Mathematics, Computing and Technology
M337 Complex Analysis

M337

Assignment Booklet I

Contents **Cut-off date**

Please send all your answers for each tutor-marked assignment (TMA) to your tutor, together with an appropriately completed assignment form (PT3).

You will find instructions on how to fill in the PT3 form in the *Assessment Handbook*. Remember to fill in the correct assignment number as listed above and to allow sufficient time in the post for the assignment to reach its destination on or before the cut-off date.

The marks allocated to each part of each question are indicated in brackets in the margin.

Questions in assignments (and in the examination) carry both *accuracy marks* and *method marks*. You should therefore, as a general practice, show all your working. As a guide, we use the following wording with the interpretations shown:

write down or *state* means 'write down without justification';

find, *determine*, *calculate*, *explain*, *derive*, *evaluate* or *solve* means that we require you to show all your working in giving an answer;

prove, *show* or *deduce* means that you should 'justify each step'; in particular, if you use a definition, result or theorem to go from one line to the next, you must state clearly which fact you are using — for example, quote the relevant unit and page, or give a Handbook reference — and check that all the necessary conditions are satisfied.

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Answer *all* questions.

Question 1 (*Unit A1*) – 25 marks

(a) Let $\alpha = -\sqrt{2} - \sqrt{2}i$ and $\beta = \sqrt{3} - \sqrt{2}i$.

(i) Evaluate each of the following, giving your answers in Cartesian form:

$$\alpha^4, \quad \alpha\overline{\beta}, \quad \frac{\alpha}{\beta}.$$

(ii) Express α in polar form, and hence determine all the cube roots of α, leaving your answers in polar form.

(iii) Write down a real quadratic polynomial that has α as one of its roots. Hence factorize $z^4 + 16$ into a product of two real quadratic polynomials. [*12*]

(b) Let

$$A = \{z : 2 \leq |z - 2i| < 3\},$$
$$B = \left\{z : \frac{\pi}{4} < \operatorname{Arg}(z + i) \leq \frac{\pi}{2}\right\}.$$

(i) Sketch the sets A, B and $A \cap B$ on separate diagrams, using the conventions on page 38 of *Unit A1*. There is no need to calculate the vertices for your sketch of $A \cap B$.

(ii) By writing $z - 3 + i = (z - 2i) + (-3 + 3i)$, prove that

$$3\sqrt{2} - 3 < |z - 3 + i| < 3 + 3\sqrt{2}, \quad \text{for } z \in A.$$

(iii) Hence find positive real numbers a and b such that

$$a < \left|\frac{z - 3 + i}{z - 2i}\right| < b, \quad \text{for } z \in A.$$ [*13*]

Question 2 (*Unit A2*) – 25 marks

(a) Let $f(z) = 2 - 3iz$.

(i) Describe the effect that f has on a typical point of $\mathbb{C}$ in terms of geometric transformations.

(ii) Let Γ be the path with parametrization

$$\gamma(t) = -3 + 4i + 5e^{it} \quad (t \in [-\pi/2, \pi/2]).$$

Sketch Γ, indicating the direction of increasing t, and identifying its initial and final points in Cartesian form.

(iii) Sketch the path $f(\Gamma)$ by applying the geometric transformations from part (a)(i) to Γ, showing the effect of each transformation in turn. Indicate the direction of $f(\Gamma)$, and identify its initial and final points in Cartesian form.

(iv) Write down the standard parametrization of $f(\Gamma)$. [*11*]

(b) The functions f, g and h are defined as follows:

$$f(z) = \cosh(2z - i),$$
$$g(z) = \operatorname{Log}(2z - i),$$
$$h(z) = (2z - i)^{\pi i}.$$

(i) Write down the domain of each of the functions f, g and h.

(ii) Let $\alpha = \frac{1}{2} + \frac{1}{2}(1 - \sqrt{3})i$. Evaluate each of the function values $f(\alpha)$, $g(\alpha)$ and $h(\alpha)$, giving your answers in Cartesian or polar form.

(iii) Explain why the function f does not have an inverse function.

(iv) Show that g has an inverse function, and find its domain and rule. [*14*]

Question 3 (*Unit A3*) – 25 marks

(a) Determine whether each of the following sequences converges, and if it does, state the limit.

(i) $\left\{\dfrac{e^{-in}}{2^n + 1}\right\}$

(ii) $\left\{\dfrac{n+1}{n+2}\sinh\left(\dfrac{n\pi i}{3}\right)\right\}$ [*7*]

(b) (i) Write down the domain A of the function

$$f(z) = i\left(\frac{z^2 - 4iz - 3}{z^2 - 2iz + 3}\right),$$

and show that $3i$ is a limit point of A.

(ii) Evaluate

$$\lim_{z \to 3i} \sqrt{i\left(\frac{z^2 - 4iz - 3}{z^2 - 2iz + 3}\right)},$$

giving your answer in Cartesian form, and justifying your method fully. [*8*]

(c) Let

$$B = \{z : 0 < \operatorname{Im} z < \operatorname{Re} z < 3\},$$
$$C = B \cup \partial B,$$
$$D = \mathbb{C} - \partial B.$$

(i) Sketch the sets B, C and D on separate diagrams. For each of these sets, state whether or not it is a region, justifying your answers.

(ii) Show that the set C is compact, and hence prove that

$$f(z) = \frac{(\sinh(z + i))^2}{(1 + e^z)^3}$$

is bounded on C. (You are NOT asked to find an upper bound for f on C.) [*10*]

Question 4 (*Unit A4*) – 25 marks

(a) Find the derivative of the function

$$f(z) = z^i \operatorname{Log}(iz),$$

and determine the domain of this derivative. [*3*]

(b) (i) Use the definition of the derivative to prove that the function

$$f(z) = 5\operatorname{Re} z - 2\operatorname{Im} z$$

is not differentiable at $1 + 3i$.

(ii) Use the Cauchy–Riemann equations to find all the points at which the following function is differentiable, and find its derivative at these points:

$$f(x + iy) = 3x^2 - x - 2y^2 + i(4xy + y).$$ [*10*]

(c) The function f is defined by

$$f(z) = \frac{iz + 3}{iz - 3}.$$

(i) Show that f is conformal.

(ii) Describe the geometric effect of f on a small disc centred at i.

(iii) Show that the paths

$$\Gamma_1 : \gamma_1(t) = -1 + \sqrt{2}e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = (t - 1) + it \quad (t \in \mathbb{R}),$$

are smooth and meet at the point i.

(iv) Determine the angle from the path $f(\Gamma_1)$ to the path $f(\Gamma_2)$ at the point $f(i)$. [*12*]

Answer *all* questions.

Question 1 (*Unit B1*) – 25 marks

(a) (i) Evaluate $\displaystyle\int_{\Gamma_1} 3\overline{z}\,dz$, where $\Gamma_1 : \gamma_1(t) = (1+i)t$ $(t \in [0,1])$.

(ii) Evaluate $\displaystyle\int_{\Gamma_2} 3\overline{z}\,dz$, where $\Gamma_2 : \gamma_2(t) = 1 + e^{it}$ $(t \in [-\pi, \pi/2])$.

(iii) Use your answers to parts (a)(i) and (a)(ii) to prove that the function $f(z) = 3\overline{z}$ cannot be the derivative of an entire function. [*11*]

(b) Use the Estimation Theorem to find an upper estimate on the modulus of the integral

$$\int_\Gamma \frac{ze^{z^2} + 2\sin z}{(z^2+7)(z-4)}\,dz,$$

where Γ is the upper half of the circle $\{z : |z| = 2\}$ traversed anticlockwise from 2 to -2. [*7*]

(c) Evaluate the following integrals, where Γ is the same contour as in part (b). In each case you should justify your method fully.

(i) $\displaystyle\int_\Gamma (\cosh z - e^{2z})\,dz$

(ii) $\displaystyle\int_\Gamma 12z^2 \sin(2z^3)\,dz$ [*7*]

Question 2 (*Unit B2*) – 25 marks

(a) Evaluate the integral

$$\int_C \frac{7z\cos \pi z}{(z-2)(3z+1)}\,dz$$

in the following cases:

(i) when $C = \{z : |z| = 1\}$;

(ii) when $C = \{z : |z - i| = 1\}$;

(iii) when $C = \{z : |z| = 3\}$. [*13*]

(b) (i) Use Liouville's Theorem to show that there is at least one value $z_0 \in \mathbb{C}$ for which

$$|\sin z_0| > 2016.$$

(ii) Show that if f is an entire function that satisfies

$$|2016i + f(z)| \geq 2016, \quad \text{for all } z \in \mathbb{C},$$

then f is constant.

(*Hint*: Consider the function $g(z) = \dfrac{2016}{2016i + f(z)}$, and apply Liouville's Theorem.)

(iii) Deduce from the result of part (b)(ii) that if f is an entire function that satisfies $\mathrm{Im}(f(z)) \geq 0$, for all $z \in \mathbb{C}$, then f is constant. [*12*]

Question 3 (*Unit B3*) – 25 marks

(a) Determine the disc of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^4}(z-1)^n. \qquad [3]$$

(b) Use Taylor's Theorem to determine the Taylor series about $2i$ for the function

$$f(z) = 1/(z-4i),$$

giving an expression for the general term of the series. Also, state the largest open disc on which the function f is represented by this Taylor series. [7]

(c) (i) Find the Taylor series about 0 for the following function (up to the term in z^6), and state the radius of convergence of the series:

$$f(z) = \exp(z - \sinh z).$$

(ii) Hence find the Taylor series about 0 for each of the following functions (up to the term in z^5):

$$g(z) = (1-\cosh z)\exp(z-\sinh z),$$
$$h(z) = (\cosh z)\exp(z-\sinh z). \qquad [10]$$

(d) Let f be an entire function such that

$$f\left(\frac{i}{n}\right) = \frac{n-i}{n^3}, \quad \text{for all } n \in \mathbb{N}.$$

Show that $f(z) = z^2(z-1)$. [5]

Question 4 (*Unit B4*) – 25 marks

(a) Locate and classify the singularities (giving the order of any poles) of the function

$$f(z) = \frac{z^2}{1-\cos \pi z}. \qquad [8]$$

(*Hint:* Use $\cos \pi z = \cos(\pi(z-2k))$, for $k \in \mathbb{Z}$.)

(b) Let $f(z) = \left(z + \frac{1}{z}\right)e^{1/z}$.

(i) Find the Laurent series about 0 for f, giving the general term of the series.

(ii) Write down a punctured open disc D, containing the circle $C = \{z : |z| = 1\}$, on which f is represented by this series.

(iii) State the nature of the singularity of f at 0.

(iv) Evaluate

$$\int_C \left(z + \frac{1}{z}\right)e^{1/z}\,dz,$$

where $C = \{z : |z| = 1\}$. [9]

(c) Find the Laurent series about 0 for the function

$$f(z) = \frac{5}{(2z-3)(z+1)}$$

on the set $\left\{z : 1 < |z| < \frac{3}{2}\right\}$. [8]