2012B MST209 exam solutions

The references to the Handbook are given as section followed by page number e.g. (5 p26)

1. Combining the logs using the rules (p11) gives $\ln((y+6)\,x^2) = \ln(2^3(x+1))$ and removing the logs gives $(y+6)x^2 = 8(x+1)$ Multiplying out the brackets and rearranging gives

$$x^2y = -6x^2 + 8x + 8 = -2(3x+2)(x-2)$$

So the answer is B.

2. The equation is not linear or separable (8 p25) so the answer is D

3. $(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = (3-2)\mathbf{i} + (-1-2)\mathbf{j} + (-4-3)\mathbf{k}$ (18 p29)
so the answer is C.

4. $\mathbf{i}$-component $= |\mathbf{F}|\cos(\pi+\theta)$ or $= |\mathbf{F}|\cos(\pi-\theta) = -|\mathbf{F}|\cos\theta$ (8 p31)
so the answer is D.

5. The direction of motion is in the **i**-direction and the resistance force opposes this and is a vector (13 p33) so the answer is C.

6. $\mathbf{H} = -2k\left(\frac{l_0}{2} - l_0\right)(-\mathbf{i}) = -kl_0\mathbf{i}$ (1 p34) so the answer is A.

7. For no inverse det **M** = 0 so $2x + 3 = 0$
so the answer is C. (42 p40)

8. Eigenvalues are $1 + \lambda^2$ (15 p41) so the answer is C.

9. Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}4(4+1) = 10$
Gravitational potential energy $= mgh = 4(-5)g$
Total energy is the sum of these so the answer is D.
(2 and 3 p35)

10 Converting $\mathbf{i} + \mathbf{j}$ to a unit vector we get $\frac{\mathbf{i}+\mathbf{j}}{\sqrt{1+1}}$ (7 p28) so the velocity of the particle is $v\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ and the momentum is $3mv\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ (9 p58) so the answer is D.

11. The options are not accelerations and so are not valid but if we ignore the *m* the answer is C. (3 p49)

12. The function is odd and has period 2 so the answer is B. (7 p61).

13. $X''T = XT'' + XT'$. Dividing by XT gives $\frac{X''}{X} = (T'' + T')/T = \alpha$ so the answer is B. (9 p63)

14. div $\mathbf{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(x^2y^2)$ (1 p66)
$= 2xy + 2yz$ so the answer is A.

15. M of I about the centre line $= \frac{1}{2}MR^2$. (6 p74)
Using the parallel axis theorem (8 p74) the required M of I $= \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$
So the answer is C.

16. $\frac{dy}{dx} = 2x(4 + y^2)$
Separating the variables $\frac{1}{(y^2+4)}\frac{dy}{dx} = 2x$ (9 p26)
Integrating $\frac{1}{2}\arctan\left(\frac{y}{2}\right) = x^2 + C$ (p24)

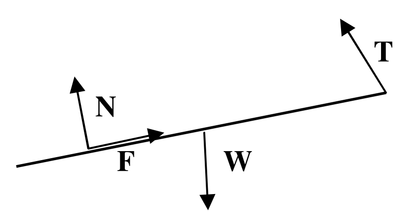
$$\frac{y}{2} = \tan(2x^2 + 2C)$$

$$y = 2\tan(2x^2 + 2C)$$

When $y = 2,\ x = 0$ so $\tan(2C) = 1$ or $2C = \frac{\pi}{4}$
The solution to the initial value problem is

$$y = 2\tan\left(2x^2 + \frac{\pi}{4}\right)$$

17.



F is the force due to the friction between the dowel and the plank
N is the normal reaction of the dowel on the plank
W is the weight of the plank
T is the tension in the cord.

18.
Length of spring $AP = x$ Extension $= x - 3$
Potential energy $= \frac{1}{2}4(x-3)^2 = 2(x-3)^2$ (2 p35)
Length of spring $PB = 3 - x$
Potential energy $= \frac{1}{2}5(3-x-1)^2 = \frac{5}{2}(2-x)^2$
Length of spring $PC = 4 - x$
Potential energy $= \frac{1}{2}3(4-x-2)^2 = \frac{3}{2}(2-x)^2$
Total potential energy $= 2(x-3)^2 + 4(2-x)^2$
Total mechanical energy
$= \frac{3}{2}\dot{x}^2 + 2(x-3)^2 + 4(2-x)^2$ (3 p35)

19. (a) $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$
(b) As the matrix is triangular the eigenvalues are 1 and -1. (8 p41) The eigenvector equations (13 p41) are given by

$$\begin{aligned}(1-\lambda)x \qquad &= 0 \quad (1)\\ x + (-1-\lambda)y &= 0 \quad (2)\end{aligned}$$

If $\lambda = 1$ these reduce to $x - 2y = 0$ so an eigenvector is $[2 \quad 1]^T$ and if $\lambda = 1$ they reduce to $x = 0$ so an eigenvector is $[0 \quad 1]^T$
(c) (11 p43)

$$\begin{bmatrix}x\\ y\end{bmatrix} = C_1\begin{bmatrix}2\\ 1\end{bmatrix}e^t + C_2\begin{bmatrix}0\\ 1\end{bmatrix}e^{-t}$$

20 a) (5 and 6 P53)
$[\mathbf{R}] = \mathrm{MLT}^{-2}$ $[\mathrm{D}] = \mathrm{L}$ $[\mathbf{v}] = \mathrm{LT}^{-1}$
So in dimensions the equation becomes

$$\mathrm{ML}T^{-2} = [c_1]\mathrm{L}^2\mathrm{T}^{-1}$$

So $[c_1] = \mathrm{ML}^{-1}\mathrm{T}^{-1}$
b) (7 p53) c_1 has SI units kg $\mathrm{m}^{-1}\mathrm{s}^{-1}$

21 (4 p54)
Length of damper $AP = x - y$
Force in damper $AP = -2(\dot{x} - \dot{y})\mathbf{i}$
Length of damper $CP = 3 - x$
Force in damper $CP = -5(-\dot{x})(-\mathbf{i}) = -5\dot{x}\mathbf{i}$

22 a) (15 p65)

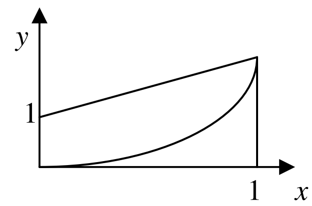
$$\mathbf{grad}\, f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

$$= z^2\sin^2\theta\,\mathbf{e}_\rho + 2z^2\sin\theta\cos\theta\,\mathbf{e}_\theta + 2\rho z\sin^2\theta\,\mathbf{e}_z$$

$$= z^2\sin^2\theta\,\mathbf{e}_\rho + z^2\sin(2\theta)\,\mathbf{e}_\theta + 2\rho z\sin^2\theta\,\mathbf{e}_z$$

b) (4 p66)
div $\mathbf{grad}\, f = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho z^2\sin^2\theta) + \frac{1}{\rho}\frac{\partial}{\partial\theta}(z^2\sin(2\theta)) + \frac{\partial}{\partial z}(2\rho z\sin^2\theta)$

$$= \frac{z^2\sin^2\theta}{\rho} + \frac{2z^2}{\rho}\cos(2\theta) + 2\rho\sin^2\theta$$

23 a)


Curves meet when $2x^2 = x + 1$ so they meet at $x = 1$ and $y = 2$.
b) Area integral $= \int_0^1\int_{y=2x^2}^{y=x+1} 8xy\,dy\,dx$ (2 p69)

$$= \int_0^1 [4xy^2]_{2x^2}^{x+1}dx$$

$$= 4\int_0^1 x(x+1)^2 - 4x^5\,dx$$

$$= 4\int_0^1 x^3 + 2x^2 + x - 4x^5\,dx$$

$$= 4\left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} - \frac{2x^6}{3}\right]_0^1 = 4\left[\frac{1}{4} + \frac{2}{3} + \frac{1}{2} - \frac{2}{3}\right] = 3$$

24 a) (6 and 5 p27)
The auxiliary equations is $\lambda^2 + 4\lambda + 4 = 0$
$(\lambda + 2)^2 = 0$ so $\lambda = -2$ twice.
The complementary function is

$$y_c = (C + Dx)e^{-2x}$$

As xe^{-2x} appears in the complementary function we need to use a trial solution of x^2e^{-2x}. (7 p27).
Combining both terms of the inhomogeneous part of the equations we can use a trial solution of the form

$$y = ax^2e^{-2x} + bx + c$$

$$\frac{dy}{dx} = 2axe^{-2x} - 2ax^2e^{-2x} + b$$

$$\frac{d^2y}{dx^2} = 2ae^{-2x} - 8axe^{-2x} + 4ax^2e^{-2x}$$

Substituting into the original equation and collecting up the terms

$$2ae^{-2x} + 4bx + 4(b + c) = 10e^{-2x} + 4x$$

Equating coefficients

$$2a = 10 \implies a = 5 \qquad 4b = 4 \implies b = 1$$

$$4(b + c) = 0 \implies c = -b = -1$$

The general solution is

$$y = (C + Dx)e^{-2x} + 5x^2e^{-2x} + x - 1$$

24 b) $y' = De^{-2x} - 2(C + Dx)e^{-2x} + 10xe^{-2x} - 10x^2e^{-2x} + 1$

$$y(0) = 0 = C - 1 \implies C = 1$$

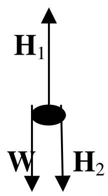
$$y'(0) = 0 = D - 2C + 1 \implies D = 2C - 1 = 1$$

The particular solution is

$$y = (1 + x)e^{-2x} + 5x^2e^{-2x} + x - 1$$

c) As $x \to \infty$ $e^{-2x} \to 0$ and $y \to x - 1$.

25 a)



$\mathbf{H}_1$ is the force in the spring AP

$\mathbf{H}_2$ is the force in the spring PB

$\mathbf{W}$ is the weight of the particle

b) Length of $AP = x$

so $\mathbf{H}_1 = -18(x-1)\mathbf{j}$ (1 p34)

Length of $BP = 1.4 - x$

$\mathbf{H}_2 = -30(1.4 - x - 2)(-\mathbf{j}) = -30(x + 0.6)\mathbf{j}$

$\mathbf{W} = 3g\mathbf{j}$

c) In equilibrium $\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W} = \mathbf{0}$

Resolving in the $\mathbf{j}$-direction

$$-18(x-1) - 30(x + 0.6) + 3g = 0$$

or $-48x + 3g = 0$ giving $x = \frac{g}{16}$.

d) Using Newton's second law

$3\ddot{x}\mathbf{j} = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W}$

Resolving in the $\mathbf{j}$-direction

$3\ddot{x} = -48x + 3g$ or $\ddot{x} + 16x = g$

e) Comparing with the standard simple harmonic equation (4 p35) $\omega^2 = 16$

so $x = \frac{g}{16} + B\cos(4t) + C\sin(4t)$

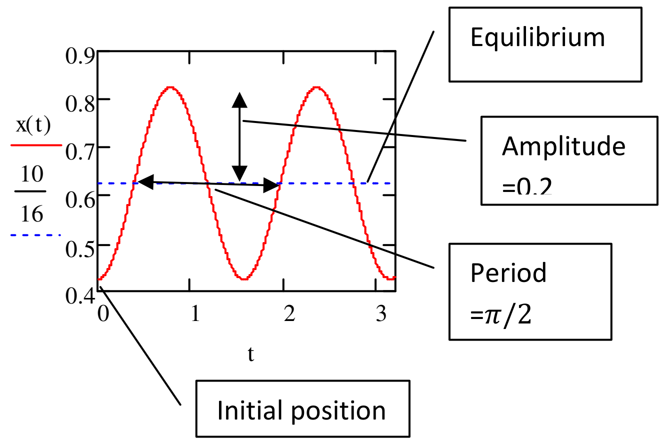
$$\dot{x} = -4B\sin(4t) + 4C\cos(4t)$$

$$\dot{x} = 0 \text{ when } t = 0 \text{ so } C = 0$$

$$x = \frac{g}{16} - 0.2 \text{ when } t = 0 \text{ so } \frac{g}{16} - 0.2 = \frac{g}{16} + B$$

so $B = -0.2$ and $x = \frac{g}{16} - 0.2\cos(4t)$

f) Using 10 for g



g) None as ω^2 depends on the sum of the stiffnesses of the springs and the mass.

26 a)

$$f = x^2y^2 - x^2 - 4y^2$$

$$f_x = 2xy^2 - 2x \qquad f_y = 2yx^2 - 8y$$

$$f_{xx} = 2y^2 - 2 \quad f_{xy} = 4yx \quad f_{yy} = 2x^2 - 8$$

b) At the stationary points (15 p45) $f_x = f_y = 0$

i.e. $2x(y^2 - 1) = 0$ (1) $2y(x^2 - 4) = 0$ (2)

Solving (1) $x = 0$ or $y = \pm 1$

Solving (2) $y = 0$ or $x = \pm 2$

So the stationary points are

$$(0,0) \ (2,1) \ (2,-1) \ (-2,1) \ (-2,-1)$$

c) (18 p46)

point	$A = 2y^2 - 2$	$B = 4xy$	$C = 2x^2 - 8$	$AC - B^2$	classification
(0,0)	-2	0	-8	16	maximum
(2,1)	0	8	0	-8^2	saddle
(2,-1)	0	-8	0	-8^2	saddle
(-2,1)	0	-8	0	-8^2	saddle
(-2,-1)	0	-8	0	-8^2	saddle

d) (14 p45) $f(2,1) = -4$

$$f_x(2,1) = f_y(2,1) = f_{xx}(2,1) = f_{yy}(2,1) = 0$$

$$f_{xy} = 8$$

So $p_2(2,1) = -4 + 8(x-2)(y-1)$

27 a) For equilibrium (6 p47)

$$x\left(1-\frac{x}{3}-\frac{y}{3}\right)=0 \ (1) \quad -y\left(1-\frac{x}{2}-\frac{y}{4}\right)=0 \ (2)$$

Solving (1) $x=0$ or $x+y=3$

Solving (2) $y=0$ or $2x+y=4$

When $x=0$ $y=0$ or $y=4$

When $y=0$ $x=0$ or $x=3$

If $x+y=3$ and $2x+y=4$ then subtracting gives

$$x=1 \text{ so } y=2$$

The equilibrium points are (0,0) (0,4) (3,0) (1,2)

b) (9 and 10 p47)

$$u=x-\frac{x^2}{3}-\frac{xy}{3} \quad v=-y+\frac{xy}{2}+\frac{y^2}{4}$$

$$J(x,y)=\begin{bmatrix}1-\frac{2x}{3}-\frac{y}{3} & -\frac{x}{3}\\ \frac{y}{2} & -1+\frac{x}{2}+\frac{y}{2}\end{bmatrix}$$

$$J(0,0)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

It is a diagonal matrix so the eigenvalues are 1 and -1.

They are real and of different signs so (0,0) is a saddle.

$$J(0,4)=\begin{bmatrix}-\frac{1}{3} & 0\\ 2 & 1\end{bmatrix}$$

It is a triangular matrix so the eigenvalues are $-\frac{1}{3}$ and 1.

They are real and of different signs so (0,4) is a saddle.

$$J(3,0)=\begin{bmatrix}-1 & -1\\ 0 & \frac{1}{2}\end{bmatrix}$$

The eigenvalues are -1 and $\frac{1}{2}$.

They are real and of different signs so (3,0) is a saddle.

$$J(1,2)=\begin{bmatrix}-\frac{1}{3} & -\frac{1}{3}\\ 1 & \frac{1}{2}\end{bmatrix}$$

The characteristic equation is $\left(-\frac{1}{3}-\lambda\right)\left(\frac{1}{2}-\lambda\right)+\frac{1}{3}=0$

Or $\lambda^2-\frac{\lambda}{6}+\frac{1}{6}=0$ or $6\lambda^2-\lambda+1=0$

Using the formula, $\lambda=\frac{1\pm\sqrt{-23}}{12}$ so eigenvalues are complex with a positive real component so the equilibrium point is a spiral source.

28.a) i) (1 p41)

$$\begin{bmatrix}-4 & 2 & 2\\ 2 & -5 & 0\\ 2 & 0 & -3\end{bmatrix}\begin{bmatrix}2\\1\\2\end{bmatrix}=\begin{bmatrix}-2\\-1\\-2\end{bmatrix}=-1\begin{bmatrix}2\\1\\2\end{bmatrix}$$

$\lambda=-1$ is the eigenvalue.

ii) (11 and 12 p41)

Trace $=-12$. Eigenvalue sum $=-7-1+\lambda=-12$ so $\lambda=-4$.

The eigenvector equations (3 p41) are

$$(-4-\lambda)x+2y+2z=0$$
$$2x-(5+\lambda)y=0$$
$$2x-(3+\lambda)z=0$$

If $\lambda=-4$ they become

$2y+2z=0, 2x-y=0, 2x+z=0$ so an eigenvector is $[1 \quad 2 \quad -2]^T$.

b) (9 p57)

$$\mathbf{x}=C_1\begin{bmatrix}-2\\2\\1\end{bmatrix}\cos(\sqrt{7}t+\phi_1)+C_2\begin{bmatrix}2\\1\\2\end{bmatrix}\cos(t+\phi_2)+ C_3\begin{bmatrix}1\\2\\-2\end{bmatrix}\cos(2t+\phi_3)$$

c) Starting from rest

$$\mathbf{x}=C_1\begin{bmatrix}-2\\2\\1\end{bmatrix}\cos(\sqrt{7}t)+C_2\begin{bmatrix}2\\1\\2\end{bmatrix}\cos(t) + C_3\begin{bmatrix}1\\2\\-2\end{bmatrix}\cos(2t)$$

The lowest frequency is 1 with eigenvector $[2 \quad 1 \quad 2]^T$

So if $x_2=0.2$ then $x_1=0.4$ and $x_3=0.4$

d) $\mathbf{x(0)}=\begin{bmatrix}0\\1\\1\end{bmatrix}=C_1\begin{bmatrix}-2\\2\\1\end{bmatrix}+C_2\begin{bmatrix}2\\1\\2\end{bmatrix}+C_3\begin{bmatrix}1\\2\\-2\end{bmatrix}$

so

$$-2C_1+2C_2+C_3=0$$
$$2C_1+C_2+2C_3=1$$
$$C_1+2C_2-2C_3=1$$

Using Gaussian elimination (or any other method)

$$\left(\begin{array}{ccc|c}-2 & 2 & 1 & 0\\ 2 & 1 & 2 & 1\\ 1 & 2 & -2 & 1\end{array}\right)\begin{array}{l}R_1\\R_2\\R_3\end{array}$$

$$\left(\begin{array}{ccc|c}-2 & 2 & 1 & 0\\ 0 & 3 & 3 & 1\\ 0 & 3 & -3/2 & 1\end{array}\right)\begin{array}{l}R_1\\R_{2a}=R_2+R_1\\R_{3a}=R_3+\frac{1}{2}R_1\end{array}$$

$$\left(\begin{array}{ccc|c}-2 & 2 & 1 & 0\\ 0 & 3 & 3 & 1\\ 0 & 0 & -9/2 & 0\end{array}\right)\begin{array}{l}R_1\\R_{2a}=R_2+R_1\\R_{3b}=R_{3a}-R_{2a}\end{array}$$

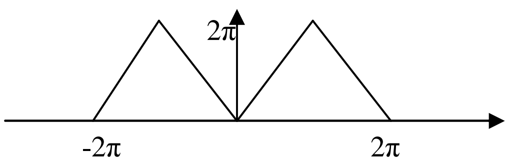
So $C_3=0$ $C_2=\frac{1}{3}$ $C_1=\frac{1}{3}$

28 d) cont

$$x_1 = -\frac{2}{3}\cos\sqrt{7}t + \frac{2}{3}\cos t$$
$$x_2 = \frac{2}{3}\cos\sqrt{7}t + \frac{1}{3}\cos t$$
$$x_3 = \frac{1}{3}\cos\sqrt{7}t + \frac{2}{3}\cos t$$

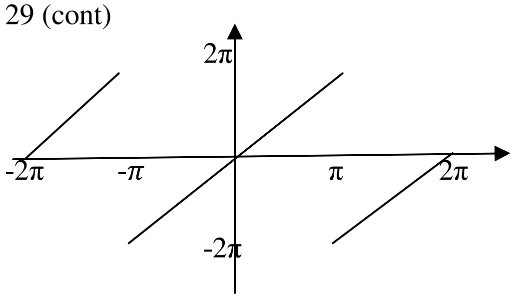
29 a) (11 p61)

$$f_{even} = \begin{cases} -2x & -\pi < x \le 0 \\ 2x & 0 \le x < \pi \end{cases} = 2|x| \quad -\pi < x < \pi$$
$$f_{even}(x + 2\pi) = f_{even}(x)$$



$$f_{odd} = 2x \quad -\pi < x < \pi$$
$$f_{even}(x + 2\pi) = f_{even}(x)$$

29 (cont)



b) (12 p61) For odd extension (as $L = \pi$)

$$B_r = \frac{2}{\pi}\int_0^{\pi} 2x\sin(rx)\,dx$$

Integrating by parts

$$B_r = \frac{4}{\pi}\left\{\left[-\frac{x}{r}\cos(rx)\right]_0^{\pi} + \int_0^{\pi}\frac{1}{\mathrm{r}}\cos(rx)\,dx\right\}$$
$$= \frac{4}{\pi}\left\{-\frac{\pi}{r}\cos(r\pi) + \left[\frac{1}{\mathrm{r}}\sin(rx)\right]_0^{\pi}\right\} = -\frac{4}{r}\cos(r\pi)$$
$$= -\frac{4}{r}(-1)^r$$
$$F_{odd} = \sum_{r=1}^{\infty} B_r\sin(rx)$$
$$= 4\sin(x) - 2\sin(2x) + \frac{4}{3}\sin(3x) + \cdots$$

For even extension

$$A_0 = \frac{1}{\pi}\int_0^{\pi} 2x\,dx = \frac{1}{\pi}[x^2]_0^{\pi} = \pi$$

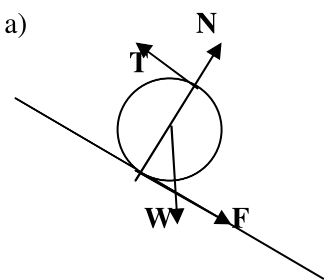
$$A_r = \frac{2}{\pi}\int_0^{\pi} 2x\cos(rx)\,dx$$

Integrating by parts

$$A_r = \frac{4}{\pi}\left\{\left[\frac{x}{r}\sin(rx)\right]_0^{\pi} - \int_0^{\pi}\frac{1}{r}\sin(rx)dx\right\}$$
$$= \frac{4}{\pi}\left[\frac{1}{\mathrm{r}^2}\cos(rx)\right]_0^{\pi} = \frac{4}{\pi r^2}(\cos(r\pi) - 1)$$
$$= \frac{4}{\pi r^2}((-1)^r - 1)$$
$$A_1 = -\frac{8}{\pi} \qquad A_2 = 0 \quad A_3 = -\frac{8}{9\pi}$$
$$F_{even} = \pi - \frac{8}{\pi}\cos(x) - \frac{8}{9\pi}\cos(3x)$$

c) The even approximation will give a better approximation as it has no discontinuities.

30 a)



b) $\mathbf{T} = |\mathbf{T}|\mathbf{i}$ $\quad \mathbf{F} = -|\mathbf{F}|\mathbf{i}$ $\quad \mathbf{N} = -|\mathbf{N}|\mathbf{j}$

$\mathbf{W} = Mg(-\sin\alpha\,\mathbf{i} + \cos\alpha\mathbf{j})$

c) Let x be the distance moved by the cylinder along the incline from the initial position then by the rolling condition $x = R\theta$ (18 p75)

d) For **W** and **N** the position vector is zero as they act through the centre of the cylinder.

$$\mathbf{r}_T = -R\mathbf{j} \quad \mathbf{r}_F = R\mathbf{j}$$

so $\mathbf{\Gamma}_W = \mathbf{\Gamma}_N = \mathbf{0}$

$\mathbf{\Gamma}_T = -R\mathbf{j} \times |\mathbf{T}|\mathbf{i} = R|\mathbf{T}|\mathbf{k}$

$\mathbf{\Gamma}_F = R\mathbf{j} \times -|\mathbf{F}|\mathbf{i} = R|\mathbf{F}|\mathbf{k}$

e) (13 p74) The torque law gives

$$I\ddot{\theta}\mathbf{k} = \sum \mathbf{\Gamma} = R(|\mathbf{T}| + |\mathbf{F}|)\mathbf{k}$$

Resolving in **k-**direction

$$I\ddot{\theta} = R(|\mathbf{T}| + |\mathbf{F}|) \quad (1)$$

f) Newton's 2nd law gives

$$M\ddot{x}\mathbf{i} = \mathbf{F} + \mathbf{N} + \mathbf{T} + \mathbf{W}$$

Resolving in the **i –**direction

$M\ddot{x} = -|\mathbf{F}| + |\mathbf{T}| - Mg\sin\alpha$ (2)

g) (6 p74) $I = \frac{1}{2}MR^2$

Substituting in (1) and rearranging

$$R\ddot{\theta} = \frac{2}{M}(|\mathbf{T}| + |\mathbf{F}|)$$

But $R\ddot{\theta} = \ddot{x}$ using (c) so $|\mathbf{F}| = \frac{M\ddot{x}}{2} - |\mathbf{T}|$

Substituting in (2)

$M\ddot{x} = -\frac{M\ddot{x}}{2} + 2|\mathbf{T}| - Mg\sin\alpha$

$\frac{3}{2}M\ddot{x} = 2|\mathbf{T}| - Mg\sin\alpha$

$\ddot{x} = \frac{4}{3M}|\mathbf{T}| - \frac{2g}{3}\sin\alpha$