M337/Specimen

Module Examination
Complex Analysis

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I (64% of the marks) you should attempt as many questions as you can.

In Part II (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is **NOT** permitted in this examination.

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3.1

PART I

(i) *You should attempt as many questions as you can in this part.*
(ii) *Each question in this part carries 8 marks.*

Question 1

Let $w = 2/(1+i)$. Determine:

(a) $\operatorname{Arg} w$; [2]

(b) all the cube roots of w, identifying the principal cube root (express your answers in polar form); [5]

(c) the smallest positive integer n such that w^n is real. [1]

Question 2

Let

$$A = \mathbb{C} - \{x \in \mathbb{R} : x \geq 0\},$$
$$B = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\},$$
$$D = \{z : |z-3| < 2\}.$$

For each of the sets

$$A \cup B, \quad A \cap D, \quad D - \{2\}, \quad \partial B,$$

write down whether the set is

(a) a region, [4]

(b) a closed set. [4]

Question 3

(a) Evaluate

$$\int_\Gamma \operatorname{Im} z \, dz,$$

where Γ is the line segment from $-i$ to i. [3]

(b) Determine an upper estimate for the modulus of

$$\int_C \frac{2\sinh z}{z^5 - 1} \, dz,$$

where C is the circle $\{z : |z| = 2\}$. [5]

Question 4

(a) Determine the disc of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{z^n}{(n-1)!}.$$ [3]

(b) Find the Taylor series about 0 (up to the term in z^3) for the function

$$f(z) = \frac{\exp(-z)}{1-z},$$

and state an open disc on which the series represents f. [5]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{z}{(3z^2 - 1)(z^2 - 3)}$$

at those poles that lie inside the unit circle $C = \{z : |z| = 1\}$. [3]

(b) Hence evaluate the real trigonometric integral

$$\int_0^{2\pi} \frac{1}{1 + 3\sin^2 t}\, dt.$$ [5]

Question 6

Use Rouché's Theorem to determine the number of solutions of the equation

$$z^7 + 5z^3 + 7 = 0$$

in the annulus $\{z : 1 < |z| < 2\}$. [8]

Question 7

Let $q(z) = \overline{z}^2$ be a velocity function.

(a) Explain why q represents a model fluid flow. [1]

(b) Determine a stream function for this flow. Hence sketch the streamline through the point $e^{i\pi/3}$, and indicate the direction of flow. [5]

(c) Find the flux of q across the unit circle. [2]

Question 8

(a) Prove that the iteration sequence

$$z_{n+1} = z_n(1 - z_n), \qquad n = 0, 1, 2, \ldots,$$

with $z_0 = \frac{1}{2}$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + 1/4, \qquad n = 0, 1, 2, \ldots,$$

with $w_0 = 0$. [2]

(b) Which of the following points c lie in the Mandelbrot set?

(i) $c = \frac{1}{2}i$

(ii) $c = 1 + \frac{1}{2}i$

Justify your answer in each case. [6]

PART II

(i) *You should attempt no more than* **TWO** *questions in this part.*
(ii) *Each question in this part carries 18 marks.*

Question 9

Let f be the function defined by $f(z) = \dfrac{\exp z}{z(z-1)^3}$.

(a) Evaluate $\displaystyle\int_\Gamma f(z)\,dz$, where

(i) $\Gamma = \{z : |z-2| = \frac{1}{2}\}$,

(ii) $\Gamma = \{z : |z-2| = \frac{3}{2}\}$. [*9*]

(b) Find a simple-closed contour Γ such that

$$\int_\Gamma f(z)\,dz = -2\pi i.$$ [*5*]

(c) Let $\phi :]1,\infty[\longrightarrow \mathbb{C}$ be the function

$$\phi(r) = \int_{C_r} f(z)\,dz,$$

where $C_r = \{z : |z-1| = r\}$.

Show that ϕ is a constant function. (There is no need to evaluate the integral.) [*4*]

Question 10

(a) Let f be the function defined by $f(z) = 4/(z^2-4)$.

(i) Locate and classify the singularities of f.

(ii) How many different Laurent series does f have about 2?

What are the annuli of convergence of these Laurent series?

(iii) Determine the Laurent series about 2 for f, on the punctured open disc $\{z : 0 < |z-2| < 1\}$.

State an expression for the general term of this series. [*10*]

(b) Let g be the function defined by $g(z) = z\cos(1/z^2)$.

(i) Explain why the only singularity of g is at 0.

(ii) Determine the Laurent series about 0 for g, giving an expression for the general term of the series.

Hence classify the singularity of g at 0.

(iii) Prove that there is a complex number z such that $\mathrm{Im}(g(z)) > 1000$. [*8*]

Question 11

(a) Given that the Laurent series about 0 for the function cosec is

$$\operatorname{cosec} z = \frac{1}{z} + \tfrac{1}{6}z + \tfrac{7}{360}z^3 + \cdots,$$

determine the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. [*7*]

(b) Let $D = \{z : |z| < 1\}$ and let f be the function

$$f(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad (z \in D).$$

(i) Use Weierstrass' Theorem to prove that f is analytic, and hence obtain a formula for the derivative of f.

(ii) Determine an analytic function g with domain $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$ that is a direct analytic continuation of f.

(iii) Write down an analytic function h that is an indirect analytic continuation of f. [*11*]

Question 12

(a) State whether each of the following assertions is true or false, *briefly* justifying your answers.

(i) All analytic functions are conformal mappings.

(ii) All generalized circles lie in $\mathbb{C}$.

(iii) All linear functions are Möbius transformations. [*6*]

(b) (i) Determine the image of $\mathcal{R} = \{z : |z| < 1,\ \operatorname{Re} z < 0\}$ under the mapping given by

$$z_1 = \frac{z+i}{-z+i}.$$

(ii) Determine the image of $\mathcal{R}_1 = \{z_1 : \operatorname{Re} z_1 > 0,\ \operatorname{Im} z_1 > 0\}$ under the mapping given by

$$z_2 = z_1^2.$$

(iii) Determine a Möbius transformation that maps $\mathcal{R}_2 = \{z_2 : \operatorname{Im} z_2 > 0\}$ onto the open unit disc $\mathcal{S}$.

(iv) Hence obtain a formula for a one-one conformal mapping f from $\mathcal{R}$ onto $\mathcal{S}$, and a formula for the corresponding inverse function f^{-1}. (In both cases, you need not simplify your answer.) [*12*]

[END OF QUESTION PAPER]