MST209 2013 exam solutions

The references to the Handbook are given as section followed by page number e.g. (5 p26)

1. Differentiating implicitly

$x^2\frac{dy}{dx}+2xy+3x^2-4y\frac{dy}{dx}=0$

Rearranging $(x^2-4y)\frac{dy}{dx}=-(2xy+3x^2)$

So the answer is A.

2. The equation is linear (13 p26) so we can use the integrating factor method. Rearranging

$\frac{dy}{dx}=x-4xy=x(1-4y)$ (9 p26) so the equation is separable making the answer C.

3. If $\mathbf{a}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, then (10 p28)

$|\mathbf{a}|=\sqrt{1+4+4}=\sqrt{9}=3$

and if $\mathbf{b}=\mathbf{7i}+4\mathbf{j}-4\mathbf{k}$, then

$|\mathbf{b}|=\sqrt{49+16+16}=\sqrt{81}=9$

and $\mathbf{a.b}=7-8-8=-9$. (15 p29)

$\cos\theta=\frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}=-\frac{1}{3}$ (16 p29) so the answer is D.

4. The force makes and angle of $\pi-\theta$ with the **i** direction so the component in the **i**-direction is $|\mathbf{F}|\cos(\pi-\theta)=-|\mathbf{F}|\cos(\theta)$ (p15 and 8 p31). The answer is D.

5. (2 and 3 p35) $U=-mgx$ and

$T=\frac{1}{2}mv^2$ so the answer is D.

6. We want zeros in the second column below the leading diagonal so the answer is C. (11 p37)

7. $\begin{bmatrix}5 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}7\\ 7\end{bmatrix}$ so one eigenvalue is 7 (1 p41).

The trace of the matrix is 9 so the other matrix is 2 (11 and 12 p41). The answer is A.

8. $\delta f\cong 5\delta x-\delta y$ (6 p45).

The maximum error in $f\cong 5|\delta x|+|\delta y|$

$=0.05+0.03=0.08$ so the answer is D.

9. Integrating (4 p32 and 3 p49) and using the initial velocity gives

$$\dot{\mathbf{x}}=-3t^2\mathbf{i}+4t\mathbf{j}+2\mathbf{i}+3\mathbf{j}$$

When $t=1$ $\dot{\mathbf{x}}=-\mathbf{i}+7\mathbf{j}$ and the speed is $\sqrt{1+49}=\sqrt{50}$ (3 p32) so the answer is B.

10. From 5 p53 the answer is A.

11. The length of the damper $l=y-x$ so $\dot{l}=\dot{y}-\dot{x}$ and $\mathbf{R}=-r(\dot{y}-\dot{x})\mathbf{i}$ (4 p54) so the answer is D.

12. Momentum $=m\mathbf{v}$ (9 p58)

$m=4$ and $\mathbf{v}=\frac{13(-5\mathbf{i}+12\mathbf{j})}{|-5\mathbf{i}+12\mathbf{j}|}=-5\mathbf{i}+12\mathbf{j}$ so the answer is C.

13. (11 p61) $g(-x)=2+x$ so the answer is B.

14 (8 p45) or (6 p64) **grad** $f=2x\mathbf{i}-\mathbf{j}$ so **grad** $f(1,-1)=2\mathbf{i}-\mathbf{j}$ so the answer is B.

15 Looking at the lines parallel to *x* first the answer is C (2 p69)

16. The auxiliary equation is (5 p26)

$$2\lambda^2+7\lambda+6=0$$

Factorising $(2\lambda+3)(\lambda+2)=0$

Giving $\lambda=-\frac{3}{2}$ and $\lambda=-2$.

The general solutions is $y=Ae^{-\frac{3}{2}t}+Be^{-2t}$

Differentiating wrt *x*

$\frac{dy}{dx}=-\frac{3}{2}Ae^{-\frac{3}{2}t}-2Be^{-2t}$

Fitting the initial condition $y(0)=1$ gives

$$1=A+B \quad (1)$$

Fitting the initial condition $\frac{dy}{dx}(0)=-1$ gives

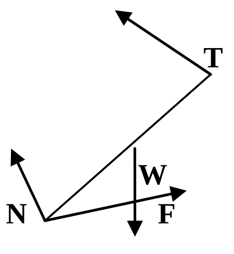
$-1=-\frac{3}{2}A-2B$ or $3A+4B=2$ (2)

Subtracting (1) times 3 from (2) gives $B=-1$

Substituting into (1) gives $A=2$ and so the required particular solution is

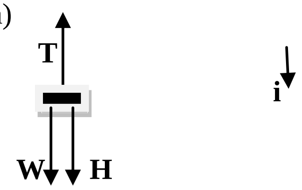
$$y=2e^{-\frac{3}{2}t}-e^{-2t}$$

17.



17.(cont) **T** is the tension in the cord acting away from the plank along the cord.
W is the weight of the plank acting vertically downwards
N is the normal reaction between the plane and the plank acting perpendicular to the plane
F is the frictional force stopping the plank slipping down the plane.

18. (a)



T is the tension in the string
W is the weight of the particle
H is the spring force

(b) Choosing the origin at the top fixed point and **i** downwards.
Length of the spring $3l_0 - L$
Using Hooke's law (1 p34)
$\mathbf{H} = -k(3l_0 - L - l_0)(-\mathbf{i}) = k(2l_0 - L)\mathbf{i}$
$\mathbf{T} = -|\mathbf{T}|\mathbf{i}$ $\quad\mathbf{W} = mg\mathbf{i}$
In equilibrium **T** + **W** + **H** = **0** (3 p30)
Resolving in the **i** direction

$$-|\mathbf{T}| + mg + k(2l_0 - L) = 0$$

Giving $|\mathbf{T}| = mg + k(2l_0 - L)$
(Alternatively you could take the origin at the bottom and **i** upwards)

19. (a) At equilibrium (6 p47)

$$x^2 - 3xy = 0 \text{ (1) and } x - y^2 - 2 = 0 \quad (2)$$

If $x = 3$ and $y = 1$
$x^2 - 3xy = 9 - 9 = 0$ so (1) is satisfied and
$x - y^2 - 2 = 3 - 1 - 2 = 0$ so (2) is satisfied. (3,1) is an equilibrium point.
(Alternatively you could evaluate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at (3, 1) directly)
b) (8 p47)
$u = x^2 - 3xy$ so $\frac{\partial u}{\partial x} = 2x - 3y$ $\frac{\partial u}{\partial y} = -3x$
$v = x - y^2 - 2$ so $\frac{\partial v}{\partial x} = 1$ $\frac{\partial v}{\partial y} = -2y$ and

$$J(x, y) = \begin{bmatrix} 2x - 3y & -3x \\ 1 & -2 \end{bmatrix}$$

At (3, 1) (9 and 10 P47)

$$J(3,1) = \begin{bmatrix} 3 & -9 \\ 1 & -2 \end{bmatrix}$$

The characteristic equation is
$\begin{vmatrix} 3 - \lambda & -9 \\ 1 & -2 - \lambda \end{vmatrix} = 0$ (13 p41)

Expanding $(3 - \lambda)(-2 - \lambda) + 9 = 0$ or
$\lambda^2 - \lambda + 3 = 0$. Using the quadratic formula

$$\lambda = 1 \pm \sqrt{1 - 12} = 1 \pm \sqrt{11}i$$

The eigenvalues are complex with positive real component so the equilibrium point is a spiral source (10 p48).

20 (a) Using 18b p44 the general solution of the equations is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(C_1 \cos t + C_2 \sin t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix}(C_3 \cos 2t + C_4 \sin 2t)$$

(b)

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(-C_1 \sin t + C_2 \cos t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix}(-2C_3 \sin 2t + 2C_4 \cos 2t)$$

At rest $\dot{\mathbf{x}}(0) = \mathbf{0}$ so $C_2 = C_4 = 0$
If we are considering the higher angular frequency then $C_1 = 0$ and so
$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} C_3 \cos 2t$. As $x_2(0) = 0.2$
$C_3 = -0.2$ giving $x_1(0) = -0.4$.
(There are other ways to approach this and I suspect if you gave the correct answer with no reason you would get full marks but I cannot say for definite)

(c) As in (b) as we are starting from rest $C_2 = C_4 = 0$ so

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(C_1 \cos t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix}(C_3 \cos 2t)$$

At $t = 0$ $\begin{bmatrix} 0.3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} C_1 + \begin{bmatrix} 2 \\ -1 \end{bmatrix} C_3$
Giving

$$0.3 = C_1 + 2C_3 \text{ (1) and } 0 = C_1 - C_3 \text{ (2)}$$

Substituting (2) into (1) gives $0.3 = 3C_3$ or $C_3 = 0.1$ and from (2) $C_1 = 0.1$ and
$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}(0.1 \cos t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix}(0.1 \cos 2t)$.
(This could be expressed as

$$\mathbf{x} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \cos t + \begin{bmatrix} 0.2 \\ -0.1 \end{bmatrix} \cos 2t$$

There are other ways of obtaining the result but I suspect mine is the most common)

21. As $\boldsymbol{\omega} = \sin \Omega t\, \mathbf{k}$ $\dot{\theta} = \sin \Omega t$ (12 p60)
(a) $\dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta = R \sin \Omega t\, \mathbf{e}_\theta$ (3 p59)

(b) The angular momentum is $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ (12 p60)
$\mathbf{l} = R\mathbf{e}_r \times m\, R \sin \Omega t\, \mathbf{e}_\theta = mR^2 \sin \Omega t\, \mathbf{k}$

c) Torque = $\dot{\mathbf{l}} = mR^2\Omega \cos \Omega t\, \mathbf{k}$ (13 p60)

22. Let $u = X(x)T(t)$ (9 p63)
Substituting in the equation

$$X'''T = \frac{1}{c^2}(XT'' + XT)$$

Divide by XT

$$\frac{X'''}{X} = \frac{1}{c^2}\left(\frac{T''}{T} + 1\right) = \mu$$

So $X''' = \mu X$ and $\frac{T''}{T} + 1 = \mu c^2$
or $X''' = \mu X$ and $T'' = (\mu c^2 - 1)T$.

23. a) Using the given unit vectors $\mathbf{W} = mg\mathbf{i}$ and $\mathbf{T} = -|\mathbf{T}|\mathbf{i}$ and their position vectors are $\mathbf{r}_W = \mathbf{0}$ and $\mathbf{r}_T = R\mathbf{j}$. The torques are $\mathbf{\Gamma}_W = \mathbf{0}$ and $\mathbf{\Gamma}_T = R\mathbf{j} \times -|\mathbf{T}|\mathbf{i} = R|\mathbf{T}|\mathbf{k}$ (13 p32)
The equation of rotational motion is (16 p75)

$$I\ddot{\theta} = R|\mathbf{T}| \quad (1)$$

d) Using Newton's second law (13 p74)

$$M\ddot{x}\mathbf{i} = \mathbf{T} + \mathbf{W}$$

Resolving in the **i**-direction gives

$$M\ddot{x} = -|\mathbf{T}| + Mg \quad (2)$$

e) If $R\dot{\theta} = \dot{x}$ then $R\ddot{\theta} = \ddot{x}$
Substituting in (1) and rearranging gives

$$|\mathbf{T}| = \frac{I}{R^2}\ddot{x}$$

Substituting in (2) and multiplying by R^2

$$MR^2\ddot{x} = -I\ddot{x} + MgR^2$$

Rearranging gives

$$\ddot{x} = \frac{MR^2}{MR^2 + I}g$$

24. Separating the variables (9 p26)

$$\frac{1}{y^2}\frac{dy}{dx} = 5\tan x + 6x^2$$

Integrating wrt x

$$\int y^{-2}dy = \int 5\tan x + 6x^2 dx$$

$$-\frac{1}{y} = -5\ln(\cos x) + 2x^3 + C$$

as $\cos x > 0$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (p24 and p12).
Substituting $y(0) = -\frac{1}{2}$ in the above gives

$$2 = -5\ln 1 + C \implies C = 2$$

Giving $-\frac{1}{y} = -5\ln(\cos x) + 2x^3 + 2$
Rearranging gives

$$y = \frac{1}{5\ln(\cos x) - 2x^3 - 2}$$

b) Using Euler's method (7 p25)

$$f(x,y) = x^3 + 4xy^2 \quad x_0 = 0 \quad Y_0 = y_0 = 1$$
$$h = 0.1 \quad x_1 = 0.1 \quad x_2 = 0.2$$
$$Y_1 = Y_0 + 0.1f(x_0, Y_0) = 1 + 0.1(0) = 1$$
$$Y_2 = Y_1 + 0.1f(x_1, Y_1) = 1 + 0.1((0.1)^3 + 0.4)$$
$$= 1 + 0.1(0.401) = 1.0401$$

The approximate solution to $y(0.2) = 1.0401$

c) The equation is linear so the integrating factor method applies. (13 p26)
Dividing by x

$$\frac{dy}{dx} - \frac{4}{x}y = x \quad (1)$$

The integrating factor is

$$p(x) = \exp\int -\frac{4}{x}dx = \exp(-4\ln x) = x^{-4}$$

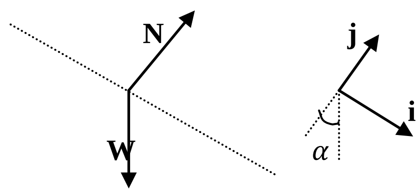
Multiply (1) by $p(x)$

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = x^{-3} \text{ or } \frac{d}{dx}(x^{-4}y) = x^{-3}$$

Integrating $x^{-4}y = -\frac{x^{-2}}{2} + C$
Multiplying by x^4 $y = -\frac{x^2}{2} + Cx^4$
$y(1) = \frac{1}{2} = -\frac{1}{2} + C$ so $C = 1$ and

$$y = x^4 - \frac{x^2}{2}$$

25 (a)
m is the mass of the luggage
g is the acceleration due to gravity
v is the speed of the luggage
N is the normal reaction of the ramp on the luggage
W is the weight of the luggage
x is the distance measured down the ramp from the top.
The unit vector **i** is pointing down the ramp and the unit vector **j** is perpendicular to the ramp and the force diagram is given as



N = |**N**|**j** $\quad$ $\mathbf{W} = mg(\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j})$
By Newton's 2nd law (10 p33) $m\mathbf{a} = \mathbf{N} + \mathbf{W}$
Resolving in the **i**-direction using $\mathbf{a} = \ddot{x}\mathbf{i}$

$$m\ddot{x} = mg\sin\alpha \text{ or } \ddot{x} = g\sin\alpha$$

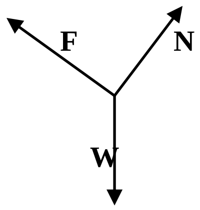
As this is constant we can use the constant acceleration formula $v^2 = v_0^2 + 2a_0x$ (7 p33)
where $a_0 = g\sin\alpha$ and $v_0 = 0$, giving

$$v^2 = 2g\sin\alpha\, x$$

25 (a) cont   At the end of the rollers $x = l_1$ so the speed at the end of the rollers is $\sqrt{2gl_1 \sin\alpha}$

(Alternatively you could have used conservation of mechanical energy as there is no friction. (2 and 3 p35) or derived the formula for v^2 by putting $\ddot{x} = v\frac{dv}{dx}$ and integrating. (6 p32))

(b) The force diagram now is



W and **N** are defined as above and **F** is the frictional force which acts along the ramp opposing the motion.

Using the sliding friction law (12 p33)

$\mathbf{F} = -|\mathbf{F}|\mathbf{i} = -\mu|\mathbf{N}|\mathbf{i}$

By Newton's 2nd law (10 p33)

$$m\mathbf{a} = \mathbf{N} + \mathbf{W} + \mathbf{F}$$

Resolving in the **i**-direction

$$m\ddot{x} = mg\sin\alpha - \mu|\mathbf{N}| \quad (1)$$

Resolving in the **j**-direction

$$0 = |\mathbf{N}| - mg\cos\alpha \quad (2)$$

Substituting (2) into (1) gives

$$m\ddot{x} = mg\sin\alpha - \mu mg\cos\alpha$$

Cancelling m $\ddot{x} = g\sin\alpha - \mu g\cos\alpha$

Again the acceleration is constant so

$v^2 = v_0^2 + 2a_0(x - x_0)$ (7 p33)

where $x_0 = l_1$ $a_0 = g\sin\alpha - \mu g\cos\alpha$

and $v_0 = \sqrt{2gl_1\sin\alpha}$, giving

$$v^2 = 2gl_1\sin\alpha + 2g(g\sin\alpha - \mu g\cos\alpha)(x - l_1)$$

At the bottom of the ramp $x = L$ and

$$v^2 = 2gl_1\sin\alpha + 2g(g\sin\alpha - \mu g\cos\alpha)(L - l_1)$$
$$= 2g(\sin\alpha - \mu\cos\alpha)L + 2g\mu l_1\cos\alpha$$

Speed at the bottom of the ramp is

$$\sqrt{2g(\sin\alpha - \mu\cos\alpha)L + 2g\mu l_1\cos\alpha}$$

c) If $v = 0$ at the end of the ramp then

$$2g(\sin\alpha - \mu\cos\alpha)L + 2g\mu l_1\cos\alpha = 0$$

Rearranging gives

$$l_1 = -\frac{(\sin\alpha - \mu\cos\alpha)}{\mu\cos\alpha}L = \left(1 - \frac{\tan\alpha}{\mu}\right)L$$

d) If μ is very large, $\frac{1}{\mu} \to 0$ and so $l_1 \to L$. This does seem reasonable as a limit as if μ is very large the luggage will come to a stop very quickly.

26 (a) The trace is 6 so the repeated eigenvector is 3 (11 and 12 p41)

(Alternatively the characteristic equation (13 p41) is $\lambda^2 - 6\lambda + 9 = 0$ giving the repeated root as $\lambda = 3$.)

The eigenvector equations (13 p41) are

$$(5 - 3)x + 4y = 0 \quad \text{and} \quad -x + (1 - 3)y = 0$$

Both give $x = -2y$ so a corresponding eigenvector is $[-2 \quad 1]^T$ (or any multiple of this).

(b) The matrix form of the equations is

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}5 & 4\\ -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-18\\ 0\end{bmatrix}$$

c) As there are repeated eigenvalues we need to use 8 p43 and find **b** such that $(\mathbf{A} - 3\mathbf{I})\mathbf{b} = \mathbf{v}$ where **A** is the matrix in part (a) and **v** is an eigenvector so

$$\begin{bmatrix}2 & 4\\ -1 & -2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{bmatrix}-2\\ 1\end{bmatrix}$$

Both equations give $b_1 + 2b_2 = -1$

Taking $b_2 = 0$ $b_1 = -1$ and $\mathbf{b} = [-1 \quad 0]^T$

(It would be just as correct to take any other value for b_2 but 0 is the simplest one)

The solution to the associated homogeneous equations (11 c) p43) is

$$\mathbf{x} = \alpha\left[\begin{bmatrix}-2\\ 1\end{bmatrix}t + \begin{bmatrix}-1\\ 0\end{bmatrix}\right]e^{3t} + \beta\begin{bmatrix}-2\\ 1\end{bmatrix}e^{3t}$$

d) For the particular integral (14 p43) we try $\mathbf{x} = [c \quad d]^T$ where c and d are constants so $\dot{\mathbf{x}} = \mathbf{0}$

Substituting in the differential equations we get

$$0 = 5c + 4d - 18 \ (1) \text{ and } 0 = -c + d \ (2)$$

Substituting (2) into (1) and rearranging gives

$9d = 18$ or $d = 2$ and so from (2) $c = 2$ and the particular integral is $\mathbf{x} = [2 \quad 2]^T$

The general solution (13 p43) is

$$\mathbf{x} = \alpha\left[\begin{bmatrix}-2\\ 1\end{bmatrix}t + \begin{bmatrix}-1\\ 0\end{bmatrix}\right]e^{3t} + \beta\begin{bmatrix}-2\\ 1\end{bmatrix}e^{3t} + \begin{bmatrix}2\\ 2\end{bmatrix}$$

e) (3 p42) Substituting in $x(0) = 3$ and $y(0) = 1$ we get $3 = -\alpha - 2\beta + 2$ (1) and $1 = \beta + 2$

so $\beta = -1$ and substituting in (1) $\alpha = 1$.

The particular solution is

$$\mathbf{x} = \left[\begin{bmatrix}-2\\ 1\end{bmatrix}t + \begin{bmatrix}-1\\ 0\end{bmatrix}\right]e^{3t} - \begin{bmatrix}-2\\ 1\end{bmatrix}e^{3t} + \begin{bmatrix}2\\ 2\end{bmatrix}$$

(or $\mathbf{x} = \begin{bmatrix}1 - 2t\\ t - 1\end{bmatrix}e^{3t} + \begin{bmatrix}2\\ 2\end{bmatrix}$)

27 (a) $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ (3 and 2 p32) so

$\mathbf{v} = \left(\frac{1}{3}t^3 - t + \frac{5}{3}\right)\mathbf{i} + \left(-gt - \frac{1}{2}t^2 + t + g\right)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = (t^2 - 1)\mathbf{i} + (-g - t + 1)\mathbf{j}$ (4 p32)

27 (b) The weight $\mathbf{W} = -mg\mathbf{j}$ and $\mathbf{N}$ is the normal reaction so from Newton's second law $m\mathbf{a} = \mathbf{N} + \mathbf{W}$ or

$$m\left((t^2-1)\mathbf{i} + (-g-t+1)\mathbf{j}\right) = \mathbf{N} - mg\mathbf{j}$$

Giving $\mathbf{N} = m\left((t^2-1)\mathbf{i} + (1-t)\mathbf{j}\right)$

c) If contact is lost with the track then $|\mathbf{N}|$ is zero. This means both components need to be zero and this can only occur if $t = 1$.

d) The initial conditions for the projectile motion are the position vector

$\mathbf{r}(1) = \left(\frac{1}{12} - \frac{1}{2} + \frac{5}{3} - \frac{1}{4}\right)\mathbf{i} + \left(-\frac{1}{2}g - \frac{1}{6} + \frac{1}{2} + g + \frac{2}{3} - \frac{1}{2}g\right)\mathbf{j} = \mathbf{i} + \mathbf{j}$ and the velocity $\dot{\mathbf{r}}(1) = \left(\frac{1}{3} - 1 + \frac{5}{3}\right)\mathbf{i} + \left(-g - \frac{1}{2} + 1 + g\right)\mathbf{j}$

$$= \mathbf{i} + \frac{1}{2}\mathbf{j}$$

The equation of motion is $\ddot{\mathbf{r}} = -g\mathbf{j}$ (5 p49)

Integrating gives $\dot{\mathbf{r}} = -gt\mathbf{j} + \mathbf{c}$. Using the initial velocity $\mathbf{c} = \mathbf{i} + \left(\frac{1}{2} + g\right)\mathbf{j}$ and $\dot{\mathbf{r}} = \mathbf{i} + \left(\frac{1}{2} + g - gt\right)\mathbf{j}$

Integrating gives $\mathbf{r} = t\mathbf{i} + \left(\frac{1}{2}t + gt - \frac{gt^2}{2}\right)\mathbf{j} + \mathbf{d}$.

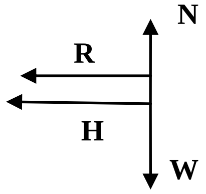
Using the initial position $\mathbf{d} = \mathbf{i} + \mathbf{j} - \mathbf{i} - \left(\frac{1}{2} + \frac{g}{2}\right)\mathbf{j}$

Substituting back gives

$\mathbf{r} = t\mathbf{i} + \left(\frac{1}{2}(t+1) + gt - \frac{gt^2}{2} - \frac{g}{2}\right)\mathbf{j}$ which could be written $\mathbf{r} = t\mathbf{i} + \left(\frac{1}{2}(t+1) - \frac{g}{2}(t-1)^2\right)\mathbf{j}$

(Alternatively you could use the equation in section 6 on p49 but you will need to give a reference and remember that *t* will have to be replaced by $t-1$ and you will need to add the initial position vector.)

28 a)



H is the force in the spring *AP* (assuming the spring is extended)

R is the force in the damper *PB* acting against the motion. (You could have it going in the other direction if *x* was decreasing.)

W is the weight of the particle acting vertically downwards.

N is the normal reaction between the track and the particle.

b) $\mathbf{W} = -3g\mathbf{j}$ $\mathbf{N} = |\mathbf{N}|\mathbf{j}$

The length of the dashpot $l = 4 - x$ so $\dot{l} = -\dot{x}$

$\mathbf{R} = -6(-\dot{x})(-\mathbf{i}) = -6\dot{x}\mathbf{i}$ (4 p54)

The length of the spring is $= x - y$

$\mathbf{H} = -12(x - y - 1)\mathbf{i}$ (1 p34)

c) Newton's second law gives

$$m\mathbf{a} = \mathbf{W} + \mathbf{N} + \mathbf{R} + \mathbf{H}.$$

As $\mathbf{a} = \ddot{x}\mathbf{i}$ and $m = 3$ resolving in the $\mathbf{i}$-direction gives

$$3\ddot{x} = -6\dot{x} - 12(x - y - 1)$$

Rearranging gives the required equation

$$3\ddot{x} + 6\dot{x} + 12x = 12(1 + y)$$

d) i) Natural frequency $\omega = \sqrt{\frac{12}{3}} = 2$ (8 p55)

ii) Damping ratio $\alpha = \frac{6}{2\sqrt{3\times 12}} = \frac{1}{2} < 1$ (7 p55)

so the system is weakly damped.

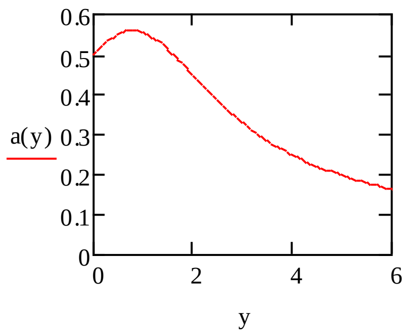
e) If $y = 1 + \frac{1}{2}\cos(\Omega t)$ the equation of motion becomes $3\ddot{x} + 6\dot{x} + 12x = 24 + 6\cos(\Omega t)$

Using $P = 6,\ k = 12, m = 3$ and $r = 6$ and using the formula in 14 p56

$$A = \frac{6}{\sqrt{(12 - 3\Omega^2)^2 + 36\Omega^2}}$$

f) If $\Omega = 2$ $A = \frac{1}{2}$. If $\Omega \to 0$ $A \to \frac{6}{12} = \frac{1}{2}$.

If $\Omega \to \infty$ $A \to 0$. Resonance will occur when $\frac{\Omega}{\omega} = \sqrt{1 - \frac{2}{4}} = \frac{1}{\sqrt{2}}$ or $\Omega = \sqrt{2}$ (16 p56)



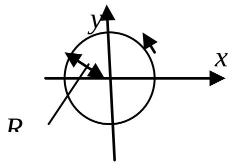
(I have produced this using Mathcad with $a(y)$being the amplitude and *y* the forcing frequency but obviously yours will be a rough sketch and does not have to be that accurate as long as it gives the general form.)

29. a)

At $t = 0$ $x = R$ and $y = 0$ and at $t = 2\pi$ $x = R$ and $y = 0$ so the curve is closed.

$x^2 + y^2 = R^2\cos^2 t + R^2\sin^2 t = R^2$ so the path is a circle radius *R* but on the other hand you might recognise the parameters as those for a circle.

29. a) cont) Either way the sketch is



b) i) Using the parameters (12 p67) we have
$\mathbf{F} = R^2\mathbf{i} + 2R\sin t\,\mathbf{j}$ $\quad \mathbf{r} = R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j}$ so

$$\frac{d\mathbf{r}}{dt} = -R\sin t\,\mathbf{i} + R\cos t\,\mathbf{j}$$

$$\mathbf{F}.\frac{d\mathbf{r}}{dt} = -R^3\sin t + 2R^2\sin t\cos t$$

$$= -R^3\sin t + R^2\sin 2t$$

$$\oint \mathbf{F}.\,d\mathbf{r} = \int_0^{2\pi} -R^3\sin t + R^2\sin 2t\,dt$$

$$= \left[R^3\cos t - \frac{R^2}{2}\cos 2t\right]_0^{2\pi} = 0$$

ii)If it is conservative the line integral needs to be zero for all paths (20 p68). We have considered only one path so we can draw no conclusion either way.

c) **curl F** = $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & 2y & 0 \end{vmatrix} = -2y\mathbf{k}$ (7 p68)

(or you could use any of the other forms for **curl**.)

d) It is not conservative as **curl F** $\neq \mathbf{0}$ at all points. (20 p68)

e) There is no contradiction as b) is only applicable to one curve not all possible curves.

30 a) Cylindrical polars would describe the system easily as the body is symmetrical about the z-axis. (11 p64) The underside is $z = 2\rho$ and the top is $z = 6 - \rho$.

b) At the intersection $2\rho = 6 - \rho$ so $\rho = 2$ and $z = 4$. The intersection is a circle of radius 2 at a height $z = 4$ and parallel to the x-y plane.

c) (12 p70) Volume $V = \int_0^2 \int_{2\rho}^{6-\rho} \int_{-\pi}^{\pi} \rho\,d\theta\,dz\,d\rho$

$$V = 2\pi \int_0^2 \int_{2\rho}^{6-\rho} \rho\,dz\,d\rho = 2\pi \int_0^2 \rho\,[z]_{2\rho}^{6-\rho} d\rho$$

$$= 2\pi \int_0^2 6\rho - 3\rho^2\,d\rho = 2\pi[3\rho^2 - \rho^3]_0^2 = 8\pi$$

Using given formula
the volume of the bottom cone is $\frac{1}{3} \times 4\pi \times 4 = \frac{16\pi}{3}$.
and the volume of the bottom cone is $\frac{1}{3} \times 4\pi \times 2 = \frac{8\pi}{3}$.
Adding these the total volume is 8π as before.

d) (14 p70)

$$M = \int_0^2 \int_{2\rho}^{6-\rho} \int_{-\pi}^{\pi} \sigma(1+\rho)\rho\,d\theta\,dz\,d\rho$$

$$= 2\pi \int_0^2 \int_{2\rho}^{6-\rho} \sigma(\rho + \rho^2)dz\,d\rho$$

$$= 2\pi \int_0^2 \sigma(\rho+\rho^2)\,[z]_{2\rho}^{6-\rho} d\rho$$

$$= 2\pi \int_0^2 \sigma(\rho+\rho^2)(6-3\rho)d\rho$$

$$= 2\pi\sigma \int_0^2 6\rho + 3\rho^2 - 3\rho^3 d\rho$$

$$= 2\pi\sigma\left[3\rho^2 + \rho^3 - \frac{3}{4}\rho^4\right]_0^2$$

$$= 2\pi\sigma[12 + 8 - 12] = 16\pi\sigma$$

e) (16 p71)

$$I = \int_0^2 \int_{2\rho}^{6-\rho} \int_{-\pi}^{\pi} \sigma(1+\rho)\rho^3\,d\theta\,dz\,d\rho$$