# M337/K

**Course Examination 2007**
**Complex Analysis**

**Friday 12 October 2007** **2.30 pm – 5.30 pm**

**Time allowed: 3 hours**

There are **TWO** parts to this paper.

In Part 1 (64% of the marks) you should attempt as many questions as you can.

In Part 2 (36% of the marks) you should attempt no more than **TWO** questions.

**At the end of the examination**

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Put all your used answer books and your question paper together with your signed desk record on top. Fix them all together using the fastener provided.

The use of calculators is **NOT** permitted in this examination.

## PART 1

*(i) You should attempt as many questions as you can in this part.*
*(ii) Each question in this part carries 8 marks.*

### Question 1

Give the Cartesian form of the following complex numbers, simplifying your answer as far as possible.

(a) $\dfrac{1+i}{2-i}$ [2]

(b) the principal cube root of $-i$ [2]

(c) $\operatorname{Log}\left(\dfrac{1+i}{\sqrt{2}}\right)$ [2]

(d) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{2-4i}$ [2]

### Question 2

Let

$$A = \{z : 1 < |z-i| < 2\} \quad \text{and} \quad B = \{z : |\operatorname{Re} z| \le 3, |\operatorname{Im} z| \le 1\}.$$

(a) Make separate sketches of the sets $A$, $B$ and $A - B$. [3]

(b) Write down which of the four sets $A$, $B$ and $A - B$, if any, is

(i) a region;

(ii) a simply-connected region;

(iii) closed;

(iv) compact. [5]

### Question 3

In this question $\Gamma$ is the line segment from $i$ to $1-i$.

(a) (i) Determine the standard parametrization for the line segment $\Gamma$.

(ii) Evaluate

$$\int_\Gamma \operatorname{Re} z \, dz.$$ [3]

(b) Determine an upper estimate for the modulus of

$$\int_\Gamma \frac{\cos z}{6+z^2}\, dz.$$ [5]

## Question 4

Evaluate the following integrals, in which $C = \{z : |z| = 1\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C \frac{\cos z}{(z-3)^3}\,dz,$ [2]

(b) $\displaystyle\int_C \frac{\cos z}{z(z-3)^2}\,dz$ [3]

(c) $\displaystyle\int_C \frac{\cos z}{z^3}\,dz$ [3]

## Question 5

(a) Find the residues of the function

$$f(z) = \frac{1}{(2z+1)(z+2)}$$

at each of the poles of $f$. [4]

(b) Hence evaluate the integral

$$\int_0^{2\pi} \frac{1}{5 + 4\cos t}\,dt.$$ [4]

## Question 6

Let $f(z) = z^5 + 3z^2 + i$.

(a) Determine the number of zeros of $f$ that lie inside:

(i) the circle $C_1 = \{z : |z| = 2\}$,

(ii) the circle $C_2 = \{z : |z| = 1\}$. [6]

(b) Show that the equation

$$z^5 + 3z^2 + i = 0$$

has exactly three solutions in the set $\{z : 1 < |z| < 2\}$. [2]

## Question 7

Let $q(z) = 2/\overline{z}$ be a velocity function on $\mathbb{C} - \{0\}$.

(a) Explain why $q$ represents a model fluid flow on $\mathbb{C} - \{0\}$. [1]

(b) Determine a complex potential function for this flow. Hence sketch the streamline through the point $i$ and the streamline through the point $1 + i$. In each case indicate the direction of flow. [5]

(c) Evaluate the flux of $q$ across the unit circle $\{z : |z| = 1\}$. [2]

**Question 8**

(a) Find the fixed points of the function $f(z) = z^2 - 4z + 6$ and classify them as (super-)attracting, repelling or indifferent. [3]

(b) Which of the following points $c$ lie in the Mandelbrot set:

(i) $c = -1 + i$;

(ii) $c = -1 - \frac{1}{8}i$.

Justify your answer in each case. [5]

## PART 2

*(i) You should attempt no more than* **TWO** *questions in this part.*
*(ii) Each question in this part carries 18 marks.*

### Question 9

(a) Let $f$ be the function

$$f(z) = \bar{z}(\operatorname{Im} z + z).$$

(i) Write $f(x + iy)$ in the form $u(x, y) + iv(x, y)$, where $u$ and $v$ are real-valued functions.

(ii) Use the Cauchy-Riemann theorem and its converse to show that $f$ is differentiable at 0, but not analytic there.

(iii) Evaluate $f'(0)$. [*9*]

(b) Let $g$ be the function $g(z) = z^3$.

(i) Show that $g$ is conformal on $\mathbb{C} - \{0\}$.

(ii) Describe the effect of $g$ on a small disc centred at $i$.

(iii) $\Gamma_1$ and $\Gamma_2$ are smooth paths meeting at $i$ and $-i$ given by

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = ti \quad (t \in \mathbb{R}).$$

Sketch these paths on a single diagram, clearly indicating their directions.

(iv) Using part (b)(ii), or otherwise, sketch the directions of $g(\Gamma_1)$ and $g(\Gamma_2)$ at $g(i)$.

(v) Explain why $g$ is not conformal at 0. [*9*]

**Question 10**

Let $f$ be the function

$$f(z) = \frac{z}{\sin z}.$$

(a) Use the Taylor series about 0 for $\sin z$ and $(1-z)^{-1}$ to show that the Laurent series about 0 for $f$ is

$$\frac{z}{\sin z} = 1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots, \qquad \text{for } 0 < |z| < \pi.$$

Hence evaluate the integral

$$\int_C \frac{1}{z^2 \sin z}\, dz,$$

where $C$ is the unit circle $\{z : |z| = 1\}$. [*8*]

(b) Write down the domain $A$ of $f$. Use the Uniqueness Theorem to show that $f$ is the only analytic function with domain $A$ such that

$$f(iy) = \frac{y}{\sinh y}, \qquad \text{for } y \in \mathbb{R},\ y > 0.$$ [*5*]

(c) Show that $f$ has singularities at points of the form $k\pi$, $k \in \mathbb{Z}$, and classify these singularities.
(Hint: You may find it helps to use $\sin z = (-1)^k \sin(z - k\pi)$, where $k \in \mathbb{Z}$.) [*5*]

**Question 11**

(a) Show that

$$\max\left\{|e^{z^4}| : |z| \le 2\right\} = e^{16}$$

and find the point, or points, at which this maximum is attained. [*9*]

(b) Show that the functions

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{5}\right)^n \qquad (|z| < 5)$$

and

$$g(z) = -\sum_{n=1}^{\infty} \left(\frac{5}{z}\right)^n \qquad (|z| > 5)$$

are indirect analytic continuations of each other. [*9*]

**Question 12**

(a) Determine the extended Möbius transformation $\widehat{f_1}$ which maps $i$ to $0$, $\infty$ to $1$ and $-i$ to $\infty$. [3]

(b) Let

$$R = \{z : |z-1| < \sqrt{2}\} \cap \{z : |z+1| < \sqrt{2}\},$$
$$S = \{z_1 : 3\pi/4 < \operatorname{Arg}_{2\pi}(z_1) < 5\pi/4\},$$
$$T = \{w : \operatorname{Re} w > 0\}.$$

(i) Sketch the regions $R$, $S$ and $T$.

(ii) Explain why $\widehat{f_1}(R) = S$.

(iii) Hence determine a one-one conformal mapping $f$ from $R$ onto $T$.

(iv) Determine a one-one conformal mapping $g$ from $R$ onto the open unit disc $D = \{z : |z| < 1\}$. (There is no need to simplify your answer.) [15]

**[END OF QUESTION PAPER]**