MST209/P

Module Examination 2011

Mathematical methods and models

Tuesday 18 October 2011 **10.00 am–1.00 pm**

Time allowed: 3 hours

Personal Identifier								
Examination No.								

You are **not** allowed to use a calculator in this examination.

There are THREE parts to this paper. In each part of the paper the questions are arranged in the order they appear in the course. There are 115 marks available, but scores greater than 100 will be rounded down to 100.

Part 1 consists of 15 questions each worth 2 marks. You are advised to spend no more than 1 hour on this part. Enter one option in each box provided on the question paper; use your answer book(s) for any rough work. Incorrect answers are not penalised. Cross out mistakes and write your answer next to the box provided.

Part 2 consists of 8 questions each worth 5 marks. You are advised to spend no more than $1\frac{1}{4}$ hours on this part.

Part 3 consists of 7 questions each worth 15 marks. Your best three marks will be added together to give a maximum of 45 marks.

In Parts 2 and 3: Write your answers in the answer book(s) provided. The marks allocated to each part of each question are given in square brackets in the margin. Unless you are directed otherwise in the question, you may use any formula or other information from the *Handbook* provided that you give a reference. Do **not** cross out any answers unless you have supplied a better alternative — everything not crossed out may receive credit.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used (as well as in the boxes above). **Failure to do so may mean that your work cannot be identified.** Use the paper fastener provided to fix together all your answer books, and the question paper, with your signed desk record on top.

PART 1

Each question in this part of the paper is worth 2 marks. Fill in the appropriate response in the box alongside the question.

Question 1

Select the option that gives y as a function of x given that

$$\frac{xy+8}{2x+y} = x - 3.$$

Options Answer:

A $\dfrac{2(x-4)(x+1)}{2x+3}$ **B** $\dfrac{2}{3}(x-4)(x+1)$

C $-\dfrac{2(x-4)(x+1)}{2x+3}$ **D** $-\dfrac{2}{3}(x-4)(x+1)$

Question 2

This question concerns the differential equation

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = \sin(x), \quad (x > 0).$$

Select the option that gives the complementary function.

Options Answer:

A $A\sin(4x) + B\cos(4x)$ **B** $(A + Bx)e^{4x}$

C $A\sin(4x)e^{-4x} + B\cos(4x)e^{-4x}$ **D** $(A + Bx)e^{-4x}$

Question 3

Select the option that gives the result of evaluating the expression $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ where

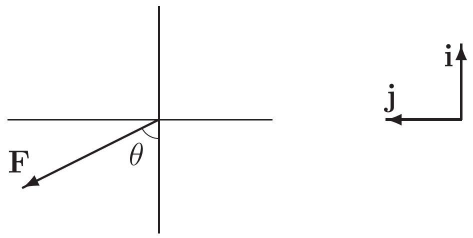
$$\mathbf{a} = \mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} \quad \text{and } \mathbf{c} = \mathbf{j} + 2\mathbf{k}.$$

Options Answer:

A 4 **B** 5 **C** 0 **D** 1

Question 4

The diagram shows the force **F** and the directions of the unit vectors, **i** and **j**.



Select the option that corresponds to the component of **F** in the **j**-direction.

Options Answer:

A $|\mathbf{F}|\sin\theta$ **B** $-|\mathbf{F}|\sin\theta$ **C** $|\mathbf{F}|\cos\theta$ **D** $-|\mathbf{F}|\cos\theta$

Question 5

A particle of mass m is suspended by a spring of stiffness $2k$ and natural length l_0 from a fixed point A, and is a vertical distance x below A. Select the option that gives the potential energy in the system, using A as the datum point for gravitational potential energy.

Options Answer:

A $mgx + 2k(x - l_0)$ **B** $-mgx + 2k(x - l_0)$

C $mgx + k(x - l_0)^2$ **D** $-mgx + k(x - l_0)^2$

Question 6

The matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

has an eigenvector $[2 \;\; -1]^T$. Select the option that gives a corresponding eigenvalue.

Options Answer:

A 0 **B** 3 **C** 2 **D** 5

Question 7

Select the option that gives the first-order Taylor polynomial approximation for the function $f(x, y) = e^x \cos(y)$ about $(0, 0)$.

Options Answer:

A $1 + x$ **B** $1 - x + y$

C $1 - x$ **D** $1 + x - y$

Question 8

A panel, 2 m high by 3 m wide and 5 mm thick, is made of material with thermal conductivity $0.1\,\mathrm{W\,m^{-1}\,K^{-1}}$. What is the rate of loss of heat through the panel when the temperature difference across it is 20^{o}C and the heat transfer coefficient on either side is $10\,\mathrm{W\,m^{-2}\,K^{-1}}$?

Options Answer:

A 30 W **B** 30 J **C** 480 W **D** 480 J

Question 9

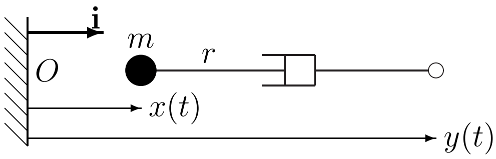
Select the option that gives the dimensions of stress (defined as force per unit area).

Options Answer:

A $\mathrm{ML^{-2}}$ **B** $\mathrm{ML^{-1}T^{-2}}$ **C** $\mathrm{kg\,m^{-2}}$ **D** $\mathrm{kg\,m^{-1}\,s^{-2}}$

Question 10

A horizontally-placed damper, with damping coefficient r, has its left end connected to a particle of mass m which is at a distance $x(t)$ from a fixed point O, and its right end is a distance $y(t)$ from O. The unit vector $\mathbf{i}$ is taken as being horizontal and points to the right as shown in the diagram below.



What is the option that gives the damping force acting on the particle?

Options Answer:

A $r\dot{x}\mathbf{i}$ **B** $-r\dot{x}\mathbf{i}$ **C** $r\,(\dot{x}-\dot{y})\,\mathbf{i}$ **D** $-r\,(\dot{x}-\dot{y})\,\mathbf{i}$

Question 11

A one-dimensional system of two particles and three springs has the differential equation of motion

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} -18 & 18 \\ 9 & -27 \end{bmatrix}$$

and $\mathbf{x}$ is the vector of the particles' displacement measured from their respective equilibrium positions at time t.

The eigenvalues of $\mathbf{A}$ are -9 and -36 and corresponding eigenvectors are $[2 \quad 1]^T$ and $[-1 \quad 1]^T$.

Select the option that gives the equation of motion of a normal mode of the system.

Options Answer:

A $\mathbf{x}(t) = -[2 \quad 1]^T \cos(9t)$ **B** $\mathbf{x}(t) = [1 \quad -1]^T \cos(3t)$

C $\mathbf{x}(t) = -[2 \quad 1]^T \cos(36t)$ **D** $\mathbf{x}(t) = [1 \quad -1]^T \cos(6t)$

Question 12

A particle of mass $2m$ is travelling with speed v in the direction given by $2\mathbf{i} + 3\mathbf{j}$. Select the option that gives the momentum of the particle.

Options Answer:

A $2mv$ **B** $2\sqrt{13}mv$ **C** $\dfrac{2}{\sqrt{13}}mv(2\mathbf{i} + 3\mathbf{j})$ **D** $\dfrac{2}{\sqrt{13}}m(2\mathbf{i} + 3\mathbf{j})$

Question 13

Consider the function $f(t) = t^4$ for $t \in [-1, 1]$. Select the option that gives a possible Fourier series for a periodic extension of f, where $A_0 \neq 0$, and A_r and B_r are **non-zero constants** (and $r = 1, 2, 3, \ldots$)

Options Answer:

A $A_0 + \sum_{r=1}^{\infty} A_r \cos(r\pi t)$ **B** $A_o + \sum_{r=1}^{\infty} B_r \sin(r\pi t)$

C $\sum_{r=1}^{\infty} A_r \cos(r\pi t)$ **D** $A_0 + \sum_{r=1}^{\infty} (A_r \cos(r\pi t) + B_r \sin(r\pi t))$

Question 14

A scalar field is defined in cylindrical polar coordinates by

$$f(\rho, \theta, z) = \rho \tan\theta.$$

Select the option that gives the $\mathbf{e}_\theta$-component of $\mathbf{grad}\, f$.

Options Answer:

A $\rho\cos\theta$ **B** $\sec^2\theta$ **C** $\rho\sec^2\theta$ **D** $\sin\theta$

Question 15

A certain numerical method is second-order. A step size $h = 0.01$ gives an approximate value at $x = a$ with an estimated global error of 0.005. Assuming h is sufficiently small, select the largest step size required to determine the approximate value at $x = a$ correct to six decimal places.

Options Answer:

A 0.001 **B** 0.0001 **C** 0.000 01 **D** 0.000 001

PART 2

Each question in this part of the paper is worth 5 marks.

Question 16

Consider the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 36x, \quad x > 0.$$

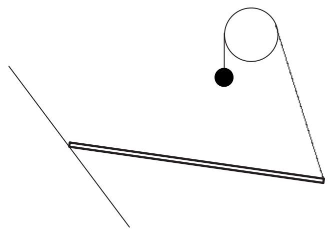
Given that the complementary function is

$$y(x) = Ae^{-3x} + Be^{-4x}$$

find a particular integral and describe the behaviour of the solution as x becomes large. [5]

Question 17

One end of a uniform plank rests on a rough slope, and the other end of the plank is held by a rope that passes over a pulley and a mass is attached to the other end of the rope as shown in the diagram to the right.
Model the rope as a model string, the pulley as a model pulley, and the plank as a model rod.
The system is in equilibrium.



Draw a force diagram showing all the forces acting on the plank. Briefly describe each force. [5]

Question 18

A particle of mass 5 kg is projected vertically upwards from a point 2 m above the ground with a speed of $3\,\mathrm{m\,s^{-1}}$. x is the vertical displacement of the particle at time t and the origin for x is at ground level; $\mathbf{i}$ is vertically upwards. The effective diameter of the particle is 0.1 m, and the quadratic model for air resistance applies. You should define any symbols that you introduce.

(a) Express the initial conditions for position vector and velocity of the particle. [1]

(b) Draw a force diagram showing all the forces acting on the particle, and express each force in vector form. [3]

(c) Determine a differential equation of motion satisfied by the particle in the upward part of its trajectory. [1]

Question 19

Use the matrix form of Gaussian elimination to find the solution of the following simultaneous equations

$$\begin{cases} 2x + 3y - z = 6 \\ 4x + 7y + z = 10 \\ 2x + 4y - 3z = 9. \end{cases}$$

[5]

Question 20

The position vector of a particle at time t is given by

$$\mathbf{r} = (t^2 - 1)\mathbf{i} + \sqrt{2}(t - 8)\mathbf{j} \quad (t > 0).$$

(a) Determine the velocity of the particle at time t. [*1*]

(b) Calculate the dot product of the velocity and the position vector, $\dot{\mathbf{r}} \cdot \mathbf{r}$, and find the time at which this dot product is zero. [*3*]

(c) What is the physical significance of $\dot{\mathbf{r}} \cdot \mathbf{r} = 0$ in the case $\dot{\mathbf{r}} \neq \mathbf{0},\ \mathbf{r} \neq \mathbf{0}$? [*1*]

Question 21

A particle of mass m is moving in a horizontal plane in a circle of radius R; its rate of rotation, anti-clockwise, $\dot{\theta}$, is given by

$$\dot{\theta} = t + \cos(2t)$$

Use plane polar unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$, in the horizontal plane, and $\mathbf{k}$ in the vertical direction, giving a right-handed coordinate system. Define any symbols that you introduce.

(a) Express the position vector, $\mathbf{r}$, of the particle in this coordinate system. [*1*]

(b) Determine the velocity of the particle in terms of t. [*1*]

(c) Find the angular momentum of the particle at time t. [*2*]

(d) Calculate the torque acting on the particle at time t. [*1*]

Question 22

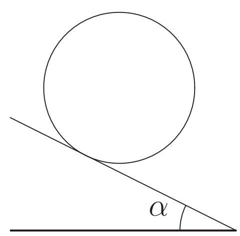
Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right).$$

Find the two ordinary differential equations that are created by the use of the separation of variables method. **Do not solve the differential equations.** [*5*]

Question 23

A cylinder, whose radius is R, mass is m and moment of inertia is I about the centre of the cylinder, is slipping and rolling down a plane inclined at an angle α to the horizontal as shown in the diagram to the right. The axis of the cylinder is horizontal, and its motion is along the line of greatest slope.



(a) Draw a force diagram showing all the forces acting on the cylinder. [*2*]

(b) Taking the centre of the cylinder as the origin, define an appropriate coordinate system, and hence determine the total torque acting on the cylinder, defining any symbols that you introduce. [*2*]

(c) Determine the equation of rotational motion. [*1*]

PART 3

Each question in this part of the paper is worth 15 marks. All of your answers will be marked and the marks from your best three answers will be added together. A maximum of 45 marks can be obtained from this part.

Question 24

(a) Solve the differential equation

$$\frac{dy}{dx} = (5 + 6x^2)y^2, \quad y(0) = 1,$$

expressing your answer in the form $y = f(x)$. [5]

(b) For the differential equation

$$\frac{dy}{dx} + 4xy = x^3, \quad y(0) = 1,$$

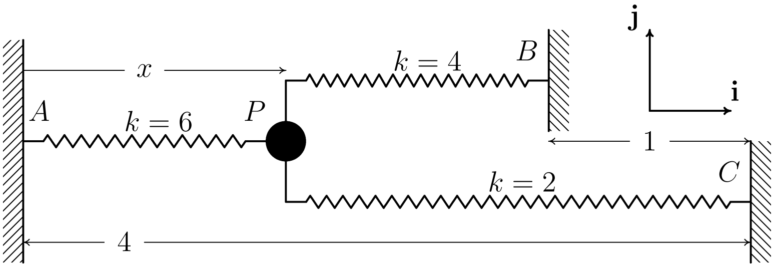
use Euler's method with a step size of $h = 0.1$ to determine an approximate solution at $x = 0.2$. [4]

(c) For the problem in part (b), solve the differential equation analytically, expressing your answer in the form $y = f(x)$.

[*You may find the following result helpful to your solution*:
$\int x^3 e^{ax^2}\, dx = \frac{1}{2a}x^2 e^{ax^2} - \frac{1}{2a^2}e^{ax^2} + C.$] [6]

Question 25

A particle P of mass 3 kg is constrained to move in a horizontal straight line along a smooth track. It has three springs, whose parameters are given in the table below, attached to it and the arrangement is shown in the diagram below. The unit vectors, $\mathbf{i}$ and $\mathbf{j}$, are defined horizontally and vertically as shown in the diagram. The distance AC is 4 m, and BC is 1 m.



Spring	stiffness	natural length
AP	$6\,\mathrm{N\,m^{-1}}$	1 m
PB	$4\,\mathrm{N\,m^{-1}}$	2 m
PC	$2\,\mathrm{N\,m^{-1}}$	3 m

At time t the displacement of P from A is $x(t)$.

(a) Draw a force diagram show all the forces acting on the particle. Define each force you introduce. *[2]*

(b) At time t, derive in full each force and express it in terms of the given unit vectors. *[5]*

(c) Find the value of x when the system is in equilibrium. *[2]*

(d) Determine a differential equation of horizontal motion when the particle is in motion, simplifying your result as far as possible. *[1]*

(e) Solve the equation of motion when it is released from rest at a distance of 1.5 m from A. *[3]*

(f) Sketch the graph of x against t for the range $0 \leq t \leq 2\pi$ showing all the salient features of the motion. *[2]*

Question 26

(a) Find the solution of the simultaneous linear differential equations

$$\frac{dx}{dt} = 3x - 4y + 4e^{-t}$$
$$\frac{dy}{dt} = -x + 3y + 2e^{-t}$$

which satisfies the initial conditions $x(0) = 0$, $y(0) = 0$. *[13]*

(b) Describe the long-term behaviour of the solution. *[2]*

Question 27

A system is represented by the pair of simultaneous first-order non-linear differential equations:

$$\frac{dx}{dt} = xy - 2x^2$$
$$\frac{dy}{dt} = 4 - 4x^2 - y^2$$

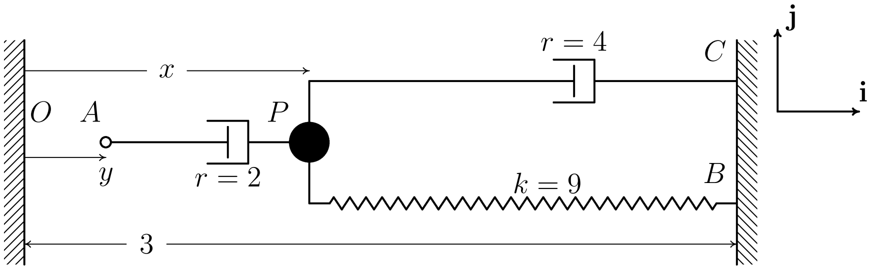
where x and y may take any real values.

(a) Find all the equilibrium points. [5]

(b) Find the linear approximation to this system of differential equations and hence classify each of the equilibrium points. [10]

Question 28

A particle P of mass 4 kg moves in a straight line along a smooth horizontal track and is connected to two model dampers, AP and CP, and one model spring, BP, as shown in the diagram below. The left-hand end, A, of the damper AP is at a distance $y(t)$ from a fixed point O, and $x(t)$ is the distance of the particle from O. The right-hand end of the damper CP and the right-hand end of the spring BP are attached to a fixed point on the right, which is a distance 3 m from O. Unit vectors $\mathbf{i}$ and $\mathbf{j}$ are defined horizontally and vertically as shown. The parameters for the dampers and the spring are given in the table below.



Object	parameters
Damper AP	Damping coefficient, $2\,\mathrm{N\,s\,m^{-1}}$
Damper CP	Damping coefficient, $4\,\mathrm{N\,s\,m^{-1}}$
Spring BP	Spring stiffness, $9\,\mathrm{N\,m^{-1}}$; natural length, 1 m

(a) Draw a force diagram show all the forces acting on the particle. Define each force that you introduce. [2]

(b) At time t, derive in full each force in terms of the given unit vectors. [5]

(c) Determine a differential equation of motion and show that it may be expressed as

$$4\ddot{x} + 6\dot{x} + 9x = 18 + 2\dot{y}.$$ [2]

(d) (i) Write down the natural angular frequency of this system. [1]

(ii) State, with justification, whether this system is strongly or weakly damped. [1]

(e) Find the amplitude of the motion in the steady-state when $y = 1 + \cos(\Omega t)$. [2]

(f) Find the value of the amplitude when $\Omega \to 0$ and $\Omega \to \infty$. Hence sketch the graph of the amplitude of the motion as a function of Ω. [2]

Question 29

(a) Consider the path defined parameterically in three parts (where R is a positive constant):

$$\begin{cases} x = Rt, & y = 0 & (0 \le t \le 1) \\ x = R\cos(\pi(t-1)), & y = R\sin(\pi(t-1)) & (1 < t \le 2) \\ x = R(t-3), & y = 0 & (2 < t \le 3) \end{cases}$$

Show that the path is continuous and closed, and sketch it. *[2]*

(b) For the vector field function,

$$\mathbf{F} = (3x^2 + 3y^2)\mathbf{i} + 6xy\mathbf{j}$$

find the value of the scalar line integral along the path given in part (a). *[8]*

(c) Determine $\mathbf{curl}\,\mathbf{F}$. *[4]*

(d) What can you deduce about the vector field function? *[1]*

Question 30

A hemisphere of radius R lies above the plane $z = 0$ with the centre of its base at the origin. The density of the hemisphere at a distance r from the origin is given by cr where c is a positive constant.

(a) Using spherical polar coordinates, and with justification, show that the mass of the hemisphere is M which is given by $M = \dfrac{\pi}{2}cR^4$. *[5]*

(b) Use spherical polar coordinates to determine the moment of inertia of the hemisphere about the z-axis in terms of M and R. *[8]*

(c) Write down the moment of inertia of the hemisphere about a vertical axis that passes through the end of a diameter of the base. *[2]*

[END OF QUESTION PAPER]