Handbook references are given in brackets

1. Combining using the rules of logs (P11)

$$\ln\left(y^2(3x+5)\right) = \ln\left(4(x-2)(x+2)\right)$$

Removing logs $y^2(3x+5) = 4(x^2-4)$
So the answer is B.

2. The answer is B

3. $(\mathbf{i}+\mathbf{j}) \times (\mathbf{j}-2\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{vmatrix}$
(P40 or use Sarrus's Rule P29)
The **i**-component is -2 so the answer is D.

4. The angle with the **j**-direction is $\frac{\pi}{2} + \theta$ and $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ (P31) so the answer is D.

5. The gravitational energy is $-mgx$ and the potential energy in the spring is $\frac{1}{2}k(x-l_0)^2$ (P35). The total potential energy is the sum of these so the answer is A.

6. $\begin{bmatrix} 3 & 0 & 6 \\ 2 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so the answer is C.

7. The answer is C.

8. $u = rx^2 + 2sxy$ and $v = -qx + py^2$

$J(x,y) = \begin{bmatrix} 2rx + 2sy & 2sx \\ -q & 2py \end{bmatrix}$ (P47)
Replacing *x* by *a* and *y* by *b* gives option C.

9. $[F] = \mathrm{MLT}^{-2}$ $[\rho] = \mathrm{ML}^{-3}$
$[v] = \mathrm{LT}^{-1}$ $[A] = \mathrm{L}^2$ (P53)

$$[F] = [\rho][v]^2[A][C]$$

So $\mathrm{MLT}^{-2} = \mathrm{ML}^{-3}\mathrm{L}^2\mathrm{T}^{-2}\mathrm{L}^2[C]$
Giving $[C] = 1$ so the answer is A.

10. The forces acting on the mass are the damping force **R** = $= -r\dot{x}\mathbf{i}$ (P54) and the spring force which is **H** $= -k(x-l_0)\mathbf{i}$ (P 34).
The equation of motion is $m\ddot{x}\mathbf{i} = \mathbf{R} + \mathbf{H}$
Resolving in the **i**-direction gives

$$m\ddot{x} = -r\dot{x} - k(x-l_0)$$

So the answer is A.

11. $\mathbf{r}_G = (4m\mathbf{0} + m\mathbf{i} + 2m(\mathbf{i}+\mathbf{j}) + 3m\mathbf{j})/10\mathrm{m}$ (P58) so the answer is D.

12. The angle between **F** and **W** is θ and **F** is in the opposite direction to $\mathbf{e}_\theta$ so

$\mathbf{W} = -mg\cos\theta\mathbf{e}_\theta - mg\sin\theta\mathbf{e}_r$ and
$\mathbf{F} = -|\mathbf{F}|\mathrm{e}_\theta$.
The equation of motion is

$$m\ddot{\mathbf{r}} = \mathbf{N} + \mathbf{W} + \mathbf{F}$$

where $\ddot{\mathbf{r}} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta$ (P59)
Resolving the equation of motion in the $\mathbf{e}_\theta$ -direction we get option A.

13. The graph is even so the answer is A.

14 . **grad** $h = \frac{\partial h}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial h}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial h}{\partial \phi}\boldsymbol{e}_\phi$ (P66) so the answer is C.

15. In spherical polars $I = \int_B \kappa(r\sin\theta)^2 dV$ (P71) and $dV = r^2\sin\theta\, d\phi d\theta dr$ (P70) so the answer is D.

16. Divide by x^2 to give $\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2} + 1$
The integrating factor is $p(x) = \exp\int\frac{1}{x}dx = x$
Multiplying by $p(x)$ gives

$$x\frac{dy}{dx} + y = \frac{d}{dx}(xy) = \frac{1}{x} + x$$

Integrating gives

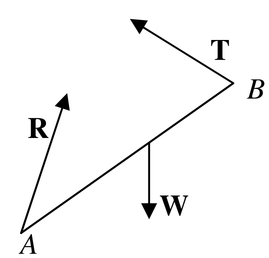
$$xy = \ln x + \frac{x^2}{2} + C$$

Dividing by *x*

$$y = \frac{1}{x}\ln x + \frac{x}{2} + \frac{C}{x}$$

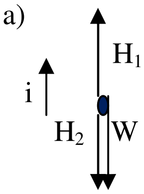
$y = 1$ when $x = 1$ gives $1 = \frac{1}{2} + C$ or $C = \frac{1}{2}$
The solution is $y = \frac{1}{x}\left(\ln x + \frac{1}{2}\right) + \frac{x}{2}$.

17.



T = tension in rope
W = weight of rod
R = reaction at the hinge

18. a)



b)

$\mathbf{H}_1$ = force in upper spring

$= -k(3l_0 - x - l_0)(-\mathbf{i}) = k(2l_0 - x)\mathbf{i}$ (P34)

$\mathbf{H}_2$ = force in lower spring $= -k(x - l_0)\mathbf{i}$

$\mathbf{W}$ = weight of mass $= -mg\mathbf{i}$

c) Using Newton's second law

$m\ddot{x}\mathbf{i} = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{W}$

Resolve in **i**-direction

$$m\ddot{x} = k(2l_0 - x) - k(x - l_0) - mg$$

Giving $m\ddot{x} + 2kx = 3kl_0 - mg$

19. The augmented matrix (P36) is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 2 & 1 & -1 & 8 \\ 3 & 1 & -2 & 11 \end{array}\right)\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Using Gaussian elimination (P37) we get

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right)\begin{array}{r} R_1 \\ R_{2a} = R_2 - 2R_1 \\ R_{3a} = R_3 - 3R_1 \end{array}$$

Then

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{r} R_1 \\ R_{2a} \\ R_{3b} = R_{3a} - R_{2a} \end{array}$$

The system reduces to

$$x_1 - x_3 = 3$$
$$x_2 + x_3 = 2$$

Let $x_3 = k$ then $x_1 = 3 + k$ and $x_2 = 2 - k$

The general solution is

$x_1 = 3 + k,\ \ x_2 = 2 - k,\ \ \ x_3 = k$

20. a) Velocity

$\dot{\mathbf{r}} = -4\pi\sin(2\pi t)\mathbf{i} - 6\pi\cos(2\pi t)\mathbf{j} - 2t\mathbf{k}$

b) Acceleration

$\ddot{\mathbf{r}} = -8\pi^2\cos(2\pi t)\mathbf{i} - 12\pi^2\sin(2\pi t)\mathbf{j} - 2\mathbf{k}$

c) By Newton's second law the total force acting on the car is $m\ddot{\mathbf{r}}$. The vertical component is in the **k**-direction so is $-2m$.

21. a) Momentum of $2m$ before the collision is $2mu\mathbf{i}$ and momentum after is $2mv\mathbf{i}$. (P58)

b) By conservation of linear momentum (P58)

$$2mu\mathbf{i} + \mathbf{0} = 2mv\mathbf{i} + mw\mathbf{i}$$

Resolving in the **i**-direction and cancelling m

$$2u = 2v + w \quad (1)$$

By Newton's law of restitution (P58)

$$v - w = -eu \ (2)$$

Adding gives $3v = (2-e)u$ so $v = \frac{(2-e)u}{3}$

From (2) $w = v + eu$ and substituting for v gives

$w = \frac{(2-e)u}{3} + eu = \frac{2}{3}(1+e)u.$

c) Energy before = $\frac{1}{2}2mu^2 = mu^2$

Energy after = $\frac{1}{2}\frac{2m(2-e)^2u^2}{9} + \frac{1}{2}m\frac{4}{9}(1+e)^2u^2$

Energy lost = $mu^2(1 - \frac{(2-e)^2}{9} - \frac{2}{9}(1+e)^2)$

$$= \frac{mu^2}{9}(9 - (4 - 4e + e^2) - 2(1 + 2e + e^2))$$

$$= \frac{mu^2}{9}(3 - 3e^2) = \frac{mu^2}{3}(1 - e^2)$$

22. a) div $\mathbf{V} = \frac{\partial V_\rho}{\partial \rho} + \frac{1}{\rho}V_\rho + \frac{1}{\rho}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$ (P66)

$$= \frac{1}{\rho}(-\rho^2\sin\theta) = -\rho\sin\theta$$

b) **curl** $\mathbf{V} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\theta & V_z \end{vmatrix}$ (P67)

$$= \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & \rho^3\cos\theta & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}(3\rho^2\cos\theta)\mathbf{e}_z = 3\rho\cos\theta\,\mathbf{e}_z$$

23.a) As the graph of Y_N against h^2 is approximately a straight line the likely order is 2. (P73)

b) Using the formula from P73 of the Handbook with $p = 2$

$$Y_{N_1} - Y_{N_2} \cong C(h_1^2 - h_2^2)$$

which gives

$1 - 0.99985 \cong C(0.02^2 - 0.01^2) = 0.0003C$

Or $C \cong \frac{0.00015}{0.0003} = 0.5$

For 6 dp accuracy $|Ch^2| \leq 0.5 \times 10^{-6}$ (P73) so

$$0.5h^2 \leq 0.5 \times 10^{-6} \quad \text{or} \quad h^2 \leq 10^{-6}$$

$$\text{or } h \leq 10^{-3} (= 0.001)$$

24.a) The auxiliary equation (p26) is

$$\lambda^2 + 2\lambda + 10 = 0$$

Using the formula (P10)

$$\lambda = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$$

The complementary function (P26) is

$$y_c = e^{-t}(C\cos 3t + D\sin 3t)$$

To find a particular integral we try

$$y = p\cos 3t + q\sin 3t$$

$$\frac{dy}{dx} = -3p\sin 3t + 3q\cos 3t$$

$$\frac{d^2y}{dx^2} = -9p\cos 3t - 9q\sin 3t$$

Substitute into the equation

$$\begin{aligned}-9p\cos 3t - 9q\sin 3t - 6p\sin 3t + 6q\cos 3t \\ + 10p\cos 3t + 10q\sin 3t \\ = 29\cos 3t - 26\sin 3t\end{aligned}$$

Collecting terms

$$\begin{aligned}(p+6q)\cos 3t + (q-6p)\sin 3t \\ = 29\cos 3t - 26\sin 3t\end{aligned}$$

Equating coefficients

$p + 6q = 29 \quad (1) \qquad q - 6p = -26 \quad (2)$

$(1)\times 6 \qquad 6p + 36q = 174 \quad (3)$

Adding (2) and (3) gives $37q = 148 \implies q = 4$

Substituting in (1) gives $p + 24 = 29 \implies p = 5$

A particular integral is $y_p = 5\cos 3t + 4\sin 3t$

The general solutions is

$$y = e^{-t}(C\cos 3t + D\sin 3t) + 5\cos 3t + 4\sin 3t$$

b) Differentiating gives

$$\begin{aligned}\dot{y} = -e^{-t}(C\cos 3t + D\sin 3t) + e^{-t}(-3C\sin 3t \\ + 3D\cos 3t) - 15\sin 3t + 12\cos 3t\end{aligned}$$

Using the initial conditions

$\dot{y}(0) = 3D - C + 12 = 4$ or $3D - C = -8$ (4)

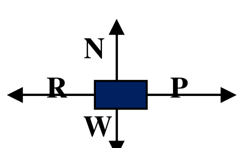
$y(0) = C + 5 = 7$ so $C = 2$

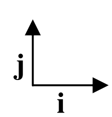
Substitute into (4) gives $D = -2$

The particular solutions is

$$y = e^{-t}(2\cos 3t - 2\sin 3t) + 5\cos 3t + 4\sin 3t$$

25. a)





P = engine force = $P\mathbf{i}$

R = resistance force = $-R(v)\mathbf{i}$

N = normal reaction = $|\mathbf{N}|\mathbf{j}$

W = weight = $-mg\mathbf{j}$

b) By Newton's 2nd law of motion

$$m\mathbf{a} = \mathbf{P} + \mathbf{R} + \mathbf{N} + \mathbf{W}$$

Resolving in the **i**-direction as motion is only in this direction.

$$ma = P - R(v) \quad (1)$$

If $v < v_0 \quad ma = P - \frac{kv}{v_0}$ but $a = v\frac{dv}{dx}$ so

$$mv\frac{dv}{dx} = \frac{Pv_0 - kv}{v_0}$$

Giving $\frac{mv_0 v}{Pv_0 + kv}\frac{dv}{dx} = 1$ as required

c) Integrating wrt x

$$mv_0\int\left(\frac{v}{Pv_0 - kv}\right)dv = x + C$$

Using the hint with $a = Pv_0$ and $b = -k$

$$mv_0\left(-\frac{v}{k} - \frac{Pv_0}{k^2}\ln(Pv_0 - kv)\right) = x + C$$

When $x = 0$ $v = 0$ so $C = -\frac{mv_0^2 P}{k^2}\ln(Pv_0)$

and $v(x)$ is given implicitly by

$$x = mv_0\left(-\frac{v}{k} - \frac{Pv_0}{k^2}\ln(Pv_0 - kv)\right) + \frac{mv_0^2 P}{k^2}\ln(Pv_0)$$

$$= \frac{mv_0^2 P}{k^2}\ln\left(\frac{Pv_0}{Pv_0 - kv}\right) - \frac{mv_0 v}{k}$$

As $v = v_0$ when $x = x_1$

$$x_1 = \frac{mv_0^2 P}{k^2}\ln\left(\frac{Pv_0}{Pv_0 - kv_0}\right) - \frac{mv_0^2}{k}$$

and the required result follows.

d) If $v > v_0$ the equation of motion becomes

$$ma = P - k$$

We can integrate this by using $a = v\frac{dv}{dx}$ or using the constant acceleration formula

$$v^2 = v_0^2 + 2a_0(x - x_0)$$

from page 33 of the Handbook where $x_0 = x_1$ and $a_0 = \frac{P-k}{m}$ in our case,

so $x = x_1 + \frac{m(v^2 - v_0^2)}{2(P-k)}$ as required.

e) $x = x_2$ when $v = v_1$ so $x_2 = x_1 + \frac{m({v_1}^2 - v_0^2)}{2(P-k)}$

Substituting for x_1

$$x_2 = \frac{mv_0^2 P}{k^2}\ln\left(\frac{P}{P-k}\right) - \frac{mv_0^2}{k} + \frac{m(v_1^2 - v_0^2)}{2(P-k)}$$

26 a) $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}4 & 2\\ 4 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-6t+5\\ -t+5\end{bmatrix}$

b) The matrix of coefficients is $\begin{bmatrix}4 & 2\\ 4 & -3\end{bmatrix}$

It's characteristic equation (P41) is

$$\begin{vmatrix}4-\lambda & 2\\ 4 & -3-\lambda\end{vmatrix} = 0$$

or $\lambda^2 - \lambda - 20 = 0$ which factorizes to $(\lambda+4)(\lambda-5) = 0$ giving $\lambda = -4$ and $\lambda = 5$

The eigenvector equations are

$$(4-\lambda)x + 2y = 0$$
$$4x - (3+\lambda)y = 0$$

If $\lambda = -4$ they reduce to $4x + y = 0$ or $y = -4x$

Taking $x = 1$ a typical eigenvector is $[1 \quad -4]^T$

If $\lambda = 5$ they reduce to $-x + 2y = 0$ or $x = 2y$

Taking $y = 1$ a typical eigenvector is $[2 \quad 1]^T$

The complementary function (P43) is

$$\mathbf{x_c} = A\begin{bmatrix}1\\ -4\end{bmatrix}e^{-4t} + B\begin{bmatrix}2\\ 1\end{bmatrix}e^{5t}$$

c) Try $x_p = a + bt$ and $y_p = c + dt$ (P43)

Substituting into the equations gives

$$b = 4a + 4bt + 2c + 2dt - 6t + 5$$
$$d = 4a + 4bt - 3c - 3dt - t + 5$$

Collecting terms

$$0 = (4b + 2d - 6)t + 4a + 2c - b + 5$$
$$0 = (4b - 3d - 1)t + 4a - 3c - d + 5$$

Equating coefficients

$$4b + 2d - 6 = 0 \ (1) \quad 4a + 2c - b + 5 = 0 \ (2)$$
$$4b - 3d - 1 = 0 \ (3) \quad 4a - 3c - d + 5 = 0 \ (4)$$

Subtracting (3) from (1)

$$5d - 5 = 0 \ \text{ so } \ d = 1$$

Substituting into (3) $4b = 4$ so $b = 1$

Substitute b and d into (2) and (4)

$$4a + 2c = -4 \ (5) \quad 4a - 3c = -4 \ (6)$$

Subtracting (6) from (5) gives $c = 0$

Substituting into (5) gives $a = -1$

A particular integral is $x_p = t - 1 \quad y_p = t$

The general solution is

$$\mathbf{x} = A\begin{bmatrix}1\\ -4\end{bmatrix}e^{-4t} + B\begin{bmatrix}2\\ 1\end{bmatrix}e^{5t} + \begin{bmatrix}t-1\\ t\end{bmatrix}$$

d) $x(0) = 0$ and $y(0) = -13$ so

$$0 = A + 2B - 1 \ (7) \text{ and } -13 = -4A + B \ (8)$$

Substituting (8) into (7) gives

$$A + 8A - 26 - 1 = 0 \ \text{ or } \ 9A = 27 \text{ or } \ A = 3$$

Substituting into (8) gives $B = -1$

The particular solution is

$$\mathbf{x} = 3\begin{bmatrix}1\\ -4\end{bmatrix}e^{-4t} - \begin{bmatrix}2\\ 1\end{bmatrix}e^{5t} + \begin{bmatrix}t-1\\ t\end{bmatrix}$$

e) In the long term the $-\begin{bmatrix}2\\ 1\end{bmatrix}e^{5t}$ term dominates so we have an increasing exponential with x twice as negative as y.

27 a) Let the temperatures at the surfaces be θ_1 at the inner surface of the pipe (radius r), θ_2 at the outer surface of the pipe (radius R) and θ_3 at the outer surface of the lagging (radius $R + b$)

For the inside convection (using hint)

$$q = 2\pi Lrh_{\text{in}}(\theta_{\text{in}} - \theta_1) \ \text{ or } \ \theta_{\text{in}} - \theta_1 = \frac{q}{2\pi Lrh_{\text{in}}} \ (1)$$

For conduction through the pipe using Fourier's law from the hint and considering r as a variable and with $A = 2\pi rL$

$$\frac{d\theta}{dr} = -\frac{q}{\kappa 2\pi Lr} = -\frac{q}{2\pi L\kappa}\frac{1}{r}$$

Integration gives $\theta = -\frac{q}{2\pi L\kappa}\ln r + C$

If $\theta = \theta_a$ when $r = r_a$ so $C = \theta_a + \frac{q}{2\pi L\kappa}\ln r_a$

If $\theta = \theta_b$ when $r = r_b$ this gives

$$\theta_b = -\frac{q}{2\pi L\kappa}\ln r_b + \theta_a + \frac{q}{2\pi L\kappa}\ln r_a$$

which can be rearranged to give

$$\theta_a - \theta_b = \frac{q}{2\pi L\kappa}\ln\left(\frac{r_b}{r_a}\right)$$

For the pipe $r_a = r$ with $\theta_a = \theta_1$ and $r_b = R$ with $\theta_b = \theta_2$ so

$$\theta_1 - \theta_2 = \frac{q}{2\pi L\kappa}\ln\left(\frac{R}{r}\right) \ (2)$$

Similarly for conduction through the lagging , we can replace κ by κ_{lag}, θ_a by θ_2, r_a by R, θ_b by θ_3, r_b by $R + b$ to give

$$\theta_2 - \theta_3 = \frac{q}{2\pi L\kappa_{\text{lag}}}\ln\left(\frac{R+b}{R}\right) \ (3)$$

For convection on the outside

$$q = 2\pi L(R+b)h_{\text{out}}(\theta_3 - \theta_{\text{out}})$$
$$\text{or} \ \ \theta_3 - \theta_{\text{out}} = \frac{q}{2\pi L(R+b)h_{\text{out}}} \ (4)$$

Adding (1), (2), (3) and (4)

$\theta_{\text{in}} - \theta_{\text{out}} = \frac{q}{2\pi L}\left\{\frac{1}{rh_{\text{in}}} + \frac{1}{\kappa}\ln\left(\frac{R}{r}\right) + \frac{1}{\kappa_{\text{lag}}}\ln\left(\frac{R+b}{R}\right) + \frac{1}{(R+b)h_{\text{out}}}\right\}$ which on rearrangement gives the required result.

27. b) For maximum q we require

$$X = \frac{1}{rh_{\text{in}}} + \frac{1}{\kappa}\ln\left(\frac{R}{r}\right) + \frac{1}{\kappa_{\text{lag}}}\ln\left(\frac{R+b}{R}\right) + \frac{1}{(R+b)h_{\text{out}}}$$

to be a minimum.

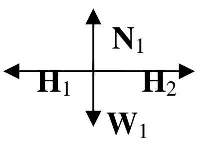
$$\frac{dX}{db} = \frac{1}{\kappa_{\text{lag}}}\frac{1}{R+b} - \frac{1}{h_{\text{out}}}\frac{1}{(R+b)^2}$$

For a minimum $\frac{dX}{db} = 0$ so $R + b = \frac{\kappa_{\text{lag}}}{h_{\text{out}}}$

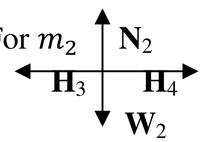
or $b = \frac{\kappa_{\text{lag}}}{h_{\text{out}}} - R$.

28.a)

For m_1



For m_2



N – normal reactions **W** – weights

H – spring forces on the particles

b) Considering the changes of the forces (P57)

$$\Delta\mathbf{H}_1 = -k_1(x_1)\hat{\mathbf{s}}_1 = -k_1x_1\mathbf{i}$$

$$\Delta\mathbf{H}_2 = -k_2(x_2 - x_1)\hat{\mathbf{s}}_2 = -k_2(x_2 - x_1)(-\mathbf{i})$$

$$\Delta\mathbf{H}_2 = k_2(x_2 - x_1)\mathbf{i}$$

By Newton's 3rd law

$$\Delta\mathbf{H}_3 = -\Delta\mathbf{H}_2 = -k_2(x_2 - x_1)\mathbf{i}$$

$$\Delta\mathbf{H}_4 = -k_3(-x_2)(-\mathbf{i}) = -k_3x_2\mathbf{i}$$

c) Using Newton's second law

$$m_1\ddot{x}_1\mathbf{i} = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2 \ (1) \quad m_2\ddot{x}_2\mathbf{i} = \Delta\mathbf{H}_3 + \Delta\mathbf{H}_4 \ (2)$$

Resolving in the **i**-direction

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$

so $m_1\ddot{x}_1 = -(k_2 + k_1)x_1 + k_2x_2$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3x_2$$

so $m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2$

Dividing by each m

$$\ddot{x}_1 = -\frac{(k_2 + k_1)}{m_1}x_1 + \frac{k_2}{m_1}x_2$$

$$\ddot{x}_2 = \frac{k_2}{m_2}x_1 - \frac{(k_2 + k_3)}{m_2}x_2$$

The dynamic matrix (P57) is as given.

d) The dynamic matrix becomes

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

and this has characteristic equation (P 41)

$$\begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 6\lambda + 8 = 0 \ \text{ or } \ (\lambda + 2)(\lambda + 4) = 0$$

So the eigenvalues are $\lambda = -2$ and $\lambda = -4$

The normal mode angular frequencies (P57) are given by

$$\omega = \sqrt{-\lambda} \ \text{ so } \ \omega_1 = \sqrt{2} \ \text{ and } \ \omega_2 = 2$$

e) The eigenvector equations (P41) are

$$(-3 - \lambda)x + y = 0$$

$$x + (-3 - \lambda)y = 0$$

For $\omega_1 = \sqrt{2}$ the equations reduce to $-x + y = 0$

or $y = x$ giving an eigenvector $[1 \quad 1]^T$

For $\omega_2 = 2$ the equations reduce to $x + y = 0$

or $y = -x$ giving an eigenvector $[1 \quad -1]^T$

The displacements of the particles (P57) are

$$x_1 = C_1\cos(\sqrt{2}\,t + \phi_1) + C_2\cos(2t + \phi_2)$$

$$x_2 = C_1\cos(\sqrt{2}\,t + \phi_1) - C_2\cos(2t + \phi_2)$$

f) The particles will move in phase-opposed motion with angular frequency 2 as the initial condition is in the form of an eigenvector associated with $\omega = 2$.

29 a) Substituting $u(x,t) = XT$ into the equation

$$kXT' = X''T$$

Dividing by XT gives $\frac{kT'}{T} = \frac{X''}{X} = \mu$

μ is a constant as the left-hand side depends only on t and the right-hand side only on x.

This gives $kT' = \mu T$ and $X'' = \mu X$ or $X'' - \mu X = 0$

As both ends of the rod are maintained at a temperature of 0°C, we have

$$u(0,t) = 0 \text{ and } u(L,t) = 0$$

so $X(0) = 0$ and $X(L) = 0$ as required.

b) When $\mu > 0$ let $\mu = h^2$ (using h as k is already used in the question). The equation for X becomes

$$X'' - h^2X = 0$$

The auxiliary equation (P26) is

$$\lambda^2 - h^2 = 0 \text{ so } \lambda = \pm h$$

so $X = Ae^{ht} + Be^{-ht}$

$$X(0) = 0 \implies A + B = 0$$

$$X(L) = 0 \implies Ae^{hL} + Be^{-hL} = 0$$

These can only be satisfied if $A = B = 0$ so we have a trivial solution.

When $\mu = 0$ $X'' = 0$ and $X = C + Dx$

$$X(0) = 0 \implies C = 0$$

$$X(L) = 0 \Rightarrow DL = 0 \Rightarrow D = 0$$

So we have a trivial solution.

When $\mu < 0$ let $\mu = -h^2$ giving $X'' + h^2 X = 0$ which has solution

$X = E\cos(hx) + F\sin(hx)$ (P35)

$X(0) = 0 \implies E = 0$ $X(L) = 0 \implies F\sin(hL) = 0$

As $F = 0$ gives a trivial solution we take

$\sin(hL) = 0 \implies hL = r\pi$ where $r = 1,2,3,\ldots.$ giving

$h = \frac{r\pi}{L}$ or $\mu = -\frac{r^2\pi^2}{L^2}$ $\quad r = 1,2,3,\ldots.$

The solution is $X_r(x) = F_r \sin\left(\frac{r\pi x}{L}\right)$

c) Substituting the value for μ in the equation for T

$$T' = \frac{\mu}{k}T = -\frac{r^2\pi^2}{kL^2}T$$

which has solution $T_r = G_r e^{-\frac{r^2\pi^2 t}{kL^2}}$ (P26)

For each r with $B_r = F_r G_r$

$u_r = X_r T_r = B_r e^{-\frac{r^2\pi^2}{kL^2}t} \sin\left(\frac{r\pi x}{L}\right)$

As the equation is linear we can add solutions (P63) over r to get the general solution

$$u(x,t) = \sum_{r=1}^{\infty} B_r\, e^{-\frac{r^2\pi^2}{kL^2}t} \sin\left(\frac{r\pi x}{L}\right)$$

d)

$$u(x,0) = x(L-x) = \sum_{r=1}^{\infty} B_r \sin\left(\frac{r\pi x}{L}\right)$$

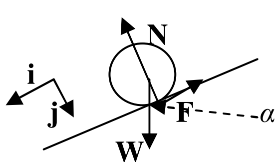
This is a Fourier sine series and so B_r is given by

$$B_r = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{r\pi x}{L}\right)dx \quad (P61)$$

30. a) Using page 74 of the handbook the moment of inertia for the tyre is $\frac{1}{2}m(R^2 + a^2)$ and that for the cylinder is $\frac{1}{2}Ma^2$. The moment of inertia of the wheel is the sum of these.

$$I = \frac{1}{2}Ma^2 + \frac{1}{2}m(R^2 + a^2)$$

b)



W = total weight of wheel

= $(M+m)g(\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j})$

N = normal reaction between wheel and plane

= $-|\mathbf{N}|\mathbf{j}$

F = frictional force between wheel and plane

= $-|\mathbf{F}|\mathbf{i}$

c) The linear motion is only in the **i**-direction so $\mathbf{a} = \ddot{x}\mathbf{i}$ and by Newton's second law

$$(M+m)\ddot{x}\mathbf{i} = \mathbf{W} + \mathbf{N} + \mathbf{F}$$

Resolving in the **i**-direction the equation of motion is

$$(M+m)\ddot{x} = (M+m)g\sin\alpha - |\mathbf{F}|$$

d) Linear kinetic energy = $\frac{1}{2}(M+m)\dot{x}^2$ and rotational kinetic energy = $\frac{1}{2}I\,\dot{\theta}^2$

so the total kinetic energy (P75) is the sum of these

$$T = \frac{1}{2}\left((M+m)\dot{x}^2 + \left(\frac{1}{2}Ma^2 + \frac{1}{2}m(R^2+a^2)\right)\dot{\theta}^2\right)$$

which is equivalent to the required expression.

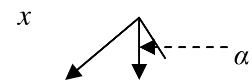
e) The rolling condition is $R\theta = x$ (P75)

Differentiation gives $\dot{\theta} = \frac{\dot{x}}{R}$. (1)

f) The potential energy of the centre of mass

$U(x) = -(M+m)g$ ×vertical distance fallen by centre of mass

$= -(M+m)gx\sin\alpha$



Using the rolling condition (1)

$$T = \frac{\dot{x}^2}{2}\left(M + m + \frac{1}{2}M\frac{a^2}{R^2} + \frac{1}{2}m\,\frac{R^2+a^2}{R^2}\right)$$

$$= \frac{\dot{x}^2}{2}\left(M + \frac{3}{2}m + \frac{1}{2}\frac{(m+M)a^2}{R^2}\right)$$

As the wheel is rolling the total energy E is constant where E is the sum of T and the $U(x)$ which gives in the expression in the question.

g) Differentiating the expression for E with respect to time

$0 = \dot{x}\ddot{x}\left(M + \frac{3}{2}m + \frac{(m+M)a^2}{2R^2}\right) - (M+m)g\dot{x}\sin\alpha$

As $\dot{x} \neq 0$ except initially, we can cancel $\dot{x}$ giving the acceleration as

$$\ddot{x} = \frac{(M+m)g\sin\alpha}{\left(M + \frac{3}{2}m + \frac{(m+M)a^2}{2R^2}\right)}$$