M337/Q

The Open University

Course Examination 2005

Complex Analysis

Tuesday 18 October 2005 2.30 pm – 5.30 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I (64% of the marks) you should attempt as many questions as you can.

In Part II (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Put all your used answer books and your question paper together with your signed desk record on top. Fix them all together using the fastener provided.

The use of calculators is **NOT** permitted in this examination.

PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

Give the Cartesian form of the following complex numbers, simplifying your answer as far as possible.

(a) $(2-2i)^4$ [2]

(b) the principal cube root of $8i$ [2]

(c) $\operatorname{Log}(1-i)$ [2]

(d) $(-1)^{-i}$ [2]

Question 2

Let

$$A = \{z : 1 \leq |z| \leq 2\} \quad \text{and} \quad B = \{z : \pi/4 < \operatorname{Arg} z < 3\pi/4\}.$$

(a) Make separate sketches of the sets A, B and $C = B - A$. [3]

(b) For each of the sets A, B and C

(i) state whether it is a region, and if not a region explain why not;

(ii) state whether it is compact, and if not compact explain why not. [5]

Question 3

In this question Γ is the line segment from i to 2.

(a) (i) Determine the standard parametrization for the line segment Γ.

(ii) Evaluate

$$\int_\Gamma \operatorname{Re} z \, dz.$$ [3]

(b) Determine an upper estimate for the modulus of

$$\int_\Gamma \frac{\cos z}{9+z^2} \, dz.$$ [5]

Question 4

Evaluate the following integrals, in which $C = \{z : |z| = 1\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C \frac{\exp z}{(3-z)^3} \, dz$ [2]

(b) $\displaystyle\int_C \frac{\exp(3-z)}{z} \, dz$ [3]

(c) $\displaystyle\int_C \frac{\exp(3-z)}{z^3} \, dz$ [3]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{1}{z^3 + 1}$$

at each of the poles of f. *[4]*

(b) Hence evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^3 + 1}\, dt.$$ *[4]*

Question 6

Let $f(z) = z^5 - 3z^3 + i$.

(a) Determine the number of zeros of f that lie inside:

(i) the circle $C_1 = \{z : |z| = 2\}$,

(ii) the circle $C_2 = \{z : |z| = 1\}$. *[6]*

(b) Show that the equation

$$z^5 - 3z^3 + i = 0$$

has exactly two solutions in the set $\{z : 1 < |z| < 2\}$. *[2]*

Question 7

Let $q(z) = 1/\bar{z}^{\,2}$ be a velocity function on $\mathbb{C} - \{0\}$.

(a) Explain why q represents a model fluid flow on $\mathbb{C} - \{0\}$. *[1]*

(b) Determine a stream function for this flow. Hence find the equation of the streamline through the point $2i$, and sketch this streamline, indicating the direction of flow. *[5]*

(c) Evaluate the flux of q across the unit circle $\{z : |z| = 1\}$. *[2]*

Question 8

(a) Find the fixed points of the function $f(z) = z^2 - 2z + 2$ and classify them as (super-)attracting, repelling or indifferent. *[3]*

(b) Which of the following points c lie in the Mandelbrot set:

(i) $c = -\frac{3}{2} - \frac{1}{2}i$;

(ii) $c = -1 + \frac{1}{6}i$.

Justify your answer in each case. *[5]*

PART II

(i) You should attempt no more than **TWO** *questions in this part.*

(ii) Each question in this part carries 18 marks.

Question 9

(a) Let f be the function

$$f(z) = z(\operatorname{Re} z + \bar{z}).$$

(i) Write $f(x+iy)$ in the form $u(x,y)+iv(x,y)$, where u and v are real-valued functions.

(ii) Use the Cauchy-Riemann theorem and its converse to show that f is differentiable at 0, but not analytic there.

(iii) Evaluate $f'(0)$. [*9*]

(b) Let g be the function $g(z) = z^2 + 3$.

(i) Show that g is conformal on $\mathbb{C} - \{0\}$.

(ii) Describe the effect of g on a small disc centred at $2i$.

(iii) Γ_1 and Γ_2 are the smooth paths meeting at $2i$ and 0 given by

$$\Gamma_1 : \gamma_1(t) = it \quad (t \in [0,2]),$$
$$\Gamma_2 : \gamma_2(t) = i + e^{it} \quad (t \in [-\pi/2, \pi/2]).$$

Sketch these paths, clearly indicating their directions.

(iv) Using part (b)(ii), or otherwise, sketch the directions of $g(\Gamma_1)$ and $g(\Gamma_2)$ at $g(2i)$.

(v) Show that g is not conformal at 0. [*9*]

Question 10

(a) Let $f(z) = \dfrac{9}{z(z-3)}$.

(i) Locate and classify the singularities of f.
(No justification required.)

(ii) Determine the Laurent series about 0 for f on the set

$$\{z : 0 < |z| < 3\},$$

giving the general term.

(iii) Determine the Laurent series about 3 for f on the set

$$\{z : |z-3| > 3\},$$

giving the general term. [*8*]

(b) Find the Taylor series about 0 (up to the term in z^4) for the function

$$g(z) = \cos(2\sin z),$$

and explain why the series represents g on $\mathbb{C}$.

Hence evaluate the integrals

$$\int_C zg(1/z)\,dz \quad \text{and} \quad \int_C z^2 g(1/z)\,dz,$$

where C is the circle $\{z : |z| = 1\}$. [*10*]

Question 11

(a) (i) Express e^{iz} in Cartesian form, where $z = x + iy$, and hence show that

$$|\exp(e^{iz})| = \exp(e^{-y}\cos x), \quad \text{for } z = x + iy \in \mathbb{C}.$$

(ii) Determine

$$\max\{|\exp(e^{iz})| : -\pi \le \operatorname{Re} z \le \pi, -1 \le \operatorname{Im} z \le 1\},$$

and find the point or points at which this maximum is attained. [*9*]

(b) Show that the functions

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n \qquad (|z| < 4)$$

and

$$g(z) = -\sum_{n=1}^{\infty} \left(\frac{4}{z}\right)^n \qquad (|z| > 4)$$

are indirect analytic continuations of each other. [*9*]

Question 12

(a) Determine the extended Möbius transformation $\hat{f}_1$ which maps i to 0, 1 to ∞ and $\frac{1}{2}(1+i)$ to 1. [*2*]

(b) Let

$$R = \{z : |z| < 1,\ \operatorname{Re} z + \operatorname{Im} z > 1\}$$

and

$$S = \{w : \operatorname{Re} w > 0,\ \operatorname{Im} w > 0\}.$$

(i) Sketch the regions R and S.

(ii) Determine and sketch the image of R under $\hat{f}_1$ of part (a).

(iii) Hence determine a conformal mapping f from R onto S.

(iv) Write down the rule of the inverse function f^{-1}. [*16*]

[END OF QUESTION PAPER]