Mathematics, Computing and Technology
M337 Complex Analysis

M337

Assignment Booklet II

Contents **Cut-off date**

Please send all your answers for each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3).

You will find instructions on how to fill in the PT3 form in the *Assessment Handbook*. Remember to fill in the correct assignment number as listed above and to allow sufficient time in the post for the assignment to reach its destination on or before the cut-off date.

The marks allocated to each part of each question are indicated in brackets in the margin.

Questions in assignments (and in the examination) carry both *accuracy marks* and *method marks*. You should therefore, as a general practice, show all your working. As a guide, we use the following wording with the interpretations shown:

write down or *state* means 'write down without justification';

find, *determine*, *calculate*, *explain*, *derive*, *evaluate* or *solve* means that we require you to show all your working in giving an answer;

prove, *show* or *deduce* means that you should 'justify each step'; in particular, if you use a definition, result or theorem to go from one line to the next, you must state clearly which fact you are using — for example, quote the relevant unit and page, or give a Handbook reference — and check that all the necessary conditions are satisfied.

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17.1

Answer *all* questions.

Question 1 (*Unit C1*) – 33 marks

(a) Let

$$f(z) = \frac{z}{(2z^2 - 5iz - 2)^2}.$$

(i) Show that f has a pole of order 2 at the point $i/2$, and evaluate the residue of f at this point. [5]

(ii) Deduce, by the strategy in Section 2, that

$$\int_0^{2\pi} \frac{1}{(5 - 4\sin t)^2}\, dt = \frac{10\pi}{27}.$$ [5]

(b) Use a theorem from Section 3 to show that

$$\int_{-\infty}^{\infty} \frac{3\sin 2t}{t(t^2+4)}\, dt = \frac{3\pi}{4}\left(1 - \frac{1}{e^4}\right).$$ [11]

(c) Use a method given in Section 4 to determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{9n^2 - 16}.$$ [12]

Question 2 (*Unit C2*) – 34 marks

(a) Sketch the path

$$\Gamma : \gamma(t) = (1 - 2t) + (1 - t)i \quad (t \in [0, 1]).$$

Write down a value $\phi \in \mathbb{R}$ such that $\operatorname{Arg}_\phi \gamma$ is a continuous argument function for Γ, and hence evaluate $\operatorname{Wnd}(\Gamma, 0)$. [4]

(b) Determine the number of zeros of the function

$$f(z) = 2z^4 + iz^3 - 7z + 3$$

in each of the following sets.

(i) $S_1 = \{z : |z| \leq 1\}$ [6]

(ii) $S_2 = \{z : 1 < |z| < 2\}$ [5]

(iii) $S_3 = \{z : 2 \leq |z|\}$ [2]

(c) Let $f(z) = 2z^2 + z\cos z$.

(i) Show that the function f is one-one near 0. [2]

(ii) Invert the Taylor series about $\alpha = 0$ for f, giving the first three non-vanishing terms. [7]

(d) Determine

$$\max\{|z^2 \exp(3 + iz^2)| : |z| \leq 2\}$$

and find all points at which the maximum is attained, giving your answers in Cartesian form. [8]

Question 3 (*Unit C3*) – 33 marks

(a) Let

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{z}{3}\right)^n \quad (|z+3| < 3)$$

and

$$g(z) = -3i \sum_{n=0}^{\infty} (1 - iz)^n \quad (|z+i| < 1).$$

Prove that the functions f and g are direct analytic continuations of each other. [*8*]

(b) Use Theorem 1.1 to evaluate the improper integral

$$\int_0^{\infty} \frac{t^{1/3}}{t^2 - 1}\, dt.$$ [*8*]

(c) Prove that the series

$$\sum_{n=1}^{\infty} \frac{\exp(z^n)}{n^2 + 2n}$$

(i) is uniformly convergent on $\{z : |z| \leq 1\}$; [*5*]

(ii) defines a function that is analytic on $\{z : |z| < 1\}$. [*3*]

(d) (i) Evaluate $\Gamma(2 - 3i)/\Gamma(-2 - 3i)$. [*4*]

(ii) Show that $\int_C \Gamma(z)\, dz = \pi i$, where $C = \{z : |z+i| = \frac{5}{2}\}$. [*5*]

Answer *all* questions.

Question 1 (*Unit D1*) – 34 marks

(a) Let f be the Möbius transformation

$$f(z) = \frac{z+1}{z+2i},$$

and let $C = \{z : |z - 2i| = 2\}$. Determine the image $f(C)$:

(i) in Apollonian form; [*7*]

(ii) in the form of Equation (1.1) of Theorem 1.2 on page 9. [*5*]

(b) Determine the point α such that α and $\beta = 1 - 2i$ are inverse points with respect to the circle $C = \{z : |z + 1 + i| = 2\}$.

Draw a sketch of the circle C that indicates clearly the relative positions of the centre of the circle and the inverse points α and β. [*4*]

(c) (i) Sketch each of the following regions:

$$\mathcal{R} = \{z : |z| < 2,\ \pi/2 < \operatorname{Arg} z < 3\pi/4\},$$
$$\mathcal{R}_1 = \{z : |z| < 16,\ \operatorname{Im} z > 0\},$$
$$\mathcal{R}_2 = \{z : |z| < 4,\ \operatorname{Re} z > 0\},$$
$$\mathcal{S} = \{z : -\pi/4 < \operatorname{Arg} z < \pi/4\}.$$
[*4*]

(ii) Write down a one-one conformal mapping f_1 from $\mathcal{R}$ onto $\mathcal{R}_1$, giving brief reasons to justify that your mapping is one-one and conformal on $\mathcal{R}$. [*2*]

(iii) Write down a one-one conformal mapping f_2 from $\mathcal{R}_1$ onto $\mathcal{R}_2$. [*1*]

(iv) Determine a Möbius transformation f_3 that maps $\mathcal{R}_2$ onto $\mathcal{S}$. [*5*]

(v) Hence obtain a one-one conformal mapping f from $\mathcal{R}$ onto $\mathcal{S}$, explaining why it is one-one and conformal on $\mathcal{R}$. [*2*]

(vi) Find the rule for the inverse mapping f^{-1} of the mapping f found in part (c)(v). [*4*]

Question 2 (*Unit D2*) – 33 marks

(a) Let $q(z) = 4 - 2i\,\overline{z}$ be a velocity function.

(i) Show that q is a model flow velocity function on $\mathbb{C}$. [*2*]

(ii) Write down a complex potential function for q, and obtain the corresponding stream function. [*3*]

(iii) Determine the equation of a general streamline for q. Hence find the equations of the streamlines through the points 0, $1 + i$ and $2 + 2i$, and sketch them on a single diagram, indicating the direction of flow in each case. [*7*]

(iv) Are there any degenerate streamlines? Give a reason for your answer. [*1*]

(v) Find the circulation of q along Γ and the flux of q across Γ, where Γ is the line segment from 0 to $1 + i$. [*3*]

(b) Consider the obstacle $K = L(-3-i, 3+i)$.

(i) Show that a one-one conformal mapping that maps $\mathbb{C} - K$ to $\mathbb{C} - K_{\sqrt{10}/2}$ is given by

$$f(z) = \tfrac{1}{2}\left(z + z\sqrt{1-(8+6i)/z^2}\right).$$ [2]

(ii) Verify that f satisfies Equation (3.5) of the Flow Mapping Theorem. [3]

(iii) Show that

$$f'(z) = \frac{f(z)}{z\sqrt{1-(8+6i)/z^2}}.$$ [4]

(iv) Deduce that the solution to the Obstacle Problem for K, with circulation 6π around K, is

$$q(z) = \overline{\left(1 - \frac{9+3i}{2z\,f(z)} - \frac{3i}{z}\right) \times \frac{1}{\sqrt{1-(8+6i)/z^2}}}.$$ [6]

(v) Verify that $\lim\limits_{z\to\infty} q(z) = 1$ for the function q in part (b)(iv). [2]

Question 3 (*Unit D3*) – 33 marks

(a) Let $f(z) = z^2 - (1-2i)z + 1 - 2i$.

(i) Find the fixed points α and β of the function f, and classify them as (super)-attracting, repelling or indifferent. [5]

(ii) Show that the iteration sequence

$$z_{n+1} = f(z_n)$$

is conjugate to the iteration sequence

$$w_{n+1} = P_{5/4}(w_n),$$

and give the conjugating function h. [3]

(iii) Verify that $h(\alpha)$ and $h(\beta)$ are the fixed points of $P_{5/4}$. [3]

(iv) Use the result of Problem 4.5(b) to find a 2-cycle of $P_{5/4}$. Hence find a 2-cycle of f, and determine its nature. [6]

(v) Determine eight distinct points that lie in the keep set $K_{5/4}$. [3]

(vi) Determine whether or not $K_{5/4}$ is connected. [2]

(b) Determine which of the following points lie in the Mandelbrot set.

(i) $c = -\frac{6}{7} - \frac{1}{7}i$ [3]

(ii) $c = \frac{7}{8} - \frac{7}{8}i$ [4]

(iii) $c = \frac{1}{6} + \frac{1}{6}i$ [4]