Question 1

(a) $\dfrac{1+i}{2-i}=\dfrac{1+i}{2-i}\,\dfrac{2+i}{2+i}=\dfrac{1+3i}{5}$. (A1, Sect. 1, Para. 6)

(b) -i = $\cos\left(-\dfrac{\pi}{2}\right)+i\sin\left(-\dfrac{\pi}{2}\right)$

The principal cube root is (A1, Sect. 3, Para. 3)

$$\cos\left(\frac{1}{3}\left(-\frac{\pi}{2}\right)\right)+i\sin\left(\frac{1}{3}\left(-\frac{\pi}{2}\right)\right)=\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}-i\frac{1}{2}.$$

(c) $\operatorname{Log}\left(\dfrac{1+i}{\sqrt{2}}\right)=\log_e\left(\left|\dfrac{1+i}{\sqrt{2}}\right|\right)+i\operatorname{Arg}\left(\dfrac{1+i}{\sqrt{2}}\right)=\log_e 1+\dfrac{\pi}{4}i=\dfrac{\pi}{4}i$ (A2, Sect. 5, Para. 1)

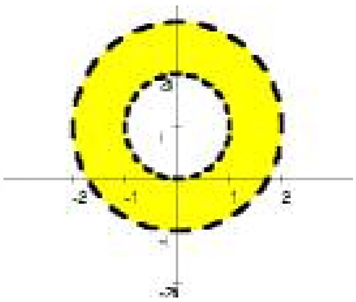
(d) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{2-4i}=\exp\left((2-4i)\operatorname{Log}\left(\dfrac{1+i}{\sqrt{2}}\right)\right)$ (A2, Sect. 5, Para. 3)

$=\exp\left((2-4i)\dfrac{\pi}{4}i\right)$ using the result from part (c)

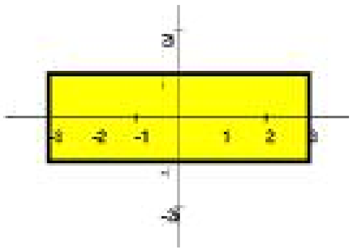
$=\exp\left(i\dfrac{\pi}{2}\right)\exp(\pi)=i\exp(\pi)$.

Question 2

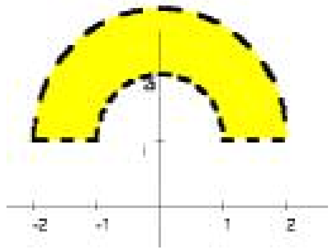
(a)



A



B



A - B

(b)(i) A and A- B are regions. (A3, Sect. 4, Para. 6)

[[B is not a region as it is not open.]]

(b)(ii) A - B is a simply-connected region. (B2, Sect 1, Para. 3)

[[In A the inside of | z - i | = 1.5 contains points not in A]]

(b)(iii) B is closed. (A3, Sect. 5, Para. 1)

(b)(iv) B is compact. (A3, Sect. 5, Para.5)

Question 3

(a)

(a)(i) The standard parametrization for the line segment Γ is (A2, Sect. 2, Para. 3)

$$\gamma(t) = (1-t)i + t(1-i) = t + i(1-2t) \qquad (t \in [0, 1])$$

(a)(ii) Since γ is a smooth path then (B1, Sect. 2, Para. 1)

$$\int_\Gamma \operatorname{Re} z\,dz = \int_0^1 \operatorname{Re}(\gamma(t))\gamma'(t)\,dt\,.$$

$$= \int_0^1 t(1-2i)\,dt = (1-2i)\left[\frac{t^2}{2}\right]_0^1 = \frac{1-2i}{2}$$

(b)

The length of Γ is $L = |(1-i)-i| = |1-2i| = \sqrt{5}$.

Using the Triangle Inequality (A1, Sect. 5, Para. 3b) then, for $z \in \Gamma$, we have

$$|\cos z| = \tfrac{1}{2}\left|e^{iz} + e^{-iz}\right| \le \tfrac{1}{2}\left\{\left|e^{iz}\right| + \left|e^{-iz}\right|\right\} = \tfrac{1}{2}\left\{e^{\operatorname{Re}(iz)} + e^{\operatorname{Re}(-iz)}\right\} \quad \text{(A2, Sect. 4, Para. 2b)}$$

$$= \tfrac{1}{2}\left\{e^{-y} + e^{y}\right\} \le \tfrac{1}{2}\left\{e^1 + e^1\right\} = e$$

Using the Backwards form of the Triangle Inequality (A1, Sect. 5, Para. 3c) then, for $z \in \Gamma$, we have

$$\left|6+z^2\right| \ge \left|\,|6| - \left|z^2\right|\right| \ge \left|6 - |1+i|^2\right| = |6-2| = 4\,.$$

Therefore $M = \left|\dfrac{\cos z}{6+z^2}\right| \le \dfrac{e}{4}$ for $z \in \Gamma$.

$f(z) = \dfrac{\cos z}{6+z^2}$ is continuous on $\mathbb{C} - \{-\sqrt{6}i, \sqrt{6}i\}$ and hence on the line Γ.

Therefore by the Estimation Theorem (B1, Sect. 4, Para. 3)

$$\left|\int_\Gamma f(z)\,dz\right| \le ML = \frac{e}{4}\sqrt{5}\,.$$

Question 4

(a)

Let $\mathcal{R}$ be the simply-connected region $\{z: |z| < 2\}$. C is a closed contour in $\mathcal{R}$, and $\frac{\cos z}{(z-3)^3}$ is analytic on $\mathcal{R}$.

By Cauchy's Theorem (B2, Sect. 1, Para. 4)

$$\int_C \frac{\cos z}{(z-3)^3}\,dz = 0\,.$$

(b)

Let $\mathcal{R}$ be the simply connected region $\{z: |z| < 2\}$. $\mathcal{R}$ is a simply-connected region and C is a simple-closed contour in ***R***. As $f(z)=\frac{\cos z}{(z-3)^2}$ is analytic on $\mathcal{R}$ then by Cauchy's Integral Formula (B2, Sect. 2, Para. 1)

$$\int_C \frac{\cos z}{z(z-3)^2}\,dz = \int_C \frac{f(z)}{z-0}\,dz = 2\pi i f(0) = 2\pi i\frac{\cos 0}{(-3)^2} = \frac{2}{9}\pi i\,.$$

(c)

Let $\mathcal{R}$ be the simply connected region $\mathbb{C}$. $\mathcal{R}$ is a simply-connected region and C is a simple-closed contour in ***R***. As $f(z) = \cos z$ is analytic on $\mathcal{R}$ then by Cauchy's n'th Derivative Formula (B2, Sect. 3, Para. 1) with $n = 2$ and $\alpha = 0$ we have

$$\int_C \frac{\cos z}{z^3}\,dz = \frac{2\pi i}{2!} f^{(2)}(0) = \pi i\,(-\cos 0) = -\pi i\,.$$

Question 5

(a)

$f(z) = \dfrac{1}{2\left(z+\frac{1}{2}\right)(z+2)}$. Therefore z has simple poles at z = -1/2 and z = -2.

$\operatorname{Res}\left(f,-\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z+\frac{1}{2}\right) f(z) = \dfrac{1}{2\left(-\frac{1}{2}+2\right)} = \dfrac{1}{3}$. [[C1, Sect. 1, Para. 1]]

$\operatorname{Res}(f,-2) = \lim_{z \to -2} (z+2) f(z) = \dfrac{1}{2\left(-2+\frac{1}{2}\right)} = -\dfrac{1}{3}$.

(b)

I shall use the strategy given in C1, Sect. 2, Para. 2.

$\displaystyle\int_0^{2\pi} \frac{1}{5+4\cos t}\,dt = \int_C \frac{1}{5+4\frac{1}{2}\left(z+z^{-1}\right)} \frac{1}{iz}\,dz$, where C is the unit circle.

$\displaystyle = -i\int_C \frac{1}{2z^2+5z+2}\,dz = -i\int_C \frac{1}{(2z+1)(z+2)}\,dz$.

The only residue inside the unit circle is the one at z = -1/2.

Therefore $\displaystyle\int_0^{2\pi} \frac{1}{5+4\cos t}\,dt = -i * 2\pi i \operatorname{Res}\left(f,-\frac{1}{2}\right) = \frac{2}{3}\pi$.

[[As it is a real integral we expect the answer will be real.]]

Question 6

(a)(i) Let $g_1(z) = z^5$.

For $z \in C_1$ then, using the Triangle Inequality (A1, Sect. 5, Para. 3),

$$| f(z) - g_1(z) | = |3z^2 + i | \leq |3z^2| + | i | = 12 + 1 < 32 = | g_1(z) |.$$

As f is a polynomial then it is analytic on the simply-connected region $\boldsymbol{R} = \mathbb{C}$. Since C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouché 's theorem (C2, Sect. 2, Para. 4) f has the same number of zeros as g_1 inside the contour C_1. Therefore f has 5 zeros inside C_1.

(a)(ii) Let $g_2(z) = 3z^2$.

On the contour C_2 we have, using the Triangle Inequality,

$$| f(z) - g_2(z) | = |z^5 + i | \leq |z^5| + | i | = 1 + 1 < 3 = | g_2(z) |.$$

As C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouché's theorem f has the same number of zeros as g_2 inside the contour C_2. Therefore f has 2 zeros inside C_2.

(b)

From part(a) f(z) has 5 - 2 = 3 solutions in the set $\{z: 1 \leq z < 2 \}$. Therefore we have to find if there are any solutions on C_2.

From part (a), on C_2 we have $| z^5 + i | \leq 2$.

Therefore, using the Backwards form of the Triangle Inequality (A1, Sect. 5, Para. 3c)

$$|f(z)| \geq | \, |3z^3| - |z^5 + i| \, | \geq | 3 - 2| = 1, \text{ on } C_2.$$

As f(z) is non-zero on C_2 then there are exactly 3 solutions in the set $\{z: 1 < z < 2 \}$.

Question 7

(a)

The conjugate velocity function $\overline{q}(z) = 2/z$.
As q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C} - \{0\}$ and $\overline{q}$ is analytic on $\mathbb{C} - \{0\}$ then q is a model fluid flow (D2, Sect. 1, Para. 14).

(b) On $\mathbb{C} - \{0\}$, $\Omega(z) = 2\,\mathrm{Log}\,z$ is a primitive of $\overline{q}$. Therefore Ω is a complex potential function for the flow (D2, Sect. 2, Para. 1).

The stream function $\Psi(x, y) = \mathrm{Im}\,\Omega(z)$ (D2, Sect. 2, Para. 4)

$$= 2\,\mathrm{Im}\left(\log_e|z| + i\,\mathrm{Arg}\,z\right) = 2\mathrm{Arg}\,z.$$

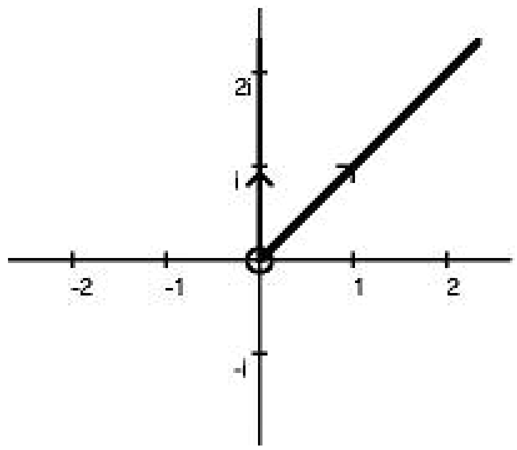
A streamline through the point i satisfies the equation

$2\mathrm{Arg}\,z = \Psi(0,1) = 2\frac{\pi}{2}$. So $\mathrm{Arg}\,z = \frac{\pi}{2}$. (D2, Sect. 2, Para. 4)

A streamline through the point 1 + i satisfies the equation

$2\mathrm{Arg}\,z = \Psi(1,1) = 2\frac{\pi}{4}$. So $\mathrm{Arg}\,z = \frac{\pi}{4}$.

Since $q(i) = \frac{2}{-i} = 2i$ the flow at i is in the y direction.
As $q(1 + i) = \frac{2}{1-i} = 1+i$ the flow at 1 + i is in the North-East direction.



(c)

The flux of q across the unit circle C = {z : |z| = 1}is (D2, Sect. 1, Para. 10)

$\mathrm{Im}\left(\int_C \overline{q}(z)\,dz\right) = \mathrm{Im}\left(\int_C \frac{2}{z}\,dz\right) = \mathrm{Im}\,(2 * 2\pi i) = 4\pi$,using Cauchy's Integral Theorem (B2, Sect. 2, Para. 1).

Question 8

(a)

If α is a fixed point of f then $f(\alpha) = \alpha^2 - 4\alpha + 6 = \alpha$ (D3, Sect. 1, Para 3).
As $\alpha^2 - 5\alpha + 6 = (\alpha - 2)(\alpha - 3) = 0$ then there are fixed points at 2 and 3.

$f'(z) = 2z - 4$.

As $f'(2) = 0$ then 2 is a super-attracting fixed point (D3, Sect. 1, Para. 5).
As $f'(3) = 2$ then 3 is a repelling fixed point.

(b)(i) [[From the diagram in Handbook looks as if point not in Mandelbrot set.]]

$P_c(0) = -1 + i$.
$P_c^2(0) = (-1+i)^2 + (-1+i) = -1 - i$.
$P_c^3(0) = (-1-i)^2 + (-1+i) = -1 + 3i$.
As $\left| P_c^3(0) \right| > 2$ then c does not lie in the Mandelbrot set (D3, Sect. 4, Para. 5).

(b)(ii)

Since $|c+1| = \left|\frac{1}{8} i\right| < \frac{1}{4}$ then P_c has an attracting 2-cycle (D3, Sect. 4, Para. 9(b)).
Therefore c belongs to the Mandelbrot set (D3, Sect. 4, Para. 8).

Question 9

(a)

(a)(i)

$f(z) = \bar{z}\,\mathrm{Im}\,z + |z|^2 = (x - iy)y + (x^2 + y^2) = u(x, y) + iv(x, y)$,
where $u(x,y) = x^2 + xy + y^2$, and $v(x,y) = -y^2$.

(a)(ii)

$\frac{\partial u}{\partial x} = 2x + y$, $\quad \frac{\partial u}{\partial y} = x + 2y$, $\quad \frac{\partial v}{\partial x} = 0$, $\quad \frac{\partial v}{\partial y} = -2y$

If f is differentiable then the Cauchy-Riemann equations hold (A4, Sect. 2, Para. 1). If they hold at (a, b) then

$$\frac{\partial u}{\partial x}(a, b) = 2a + b = -2b = \frac{\partial v}{\partial y}(a, b), \text{ and}$$
$$\frac{\partial v}{\partial x}(a, b) = 0 = -(a + 2b) = -\frac{\partial u}{\partial y}(a, b)$$

Therefore the Cauchy-Riemann equations only hold at (0, 0).

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at (0, 0).
3. satisfy the Cauchy-Riemann equations at (0, 0)

then, by the Cauchy-Riemann Converse Theorem (A4, Sect. 2, Para. 3), f is differentiable at 0.

As the Cauchy-Riemann equations only hold at (0, 0) then f is not differentiable on any region surrounding 0. Therefore f is not analytic at 0. (A4, Sect. 1, Para. 3)

(a)(iii)

$f'(0, 0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0$ (A4, Sect. 2, Para. 3).

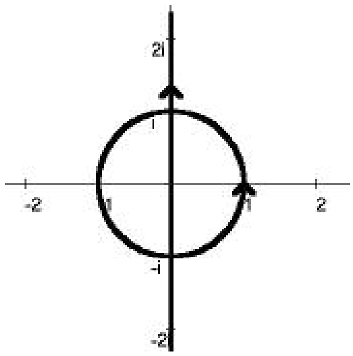
(b)(i)

The domain of g is $\mathbb{C}$ (A4, Sect. 1, Para. 7) and its derivative $g'(z)=3z^2$ also has domain $\mathbb{C}$ (A4, Section 3, Para. 4) . Therefore g is analytic on $\mathbb{C}$. Since $g'(z) \neq 0$ when $z \neq 0$ then g is conformal on $\mathbb{C} - \{0\}$ (A4, Sect. 4, Para. 6).

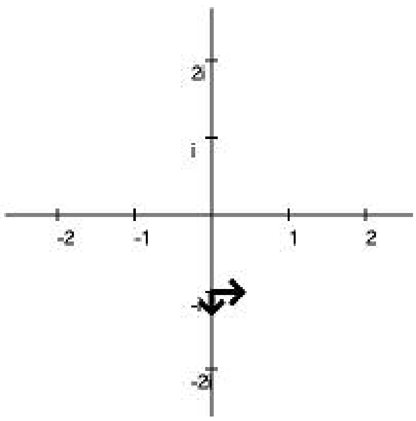
(b)(ii)

As g is analytic on $\mathbb{C}$ and $g'(i) \neq 0$ then a small disc centred at i is mapped approximately (A4, Sect. 1, Para. 11) to a small disc centred at g(i) = -i. The disc is rotated by Arg $(g'(i))$ = Arg (-3) = π, and scaled by a factor $|g'(i)| = 3$.

(b)(iii) Γ_1 is the circle and Γ_2 the vertical line.



(b)(iv) The horizontal line in the diagram below is $g(\Gamma_1)$. (A4, Sect. 4, Para. 4)



(b)(v)

Let Γ_3 be the smooth path given by $\Gamma_3 : \gamma_3(t) = t$ $(t \in \mathbb{R})$.
Γ_2 and Γ_3 meet at a right-angle at z = 0.

For i = 2, 3 we have $(g \circ \gamma_i)'(t) = \gamma_i'(t) g'(\gamma_i(t)) = \gamma_i'(t) 3(\gamma_i(t))^2$. So $(g \circ \gamma_i)'(0) = 0$.

As the images of Γ_1 and Γ_2 do not meet at right-angles at 0 then g is not conformal at 0.

Question 10

(a)

$\sin z = z\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots\right)$, for $z \in \mathbf{C}$. (B3, Sect. 3, Para. 5)

Since $0<\left|\frac{\sin z}{z}\right|<1$ when $0 < |z| < \pi$ then $\left|-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots\right|<1$.

$$\frac{z}{\sin z}=\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots\right)^{-1}$$
$$=1+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\ldots\right)+\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\ldots\right)^2+\ldots$$
$$=1+\frac{z^2}{6}+z^4\left(-\frac{1}{120}+\frac{1}{36}\right)+\ldots$$

Therefore the Laurent series about 0 for f is $1+\frac{z^2}{6}+\frac{7}{360}z^4+\ldots$ for $0 < |z| < \pi$

$\frac{1}{z^2\sin z}=\frac{f(z)}{z^3}$ is analytic on the punctured disc $\mathbb{C}$ - {0}. The Laurent series about 0 is

$$\frac{1}{z^3}+\frac{1}{6z}+\frac{7}{360}z+\ldots=\sum_{n=-\infty}^{\infty}a_n z^n$$

As C is a circle with centre 0 then (B4, Sect. 4, Para. 2)

$$\int_C \frac{f(z)}{z^3}dz = 2\pi i a_{-1} = 2\pi i\left(\frac{1}{6}\right)=\frac{\pi i}{3} .$$

(b)

The domain of A = $\mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$.

Suppose that g is another analytic function with domain A which agrees with f on $\{iy : y > 0\}$

The set $S=\left\{i\left(1+\frac{1}{n}\right): n=1,2,3,\ldots\right\} \subseteq$ A and has the limit point $i \in$ A.

f agrees with g throughout the set $S \subseteq A$ and S has a limit point which is in A. By the Uniqueness theorem (B3, Sect. 5, Para. 7) f agrees with g throughout A. Hence f is the only analytic function with domain A such that $f(iy) = \dfrac{y}{\sinh y}$ for $y > 0$.

(c) 6 marks

Since $\sin z = 0$ when $z = 0$, $z = \pm\pi$, $z = \pm 2\pi$, …. Then f(z) has singularities of the form $k\pi$, $k \in \mathbb{Z}$.

Singularity at z = 0.

At z = 0 we can use the Laurent series found in part (a). Since f(0) = 1 then the singularity at 0 is a removable singularity.

Singularities at $z = k\pi$ where $k \in \mathbb{Z} - \{0\}$.

$\sin(z - k\pi) = \sin z * \cos k\pi - \cos z * \sin k\pi = (-1)^k \sin z$.

Therefore $f(z) = \dfrac{z}{\sin z} = (-1)^k \dfrac{z}{\sin(z - k\pi)} = (-1)^k \left\{ \dfrac{z - k\pi}{\sin(z - k\pi)} + \dfrac{k\pi}{\sin(z - k\pi)} \right\}$

$$\lim_{z \to k\pi} \frac{z - k\pi}{\sin(z - k\pi)} = 1$$

As $\displaystyle\lim_{z \to k\pi} (z - k\pi)\frac{k\pi}{\sin(z - k\pi)} = k\pi$ then there is a simple pole at $z = k\pi$ (B4, Sect. 3, Para. 2).

Therefore there f has simple poles at $k\pi$ where $k \in \mathbb{Z} - \{0\}$.

Question 11

(a)

Let $f(z) = \exp(z^4)$ and $R = \{z : |z| \le 2\}$.

As f is analytic on the bounded region R and continuous on $\overline{R}$ then by the Maximum Principle (C2, Sect. 4, Para. 4) there exists an $\alpha \in \partial R = \{ z : |z| = 2\}$ such that $|f(z)| \le |f(\alpha)|$ for $z \in \overline{R}$.

When $z \in \partial R$ we can write it in the form $z = 2\exp(i\theta)$, where θ is real.
Then $|\exp(z^4)| = \exp(\operatorname{Re}(z^4)) = \exp(16\cos 4\theta)$.
This is a maximum when 4θ is a multiple of 2π and it equals e^{16}.

Therefore $\max\{ |\exp(z^4)| : |z| \le 2 \} = e^{16.}$ The maximum is attained when
$z = 2e^0 = 2$, $z = 2e^{\pi/2} = 2i$, $z = 2e^{\pi} = -2$, and $z = 2e^{3\pi/4} = -2i$.

(b)

Let $D_f = \{z: |z| < 5\}$ and $D_g = \{z: |z| > 5\}$.
Since $D_f \cup D_g = \varnothing$ then f and g are not direct analytic continuations of each other.

Let $h(z) = \dfrac{1}{1-\frac{z}{5}} = \dfrac{5}{5-z}$ on D_h, where $D_h = \mathbf{C} - \{5\}$.

When $z \in D_f$ then $|z|/5 < 1$ and the geometric series $\sum_{n=0}^{\infty}\left(\frac{z}{5}\right)^n$ is convergent and has the sum

$$\frac{1}{1-\frac{z}{5}} = \frac{5}{5-z}. \quad \text{(B3, Sect. 3, Para. 5)}$$

Since $f = h$ when $z \in D_f \subseteq D_f \cup D_h$ then h is an analytic continuation of f.

When $z \in D_g$ then $5/|z| < 1$ and the geometric series $\sum_{n=0}^{\infty}\left(\frac{5}{z}\right)^n$ is convergent and has the sum $\dfrac{1}{1-\frac{5}{z}} = \dfrac{z}{z-5}$.

Therefore $-\sum_{n=1}^{\infty}\left(\frac{5}{z}\right)^n = -\frac{5}{z}\sum_{n=0}^{\infty}\left(\frac{5}{z}\right)^n = -\frac{5}{z-5}$ when $z \in D_g$.

Since $g = h$ when $z \in D_g \subseteq D_g \cup D_h$ then g is an analytic continuation of h.

Since (f, D_f), (g, D_g), (h, D_h) form a chain then f and g are indirect analytic continuations of each other.

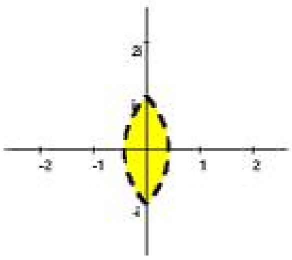
Question 12

(a)

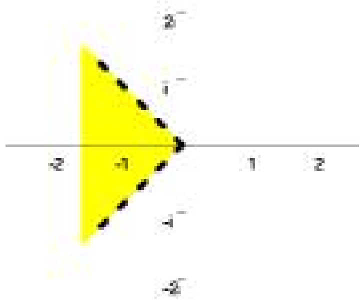
Using the formula for a transformation mapping points to the standard triple (D1, Sect. 2, Para. 11) then the Möbius transformation $\hat{f}_1$ which maps i, ∞ , and - i to the standard triple of points 0, 1, and ∞ respectively is

$$f_1(z)=\frac{(z-i)}{(z+i)}\frac{(\infty+i)}{(\infty-i)}=\frac{z-i}{z+i}$$

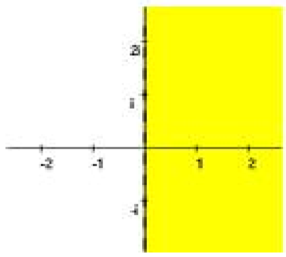
(b)(i)



R



S



T

(b)(ii) Since $\hat{f}_1$ maps i to 0 and -i to ∞ then the curved boundaries of R are mapped to extended lines originating at the origin. As the angle between the arcs of the circles meeting at i is $\pi/2$ and the mapping $\hat{f}_1$ is conformal then the angle between the extended lines at the origin is also $\pi/2$.

Points in R, which lie on the imaginary axis, can be written as

$z = i\,(1 - \delta)$ where $0 < \delta < 2$.

For these points we have $f_1(z)=\dfrac{i\,(1-\delta-1)}{i\,(2-\delta)}=-\dfrac{\delta}{2-\delta}$.

Therefore $\hat{f}_1$ maps these points to the negative real-axis. In R this line is at an angle of $\pi/4$ to the two arcs of the circles so, as it is a conformal mapping, then in the image of R this is also true.

Therefore the image of R under $\hat{f}_1$ is R_1 is S.

(b)(iii) A conformal mapping from S onto T is the power function $w = z^2$.

Since the combination of conformal mapping is also conformal then a conformal mapping from R to T is

$$f(z) = \left(\frac{z-i}{z+i}\right)^2.$$

(b)(iv)

D1, Sect. 4, Para. 5 gives

$$h(z) = \frac{z-1}{z+1}$$

as a conformal mapping from T onto the open unit disc.

Therefore a one-one conformal mapping from R onto the open unit disc is

$$g(z) = \frac{\left(\frac{z-i}{z+i}\right)^2 - 1}{\left(\frac{z-i}{z+i}\right)^2 + 1}.$$