Based on tutorials of dr. Paul Reed and the OUSA revision weekend 1998, and my own
Typed by I.R. van de Stadt BSc. (Hons)

# Part I

**Question 1**

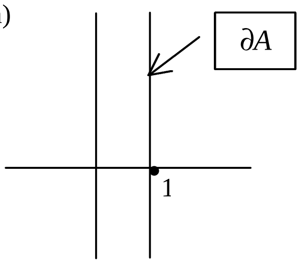
(a) (i) $w = \dfrac{(i+1)}{(i-1)(i+1)} = -\frac{1}{2}(i+1) \Rightarrow \operatorname{Arg} w = -\frac{3p}{4}$

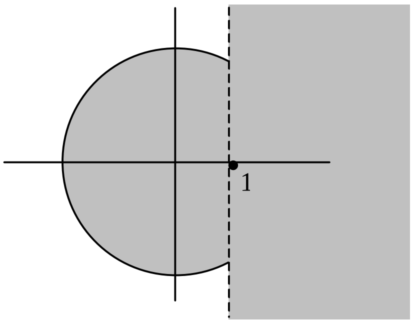
(a) (ii) $w = -\frac{1}{2}(i+1) = \frac{1}{\sqrt{2}}e^{-i3p/4} \Rightarrow \left(2^{-1/8}e^{-i3p/4}\right)^{1/4} = 2^{-1/8}e^{-i3p/16} = 2^{-1/8}\left(\cos\frac{3p}{16} + i\sin\frac{3p}{16}\right)$

(b) $i = e^{ip/2} \Rightarrow i^{-i} = \left(e^{ip/2}\right)^{-i} = e^{p/2}$ (Principal argument)

**Question 2**

(a)




(b) $A \cup \{1\}$ is not a region. $A \cap B$ is not a region.

(c) $B - A$ is compact. $\overline{A}$ is not compact.

**Question 3**

(a) The domain of the function $f$ is ( . $F$ is continuous by the Composition Rule, since the functions Im$z$ and $z^2$ are basic continuous functions.

(b) $a = 0, b = 1+i \Rightarrow g(t) = (1+i)t, \quad t \in [0,1] \Rightarrow g'(t)(1+i)$, $\operatorname{Im} z^2 \to \operatorname{Im}(1+i)^2 t^2 = 2t^2$

So, $\int_\Gamma f(z)dz = \int_0^1 2t^2(1+i)dt = (1+i)\left[\frac{2}{3}t^2\right]_0^1 = \frac{2}{3}(1+i)$

**Question 4**

(a) Taylor series is $f(0) + zf'(0) + \dfrac{z^2}{2!}f''(0) + \dfrac{z^3}{3!}f'''(0) + \dfrac{z^4}{4!}f''''(0) + \cdots$

Since $f(z) = e^z \cos z$, we have $\quad f(0) = 1$

$f'(z) = e^z\cos z - e^z \sin z \Rightarrow \quad f'(0) = 1$

$f''(z) = e^z\cos z - e^z\sin z - e^z \sin z - e^z\cos z = -2e^z\sin z \Rightarrow \quad f''(0) = 0$

$f'''(z) = -2e^z\sin z - 2e^z\cos z \quad f''(0) = -2$

$f''''(z) = -2\left(e^z\sin z + e^z\cos z + e^z\cos z - e^z\sin z\right) = -4e^z\cos z \Rightarrow \quad f''''(0) = -4$

So up to $z^4$ we have

$$f(z) = 1 + z - \frac{2}{3!}z^3 - \frac{4}{4!}z^4 + \cdots$$

Since $f(z)$ is analytic for all discs $0 \le z \le r$, the Taylor series represents $f(z)$ for all discs $0 \le z \le r$ (i.e. all of ( )

(i) $$\int_{|z|=1}\frac{f(z)}{z^2}dz = \int_{|z|=1}\frac{1}{z^2}\left(1 + z - \frac{2}{3!}z^3 - \frac{4}{4!}z^4 + \cdots\right)dz = \int_{|z|=1}\left(\frac{1}{z^2} + \frac{1}{z} - \frac{2}{3!}z - \frac{4}{4!}z^2 + \cdots\right)dz$$

(only this term is important since from it we find the residue at $z = 0$ )

So,

$$\int_{|z|=1}\frac{f(z)}{z^2}dz = 2pi\operatorname{Res}(f,0) = 2pi \cdot 1 = 2pi$$

(ii) Note that $g(z) = \frac{d}{dz} f(z)$. Since $f(z)$ is represented by its Taylor series in all discs we have

$$g(z) = \frac{d}{dz}\left(1 + z - \frac{2}{3!}z^3 - \frac{4}{4!}z^4 + \cdots\right) = 1 - z^2 - \frac{4}{3!}z^3 + \cdots$$

(by the Differentiation Rule). The result is the first 3 terms of the Taylor series for $g(z)$.

**Question 5**

(a) $$f(z) = \frac{e^{3iz}}{z^2+4} = \frac{e^{3iz}}{(z-2i)(z+2i)}$$ Note this has **simple** poles at $\pm 2i$

To find Residue use **Cover up** rule (Theorem 1.1 unit C1)

Residue at $2i$

$$\lim_{z\to 2i}(z-2i)\frac{e^{3iz}}{(z-2i)(z+2i)} = \frac{e^{-6}}{4i} = \frac{-ie^{-6}}{4}$$

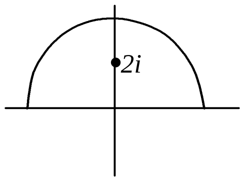
Residue at -2$i$

$$\lim_{z\to -2i}(z+2i)\frac{e^{3iz}}{(z-2i)(z+2i)} = \frac{e^{6}}{-4i} = \frac{ie^{6}}{4}$$

[Note the cover up rule **only** works for simple poles. For multiple poles which have a term like $(z-a)^n, \quad n = 2,3,4,\ldots$ in the denominator then other methods need to be used]

(b) We have

$$\int_{-\infty}^{\infty}\frac{\cos 3t}{t^2+4}dt = \mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{e^{3ti}}{t^2+4}dt\right)$$




Theorem 3.4 (C1) satisfied.

Since there are no simple poles on the real axis, we have

$$\int_{-\infty}^{\infty}\frac{e^{3ti}}{t^2+4}dt = 2pi\{\text{sum of residues in upper half plane}\}$$

So we have (using the result of part (a)

$$\int_{-\infty}^{\infty}\frac{\cos 3t}{t^2+4}dt = \mathrm{Re}\,2pi\left(\frac{-ie^{-6}}{4}\right) = \tfrac{1}{2}pe^{-6}$$

**Question 6**

(a) We have $\left|e^z\right| = \left|e^{x+iy}\right| = e^x$. Since $|z| = 1$ we have $\left|e^z\right| \geq e^{-1}$

[Here you are expected to know that $e < 3$ ]

So we have $\left|e^z\right| \geq e^{-1} > \frac{1}{3}$

(b) Rouchés Theorem (page 19 Unit C1)

We have $\left|f - e^z\right| = \left|-\frac{1}{3}z^4\right| = \frac{1}{3}$

So we have $\left|f - e^z\right| = \frac{1}{3} \leq \left|e^z\right|, \quad \text{for } z \in |z| = 1$

So $f$ has the same number of zeros inside $|z| = 1$ as $e^z$. However $e^z$ has no zeros and hence $f$ has no zeros inside $|z| = 1$. This is Rouchés Theorem.

(c) $$\int_\Gamma \frac{1}{f(z)}dz = 0$$

This is because $f(z)$ is analytic an non-zero inside $\Gamma$. So $1/f(z)$ is analytic and so the integral is zero by Cauchy's Theorem.

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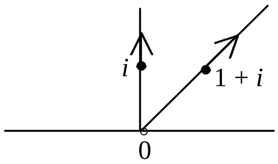
**Question 7**

(a) $\overline{q}(z) = \frac{2}{z}$ is analytic on ( $-\{0\}$ so $q$ represents a model fluid flow (See HB. 1.14 p.38)

(b) A primitive of $\overline{q}(z) = \frac{2}{z}$ is $\Omega(z) = 2\,\mathrm{Log}\,z, \quad z \in$ ( $-\{x \in$ R $: x \le 0\}$
is a complex potential function for the flow.
Streamlines satisfy the equation
$\mathrm{Im}\,\Omega(z) = \mathrm{Im}\,2\,\mathrm{Log}\,z = \mathrm{Im}\left(2\log_e|z| + i2\,\mathrm{Arg}\,z\right) = 2\,\mathrm{Arg}\,z = k$




$q(i) = \frac{2}{\overline{i}} = 2i$ ; $q(1+i) = \frac{2}{\overline{1+i}} = \frac{2(1+i)}{(1-i)(1+i)} = (1+i)$

(c) $\mathcal{F} = \mathrm{Im}\int_C \overline{q}(z)dz = \mathrm{Im}(2pi \times 2) = 4p$ by Cauchy's Integral Formula, since the function $f(z) = 2$ is analytic on ( . ( is a simply-connected region and the unit circle isis a simple-closed contour in (
and 0 is a point inside the unit circle.

**Question 8**

(a) See HB 2.1 p. 41. $z_{n+1} = 2z_n - 2z_n^2$, with $a = -2, b = 2, c = 0 \Rightarrow d = -2\times 0 + \frac{1}{2}\times 2 - \frac{1}{4}\times 2^2 = 0$
So the sequence is conjugate ot the iteration sequence $w_{n+1} = w^2$ .
Then the conjugating function is $h(z) = -2z + 1 \Rightarrow w_0 = -2z_0 + 1 = -2\times -1 + 1 = 3$
As required.

(b) (i) $P_{-1\ \ i}(0) = -1 - i; P_{-1-}\ (0) = (-1-i)^2 - 1 - i = -1 + i; P_{\ 1-i}(0) = (-1+i)^2 - 1 - i = -1 - 3i \Rightarrow$
$\left| \ {}^{3}_{-1-i}(0)\right| \quad \sqrt{10} >$ , so $c \notin M$ (See HB 4.5 p. 42).

(b) (ii) $i$ is in the large cardioid. Hence we check HB 4.9 (a) p. 43:
$\left(8\left|\frac{-i}{4}\right| \quad -\frac{3}{2}\right) + 8 \quad (i) = \left(\frac{1}{2} - \frac{}{2}\right)^2 = 1 < 3$
So by HB 4.3 p 43 $c \in M$ .

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## Part II

**Question 9**

(a) (i) $\sin(\quad + iy) = \sin x \cos \quad + \cos x \sin iy = \sin \quad \cosh y + i \cos \quad \sinh z \Rightarrow$

$$|\sin(x+iy)|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 z = \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y$$

$$= \sin^2 x + \sin^2 x \sinh^2 y + \sinh^2 y - \sin^2 x \sinh^2 y = \sin^2 x + \sinh^2 y$$

as required.

$\cos(x+iy) = \cos x \cos iy + \sin x \sin iy = \cos x \cosh y + i \sin x \sinh z \Rightarrow$

$$|\cos(x+iy)|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 z = \cos^2 x \cosh^2 y + (1-\cos^2 x)(\cosh^2 x - 1)$$

$$= \cos^2 x \cosh^2 y + \cosh^2 x - 1 - \cos^2 x \cosh^2 y + \cos^2 x = \cosh^2 x - 1 + \cos^2 x$$

$$= \sinh^2 x + \cos^2 x$$

(a) (ii) $|\tan z|^2 = \dfrac{|\sin z|^2}{|\cos z|^2} = \dfrac{\sinh^2 x + \sin^2 x}{\sinh^2 x + \cos^2 x} \le 1 \Rightarrow \sinh^2 x + \sin^2 x \le \sinh^2 x + \cos^2 x \Rightarrow$

$\sin^2 x \le \cos^2 x \Rightarrow -\frac{1}{4}p \le x \le \frac{1}{4}p$

(b) $f$ is defined on ( . $u(x,y) = x^2 + by^2; v(x,y) = 2axy$ . Now,

$$\frac{\partial u}{\partial x} = 2x; \quad \frac{\partial v}{\partial y} = 2ax; \quad \frac{\partial v}{\partial x} = 2ax; \quad \frac{\partial u}{\partial y} = 2by$$

all exist on ( , are all continuous on ( , since they are all multiples (Multiple Rule) of the basic continuous function $x$. The Cauchy-Riemann equations are statisfied on all $z \in$ ( when

$2x = 2ax \Rightarrow a = 1; \quad 2ax = -2by \Rightarrow b = -1$

Then $f$ is differentiable on all of ( when $a = 1$ and $b = -1$, and, because its domain ( is a region $f$ is analytic on ( . (See HB 2.3 p.19 and HB 1.3 p.18)

**Question 10**

(a) $f(x)=\dfrac{\sin z}{z(z-3i)(z+3i)}$ has a removable singularity at $z=0$ and simple poles at $z=3i$ and $z=-3i$. So by the Cover-Up Rule

$$\operatorname{Res}(f,0)=\frac{\sin 0}{(-3i)(3i)}=0;\quad \operatorname{Res}(f,3i)=\frac{\sin 3i}{3i\times 6i}=-\tfrac{1}{18}i\sinh 3$$

$$\operatorname{Res}(f,-3i)=\frac{\sin(-3i)}{-3i(-6i)}=\frac{-\sin 3i}{3i\times 6i}=\tfrac{1}{18}i\sinh 3$$

(b) (i) $f$ is analytic on the simply-connected region ( except for 3 singularities. $\Gamma$ is a simple-closed contour in ( , not passing through any of the singularities. So, since only 0 is inside $\Gamma$ and the other singularities are outside $\Gamma$

$$\int_\Gamma f(z)dz=2pi\operatorname{Res}(f,0)=0 \text{ (see part (a))}$$

by Cauchy's Residue Theorem

(b) (ii) $f$ is analytic on the simply-connected region ( except for 3 singularities. $\Gamma$ is a simple-closed contour in ( , not passing through any of the singularities. So, all three singularies are inside $\Gamma$

$$\int_\Gamma f(z)dz=2pi(\operatorname{Res}(f,0)+\operatorname{Res}(f,3i)+\operatorname{Res}(f,-3i))=2pi\left(0-\tfrac{1}{18}i\sinh 3+\tfrac{1}{18}i\sinh 3\right)=0 \text{ (see part (a))}$$

by Cauchy's Residue Theorem

(c) $f$ is analytic on the simply-connected region ( except for 3 singularities. Let $\Gamma=|z-3i|=1$.Then $\Gamma$ is a simple-closed contour in ( , not passing through any of the singularities and only 3i is inside $\Gamma$ and the other singularities are outside $\Gamma$. Hence, by Cauchy's Residue Theorem

$$\int_\Gamma f(z)dz=2pi\operatorname{Res}(f,3i)=2pi\left(-\tfrac{1}{18}i\sinh 3\right)=\frac{p\sinh 3}{9} \text{ (see part (a))}$$

as required.

(d) $\dfrac{1}{zf(z)}=\dfrac{z^2+9}{\sin z}$ in analytic on $\{z:|z|<p\}$, except for a simple pole at 0

$$\operatorname{Res}\left(\frac{1}{zf(z)},0\right)=\frac{0^2+9}{\cos 0}=9,$$

by the g/h Rule, since $z^2+9$ and $\dfrac{d\sin z}{dz}=\cos z$ are analytic at 0 and $\cos 0=1\neq 0$

**Question 11**

(a) (i) $\left|\exp\left(e^{-z}\right)\right| = \exp\left(\mathrm{Re}\left(e^{-z}\right)\right) = \exp\left(e^{-x}(\cos y)\right)$ as required.

(a) (ii) The interval $R = \{z : -1 < x < 1, -p < y < p\}$ is a bounded region. The function $f(z) = \exp\left(e^{-z}\right)$ is analytic on $R$ and continuous and non-zero on $\overline{R}$, then by the Maximum Principle (HB. p. 31) there is a maximum is on $\partial R$

On $\{z : x = -1, -p \le y \le p\}, \max\left|\exp\left(e^{1}(\cos y)\right)\right| = \exp\left(e^{1}(\cos 0)\right) = e^{e}$

On $\{z : x = 1, -p \le y \le p\}, \max\left|\exp\left(e^{-1}(\cos y)\right)\right| = \exp\left(e^{-1}(\cos 0)\right) = e^{-e}$

On $\{z : -1 \le x \le 1, y = -p\}, \max\left[\exp\left(e^{1}(\cos(-p))\right)\right] = \exp\left(-e^{-1}\right) = e^{-e^{-1}}$

On $\{z : -1 \le x \le 1, y = p\}, \max\left[\exp\left(e^{1}(\cos(p))\right)\right] = \exp\left(-e^{-1}\right) = e^{-e^{-1}}$

So $\max\left\{\left|\exp\left(e^{-z}\right)\right| : -1 \le \mathrm{Re}\, z \le 1, -p \le \mathrm{Im}\, z \le p\right\}$ is $e^{e}$ which is attained at $z = -1$

(b) Let $h(z) = \dfrac{r}{r - z}, \quad z \in ( \; - \{r\}$

Now *f* and *g* are Basic Taylor Series (HB 3.5 p. 25)

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{z}{r}\right)^{n} = \left(1 - \tfrac{z}{r}\right)^{-1}, \quad |z| < r \Rightarrow f(z) = \frac{r}{(r - z)} = h(z), \quad |z| < r$$

$$g(z) = -\sum_{n=1}^{\infty}\left(\frac{r}{z}\right)^{n} = -\sum_{n=0}^{\infty}\left(\frac{r}{z}\right)^{n} + 1 = -\left(1 - \tfrac{r}{z}\right)^{-1} + 1, \quad |z| > r \Rightarrow$$

$$g(z) = -\frac{z}{(z - r)} + 1 = \frac{z}{(r - z)} + 1 = \frac{z + r - z}{(r - z)} = \frac{r}{(r - z)} = h(z), \quad |z| > r$$

*f* , *g* and *h* are analytic on their domains.

The region $|z| < r$, the domain of *f* overlaps with the region $z \in ( \; - \{r\}$, the domain of *h*. Also the region $|z| > r$, the domain of *g* overlaps with $z \in ( \; - \{r\}$. Hence *f* and *h* are direct continuations of each other (or *f* is a direct continuation of *h* by Taylor series (HB 2.1 p.33) and so are *h* and *g* likewise. The regions $|z| < r$ and $|z| > r$ do not overlap, and so *f* and *g* are indirect continuations of each other. (HB 1.1 p. 33 and HB 2.3 p.33)

**Question 12**

(a) (i) The inverse mappings are to the standard triple of points 0,1,∞ (HB 2.11 p.36)

$\alpha = -i \to 0$

$\beta = 1 \to 1$

$\gamma = i \to \infty$

So $\hat{f}^{-1}$ corresponds to

$$\hat{f}^{-1}(z) = \frac{(z+i)(1-i)}{(z-i)(1+i)} = \frac{(z+i)(1-i)^2}{(z-i)(1+i)(1-i)} = \frac{-i(z+i)}{(z-i)} = \frac{-iz+1}{z-i}$$

and so the extended Möbius transformation $\hat{f}$ that maps 0 to $-i$, 1 to 1 and ∞ to $i$ is

$$\hat{f}(z) = \frac{-iz-1}{-z-i} = \frac{iz+1}{z+i}$$

(a) (ii) The extended imaginary axis is the set $\{z : \operatorname{Re} z = 0\} \cup \{\infty\}$

We use the three point trick. The imaginary axis passes through the points $-i$, 0 and $i$.
Now

$$\hat{f}(-i) = \frac{-i^2+1}{-i+i} = \frac{2}{0} = \infty\ ,\quad \hat{f}(0) = \frac{i\cdot 0+1}{0+i} = \frac{1}{i} = -i \ \text{ and }\ \hat{f}(i) = \frac{i^2+1}{i+i} = \frac{0}{2i} = 0$$

So the image of the imaginary axis is the generalized circle that passes throught the extended points ∞ , -*i*, and 0. Thus the image is the imaginary axis, as required.

Maybe the following is a safer proof.

The Apollonian form of the imaginary axis is $\{z : |z+1| = |z-1|\} \cup \{\infty\}$

Now from part (a) (i) we know already that

$$z = \hat{f}^{-1}(w) = \frac{-iw+1}{w-i}$$

Since $w \in \hat{f}(\operatorname{Im} axis) \Leftrightarrow \hat{f}^{-1}(w) \in \operatorname{Im} axis$ . So, $w = \infty, w = i$ or

$$\left|\frac{-iw+1}{w-i}+1\right| = \left|\frac{-iw+1}{w-i}-1\right| \Leftrightarrow \left|\frac{-iw+1+w-i}{w-i}\right| = \left|\frac{-iw+1-w+i}{w-i}\right| \Leftrightarrow$$

$$|(1-i)w+1-i| = |-(1+i)w+1+i| = |1-i||w+1| = |1+i||-w+1| \Leftrightarrow |w+1| = |w-1|$$

Therefore the image is $\{w : |w+1| = |w-1|\} \cup \{\infty\}$, which is the imaginary axis as required.

(b) (i) HB 1.10 p. 12 and HB 5.3 p. 14 .

Let $h(z) = \sqrt{z} = z^{1/2} = \exp\left(\tfrac{1}{2}\operatorname{Log} z\right) = \exp\left(\tfrac{1}{2}(\log|z| + i\operatorname{Arg} z)\right) = \left(\sqrt{|z|}\right)e^{i\operatorname{Arg} z/2}$

So, if $\left(\sqrt{|z_1|}\right)e^{i\operatorname{Arg} z_1/2} \neq \left(\sqrt{|z_2|}\right)e^{i\operatorname{Arg} z_2/2} \Leftrightarrow |z_1|e^{i\operatorname{Arg} z_1} \neq |z_2|e^{i\operatorname{Arg} z_2} \Leftrightarrow z_1 = z_2$

Hence $h(z)$ is a one-one function. $h(z)$ is analytic on $\{z : \operatorname{Re} z > 0\}$, with $h'(z) = \frac{1}{2\sqrt{z}} \neq 0$ for all *z* in $\{z : \operatorname{Re} z > 0\}$. By HB 2.6 p. 36 the Möbius transformation *f* is one-one on $\{z : \operatorname{Re} z > 0\}$ since $i\cdot i - 1\cdot 1 = -2 \neq 0$ and by HB 2.2 *f* is conformal. Now $g(z) = h(f(z)) = \sqrt{f(z)}$, so by HB 4.5 p. 37 *g* is a one-one conformal mapping.

De domain is at the left of the points $-i$, 0 and $i$, which $\hat{f}$ maps to 0, -*i*, and ∞ with the image at the left. This shows that $\hat{f}$ maps $\{z : \operatorname{Re} z > 0\}$ to $\{z : \operatorname{Re} z > 0\}$. Using the three point trick to show where $\hat{h}(z)$ maps the imaginary axis onto. The imaginary axis passes through the points $-i$, 0 and $i$ with the domain at the left, which $\hat{h}(z)$ maps to $\frac{1}{\sqrt{2}}(1-i)$, 0, $\frac{1}{\sqrt{2}}(1-i)$, with the image at the left, which shows that $\hat{h}(z)$ maps $\{z : \operatorname{Re} z > 0\}$ to $\left\{z : -\tfrac{1}{4}\pi < \operatorname{Arg} z < -\tfrac{1}{4}\pi\right\}$

(b) (ii) *f* and *h* are one-one and conformal on these regions. $h^{-1}(z) = z^2$ Hence since $(h \circ f)^{-1} = f^{-1} \circ h^{-1}$,

$$g^{-1}(z) = \frac{-iz^2+1}{z^2-i} = \frac{z^2+i}{iz^2+1}$$