MST209/H

Module Examination 2012

Mathematical methods and models

Monday 15 October 2012 **2.30 pm–5.30 pm**

Time allowed: 3 hours

Personal Identifier								
Examination No.								

You are **not** allowed to use a calculator in this examination.

There are THREE parts to this paper. In each part of the paper the questions are arranged in the order they appear in the course. There are 115 marks available, but scores greater than 100 will be rounded down to 100.

Part 1 consists of 15 questions each worth 2 marks. You are advised to spend no more than 1 hour on this part. Enter one option in each box provided on the question paper; use your answer book(s) for any rough work. Incorrect answers are not penalised. Cross out mistakes and write your answer next to the box provided.

Part 2 consists of 8 questions each worth 5 marks. You are advised to spend no more than $1\frac{1}{4}$ hours on this part.

Part 3 consists of 7 questions each worth 15 marks. Your best three marks will be added together to give a maximum of 45 marks.

In Parts 2 and 3: Write your answers in the answer book(s) provided. The marks allocated to each part of each question are given in square brackets in the margin. Unless you are directed otherwise in the question, you may use any formula or other information from the *Handbook* provided that you give a reference. Do **not** cross out any answers unless you have supplied a better alternative — everything not crossed out may receive credit.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used (as well as in the boxes above). **Failure to do so may mean that your work cannot be identified.** Use the paper fastener provided to fix together all your answer books, and the question paper, with your signed desk record on top.

PART 1

Each question in this part of the paper is worth 2 marks. Fill in the appropriate response in the box alongside the question.

Question 1

Given that

$$\ln(y+6) + 2\ln(x) = 3\ln 2 + \ln(x+1)$$

and that $x > 0$ and $y > -6$, select the option that gives y as a function of x.

Options Answer:

A $y = \dfrac{2(3x+2)(1-x)}{x^2}$ **B** $y = \dfrac{2(3x+2)(2-x)}{x^2}$

C $y = \dfrac{2(3x+2)(x-1)}{x^2}$ **D** $y = \dfrac{2(3x+2)(x-2)}{x^2}$

Question 2

This question concerns the differential equation

$$\frac{dy}{dx} = x^2 + y^2.$$

Which of the following options is correct?

Options Answer:

A The differential equation may be solved using the separation of variables method but not the integrating factor method.

B The differential equation may be solved using the integrating factor method but not the separation of variables method.

C The differential equation may be solved using either the integrating factor method or the separation of variables method.

D The differential equation cannot be solved using either the integrating factor method or the separation of variables method.

Question 3

Select the option that gives the cross product

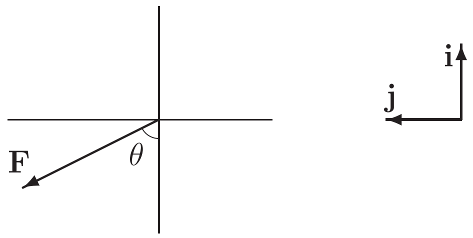
$$(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

Options Answer:

A $\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$ **B** $5\mathbf{i} + \mathbf{j} - \mathbf{k}$ **C** $\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$ **D** $5\mathbf{i} - \mathbf{j} + \mathbf{k}$

Question 4

The diagram shows the force **F** and the directions of the unit vectors, **i** and **j**.



Select the option that corresponds to the component of **F** in the **i**-direction.

Options Answer:

A $|\mathbf{F}|\sin\theta$ **B** $-|\mathbf{F}|\sin\theta$ **C** $|\mathbf{F}|\cos\theta$ **D** $-|\mathbf{F}|\cos\theta$

Question 5

A ball of diameter D is moving vertically upwards with speed v; the ball is subject to air-resistance for which the quadratic model is appropriate. The **i**-axis is vertically upwards. Choose the option that gives the resistance force acting on the ball, where c_2 is the appropriate constant.

Options Answer:

A $c_2D^2v^2\,\mathbf{i}$ **B** $c_2D^2v^2$ **C** $-c_2D^2v^2\,\mathbf{i}$ **D** $-c_2D^2v^2$

Question 6

A horizontal spring whose natural length is l_0 and whose stiffness is $2k$ is compressed so that its length is half the natural length. The **i**-axis is horizontal and its positive direction is to the right. Choose the option that gives the force acting on the left-hand support.

Options Answer:

A $-kl_0\,\mathbf{i}$ **B** $-kl_0$ **C** $kl_0\,\mathbf{i}$ **D** kl_0

Question 7

For what value of x does the matrix $\mathbf{M} = \begin{bmatrix} x & -1 \\ 3 & 2 \end{bmatrix}$ not have an inverse?

Options Answer:

A 0 **B** $\frac{3}{2}$ **C** $-\frac{3}{2}$ **D** 6

Question 8

The matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ has eigenvalues -1 and 4. What are the eigenvalues of $(\mathbf{I} + \mathbf{A}^2)$, where $\mathbf{I}$ is the identity matrix?

Options Answer:

A $-1, 4$ **B** $0, 17$ **C** $2, 17$ **D** $0, 5$

Question 9

Consider a particle of mass 4 kg moving with velocity $2\mathbf{i} + \mathbf{j}\,\mathrm{m\,s^{-1}}$ at a height of 5 m below the datum line, in which the only force acting is the gravitational force. Which of the following options gives the total mechanical energy of the particle where g is the magnitude of the acceleration due to gravity?

Options Answer:

A $10 + 20g$ Joules **B** $20 + 20g$ Joules

C $20 - 20g$ Joules **D** $10 - 20g$ Joules

Question 10

A particle of mass $3m$ is moving in the direction $\mathbf{i} + \mathbf{j}$ with speed v. Choose the option that gives the momentum of the particle.

Options Answer:

A $\frac{3}{2}mv^2$ **B** $3mv$ **C** $3mv(\mathbf{i} + \mathbf{j})$ **D** $\frac{3}{\sqrt{2}}mv(\mathbf{i} + \mathbf{j})$

Question 11

A particle of mass m is moving in a circle of radius R in the vertical plane. Choose the option that gives the component of acceleration of the particle in the $\mathbf{e}_r$-direction.

Options Answer:

A $mR\dot{\theta}^2$ **B** $mR\ddot{\theta}$ **C** $-mR\dot{\theta}^2$ **D** $-mR\ddot{\theta}$

Question 12

Consider the function $f(t) = t^3 + t$ for $t \in [-1, 1]$. Select the option that gives a possible Fourier series for a periodic extension of f, where A_0, A_r and B_r are **non-zero constants** (and $r = 1, 2, 3, \ldots$).

Options Answer:

A $A_0 + \sum_{r=1}^{\infty} A_r \cos(r\pi t)$ **B** $\sum_{r=1}^{\infty} B_r \sin(r\pi t)$

C $\sum_{r=1}^{\infty} A_r \cos(r\pi t)$ **D** $A_0 + \sum_{r=1}^{\infty} (A_r \cos(r\pi t) + B_r \sin(r\pi t))$

Question 13

The method of separation of variables is applied to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}.$$

Assume that $u(x,t) = X(x)T(t)$. Select the option that gives the pair of resulting ordinary differential equations (where α is a non-zero constant).

Options Answer:

A $X' = \alpha X$, $T' + T = \alpha$ **B** $X'' = \alpha X$, $T'' + T' = \alpha T$ **C** $X' = \alpha X$, $T' = \alpha T$ **D** $X'' = \alpha X$, $T'' = \alpha T$

Question 14

A vector field in cartesian coordinates is given by

$$\mathbf{F}(x,y,z) = x^2 y\,\mathbf{i} + y^2 z\,\mathbf{j} + x^2 y^2\,\mathbf{k}.$$

Which option gives $\operatorname{div}\mathbf{F}$?

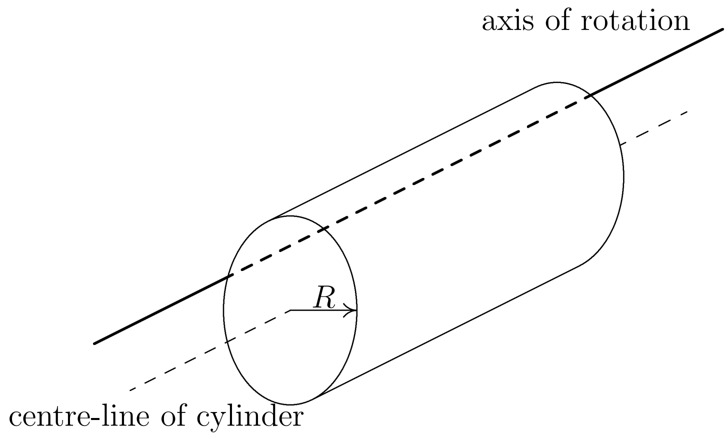
Options Answer:

A $2xy + 2yz$ **B** $x^2y + y^2z + x^2y^2$

C $2xy + 2yz + x^2y^2$ **D** $2xy\,\mathbf{i} + 2yz\,\mathbf{j}$

Question 15

A solid cylinder of mass M and radius R rotates about an axis that is parallel to the centre-line of the cylinder and is on the surface of the cylinder as is shown in the diagram below, where the solid line (dotted where it is hidden by the cylinder) shows the axis of rotation and the centre-line of the cylinder, where visible, is indicated by thinner dotted lines.



Choose the option that gives the moment of inertia of the cylinder about the axis of rotation.

Options Answer:

A $\frac{1}{2}MR^2$ **B** MR^2 **C** $\frac{3}{2}MR^2$ **D** $2MR^2$

PART 2

Each question in this part of the paper is worth 5 marks.

Question 16

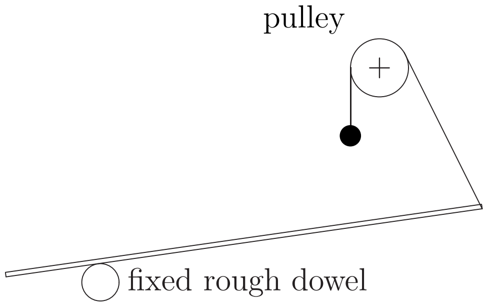
Solve the initial value problem

$$\frac{dy}{dx} = 8x + 2xy^2, \quad y(0) = 2,$$

expressing your answer in the form $y = f(x)$ [5]

Question 17

A plank rests at an incline on top of a rough dowel whose axis is horizontal; the axis of the plank is perpendicular to the peg. A cord is attached to the right-hand end of the plank, and passes over a pulley with a mass hanging down on the other side of the pulley as shown in the diagram on the right. The system is in equilibrium.

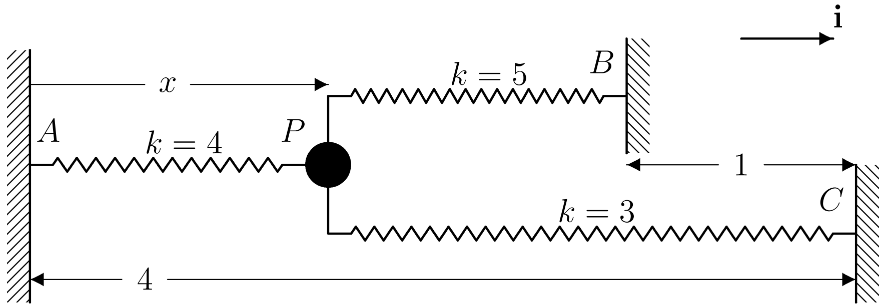


The plank can be taken as a model rod, the pulley as a model pulley, and the cord as a model string.

Draw a force diagram showing all the forces acting on the plank, and define each force identified. [5]

Question 18

A particle P of mass 3 kg is constrained to move in a horizontal straight line along a smooth track. Attached to the particle are three springs whose parameters are given in the table below. The arrangement is shown in the diagram below. The unit vector, $\mathbf{i}$, is defined to be horizontal as shown in the diagram. The distance AC is 4 m, and BC is 1 m.



Spring	stiffness	natural length
AP	$4\,\mathrm{N\,m^{-1}}$	3 m
PB	$5\,\mathrm{N\,m^{-1}}$	1 m
PC	$3\,\mathrm{N\,m^{-1}}$	2 m

What is the total potential energy in the three springs at time t when the displacement of P from A is $x(t)$? Hence write down an expression for the total mechanical energy of the spring system. [5]

Question 19

Consider the following simultaneous linear differential equations.

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = x - y$$

(a) Express the system of equations in matrix form. [1]

(b) Find the eigenvalues and eigenvectors of the matrix of coefficients. [3]

(c) Hence or otherwise find the general solution to this system. [1]

Question 20

Consider the linear model of air resistance, where the force $\mathbf{R}$ is given by

$$\mathbf{R} = -c_1 D\mathbf{v},$$

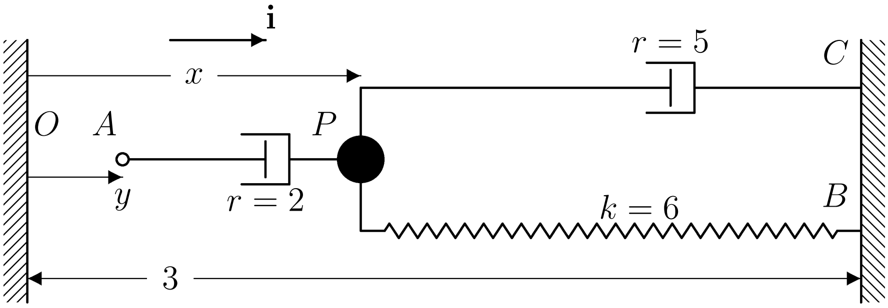
where D is the effective diameter of the particle, $\mathbf{v}$ its velocity, and c_1 a constant.

(a) Use dimensional consistency to determine the dimensions of c_1. [4]

(b) Give the SI units for c_1. [1]

Question 21

A particle P of mass 4 kg moves in a straight line along a smooth horizontal track and is connected to two model dampers, AP and CP, and one model spring, BP, as shown in the diagram below. At time t the left-hand end, A, of the damper AP is at a distance $y(t)$ from a fixed point O, and $x(t)$ is the distance of the particle from O. The right-hand end of the damper CP and the right-hand end of the spring BP are attached to a fixed point on the right, which is a distance 3 m from O. The unit vector $\mathbf{i}$ is defined to be horizontal as shown in the diagram. The parameters for the dampers and spring are given in the table below.



Object	parameters
Damper AP	Damping coefficient, $2\,\mathrm{N\,s\,m^{-1}}$
Damper PC	Damping coefficient, $5\,\mathrm{N\,s\,m^{-1}}$
Spring PB	Stiffness, $6\,\mathrm{N\,m^{-1}}$; natural length, 1 m

Find expressions for the damping forces acting on the particle. [*5*]

Question 22

A scalar field in cylindrical polar coordinates is given by

$$f(\rho, \theta, z) = \rho z^2 \sin^2 \theta.$$

(a) Find $\mathbf{grad}\, f$. [*3*]

(b) Find $\operatorname{div} \mathbf{grad}\, f$. [*2*]

Question 23

S is the region bounded by the curves $x = 0$, $y = 2x^2$ and $y = x + 1$.

(a) Sketch the region S defined above. [*1*]

(b) Find the value of the area integral of the function $f(x, y) = 8xy$ over the region S. [*4*]

PART 3

Each question in this part of the paper is worth 15 marks. All of your answers will be marked and the marks from your best three answers will be added together. A maximum of 45 marks can be obtained from this part.

Question 24

Consider the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 10e^{-2x} + 4x.$$

(a) Find the general solution of this equation. [*10*]

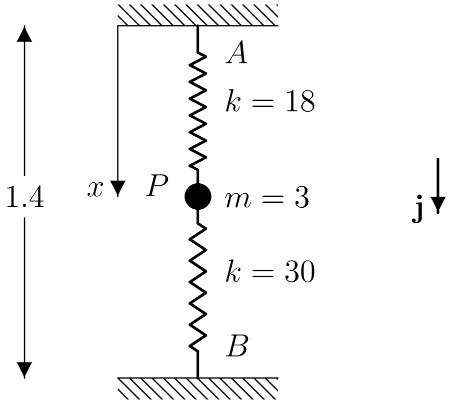
(b) Hence find the particular solution that satisfies $y(0) = 0$ and $y'(0) = 0$. [*4*]

(c) Describe briefly the behaviour of y as $x \to \infty$. [*1*]

Question 25

A particle P of mass 3 kg is constrained to move in a vertical straight line between the fixed points, A and B. Attached to the particle are two springs whose parameters are given in the table below. The arrangement is shown in the diagram below. Assume that air-resistance is negligible. The unit vectors, $\mathbf{i}$ and $\mathbf{j}$, are defined horizontally and vertically as shown in the diagram. The distance AB is 1.4 m. At time t the displacement of P from A is $x(t)$.

Spring	stiffness	natural length
AP	$18\,\mathrm{N\,m^{-1}}$	1 m
PB	$30\,\mathrm{N\,m^{-1}}$	2 m



(a) Draw a force diagram show all the forces acting on the particle. Define each force you introduce. [*2*]

(b) At time t, express each force in terms of the given unit vectors. [*4*]

(c) Find the value of x in terms of g, the magnitude of the acceleration due to gravity, when the system is in equilibrium. [*2*]

(d) Determine a differential equation of vertical motion when the particle is in motion, simplifying your result as far as possible. [*1*]

(e) Solve the equation of motion when it is released from rest at a distance of 0.2 m above the equilibrium position found in part (c). [*3*]

(f) Sketch the graph of x against t for the range $0 \le t \le \pi$ showing all the salient features of the motion. [*2*]

(g) If the two springs are inter-changed, what effect will this have on the period of the motion? Briefly justify your answer. [*1*]

Question 26

Given the function

$$f(x, y) = x^2y^2 - x^2 - 4y^2$$

(a) Calculate the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$. [5]

(b) Find the positions of the stationary points of $f(x, y)$. [4]

(c) Classify each of the stationary points as a local maximum, minimum or saddle point. [3]

(d) Find the second-order Taylor polynomial of f about (2,1). [3]

Question 27

Two populations X and Y co-exist according to the model

$$\frac{dx}{dt} = x\left(1 - \tfrac{1}{3}x - \tfrac{1}{3}y\right)$$
$$\frac{dy}{dt} = -y\left(1 - \tfrac{1}{2}x - \tfrac{1}{4}y\right)$$

where x and y give the size of the populations X and Y respectively in some appropriate units. The model is valid for $x \geq 0$ and $y \geq 0$.

(a) Find all the equilibrium points of this system. [6]

(a) Classify each of the equilibrium points. [9]

Question 28

(a) The matrix

$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -5 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

has an eigenvector $[-2 \quad 2 \quad 1]^T$ and the corresponding eigenvalue is -7.

(i) Another eigenvector is $[2 \quad 1 \quad 2]^T$; find the corresponding eigenvalue.

(ii) Find the remaining eigenvalue and its corresponding eigenvector. [5]

The differential equation of motion for a system of three particles connected by springs is

$$\ddot{\mathbf{x}} = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -5 & 0 \\ 2 & 0 & -3 \end{bmatrix} \mathbf{x}$$

where x_1, x_2 and x_3 are the displacements of the three particles from their respective equilibrium positions and $\mathbf{x} = [x_1 \; x_2 \; x_3]^T$.

(b) Write down the general solution for $\mathbf{x}$. [4]

(c) The system is at rest initially and $x_2(0) = 0.2$. What should be the displacements of the other two particles for normal mode motion at the lowest frequency? [2]

(d) The particles are at rest and they are given an initial displacement $\mathbf{x}(0) = [0 \quad 1 \quad 1]^T$. Determine the expressions for the displacement of each particle at time t. [4]

Question 29

Consider the function

$$f(x) = 2x, \quad (0 \le x < \pi).$$

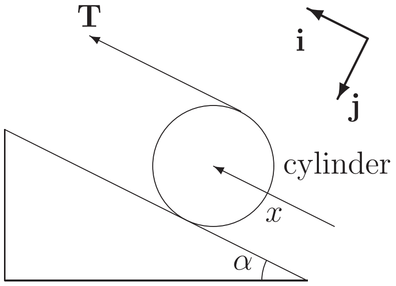
(a) Write down the even and odd periodic extensions of f and sketch each over the range $[-2\pi, 2\pi]$. *[4]*

(b) For each of the above extensions, determine symbolically the coefficients of the general term of the Fourier series. Hence, for each extension, write down the first three (non-zero) terms of the Fourier series approximation. *[9]*

(c) State, with a reason, which Fourier series would provide a better approximation to the approximated function. *[2]*

Question 30

A uniform cylinder, whose radius is R and whose mass is M, rolls without slipping up a rough plane inclined at an angle α to the horizontal as shown in the diagram below. It has a string wound round it and the free end of the string has a constant tension, $\mathbf{T}$. The axis of the cylinder is horizontal, and its motion is along the line of greatest slope. The inclination of the string is such that the string is always parallel to the plane. x is the displacement of the cylinder parallel to the plane from a fixed position; the unit axes are as defined in the diagram, $\mathbf{i}$ being parallel to the plane, and $\mathbf{j}$ into the plane.



(a) Draw a force diagram showing all the forces acting on the cylinder. *[2]*

(b) Express all the forces as components of the given unit vectors. *[2]*

(c) Define the relationship between the angle of rotation, θ, of the cylinder and the distance the cylinder has travelled along the incline, defining any variables you introduce. *[2]*

(d) Taking the centre of the cylinder as the origin, find a position vector on the line of action of each force in terms of the unit vectors, and hence find the torques due to the forces acting on the cylinder. *[3]*

(e) Apply the torque law about the axis of the cylinder to find a differential equation of motion for the rotation of the cylinder. *[2]*

(f) Apply Newton's second law to obtain a differential equation of motion for the translation of the cylinder. *[2]*

(g) Write down the moment of inertia of the cylinder about its axis of symmetry and hence show that

$$\ddot{x} = \frac{4|\mathbf{T}|}{3M} - \frac{2g\sin\alpha}{3}.$$ *[2]*

[END OF QUESTION PAPER]