M337 Solutions to the Specimen Examination Paper

This is a rough guide to the type of written solutions required. We do not expect your solutions to be as neatly laid out as these and, of course, for some questions there are alternative ways of doing them. Any correct method receives full marks.

A mark scheme is provided so that you can mark your own attempts. This uses accuracy marks (A-marks) and method marks (M-marks), with some indication about how these are awarded. Note that results used may be referred to by name, handbook (HB) reference or unit reference.

A few comments are included to try to help you with your revision and examination technique. In general, we recommend that you draw diagrams where relevant, as this should help you in devising solutions.

Solutions to Part I

Mark scheme

Question 1

(a) Since $w = 2(1-i)/2 = 1-i$, we have — 1A

$$\operatorname{Arg} w = -\pi/4.$$ — 1A

(b) Since $w = \sqrt{2}e^{-i\pi/4}$, the cube roots of w are — 1A

$$z_k = (\sqrt{2}\,)^{1/3}\left(\cos\left(-\frac{\pi}{12}+k\frac{2\pi}{3}\right)+i\sin\left(-\frac{\pi}{12}+k\frac{2\pi}{3}\right)\right),$$

$k = 0, 1, 2;$ — 1M

that is,

$$z_0 = 2^{1/6}(\cos(-\pi/12)+i\sin(-\pi/12)),$$
$$z_1 = 2^{1/6}(\cos 7\pi/12 + i\sin 7\pi/12),$$
$$z_2 = 2^{1/6}(\cos 5\pi/4 + i\sin 5\pi/4).$$

2A ($-\frac{1}{2}$ for each error)

(The form $z_0 = 2^{1/6}e^{-\pi i/12}$, etc., is also acceptable.)

Since $-\pi/4$ is the principal argument of w, z_0 is the principal cube root of w. — 1A

(c) $n = 4$ (since $\operatorname{Arg} w = -\pi/4$). — 1A

WEB 03095 3

Question 2

	$A \cup B$	$A \cap D$	$D - \{2\}$	∂B	
(a) region	✓	×	✓	×	1A for each answer
(b) closed	×	×	×	✓	

(Here you probably found that sketching the given sets A, B and D helped you to answer the question.)

Question 3

(a) The standard parametrization for the given line segment is

$$\begin{aligned}\gamma(t) &= (1-t)(-i) + ti \\ &= -i + 2ti \qquad (t \in [0,1]).\end{aligned}$$ 1A

Thus

$$\int_\Gamma \operatorname{Im} z\, dz = \int_0^1 \operatorname{Im}(\gamma(t))\, \gamma'(t)\, dt$$ 1M

$$\begin{aligned} &= \int_0^1 (2t-1) 2i\, dt \\ &= 2i \left[t^2 - t\right]_0^1 \\ &= 0. \end{aligned}$$ 1A

(*Alternatively*, use the easier equivalent parametrization $\gamma(t) = it$ $(t \in [-1,1])$.)

(b) The length of the contour is $L = 4\pi$. $\frac{1}{2}$A

By the backwards form of the Triangle Inequality, $\frac{1}{2}$M

$$\begin{aligned}|z^5 - 1| &\geq |z|^5 - 1 \\ &= 2^5 - 1 = 31, \qquad \text{for } z \in C.\end{aligned}$$ $\frac{1}{2}$A

Also by the Triangle Inequality,

$$\begin{aligned}|\sinh z| &= \left| \frac{e^z - e^{-z}}{2} \right| \\ &\leq \tfrac{1}{2}\left(|e^z| + |e^{-z}|\right) \\ &\leq e^{|z|} \qquad \left(\text{since } |e^z| \leq e^{|z|}\right) \\ &= e^2, \qquad \text{for } z \in C.\end{aligned}$$ 1A, 1M

Hence, by the Estimation Theorem (which applies since the integrand is continuous on C because $z^5 - 1 \neq 0$ on C), 1M for theorem

$$\left| \int_C \frac{2 \sinh z}{z^5 - 1}\, dz \right| \leq \frac{2e^2}{31} \times 4\pi = \frac{8\pi e^2}{31}.$$ $\frac{1}{2}$A

Question 4

(a) Let $z_n = z^n/(n-1)!$, $n = 2, 3, \ldots$. Then, for all $z \neq 0$,

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|z|}{n} \to 0 \qquad \text{as } n \to \infty.$$ 1A

Hence the radius of convergence is ∞, by the Ratio Test, and so the disc of convergence of the power series is $\mathbb{C}$. 1M for test 1A

(b) Since

$$\exp(-z) = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots, \qquad \text{for } z \in \mathbb{C},$$ $\frac{1}{2}$A

and

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, \qquad \text{for } |z| < 1,$$ $\frac{1}{2}$A

we deduce, by the Product Rule, that 1M for rule

$$\begin{aligned} f(z) &= \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right)(1 + z + z^2 + z^3 + \cdots) \\ &= 1 + (1-1)z + \left(1 - 1 + \frac{1}{2!}\right)z^2 \\ &\quad + \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!}\right)z^3 + \cdots \\ &= 1 + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 + \cdots, \qquad \text{for } |z| < 1. \end{aligned}$$

1M for manipulation

1A for series

This Taylor series represents f on $\{z : |z| < 1\}$. 1A for disc

Question 5

(a) The function f has simple poles at $\pm 1/\sqrt{3}$ and $\pm\sqrt{3}$, of which only those at $\pm 1/\sqrt{3}$ lie inside the unit circle. By the g/h Rule (with $g(z) = z/(z^2-3)$, $h(z) = 3z^2 - 1$, $h'(z) = 6z$), we deduce that 1A, 1M for rule

$$\begin{aligned} \operatorname{Res}(f, 1/\sqrt{3}\,) &= \frac{g(1/\sqrt{3}\,)}{h'(1/\sqrt{3}\,)} \\ &= \frac{1/\sqrt{3}}{(6/\sqrt{3}\,)(1/3 - 3)} \\ &= -\tfrac{1}{16} \end{aligned}$$ $\frac{1}{2}$A

and

$$\begin{aligned} \operatorname{Res}(f, -1/\sqrt{3}\,) &= \frac{g(-1/\sqrt{3}\,)}{h'(-1/\sqrt{3}\,)} \\ &= \frac{-1/\sqrt{3}}{(-6/\sqrt{3}\,)(1/3 - 3)} \\ &= -\tfrac{1}{16}. \end{aligned}$$ $\frac{1}{2}$A

(b) Using the strategy for trigonometric integrals, we have

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1+3\sin^2 t}\,dt &= \int_C \frac{1}{1+3((z-z^{-1})/2i)^2}\,\frac{1}{iz}\,dz && \text{1M}\\
&= \int_C \frac{1}{iz\left(1-\frac{3}{4}(z^2-2+z^{-2})\right)}\,dz\\
&= -4i\int_C \frac{z}{z^2(4-3(z^2-2+z^{-2}))}\,dz\\
&= 4i\int_C \frac{z}{3z^4-10z^2+3}\,dz\\
&= 4i\int_C \frac{z}{(3z^2-1)(z^2-3)}\,dz, && \text{2A for manipulation}
\end{aligned}$$

where C is the unit circle.

Thus, by part (a) and the Residue Theorem, 1M for theorem

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{1+3\sin^2 t}\,dt &= 4i\cdot 2\pi i\left(\operatorname{Res}(f,1/\sqrt{3}\,)+\operatorname{Res}(f,-1/\sqrt{3}\,)\right)\\
&= -8\pi\left(-\tfrac{1}{16}-\tfrac{1}{16}\right)\\
&= \pi. && \text{1A}
\end{aligned}$$

Question 6

Let $\Gamma_1=\{z:|z|=2\}$ and $f(z)=z^7+5z^3+7$, and choose $g(z)=z^7$. $\frac{1}{2}$A for g
Then $f(z)-g(z)=5z^3+7$ and, for $z\in\Gamma_1$,

$$\begin{aligned}
|f(z)-g(z)| &= |5z^3+7|\\
&\le 5|z|^3+7 \qquad \text{(Triangle Inequality)} && \text{1M}\\
&= 47,
\end{aligned}$$

whereas

$$|g(z)| = |z|^7 = 128 > 47, \qquad \text{for } z\in\Gamma_1. \qquad \tfrac{1}{2}\text{A}$$

Hence, by Rouché's Theorem, f has the same number of zeros as g inside Γ_1, namely 7. 1A

Let $\Gamma_2=\{z:|z|=1\}$ and choose $g(z)=7$. Then $\frac{1}{2}$A for g
$f(z)-g(z)=z^7+5z^3$ and, for $z\in\Gamma_2$,

$$\begin{aligned}
|f(z)-g(z)| &= |z^7+5z^3|\\
&\le |z|^7+5|z|^3 \qquad \text{(Triangle Inequality)} && \text{1M}\\
&= 6,
\end{aligned}$$

whereas

$$|g(z)| = 7 > 6, \qquad \text{for } z\in\Gamma_2. \qquad \tfrac{1}{2}\text{A}$$

Hence, by Rouché's Theorem, f has the same number of zeros as g inside Γ_2, namely 0. 1A

Also, f has no zeros on Γ_2, since $|f(z)-g(z)|<|g(z)|$ for $z\in\Gamma_2$, and 1M
so f has 7 zeros in $\{z:1<|z|<2\}$. 1A

Question 7

(a) The function q is a model flow velocity function because the function $\overline{q}(z) = z^2$ is analytic (on $\mathbb{C}$). 1M

(b) A complex potential function for q is

$$\Omega(z) = \tfrac{1}{3}z^3, \qquad \text{1A}$$

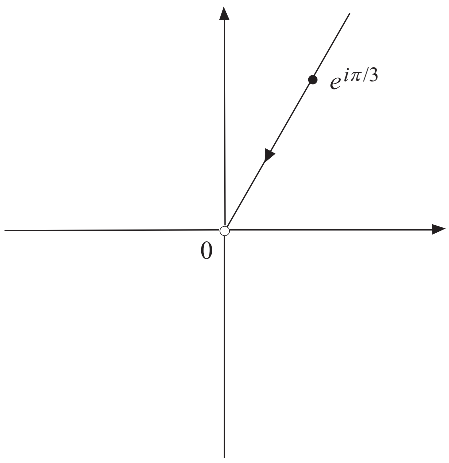
because this is a primitive of $\overline{q}$. The corresponding stream function is

$$\begin{aligned}\operatorname{Im}\Omega(z) &= \tfrac{1}{3}\operatorname{Im}(x+iy)^3 \qquad (z = x+iy) \qquad \text{1M}\\ &= \tfrac{1}{3}(3x^2y - y^3)\\ &= x^2y - \tfrac{1}{3}y^3. \qquad \text{1A}\end{aligned}$$

The streamline through the point $e^{i\pi/3} = \frac{1}{2}(1 + i\sqrt{3})$ satisfies

$$x^2y - \tfrac{1}{3}y^3 = \left(\tfrac{1}{2}\right)^2\sqrt{3}/2 - \tfrac{1}{3}(\sqrt{3}/2)^3 = 0. \qquad \tfrac{1}{2}\text{M}$$

Since $x^2y - \frac{1}{3}y^3 = \frac{1}{3}y(3x^2 - y^2) = 0$ is satisfied on $y = 0$, $y = \sqrt{3}x$ and $y = -\sqrt{3}x$, and 0 is a stagnation point, the streamline is as shown below. $\frac{1}{2}$M



1A ($\frac{1}{2}$A for streamline, $\frac{1}{2}$A for direction)

(c) Since q is a model flow velocity function on $\mathbb{C}$, it is locally flux-free; in particular, the flux across the unit circle is 0. 1A, 1M

Question 8

(a) We deduce from HB page 41, item 2.1 (*Unit D3*, Theorem 2.1), that 1M

$$z_{n+1} = z_n(1 - z_n) = -z_n^2 + z_n, \qquad n = 0, 1, 2, \ldots,$$

is conjugate to

$$w_{n+1} = w_n^2 + d, \qquad n = 0, 1, 2, \ldots,$$

where $d = -1 \times 0 + \frac{1}{2} \times 1 - \frac{1}{4} \times 1^2 = \frac{1}{4}$, with $\frac{1}{2}$A

$$w_0 = h(z_0) = -z_0 + \tfrac{1}{2} = 0, \qquad \tfrac{1}{2}\text{A}$$

since $z_0 = \frac{1}{2}$.

(b) (i) The point $\frac{1}{2}i$ appears to lie inside the main cardioid (HB page 42, figure). If $c = \frac{1}{2}i$, then $|c|^2 = \frac{1}{4}$ and $\operatorname{Re} c = 0$, so

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8\operatorname{Re} c = \left(2 - \tfrac{3}{2}\right)^2 = \tfrac{1}{4} < 3. \qquad \text{1A}$$

Hence $P_c(z) = z^2 + c$ has an attracting fixed point, and so $c \in M$, by HB page 43, item 4.8. 1M, 1A

(ii) The point $1 + \frac{1}{2}i$ appears to lie outside M (HB page 42, figure). If $c = 1 + \frac{1}{2}i$, then the first few terms of $\{P_c^n(0)\}$ are

$$1 + \tfrac{1}{2}i,$$

$$\begin{aligned}\left(1 + \tfrac{1}{2}i\right)^2 + \left(1 + \tfrac{1}{2}i\right) &= 1 + i - \tfrac{1}{4} + 1 + \tfrac{1}{2}i \qquad \text{1A}\\ &= \tfrac{7}{4} + \tfrac{3}{2}i.\end{aligned}$$

Since $\left|\frac{7}{4} + \frac{3}{2}i\right| = \sqrt{\frac{85}{16}} > 2$, we deduce that $c \notin M$, by HB page 42, item 4.5. 1A, 1M

Solutions to Part II

Question 9

(a) (i) The function f is analytic on $\mathbb{C} - \{0, 1\}$, so it is analytic on the simply-connected region $\{z : \operatorname{Re} z > 1\}$, which contains the given closed contour $\Gamma = \left\{z : |z - 2| = \frac{1}{2}\right\}$. Hence, for this Γ, by Cauchy's Theorem, 1M for details 1M for theorem

$$\int_\Gamma f(z)\,dz = 0. \qquad \text{1A}$$

(ii) The function $g(z) = (\exp z)/z$ is analytic on the simply-connected region $\{z : \operatorname{Re} z > 0\}$, which contains the given simple-closed contour $\Gamma = \left\{z : |z - 2| = \frac{3}{2}\right\}$. Hence, for this Γ, by Cauchy's nth Derivative Formula, with $n = 2$, 1M for approach 1M for details 1M for formula

$$\begin{aligned}\int_\Gamma f(z)\,dz &= \int_\Gamma \frac{g(z)}{(z-1)^3}\,dz\\ &= \frac{2\pi i}{2!}\,g^{(2)}(1) \qquad \text{(since 1 lies inside } \Gamma\text{)}. \qquad \text{1A}\end{aligned}$$

Now

$$g'(z) = \frac{e^z(z-1)}{z^2},$$

so

$$\begin{aligned}g''(z) &= \frac{z^2(e^z + e^z(z-1)) - 2ze^z(z-1)}{z^4}\\ &= \frac{(z^2 - 2z + 2)e^z}{z^3}. \qquad \text{1A}\end{aligned}$$

Thus

$$\int_\Gamma f(z)\,dz = \frac{\pi i(1 - 2 + 2)e^1}{1} = e\pi i. \qquad \text{1A}$$

(b) If Γ is any simple-closed contour, such as $\left\{z : |z| = \frac{1}{2}\right\}$, that surrounds 0 but not 1, then 1M for approach

$$\int_\Gamma f(z)\, dz = \int_\Gamma \frac{h(z)}{z}\, dz,$$ 1M

where the function $h(z) = (\exp z)/(z-1)^3$ is analytic on a simply-connected region containing Γ. Hence, by Cauchy's Integral Formula, for such a Γ, 1M 1M

$$\int_\Gamma f(z)\, dz = 2\pi i\, h(0) = -2\pi i.$$ 1A

(c) The function f is analytic on $\mathbb{C}$ except for the two singularities at $0, 1$, and each C_r, for $r > 1$, is a simple-closed contour surrounding both of these singularities. Hence, by the Residue Theorem, for each $r > 1$, 1M 1M for theorem

$$\phi(r) = \int_{C_r} f(z)\, dz = 2\pi i(\operatorname{Res}(f,0) + \operatorname{Res}(f,1)).$$ 1A

Thus ϕ is a constant function. 1M for deduction

Question 10

(a) (i) The function f is analytic everywhere except at 2 and -2. Hence it is analytic on the punctured discs

$$D_1 = \{z : 0 < |z-2| < 2\},$$
$$D_2 = \{z : 0 < |z+2| < 2\},$$

so the only singularities of f are at 2 and -2. $\frac{1}{2}$A, $\frac{1}{2}$M

Since

$$f(z) = \frac{g_1(z)}{z-2}, \qquad \text{for } z \in D_1,$$ 1M

where the function $g_1(z) = 4/(z+2)$ is analytic on D_1 with $g_1(2) \neq 0$, f has a simple pole at 2. 1A

Similarly,

$$f(z) = \frac{g_2(z)}{z+2}, \qquad \text{for } z \in D_2,$$

where the function $g_2(z) = 4/(z-2)$ is analytic on D_2 with $g_2(-2) \neq 0$, so f has a simple pole at -2. 1A

(ii) The function f has two Laurent series about the point 2, with annuli of convergence

$$\{z : 0 < |z-2| < 4\} \quad \text{and} \quad \{z : 4 < |z-2|\}.$$ 2A

(iii) Put $h = z - 2$. Then

$$\begin{aligned} f(z) &= \frac{4}{z^2 - 4} \\ &= \frac{4}{(z-2)(z+2)} \\ &= \frac{4}{h(h+4)} \\ &= \frac{1}{h(1 + h/4)} && \text{1M} \\ &= \frac{1}{h}\left(1 - \frac{h}{4} + \left(\frac{h}{4}\right)^2 - \cdots\right), \qquad \text{for } 0 < |h| < 4, && \tfrac{1}{2}\text{M} \\ &= \frac{1}{z-2}\left(1 - \frac{z-2}{4} + \left(\frac{z-2}{4}\right)^2 - \cdots\right), \\ &\quad \text{for } 0 < |z-2| < 4. && \text{1A} \end{aligned}$$

Since $\{z : 0 < |z-2| < 1\} \subseteq \{z : 0 < |z-2| < 4\}$, this is the Laurent series about 2 on $\{z : 0 < |z-2| < 1\}$. $\tfrac{1}{2}$M

The general term of this series is of the form

$$\frac{(-1)^n}{z-2}\left(\frac{z-2}{4}\right)^n = \frac{(-1)^n}{4^n}(z-2)^{n-1}, \qquad \text{for } n = 0, 1, 2, \ldots. \qquad \text{1A}$$

(b) (i) The function g is analytic on $\mathbb{C} - \{0\}$, but not at 0, so 0 is the only singularity of g. 1M

(ii) We have

$$\begin{aligned} g(z) &= z\cos(1/z^2) \\ &= z\left(1 - \frac{1}{2!}\left(\frac{1}{z^2}\right)^2 + \frac{1}{4!}\left(\frac{1}{z^2}\right)^4 - \cdots\right) && \text{1M} \\ &= z - \frac{1}{2!}\cdot\frac{1}{z^3} + \frac{1}{4!}\cdot\frac{1}{z^7} - \cdots, \qquad \text{for } z \in \mathbb{C} - \{0\}. && \text{1A} \end{aligned}$$

The general term of this series is of the form

$$z\frac{(-1)^n}{(2n)!}\left(\frac{1}{z^2}\right)^{2n} = \frac{(-1)^n}{(2n)!}\cdot\frac{1}{z^{4n-1}}, \qquad \text{for } n = 0, 1, 2, \ldots. \qquad \text{1A}$$

Since the Laurent series for g about 0 has infinitely many negative powers, g has an essential singularity at 0. $\tfrac{1}{2}$M, $\tfrac{1}{2}$A

(iii) Since g has an essential singularity at 0, we can apply the Casorati–Weierstrass Theorem. Take $\alpha = 0$, $\delta = 1$, and choose $\beta = 1001i$ and $\varepsilon = 1$, so that the disc with centre β and radius ε lies in 1M

$$\{w : \operatorname{Im} w > 1000\}.$$

By the Casorati–Weierstrass Theorem, there exists z such that 1M

$$0 < |z| < 1 \quad \text{and} \quad |g(z) - \beta| < \varepsilon.$$

Hence $\operatorname{Im}(g(z)) > 1000$, as required. 1M

Question 11

(a) The function $\phi(z) = 1/z^2$ is even and analytic on $\mathbb{C}$, apart from a pole at 0, and 1M

$$\begin{aligned} f(z) &= \frac{\pi \operatorname{cosec} \pi z}{z^2} \\ &= \frac{\pi}{z^2}\left(\frac{1}{\pi z} + \frac{\pi z}{6} + \frac{7}{360}(\pi z)^3 + \cdots\right) \\ &= \frac{1}{z^3} + \frac{\pi^2}{6}\frac{1}{z} + \cdots, \end{aligned}$$ 1M

so that $\operatorname{Res}(f, 0) = \pi^2/6$. 1A

Also, if S_N is the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$, then

$$|\operatorname{cosec} \pi z| \le 1, \qquad \text{for } z \in S_N,$$ 1M

and $|z| \ge N + \frac{1}{2}$, for $z \in S_N$, so that, by the Estimation Theorem,

$$\left| \int_{S_N} f(z)\, dz \right| \le \frac{\pi}{\left(N + \frac{1}{2}\right)^2} 4(2N+1) = \frac{16\pi}{2N+1},$$ $\frac{1}{2}$A, $\frac{1}{2}$M

which tends to 0 as $N \to \infty$. Hence, by HB page 30, item 4.3 (*Unit C1*, Theorem 4.2), 1M

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\tfrac{1}{2} \operatorname{Res}(f, 0) = -\frac{\pi^2}{12}.$$ 1A

(b) (i) To apply Weierstrass' Theorem, we need to show that the series defining f is uniformly convergent on each closed disc in $D = \{z : |z| < 1\}$.

Let $E = \{z : |z| \le r\}$, where $0 < r < 1$, and put $\phi_n(z) = z^n/n$, for $n = 1, 2, 3, \ldots$. Then 1M for approach using M-test

$$|\phi_n(z)| \le \frac{r^n}{n} \le r^n, \qquad \text{for } n = 1, 2, 3, \ldots,$$

so that Assumption 1 of the M-test holds with $M_n = r^n$. Also, 1M for Assumption 1

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n \text{ is convergent (since } r < 1),$$ 1M for Assumption 2

so Assumption 2 also holds. Thus, by the M-test, $\sum_{n=1}^{\infty} \phi_n(z)$ is uniformly convergent on E, and so on each closed disc in $\{z : |z| < 1\}$. 1M for M-test

Therefore, by Weierstrass' Theorem, f is analytic on $\{z : |z| < 1\}$, and its derivative can be obtained from term by term differentiation: 1M for correct application

$$f'(z) = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n} = \sum_{n=1}^{\infty} z^{n-1} = 1 + z + z^2 + \cdots.$$ 1A

(ii) The function $g(z) = -\operatorname{Log}(1-z)$ is analytic on $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$ and its Taylor series about 0 is 1A

$$g(z) = -\left((-z) - \frac{(-z)^2}{2} + \frac{(-z)^3}{3} - \cdots\right) \qquad \text{1M, 1A}$$
$$= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots, \qquad \text{for } |z| < 1.$$

Therefore g agrees with f on $\{z : |z| < 1\}$, so g is a direct analytic continuation of f. 1M

(iii) The function

$$h(z) = -\operatorname{Log}(1-z) \qquad (|z+3| < 1) \qquad \text{1A}$$

is a direct analytic continuation of g, but not of f, so it is an indirect analytic continuation of f.

Question 12

(a) (i) False. The analytic function $f(z) = z^2$ is not conformal at 0, since $f'(0) = 0$. 1A, 1M

(ii) False. The extended line $\{z : \operatorname{Im} z = 0\} \cup \{\infty\}$ is a generalized circle, but it does not lie in $\mathbb{C}$. 1A, 1M

(iii) True. A linear function is of the form 1A, 1M

$$f(z) = az + b,$$

where $a \neq 0$, and this takes the form of a Möbius transformation:

$$f(z) = \frac{az+b}{cz+d},$$

where $c = 0$, $d = 1$ and $ad - bc = a \neq 0$.

(b) (i) The extended Möbius transformation corresponding to 1M

$$z_1 = (z+i)/(-z+i),$$

maps $-i$ to 0, 0 to 1, and i to ∞. Also, $\partial\mathcal{R}$ has a right-angled corner at $-i$, so the image of $\partial\mathcal{R}$ has a right-angled corner at 0. Using the preservation of boundary orientation, we find that the image of $\mathcal{R}$ is 1M 1M

$$\mathcal{R}_1 = \{z_1 : \operatorname{Re} z_1 > 0,\ \operatorname{Im} z_1 > 0\}. \qquad \tfrac{1}{2}\text{A}$$

(ii) The mapping $z_2 = z_1^2$ squares the modulus and doubles the argument, so the image of $\mathcal{R}_1$ is 1M

$$\mathcal{R}_2 = \{z_2 : \operatorname{Im} z_2 > 0\}. \qquad \tfrac{1}{2}\text{A}$$

(iii) We choose a Möbius transformation whose extension to $\widehat{\mathbb{C}}$ maps $i, -i$ (which are inverse points with respect to $\partial\mathcal{R}_2$) to $0, \infty$ (which are inverse points with respect to $\partial\mathcal{S}$): 1M

$$w = \frac{z_2 - i}{z_2 + i}. \qquad \text{1A}$$

Since this maps 0 to $-1 \in \partial\mathcal{S}$, we deduce that it maps $\partial\mathcal{R}_2$ onto $\partial\mathcal{S}$ and hence $\mathcal{R}_2$ onto $\mathcal{S}$. 1M

(iv) Composing the above mappings, we deduce that

$$f(z) = w = \frac{z_2 - i}{z_2 + i} = \frac{z_1^2 - i}{z_1^2 + i} = \frac{\left(\dfrac{z+i}{-z+i}\right)^2 - i}{\left(\dfrac{z+i}{-z+i}\right)^2 + i} \qquad \text{1M, } \tfrac{1}{2}\text{A}$$

is a one-one conformal mapping (because each of the constituent mappings is one-one and conformal) from $\mathcal{R}$ onto $\mathcal{S}$. The corresponding inverse function is

$$f^{-1}(w) = z = \frac{iz_1 - i}{z_1 + 1} = \frac{i\sqrt{z_2} - i}{\sqrt{z_2} + 1} = \frac{i\sqrt{\dfrac{iw+i}{-w+1}} - i}{\sqrt{\dfrac{iw+i}{-w+1}} + 1}, \qquad 1\tfrac{1}{2}\text{M, 1A}$$

where the formula for the inverse function of a Möbius transformation has been used twice.

(Drawing diagrams will have helped you to answer this question.)