M337/H

The Open University

Course Examination 2009
Complex Analysis

Thursday 15 October 2009 2.30 pm – 5.30 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part 1 (64% of the marks) you should attempt as many questions as you can.

In Part 2 (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is **NOT** permitted in this examination.

PART 1

(i) You should attempt as many questions as you can in this part.
(ii) Each question in this part carries 8 marks.

Question 1

Give the Cartesian form of the following complex numbers, simplifying your answer as far as possible.

(a) $\dfrac{7-4i}{2+i}$ [2]

(b) $2e^{-\pi i/6}$ [2]

(c) $(\sqrt{3}-i)^6$ [2]

(d) i^{-2i} [2]

Question 2

Let

$$A = \{z : -1 < \operatorname{Re} z < 1\} \quad \text{and} \quad B = \{z : 2 \le |z| \le 4\}.$$

(a) Make separate sketches of the sets A, B and $C = A - B$. [3]

(b) For each of the sets A, B and C:

(i) state whether it is a region, and if not a region explain why not;

(ii) state whether it is compact, and if not compact explain why not. [5]

Question 3

Let Γ be the line segment from -1 to i.

(a) Write down the standard parametrization for Γ. [1]

(b) Evaluate

$$\int_\Gamma (\operatorname{Im} z)^2 \, dz. \quad [2]$$

(c) Determine an upper estimate for the modulus of

$$\int_\Gamma \frac{3e^{\overline{z}}}{3+z^5} \, dz. \quad [5]$$

Question 4

Evaluate the following integrals, in which $C = \{z : |z| = \frac{1}{2}\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C \frac{\cos 2z + \sin 2z}{(z-i)^2}\,dz$ [2]

(b) $\displaystyle\int_C \frac{\cos 2z + \sin 2z}{z(z-i)}\,dz$ [3]

(c) $\displaystyle\int_C \frac{\cos 2z + \sin 2z}{z^2}\,dz.$ [3]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{z^2+1}{z(z-3)(3z-1)}$$

at each of the poles of f. [4]

(b) Hence evaluate the integral

$$\int_0^{2\pi} \frac{\cos t}{5 - 3\cos t}\,dt.$$ [4]

Question 6

Let $f(z) = z^7 + 4z^3 - 2i$.

(a) Determine the number of zeros of f that lie inside:

(i) the circle $C_1 = \{z : |z| = 2\}$,

(ii) the circle $C_2 = \{z : |z| = 1\}$. [6]

(b) Show that the equation

$$z^7 + 4z^3 - 2i = 0$$

has exactly four solutions in the set $\{z : 1 < |z| < 2\}$. [2]

Question 7

Let $q(z) = \overline{z} - i$ be a velocity function.

(a) Explain why q represents a model fluid flow. [1]

(b) Determine a complex potential function for this flow. Hence sketch the streamline through the point 1 and indicate the direction of flow. [5]

(c) Evaluate the circulation of q along the path

$$\Gamma : \gamma(t) = t \quad (t \in [0, 2]).$$ [2]

Question 8

(a) Find the fixed points of the function $f(z) = z^2 + 4z + 2$ and classify them as (super-)attracting, repelling or indifferent. [3]

(b) Which of the following points c lie in the Mandelbrot set.

(i) $c = -\frac{3}{2} + \frac{1}{2}i$

(ii) $c = -1 - \frac{1}{5}i$

Justify your answer in each case. [5]

PART 2

(i) You should attempt no more than **TWO** *questions in this part.*
(ii) Each question in this part carries 18 marks.

Question 9

(a) Let f be the function

$$f(z) = z(2 + \overline{z}).$$

(i) Write $f(x + iy)$ in the form $u(x, y) + iv(x, y)$, where u and v are real-valued functions.

(ii) Use the Cauchy-Riemann theorem and its converse to show that f is differentiable at 0, but not analytic there.

(iii) Evaluate $f'(0)$. [*8*]

(b) Let g be the function $g(z) = \dfrac{i}{z^3}$.

(i) Show that g is conformal at i.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = (1 - t)i - t \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point i, and sketch Γ_1 and Γ_2 on the same diagram.

(iii) Describe the effect of g on a small disc centred at i and hence make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i)$. [*10*]

Question 10

(a) Let f be the function

$$f(z) = \frac{\sin z}{z(z-3)^4}.$$

Write down the singularities of f and determine their nature. [5]

(b) (i) Write down the Laurent series about 0 for the function

$$g(z) = \sin(1/z),$$

giving an expression for the general term of the series, and state its annulus of convergence. [3]

(ii) Hence evaluate the integral

$$\int_C z^4 \sin(1/z)\, dz,$$

where C is the unit circle $\{z : |z| = 1\}$. [3]

(c) (i) Determine the first three non-zero terms of the Taylor series about 0 for the function

$$h(z) = \operatorname{Log}(\cos z).$$

(ii) Hence determine the first three non-zero terms of the Taylor series about 0 for tan. [7]

Question 11

(a) Find the residues of the function

$$f(z) = \frac{\pi \cot \pi z}{16z^2 + 1}$$

at each of the points 0, $\frac{1}{4}i$, $-\frac{1}{4}i$. [6]

(b) Hence determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 + 1}.$$ [8]

(c) Use your result from part (b) to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{16n^2 + 1} = \frac{\pi}{4} \coth \frac{\pi}{4}.$$ [4]

Question 12

(a) Determine the extended Möbius transformation $\widehat{f_1}$ which maps 1 to 0, $-i$ to 1 and -1 to ∞. *[2]*

(b) Let

$$R = \{z : |z| < 1, \operatorname{Im} z < 0\}$$

and

$$S = \{w : \operatorname{Im} w < 0\}.$$

(i) Sketch the regions R and S.

(ii) Determine and sketch the image of R under $\widehat{f_1}$ of part (a).

(iii) Hence determine a conformal mapping f from R onto S.

(iv) Write down the rule of the inverse function f^{-1}. *[16]*

[END OF QUESTION PAPER]