M337/B

Module Examination 2013
Complex Analysis

Monday 10 June 2013 **2.30 pm – 5.30 pm**

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part 1 (64% of the marks) you should attempt as many questions as you can.

In Part 2 (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is **NOT** permitted in this examination.

PART 1

(i) *You should attempt as many questions as you can in this part.*
(ii) *Each question in this part carries 8 marks.*

Question 1

Determine each of the following complex numbers in Cartesian form, simplifying your answers as far as possible.

(a) $\exp(3 + \frac{1}{4}\pi i)$ [2]

(b) The principal cube root of -8. [2]

(c) i^{1-2i} [2]

(d) $\cos(i \log_e 2)$ [2]

Question 2

Let

$$A = \{z : 1 < |z| \leq 4\} \quad \text{and} \quad B = \{z : \operatorname{Re} z > -1, \operatorname{Im} z > -1\}.$$

(a) Make separate sketches of the sets A, B, $C = A - B$ and $D = \partial B$. [4]

(b) For each of the sets A, B, C and D:

(i) state whether it is a region, and if not explain why not;

(ii) state whether it is compact, and if not explain why not. [4]

Question 3

In this question Γ is the circle $\{z : |z| = 2\}$.

(a) Write down the standard parametrization for Γ.

(b) Evaluate

$$\int_\Gamma \overline{z}\, dz.$$ [3]

(c) Determine an upper estimate for the modulus of

$$\int_\Gamma \frac{2 \sin z}{\overline{z}^2 + 1}\, dz.$$ [5]

Question 4

Evaluate the following integrals, in which $C = \{z : |z| = 1\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C \frac{\operatorname{Log}(2-z)}{z^2+4}\, dz$ [2]

(b) $\displaystyle\int_C \frac{\operatorname{Log}(2-z)}{z(z-2)}\, dz$ [3]

(c) $\displaystyle\int_C \frac{\operatorname{Log}(2-z)}{z^3}\, dz.$ [3]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{z^2+1}{z(5z-1)(z-5)}$$

at each of the poles of f. *[4]*

(b) Hence evaluate the integral

$$\int_0^{2\pi} \frac{\cos t}{13 - 5\cos t}\, dt.$$ *[4]*

Question 6

Let $f(z) = iz^5 + 5z^2 - 3i$.

(a) Determine the number of zeros of f that lie inside:

(i) the circle $C_1 = \{z : |z| = 2\}$, *[3]*

(ii) the circle $C_2 = \{z : |z| = 1\}$. *[3]*

(b) Show that the equation

$$iz^5 + 5z^2 - 3i = 0$$

has exactly 3 solutions in the set $\{z : 1 < |z| < 2\}$. *[2]*

Question 7

Let $q(z) = \overline{z} + 1 - i$ be a velocity function.

(a) Explain why q represents a model fluid flow. *[1]*

(b) Determine a complex potential function for this flow. Hence sketch the streamline through the point 1 and indicate the direction of flow. *[5]*

(c) Evaluate the circulation of q along the path

$$\Gamma : \gamma(t) = t \quad (t \in [0, 4]).$$ *[2]*

Question 8

(a) Show that the iteration sequence

$$z_{n+1} = 15z_n^2 + 3z_n + \tfrac{1}{16}, \qquad n = 0, 1, 2, \ldots,$$

with $z_0 = 0$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + \tfrac{3}{16}, \qquad n = 0, 1, 2, \ldots,$$

with $w_0 = \frac{3}{2}$. *[3]*

(b) Find the fixed points of $P_{\frac{3}{16}}(z) = z^2 + \frac{3}{16}$ and determine their nature. *[3]*

(c) Determine whether or not $-\frac{3}{2} + i$ lies in the Mandelbrot set M. *[2]*

PART 2

(i) *You should attempt no more than* **TWO** *questions in this part.*
(ii) *Each question in this part carries 18 marks.*

Question 9

(a) Let f be the function

$$f(z) = z(3 + \overline{z}) + \operatorname{Re} z.$$

(i) Write $f(x + iy)$ in the form $u(x, y) + iv(x, y)$, where u and v are real-valued functions. [*2*]

(ii) Use the Cauchy–Riemann theorem and its converse to show that f is differentiable at $-\frac{1}{2}$, but not analytic there. [*4*]

(iii) Evaluate $f'\left(-\frac{1}{2}\right)$. [*1*]

(b) Let g be the function $g(z) = z + \dfrac{i}{z}$.

(i) Show that g is conformal at 1. [*1*]

(ii) Describe the effect of g on a small disc centred at 1. [*3*]

(iii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = (t - 1)i + t \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point 1, and find the angle from Γ_1 to Γ_2 at this point of intersection. [*2*]

(iv) Sketch the paths Γ_1 and Γ_2 on the same diagram, clearly indicating their directions. [*2*]

(v) Using part (b)(ii), or otherwise, sketch the directions of $g(\Gamma_1)$ and $g(\Gamma_2)$ at $g(1)$. [*1*]

(vi) Find the image of the unit circle under g. Why is g not conformal at $e^{i\pi/4}$? [*2*]

Question 10

(a) Let f be the function $f(z) = \dfrac{1}{(z-1)(z-5)}$.

(i) Locate and classify the singularities of f. [2]

(ii) Find the Laurent series about 2 for f on the annulus $\{z : 1 < |z-2| < 3\}$,

giving the constant term and two terms on either side of the constant term. [7]

(b) (i) Find the Taylor series about 0 (up to the term in z^4) for the function

$$g(z) = \exp(z \sin z),$$

and explain why the series represents g on $\mathbb{C}$. [4]

(ii) Hence evaluate the integral

$$\int_C z^3 g(1/z)\, dz,$$

where C is the circle $\{z : |z| = 2\}$. [5]

Question 11

(a) Find the residues of the function

$$f(z) = \frac{\pi \cot \pi z}{16z^2 + 9}$$

at each of the points 0, $\frac{3}{4}i$, $-\frac{3}{4}i$. [6]

(b) Hence determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 + 9}.$$ [8]

(c) Use your result from part (b) to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{16n^2 + 9} = \frac{\pi}{12} \coth \frac{3\pi}{4}.$$ [4]

Question 12

(a) Determine the extended Möbius transformation $\widehat{f_1}$ which maps -1 to 0, ∞ to 1 and $-i$ to ∞. *[3]*

(b) Let

$$R = \{z : |z| < 1\} \cap \{z : |z + 1 + i| < 1\},$$
$$S = \{z_1 : 3\pi/4 < \operatorname{Arg}_{2\pi}(z_1) < 5\pi/4\},$$
$$T = \{w : \operatorname{Re} w > 0\}.$$

(i) Sketch the regions R, S and T. *[3]*

(ii) Explain why $\widehat{f_1}(R) = S$. *[6]*

(iii) Hence determine a one-one conformal mapping f from R onto T. *[4]*

(iv) Write down the rule of the inverse function f^{-1}. *[2]*

[END OF QUESTION PAPER]