
When $t=0$ $y=1$
So $1=C$
The particular solution is

$$y=\cos\left(\frac{5t}{2}\right)+4\sin\left(\frac{5t}{2}\right)+t\left(\sin\left(\frac{5t}{2}\right)-2\cos\left(\frac{5t}{2}\right)\right)$$

25.a)



N is the normal reaction between the ice and the stone and **W** is the weight of the stone. As the motion is in the **i**-direction $\mathbf{v} = v\mathbf{i}$ so $\mathbf{F}_1 = -k_1 v\mathbf{i}$ and $\mathbf{F}_2 = -k_2 v^2\mathbf{i}$ as v is positive. Also $\mathbf{W} = mg\mathbf{k}$ and $\mathbf{N} = |\mathbf{N}|\mathbf{k}$

b) As the motion is in the **i**-direction the acceleration is $\ddot{x}\mathbf{i}$ and the equation of motion is

$$M\ddot{x}\mathbf{i} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{N} + \mathbf{W}$$

Resolving in the **i**-direction

$$M\ddot{x} = -k_1 v - k_2 v^2$$

c) Writing $\ddot{x}$ as $v\frac{dv}{dx}$ the equation becomes

$$Mv\frac{dv}{dx} = -k_1 v - k_2 v^2$$

Dividing by v $(\neq 0)$ gives $M\frac{dv}{dx} = -(k_1 + k_2 v)$

Separating the variables (p26)

$$\frac{1}{k_1 + k_2 v}\frac{dv}{dx} = -\frac{1}{M}$$

Integrating $\frac{1}{k_2}\ln(k_1 + k_2 v) = -\frac{1}{M}x + C$

$$v = v_0 \text{ when } x = 0 \text{ so } C = \frac{1}{k_2}\ln(k_1 + k_2 v_0)$$

Substituting in (1) and rearranging gives

$$\frac{x}{M} = \frac{1}{k_2}\ln(k_1 + k_2 v_0) - \frac{1}{k_2}\ln(k_1 + k_2 v)$$

or $x = \frac{M}{k_2}\ln\left(\frac{k_1 + k_2 v_0}{k_1 + k_2 v}\right)$

d) When the stone comes to rest $v = 0$ and so

$$x_{\max} = \frac{M}{k_2}\ln\left(\frac{k_1 + k_2 v_0}{k_1}\right)$$

e) It remains stationary as $\mathbf{F}_1$ and $\mathbf{F}_2$ are both zero and so no resultant force acts on the stone.

26. a) In matrix form the equations are

$\dot{x} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ -26\sin t \end{bmatrix}$ if $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

b) The matrix of coefficients is $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

For the eigenvalues $\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0$ (p 41)

So $\lambda^2 - 6\lambda + 5 = 0$

Giving $\lambda = 5$ and $\lambda = 1$ as eigenvalues

For eigenvectors

$$(2-\lambda)x + 3y = 0$$
$$x + (4-\lambda)y = 0$$

When $\lambda = 5$, $-3x + 3y = 0$ and $x - y = 0 \implies x = y$

An eigenvector for $\lambda = 5$ is $[1 \;\; 1]^T$

When $\lambda = 1$ the eigenvector equations become

$$x + 3y = 0 \implies x = -3y$$

An eigenvector for $\lambda = 1$ is $[-3 \;\; 1]^T$

The complementary function of the equations(p 42/43) is

$$\mathbf{x} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{5t} + B\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{t}$$

c) For a particular integral (p 43) we try

$$x = a\cos t + b\sin t$$
$$y = c\cos t + d\sin t$$

Substitute into the equation

$$-a\sin t + b\cos t$$
$$= 2a\cos t + 2b\sin t + 3c\cos t + 3d\sin t$$

$$-c\sin t + d\cos t$$
$$= a\cos t + b\sin t + 4c\cos t + 4d\sin t - 26\sin t$$

Collecting terms

$$(2a + 3c - b)\cos t + (2b + 3d + a)\sin t = 0$$
$$(a + 4c - d)\cos t + (b + 4d + c)\sin t = 26\sin t$$

Equating coefficients

$$\begin{aligned} a + 2b \quad + 3d &= 0 \\ 2a - b + 3c \quad &= 0 \\ a \quad + 4c - d &= 0 \\ b + c + 4d &= 26 \end{aligned}$$

These can be solved in any way you like but I have used an adapted version of Gaussian elimination.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 2 & -1 & 3 & 0 & 0 \\ 1 & 0 & 4 & -1 & 0 \\ 0 & 1 & 1 & 4 & 26 \end{bmatrix}$$

$R2 - 2R1, \qquad R3 - R1$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & -5 & 3 & -6 & 0 \\ 0 & -2 & 4 & -4 & 0 \\ 0 & 1 & 1 & 4 & 26 \end{bmatrix}$$

New $R3/{-2}$ and swopped with $R2$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & -5 & 3 & -6 & 0 \\ 0 & 1 & 1 & 4 & 26 \end{bmatrix}$$

New $R3 + 5R2$ and $R4 - R3$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & -7 & 4 & 0 \\ 0 & 0 & 3 & 2 & 26 \end{bmatrix}$$

26. (cont) New $R4 + \frac{3}{7}R3$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & -7 & 4 & 0 \\ 0 & 0 & 0 & \dfrac{26}{7} & 26 \end{bmatrix}$$

Using back substitution

$$d = 7, c = 4, b = -6, a = -9$$

So the particular integral is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9\cos t - 6\sin t \\ 4\cos t + 7\sin t \end{bmatrix}$$

And the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{5t} + B\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{t} + \begin{bmatrix} -9\cos t - 6\sin t \\ 4\cos t + 7\sin t \end{bmatrix}$$

d) $x(0) = 1$ and $y(0) = 2$ gives

$$1 = A - 3B - 9$$
$$2 = A + B + 4$$

Or

$$A - 3B = 10$$
$$A + B = -2$$

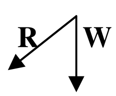
Giving $B = -3$ and $A = 1$

The particular solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{5t} + \begin{bmatrix} 9 \\ -3 \end{bmatrix}e^{t} + \begin{bmatrix} -9\cos t - 6\sin t \\ 4\cos t + 7\sin t \end{bmatrix}$$

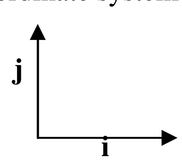
e) As e^{5t} is the dominant term the long term behaviour is $\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{5t}$

27.a)



W = weight
R = resistance force

b) The coordinate system could be



Then **v** = $\dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ and the forces are **W** = $-mg\mathbf{i}$ and **R** = $-km\mathbf{v} = -km(\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$

c) Newton's 2nd law gives $m\mathbf{a} = \mathbf{W} + \mathbf{R}$ (1)

where **a**= $\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

Resolving (1) in the **i**-direction gives

$m\ddot{x} = -mk\dot{x}$ or $\ddot{x} = -k\dot{x}$ (2)

d) Resolving (1) in the **j**-direction gives

$m\ddot{y} = -mk\dot{y} - mg$ or $\ddot{y} = -k\dot{y} - g$ (3)

e) Integrating (2) with respect to t gives

$$\dot{x} = -kx + C$$

As $\dot{x} = v_0\cos\theta$ when $x = 0$ and $t = 0$, $C = v_0\cos\theta$

Giving $\dot{x} = -kx + v_0\cos\theta$ (4)

Integrating (3) with respect to t gives

$$\dot{y} = -ky - gt + D$$

As $\dot{y} = v_0\sin\theta$ when $y = 0$ and $t = 0$, $D = v_0\sin\theta$

$\dot{y} = -ky - gt + v_0\sin\theta$ (5)

which results in the given set of equations.

f) As the equations are not connected we can solve them individually. (You could use the methods of unit 11 instead or any other method as none is specified)

Equation (4) can be written as $\dot{x} + kx = v_0\cos\theta$

The integrating factor is e^{kt} so $\frac{d}{dt}\left(e^{kt}x\right) = v_0\cos\theta\, e^{kt}$

Integrating gives $e^{kt}x = \frac{v_0\cos\theta}{k}e^{kt} + C$

Or $x = \frac{v_0\cos\theta}{k} + Ce^{-kt}$ (6)

Equation (5) can be written as $\dot{y} + ky = -gt + v_0\sin\theta$ (You could solve it using the integrating factor or as below). The homogeneous equation is $\dot{y} + ky = 0$ which has solution $y_c = De^{-kt}$. The particular integral can be found by substituting $y = pt + q$ into the equation to give

$$p + kpt + kq = -gt + v_0\sin\theta$$

Equating coefficients gives

$kp = -g$ and $p + kq = v_0\sin\theta$

So $p = -\frac{g}{k}$ and $q = \frac{v_0\sin\theta + \frac{g}{k}}{k}$

The particular integral is $y_p = -\frac{g}{k}t + \frac{kv_0\sin\theta + g}{k^2}$

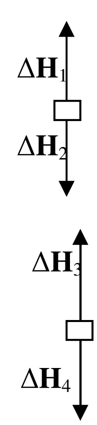
The general solution is

$y = De^{-kt} - \frac{g}{k}t + \frac{kv_0\sin\theta + g}{k^2}$ (7)

As $x(0) = 0$ and y(0)=0, (6) gives $C = -\frac{v_0\cos\theta}{k}$ and (7) gives $D = -\frac{kv_0\sin\theta + g}{k^2}$, the particular solutions are

$$x = \frac{v_0\cos\theta}{k}\left(1 - e^{-kt}\right)$$
$$y = \frac{kv_0\sin\theta + g}{k^2}\left(1 - e^{-kt}\right) - \frac{g}{k}t$$

28. a)



28(cont) The forces indicated are the spring forces over and above those required for equilibrium.

b) $\Delta\mathbf{H}_1 = -kx_1\mathbf{i}$

$\Delta\mathbf{H}_2 = -4k(x_2 - x_1)(-\mathbf{i}) = 4k(x_2 - x_1)\mathbf{i}$

$\Delta\mathbf{H}_3 = -4k(x_2 - x_1)\mathbf{i}$

$\Delta\mathbf{H}_4 = -k(-x_2)(-\mathbf{i}) = -kx_2\mathbf{i}$

c) Using Newton's second law we get

$m\ddot{x}_1\mathbf{i} = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2$ and $m\ddot{x}_2\mathbf{i} = \Delta\mathbf{H}_3 + \Delta\mathbf{H}_4$

Resolving in the **i**-direction gives

$$m\ddot{x}_1 = -kx_1 + 4k(x_2 - x_1) = -5kx_1 + 4kx_2$$

$$m\ddot{x}_2 = -4k(x_2 - x_1) - kx_2 = 4kx_1 - 5kx_2$$

Dividing by m and putting in matrix form gives

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{5k}{m} & \frac{4k}{m}\\ \frac{4k}{m} & -\frac{5k}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Giving the required dynamic matrix.

d) The eigenvalues for the dynamic matrix will be $\frac{k}{m} = 4$ times the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix}-5 & 4\\ 4 & -5\end{bmatrix}$ and the eigenvectors will be the same (p41). (You can use the full equation but the arithmetic gets a little more complicated)

The eigenvalues of **A** are given by $(-5-\lambda)^2 - 16 = 0$ or $\lambda^2 + 10\lambda + 9 = 0$

Factorising gives $\lambda = -9$ and $\lambda = -1$. The eigenvalues of the dynamic matrix are 4 times these i.e. -36 and -4 so the normal mode angular frequencies are

$\omega_1 = \sqrt{36} = 6$ and $\omega_2 = \sqrt{4} = 2$ (p57).

e) The eigenvector equations (p41) for **A** are

$$(-5-\lambda)x + 4y = 0 \text{ and } 4x + (-5-\lambda)y = 0$$

For $\omega_1 = 6$ $\lambda = -9$ and the eigenvector equations become $4x + 4y = 0$ or $y = -x$ so a possible eigenvector is $[1 \quad -1]^T$

For $\omega_2 = 2$ $\lambda = -1$ and the eigenvector equations become $-4x + 4y = 0$ or $y = x$ so a possible eigenvector is $[1 \quad 1]^T$

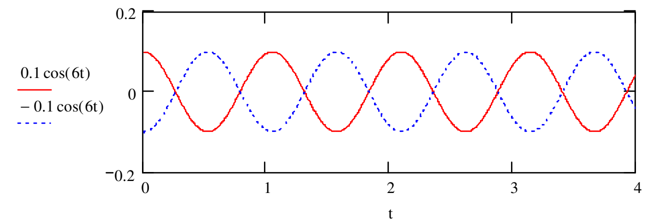
These give (p57)

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = C_1\begin{bmatrix}1\\ -1\end{bmatrix}\cos(6t + \phi_1) + C_2\begin{bmatrix}1\\ 1\end{bmatrix}\cos(2t + \phi_2)$$

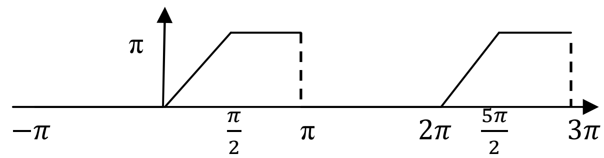
f) The starting position is in the form of the first normal mode so the motion will be phase-opposed as follows where $x_1 = 0.1\cos(6t)$ and $x_2 = -0.1\cos(6t)$ (as they are starting from rest) (p56)

or

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}(A\cos(6t) + B\sin(6t)) + \begin{bmatrix}1\\ 1\end{bmatrix}(C\cos(2t) + D\sin(2t))$$



29. a)



The function is neither even nor odd.

b) Using section 9 page 61 with n replacing r (or you could use r throughout) and $\tau = 2\pi$ we get

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\left\{\int_0^{\frac{\pi}{2}} 2t\,dt + \int_{\frac{\pi}{2}}^{\pi} \pi\,dt\right\}$$

$$= \frac{1}{2\pi}\left\{[t^2]_0^{\frac{\pi}{2}} + \frac{\pi^2}{2}\right\} = \frac{3\pi}{8}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$

So $A_n = \frac{1}{\pi}\left\{2\int_0^{\frac{\pi}{2}} t\cos(nt)dt + \pi\int_{\frac{\pi}{2}}^{\pi}\cos(nt)dt\right\}$

The first integral will have to be integrated by parts (p23) where $g(t) = t$ and $h'(t) = \cos(nt)$ then $g'(t) = 1$ and $h(t) = \frac{\sin(nt)}{n}$ and the integral becomes

$$\int_0^{\frac{\pi}{2}} t\cos(nt)dt = \left[\frac{t\sin(nt)}{n}\right]_0^{\frac{\pi}{2}} - \frac{1}{n}\int_0^{\frac{\pi}{2}}\sin(nt)dt$$

$$= \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n}\left[-\frac{\cos(nt)}{n}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

So $A_n = \frac{1}{n}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{\pi n^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) + \left[\frac{\sin(nt)}{n}\right]_{\frac{\pi}{2}}^{\pi}$

$$= \frac{1}{n}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{\pi n^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) - \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{2}{\pi n^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

$$A_1 = -\frac{2}{\pi} \quad A_2 = -\frac{1}{\pi} \quad A_3 = -\frac{2}{9\pi^2} \quad A_4 = 0$$

$-\frac{2}{9\pi}$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

So $B_n = \frac{1}{\pi}\left\{2\int_0^{\frac{\pi}{2}} t\sin(nt)\,dt + \pi\int_{\frac{\pi}{2}}^{\pi}\sin(nt)\,dt\right\}$

29. (cont) Again the first integral will have to be integrated by parts where $g(t) = t$ and $h'(t) = \sin(nt)$ then

$g'(t) = 1$ and $h(t) = -\frac{\cos(nt)}{n}$ and the integral becomes

$$\int_0^{\frac{\pi}{2}} t\sin(nt)\,dt = \left[-\frac{t\cos(nt)}{n}\right]_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos(nt)\,dt$$

$$= -\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n}\left[\frac{\sin(nt)}{n}\right]_0^{\frac{\pi}{2}}$$

so

$$B_n = -\frac{1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) + \left[-\frac{\cos(nt)}{n}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{1}{n}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n}\cos(n\pi) + \frac{1}{n}\cos\left(\frac{n\pi}{2}\right)$$

$$B_n = \frac{2}{\pi n^2}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n}\cos(n\pi)$$

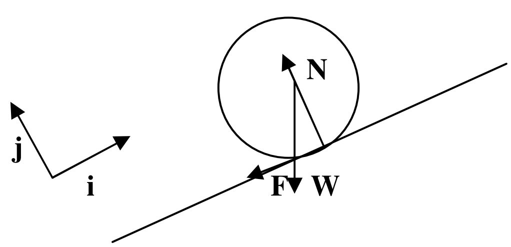
$B_1 = \frac{2}{\pi} + 1$ $B_2 = -\frac{1}{2}$ $B_3 = -\frac{2}{9\pi} + \frac{1}{3}$ $B_4 = -1/4$

The Fourier series is

$$\frac{3\pi}{8} - \frac{2}{\pi}\cos t + \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{\pi}\cos(2t) - \frac{1}{2}\sin(2t) - \frac{2}{9\pi}\cos(3t) + \left(\frac{1}{3} - \frac{2}{9\pi}\right)\sin(3t) - \frac{1}{4}\sin(4t) + \cdots.$$

30.a) Moment of Inertia of sphere is $I = \frac{2}{5}MR^2$ (p74)

b)



W is the weight of the sphere, **F** is the frictional force between the sphere and the plane and **N** is the normal reaction of the plane on the sphere.

c) $\mathbf{W} = -Mg(\sin\alpha\,\mathbf{i} + \cos\alpha\,\mathbf{j})$ $\mathbf{N} = |\mathbf{N}|\mathbf{j}$

$\mathbf{F} = -|\mathbf{F}|\mathbf{i} = -\mu'|\mathbf{N}|\mathbf{i}$

The acceleration of the centre of mass is in the **i**-direction so $\mathbf{a} = \ddot{x}\mathbf{i}$.

Newton's 2nd law gives $M\mathbf{a} = \mathbf{W} + \mathbf{N} + \mathbf{F}$ for the motion of the centre of mass.

Resolving in **j**-direction gives $|\mathbf{N}| = Mg\cos\alpha$ (1)

Resolving in **i**-direction gives

$M\ddot{x} = -Mg\sin\alpha - \mu'|\mathbf{N}| = -Mg(\sin\alpha + \mu'\cos\alpha)$

using (1). Dividing by M gives

$$\ddot{x} = -g(\sin\alpha + \mu'\cos\alpha) \quad (2)$$

d) Relative to the centre of mass, the frictional force has position vector $\mathbf{r} = -R\mathbf{j}$ and the other two forces go through the centre of mass so have zero position vectors. The external torque (p32) relative to the centre of mass is

$\mathbf{\Gamma} = -R\mathbf{j} \times \mathbf{F} = -R\mathbf{j} \times -\mu' Mg\cos\alpha\,\mathbf{i} = -\mu' RMg\cos\alpha\,\mathbf{k}$

Using the equation of relative rotational motion (p75)

$$I\ddot{\theta} = -\mu' RMg\cos\alpha$$

Substituting for I and rearranging gives

$$\ddot{\theta} = -\frac{5\mu' g\cos\alpha}{2R} \quad (3)$$

e)

Due to the linear motion the point of contact is moving up the plane with speed $\dot{x}$ and due to the rotational motion it is moving up the plane with speed $R\dot{\theta}$. The resultant of these two gives the velocity of the point of contact as $\mathbf{v} = (\dot{x} + R\dot{\theta})\mathbf{i}$

f) Integrating (2) with respect to t gives

$$\dot{x} = -g(\sin\alpha + \mu'\cos\alpha)t + B$$

But $\dot{x} = v_0$ when $t = 0$ so $B = v_0$ and

$\dot{x} = v_0 - g(\sin\alpha + \mu'\cos\alpha)t$ (4)

Integrating (3) with respect to t gives

$$\dot{\theta} = -\frac{5\mu' g\cos\alpha}{2R}t + D$$

But $\dot{\theta} = \omega_0$ when $t = 0$ so $D = \omega_0$ and

$$\dot{\theta} = \omega_0 - \frac{5\mu' g\cos\alpha}{2R}t \quad (5)$$

Substituting (4) and (5) in the expression for **v** gives

$$\mathbf{v} = \left(v_0 - g(\sin\alpha + \mu'\cos\alpha)t + R\,\omega_0 - \frac{5\mu' g\cos\alpha}{2}t\right)\mathbf{i}$$

$$= \left(v_0 + R\omega_0 - g\left(\sin\alpha + \frac{7}{2}\mu'\cos\alpha\right)t\right)\mathbf{i}$$

When slipping stops $\mathbf{v} = \mathbf{0}$ so

$$v_0 + R\omega_0 - g\left(\sin\alpha + \frac{7}{2}\mu'\cos\alpha\right)t = 0$$

And

$$t = \frac{v_0 + \omega_0}{g\left(\sin\alpha + \frac{7}{2}\mu'\cos\alpha\right)}$$

As required.