MST209 2008

(Handbook references are given as page numbers)

1. ln(x) is defined only for $x > 0$ (page 11) so $f(x)$ is defined for $2 - x > 0$ and $3 + x > 0$ giving $2 > x$ and $x > -3$. Option C.

2. Equation 1 is a linear equation and the variables do not separate. Equation 2 is not linear and the variables separate (page 26). Option C.

3. Either using the formula or Sarrus's rule, (page 29) the cross-product is
$(1 - (-2 \times 2))\mathbf{i} + (2 - 2)\mathbf{j} + ((2 \times -2) - 1)\mathbf{k}$
$= 5\mathbf{i} - 5\mathbf{k}$ Option A

4. Using page 31, the angle between **W** and **i** is $\frac{\pi}{2} + \theta$ giving the **i**-component of **W** as
$|\mathbf{W}|\cos\left(\frac{\pi}{2} + \theta\right) = -|\mathbf{W}|\sin(\theta)$. Option D.

5. Each spring has length l_0 and natural length $2l_0$ so their length is reduced by l_0. From page 35
PE in a spring is ½ x stiffness x (deformation)^2.
For left-hand spring PE $= \frac{1}{2}4k(l_0)^2 = 2kl_0^2$
For right-hand spring PE$= \frac{1}{2}2k(l_0)^2 = kl_0^2$
Adding these gives total PE = $3kl_0^2$ Option C.

6. From first two bullet points at bottom of page 41, the eigenvalues of $\boldsymbol{A}^{-3}$ are $\frac{1}{\lambda^3}$ so we have option D

7. Referring to page 43 the correct option is B.

8. The correct option is A. (page 44)

9. Referring to page 53, [power] = [energy]/[time] = $\frac{\mathrm{ML^2T^{-2}}}{\mathrm{T}} = \mathrm{ML^2T^{-3}}$ Option D.

10. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so the particles will move in the second normal mode which is phase-opposed (page 57) Option C.

11. Mass of 3 kg is at $L\mathbf{i}$, mass of 2 kg is at $L\mathbf{i} + L\mathbf{j}$ and mass 1 kg is at $L\mathbf{j}$ so centre of mass (page 58) is
$\frac{3L\mathbf{i}+2L\mathbf{i}+2L\mathbf{j}+L\mathbf{j}}{3+2+1} = \frac{5L}{6}\mathbf{i} + \frac{3L}{6}\mathbf{j}$. Option A

12. $\mathbf{r}(t) = 2\mathbf{i} + t\mathbf{k}$ and $\dot{\mathbf{r}}(t) = \mathbf{k}$
From page 60, angular momentum
$= \mathbf{r} \times m\dot{\mathbf{r}} = (2\mathbf{i} + t\mathbf{k}) \times m\mathbf{k} = -2m\mathbf{k}$ Option C.

13. From page 64,
$$\mathbf{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$
Option C

14. Using page 71
M of I = $\int_B f d^2 dV = \int_B c\rho\,\rho^2 dV$
$= \int_0^h \int_{-\pi}^{\pi} \int_0^R c\rho^3\rho \, d\rho d\theta dz$
(using page 70 but the integrations are in a different order which is permissible) We need the integral the whole way round the cylinder so the limits for θ are $-\pi$ to π. Option C.

15. Using item 22 page 73, the graph is linear so the order is 1. Option A.

16. The equation is linear so we can use the integrating factor method ((13) page 26)
$$\frac{dy}{dx} + y\tan x = x\cos x$$
$$p = \exp\left(\int \tan x \, dx\right) = \exp(-\ln(\cos x)) = \frac{1}{\cos x}$$
$$= \sec x$$
Multiply equation by sec x
$$\sec x\frac{dy}{dx} + y\sec x\tan x = x$$
$$\frac{d}{dx}(y\sec x) = x$$
Integrating $y\sec x = \frac{x^2}{2} + C$
$$y = \frac{x^2}{2}\cos x + C\cos x$$
$$y(0) = 1 \implies C = 1$$
So $y = \left(\frac{x^2}{2} + 1\right)\cos x$.

17\.

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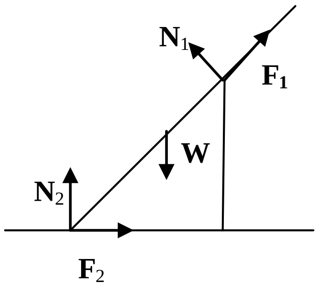
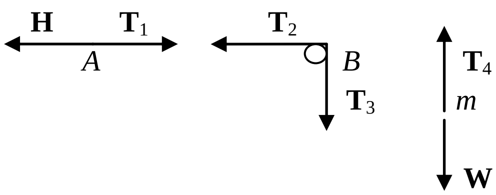


$\mathbf{N}_1$ is the normal reaction at the wall
$\mathbf{F}_1$ is the frictional force at the wall
W is the weight of the rod
$\mathbf{N}_2$ is the normal reaction at the floor
$\mathbf{F}_2$ is the frictional force at the floor.

18 a) $AB = l - x$ and $y + AB = d$
so $y = d - l + x$.

b)



$\mathrm{H} = -k(y - l_0)\mathbf{j}$. By Newton's 3rd law, $|\mathbf{T}_1| = |\mathbf{H}|$
As it is a model pulley, $|\mathbf{T}_2| = |\mathbf{T}_3|$
As it is a model string, $|\mathbf{T}_1| = |\mathbf{T}_2|$ and $|\mathbf{T}_3| = |\mathbf{T}_4|$
So $\mathbf{T}_4 = -k(y - l_0)\mathbf{i}$ $\mathbf{W} = mg\mathbf{i}$
The equation of motion of the mass is
$m\ddot{x}\mathbf{i} = \boldsymbol{W} + \mathbf{T}_4 = mg\mathbf{i} - k(y - l_0)\mathbf{i}$
Resolving in the **i**-direction and rearranging gives

$$m\ddot{x} = mg + kl_0 - k(d - l + x)$$

19\. (See (11) page 37) The augmented matrix is

$$\begin{array}{cl}\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 4 & 9 & 3 & 8 \\ -3 & -4 & 8 & 7 \end{array}\right) & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}\end{array}$$

$$\begin{array}{cl}\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) & \begin{array}{l} R_1 \\ R_2 - 4R_1 = R_{2a} \\ R_3 + 3R_1 = R_{3a} \end{array}\end{array}$$

$$\begin{array}{cl}\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) & \begin{array}{l} R_1 \\ R_{2a} \\ R_{3a} - 2R_{2a} \end{array}\end{array}$$

$$\begin{aligned} x_1 + 2x_2 \quad &= 1 \\ x_2 + 3x_3 &= 1 \\ 2x_3 &= 2 \end{aligned}$$

So $x_3 = 1,\ x_2 = 4 - 3 = 1$ and $x_1 = 1 - 2 = -1$
(Giving $x_1 = -1,\ x_2 = 1$ and $x_3 = 1$)

20\. $h_{\text{in}} = 10,\ h_{\text{out}} = 100,\ \kappa = 0.5,\ b = 0,$
$h_a = 10\ \ \kappa_l = 0.2\ \ b_l = x \times 10^{-3}$

(a) Using the formula from page 51

$$U = \left(\frac{1}{h_{\text{in}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{out}}}\right)^{-1} = \left(\frac{1}{10} + \frac{0.1}{0.5} + \frac{1}{100}\right)^{-1}$$

$$= (0.1 + 0.2 + 0.01)^{-1} = (0.31)^{-1} = \frac{100}{31}$$

b) Adding the lining

$$U_l = \left(\frac{1}{h_{in}} + \frac{b}{\kappa} + \frac{1}{h_a} + \frac{b_l}{\kappa_l} + \frac{1}{h_{out}}\right)^{-1}$$

$$U_l = \left(0.31 + 0.1 + \frac{0.00x}{0.2}\right)^{-1}$$

$$= (0.41 + .005x)^{-1}$$

(c) $U_l = \frac{U}{2}$ so $(0.41 + 0.005x)^{-1} = \frac{(0.31)^{-1}}{2}$

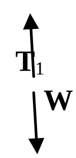
$$0.41 + 0.005x = 2(0.31)$$

$$0.005x = 0.21 \text{ or } x = 42\text{mm}$$

21\. (a) Acceleration $= -R\dot{\theta}^2\mathbf{e}_r = -\frac{v_0^2}{R}\mathbf{e}_r$ as we have motion with constant speed. (page 59)

(b) The only force on the car is the tension in the string so $\mathbf{T} = -T\mathbf{e}_r$ if T is the magnitude of the tension in string. Applying Newton's second law to the car
$-T\boldsymbol{e}_r = -\frac{mv_0^2}{R}\mathbf{e}_r$ so $T = \frac{mv_0^2}{R}$.

For mass the force diagram is
where $\mathrm{W} = -Mg\mathbf{k}$ and $\mathrm{T}_1 = T\mathbf{k}$
as the string is a model string and so $|\mathbf{T}| = |\mathbf{T}_1|$.
As the mass is stationary $\mathbf{W} + \mathbf{T}_1 = \mathbf{0}$.



So $T = \frac{mv_0^2}{R} = Mg$ or $v_0 = \sqrt{\frac{MRg}{m}}$.

22 (a)

div $\mathbf{V} = \frac{\partial V_\rho}{\partial \rho} + \frac{1}{\rho}V_\rho + \frac{1}{\rho}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$ (page 66)

$$= -\sin\theta - \sin\theta - \sin\theta = -3\sin\theta$$

(as $V_\rho = -\rho\sin\theta,\ V_\theta = \rho\cos\theta,\ V_z = 1$)

(b) **curl V**$= \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_\rho & \rho\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ -\rho\sin\theta & \rho^2\cos\theta & 1 \end{vmatrix}$ using the formula from page 67.

$$= \frac{1}{\rho}\big((0-0)\mathbf{e}_\rho - (0-0)\mathbf{e}_\theta + (2\rho\cos\theta + \rho\cos\theta)\boldsymbol{e}_z\big)$$

$= 3\cos\theta\ \mathbf{e}_z$

23 (a) Surface area S_a is given by (page 71)

$$\int_S \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA = \int_S \sqrt{1+(2x)^2+(2y)^2}\,dA$$

as $f(x,y)=x^2+y^2$.

$S_a=\int_S \sqrt{1+4(x^2+y^2)}\,dA$

Using polar co-ordinates $x^2+y^2=r^2 \le 2$

so $dA = r dr d\theta$ and $0\le r\le\sqrt{2}$ and $-\pi<\theta\le\pi$

$S_a=\int_{-\pi}^{\pi}\int_0^{\sqrt{2}} r\sqrt{1+4r^2}\,drd\theta$

(b) $S_a=\int_0^{\sqrt{2}}\int_{-\pi}^{\pi} r(1+4r^2)^{\frac{1}{2}}d\theta dr$

(as we can swop the order of integration in multiple integrals)

$S_a=2\pi\int_0^{\sqrt{2}} r(1+4r^2)^{\frac{1}{2}}dr$

Integrations can be done by inspection or by using the substitution (page 23). Let $u=1+4r^2$

then $\frac{du}{dr}=8r$. If $r=0,\ u=1$ and if $r=\sqrt{2},\ u=9$ so

$$S_a=2\pi\int_1^9 \frac{1}{8}\,u^{\frac{1}{2}}du=\frac{\pi}{4}\left[\frac{2u^{\frac{3}{2}}}{3}\right]_1^9=\frac{\pi}{6}(27-1)=\frac{13\pi}{3}.$$

24. (a) Auxiliary equation (page 26) is $\lambda^2+6\lambda+9=0$ or $(\lambda+3)^2=0$ giving $\lambda=-3$ twice

so the complementary function is $y_c=(C+Dt)e^{-3t}$

For particular integral (page 27) a trial solution for $25e^{-2t}$ is ae^{-2t} and a trial solution for $27t$ is $bt+c$ so we can use the trial solution

$$y=ae^{-2t}+bt+c$$

$$\frac{dy}{dx}=-2ae^{-2t}+b \qquad \frac{d^2y}{dx^2}=4ae^{-2t}$$

Substituting in the equation and collecting up the terms

$$ae^{-2t}+9bt+6b+9c=25e^{-2t}+27t$$

Comparing coefficients gives

$$a=25,\ \ 9b=27 \text{ and } 6b+9c=0$$

$$\text{so } b=3 \text{ and } c=-2$$

The particular integral is $y_p=25e^{-2t}+3t-2$

The general solution is

$$y=y_c+y_p=(C+Dt)e^{-3t}+25e^{-2t}+3t-2$$

(b) $\dot{y}=-3(C+Dt)e^{-3t}+De^{-3t}-50e^{-2t}+3$

$y(0)=0 \Rightarrow C+25-2=0$ so $C=-23$

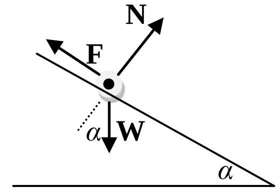
$\dot{y}(0)=0 \Rightarrow -3C+D-50+3=0 \Rightarrow -3C+D=47$

So $D=47+3C=47-69=-22$

$$y=-(23+22t)e^{-3t}+25e^{-2t}+3t-2$$

(c) As $t\to\infty$ $y\to 3t-2$ so y increases linearly with t for large t.

25. (a)



$\mathbf{W}=mg\sin\alpha\,\mathbf{i}-mg\cos\alpha\,\mathbf{j}$ $\quad N=|\mathbf{N}|$

$\mathbf{F}=-|\mathbf{F}|\mathbf{i}=-kv\mathbf{i}$

(b) $m\mathbf{a}=\mathbf{W}+\mathbf{N}+\mathbf{F}$

Resolving in the **i**-direction gives

$$ma=mg\sin\alpha-kv$$

or $m\frac{dv}{dt}=mg\sin\alpha-kv$ (1) as $a=\frac{dv}{dt}$.

(c) Terminal speed (page 34) occurs when $a=0$ so

$$v_T=\frac{mg\sin\alpha}{k}$$

(d) Separating the variables of (1) (page 26) (or could use integrating factor)

$$\frac{1}{mg\sin\alpha-kv}\frac{dv}{dt}=\frac{1}{m}$$

Integrating

$$\int 1/(mg\sin\alpha-kv)dv=\frac{t}{m}+C$$

$$-\frac{1}{k}\ln(mg\sin\alpha-kv)=\frac{t}{m}+C$$

as $v=v_0<v_T=\frac{mg\sin\alpha}{k}$ at $t=0$ and so

$$mg\sin\alpha-kv>0$$

Also $C=\frac{1}{k}\ln(mg\sin\alpha-kv_0)$, so

$$\frac{t}{m}=\frac{1}{k}(\ln(mg\sin\alpha-kv_0)-\ln(mg\sin\alpha-kv)$$

$$\frac{kt}{m}=\ln\left(\frac{mg\sin\alpha-kv_0}{mg\sin\alpha-kv}\right)$$

$$\frac{mg\sin\alpha-kv_0}{mg\sin\alpha-kv}=e^{\frac{kt}{m}}$$

$$mg\sin\alpha-kv=(\,mg\sin\alpha-kv_0)e^{-\frac{kt}{m}}$$

$$v=\frac{1}{k}\left(mg\sin\alpha-(mg\sin\alpha-kv_0)e^{-\frac{kt}{m}}\right)$$

(e)

$$v=\frac{dx}{dt}=\frac{1}{k}\left(mg\sin\alpha-(mg\sin\alpha-kv_0)e^{-\frac{kt}{m}}\right)$$

Integrating

$$x = \frac{mgt}{k}\sin\alpha + \frac{m}{k^2}(mg\sin\alpha - kv_0)e^{-\frac{kt}{m}} + D$$

When $t = 0,\ x = 0$ so $D = -\frac{m}{k^2}((mg\sin\alpha - kv_0)$

$$x = \frac{mgt}{k}\sin\alpha + \frac{m}{k^2}(mg\sin\alpha - kv_0)\left(e^{-\frac{kt}{m}} - 1\right)$$

26. (a)

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}5 & -2\\ -10 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$ This is of the correct form

if $\mathbf{x} = \begin{bmatrix}x\\ y\end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix}5 & -2\\ -10 & 4\end{bmatrix}$.

(b) Eigenvalues of **A** (page 41) are given by

$$\begin{vmatrix}5-\lambda & -2\\ -10 & 4-\lambda\end{vmatrix} = 0 \text{ or } (5-\lambda)(4-\lambda) - 20 = 0$$

Giving $-9\lambda + \lambda^2 = 0$ or $\lambda(\lambda - 9) = 0$

The eigenvalues are $\lambda = 0$ and $\lambda = 9$

For the eigenvectors

$$(5-\lambda)x - 2y = 0$$
$$-10x + (4-\lambda)y = 0$$

With $\lambda = 0$ these reduce to

$$5x - 2y = 0 \text{ and } -10x + 4y = 0$$

Both give $y = \frac{5}{2}x$ and so an eigenvalue is $[2 \quad 5]^T$.

With $\lambda = 9$ these reduce to

$$-4x - 2y = 0 \text{ and } -10x - 5y = 0.$$

Both give $y = -2x$ and so an eigenvalue is $[1 \quad -2]^T$

From page 43 the solution becomes

$$\mathbf{x} = C_1\begin{bmatrix}2\\ 5\end{bmatrix} + C_2\begin{bmatrix}1\\ -2\end{bmatrix}e^{9t}$$

(c) As $x = 6$ and $y = 6$ when $t = 0$, we get

$6 = 2C_1 + C_2$ (1) and $6 = 5C_1 - 2C_2$ (2)

(1) times 2 plus (2) gives $9C_1 = 18$ or $C_1 = 2$

From (1) $C_2 = 6 - 4 = 2$ giving

$$x = 4 + 2e^{9t} \text{ and } y = 10 - 4e^{9t}$$

(d) The chemical Y reduces to zero when $10 = 4e^{9t}$ or

$$t = \frac{1}{9}\ln\left(\frac{5}{2}\right)$$

(e) At this time $x = 4 + 2 \times \frac{5}{2} = 9$ as $e^{9t} = \frac{5}{2}$. This is an increase of 50% from 6.

27. (a) At equilibrium (page 47)

$$x\left(1 - \frac{x}{X} - \frac{y}{Y}\right) = 0 \quad (1)$$
$$y\left(1 - \frac{x}{2X} - \frac{2y}{Y}\right) = 0 \quad (2)$$

From (1) $x = 0$ or $x = X\left(1 - \frac{y}{Y}\right)$

Substituting $x = 0$ in (2) gives $y\left(1 - \frac{2y}{Y}\right) = 0$

So $y = 0$ or $y = \frac{Y}{2}$ giving points $(0,0)$ and $\left(0, \frac{Y}{2}\right)$.

Substituting $x = X\left(1 - \frac{y}{Y}\right)$ (3) into (2) gives

$y\left(\frac{1}{2} - \frac{3y}{2Y}\right) = 0$ or $y = 0$ and $y = \frac{Y}{3}$

Using (3) when $y = 0$ $x = X$ and when $y = \frac{Y}{3}$ $x = \frac{2X}{3}$

giving points $(X, 0)$ and $\left(\frac{2X}{3}, \frac{Y}{3}\right)$

The equilibrium points are $(0,0)$ $\left(0, \frac{Y}{2}\right)$ $(X, 0)$ $\left(\frac{2X}{3}, \frac{Y}{3}\right)$.

(b) Taking $u = x - \frac{x^2}{X} - xy$ and $v = -y + \frac{xy}{2X} + \frac{2y^2}{Y}$

$$\frac{\partial u}{\partial x} = 1 - \frac{2x}{X} - \frac{y}{Y} \qquad \frac{\partial v}{\partial x} = \frac{y}{2X}$$
$$\frac{\partial u}{\partial y} = -\frac{x}{Y} \qquad \frac{\partial v}{\partial y} = -1 + \frac{x}{2X} + \frac{4y}{Y}$$

$$\mathbf{J}(x,y) = \begin{bmatrix}1 - \frac{2x}{X} - \frac{y}{Y} & -\frac{x}{Y}\\ \frac{y}{2X} & -1 + \frac{x}{2X} + \frac{4y}{Y}\end{bmatrix}$$

$$\mathbf{J}(0,0) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

has eigenvalues 1 and -1. (The eigenvalues of a triangular matrix are the diagonal matrix. Page 41) These are real and distinct and of opposite sign and so $(0,0)$ is a saddle point. (page 47)

$$\mathbf{J}\left(0, \frac{Y}{2}\right) = \begin{bmatrix}\frac{1}{2} & 0\\ \frac{Y}{4X} & 1\end{bmatrix}$$

has eigenvalues $\frac{1}{2}$ and 1. These are real and distinct and both positive and so $\left(0, \frac{Y}{2}\right)$ is a source.

$$\mathbf{J}(X, 0) = \begin{bmatrix}-1 & -\frac{X}{Y}\\ 0 & -\frac{1}{2}\end{bmatrix}$$

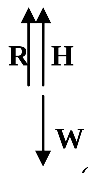
has eigenvalues -1 and $-\frac{1}{2}$. These are real and distinct and both negative so $(X, 0)$ is a sink.

$$\mathbf{J}\left(\frac{2X}{3}, \frac{Y}{3}\right) = \begin{bmatrix}-\frac{2}{3} & -\frac{2X}{3Y}\\ \frac{Y}{6X} & \frac{2}{3}\end{bmatrix}$$

The characteristic equation is

$\left(-\frac{2}{3}-\lambda\right)\left(\frac{2}{3}-\lambda\right)+\frac{1}{9}=0$. This simplifies to $\lambda^2=\frac{1}{3}$ so $\lambda=\pm\frac{1}{\sqrt{3}}$. These are real and distinct and of opposite sign and so $\left(\frac{2X}{3},\frac{Y}{3}\right)$ is a saddle point.

28 (a)



$\mathbf{W}=mg\mathbf{i}$

$\mathbf{H}=-k(x-l_0)\mathbf{i}$ (page 34)

$\mathbf{R}=-r(\dot{l})(-\mathbf{i})=r(\dot{y}-\dot{x})\mathbf{i}$

as the length of the damper is $l=y-x$. (page 54)

(b) The equation of motion is

$$m\ddot{x}\mathbf{i}=\mathbf{W}+\mathbf{H}+\mathbf{R}$$

Resolving in the **i**-direction gives

$$m\ddot{x}=mg-k(x-l_0)+r(\dot{y}-\dot{x})$$

So $m\ddot{x}+r\dot{x}+kx=mg+kl_0+r\dot{y}$

(c) Using the given values

$$\ddot{x}+4\dot{x}+13x=g+13+4\dot{y}$$

and with $\dot{y}=\sin t$, we get

$$\ddot{x}+4\dot{x}+13x=g+13+4\sin t$$

A possible trial solution for the particular integral is (using page 27) $\quad x=a+b\sin t+c\cos t$

$$\dot{x}=b\cos t-c\sin t \qquad \ddot{x}=-b\sin t-c\cos t$$

Substituting in the equation

$$\begin{aligned}-b\sin t-c\cos t+4b\cos t-4c\sin t+13a&\\+13b\sin t+13c\cos t&\\=g+13+4\sin t&\end{aligned}$$

$$\begin{aligned}13a+(12b-4c)\sin t+(12c+4b)\cos t&\\=g+13+4\sin t&\end{aligned}$$

Equating coefficients $a=\frac{g}{13}+1$

$$12b-4c=4 \quad 12c+4b=0 \quad \Longrightarrow b=-3c$$

Substituting in the other equation gives $-40c=4$

So $c=-\frac{1}{10}$ and $b=\frac{3}{10}$ giving

$$x=\frac{3}{10}\sin t-\frac{1}{10}\cos t+1+\frac{g}{13}$$

The amplitude (page 35) is given by

$$\sqrt{\left(\frac{3}{10}\right)^2+\left(\frac{1}{10}\right)^2}=\frac{1}{\sqrt{10}}$$

29. (a)

As $\mu>0$ let $\mu=k^2 \quad k>0$ then the equation becomes

$\frac{d^2X}{dx^2}+k^2X=0$,

which has general solution (page 35)

$$X=B\cos(kx)+A\sin(kx)$$

$$X'=-kB\sin(kx)+kA\cos(kx)$$

$$X(0)=0 \Rightarrow B=0 \quad \frac{dX}{dx}(L)=0 \Longrightarrow kA\cos(kL)=0$$

As $k\neq 0$ and $A=0$ gives the trivial solution,

$\cos(kL)$=0 so $k=\frac{(2n-1)\pi}{2L}$ giving $\mu=\frac{(2n-1)^2\pi^2}{4L^2}$

And $X(x)=A_n\sin\left(\frac{(2n-1)\pi x}{2L}\right) \quad n=1,2,3,\ldots$

b) Using the value for μ above the equation for T is

$$\frac{d^2T}{dt^2}+\frac{(2n-1)^2\pi^2c^2}{4L^2}T=0$$

Let $b=\frac{(2n-1)\pi c}{2L}$ for convenience then $\frac{d^2T}{dt^2}+b^2T=0$

This has general solution $T=C\cos(bt)+D\sin(bt)$

$$\frac{dT}{dt}=-bC\sin(bt)+bD\cos(bt)$$

$\frac{dT}{dt}(0)=0 \quad \Longrightarrow D=0$ so $T(t)=C\cos(bt)$

$$\begin{aligned}u(x,t)&=X(x)T(t)\\&=A_n\sin\left(\frac{(2n-1)\pi x}{2L}\right)\cos\left(\frac{(2n-1)\pi ct}{2L}\right)\end{aligned}$$

incorporating the C into A_n.

Adding solutions of this type (page 63 (d))

$$u=\sum_{n=1}^{\infty}A_n\sin\left(\frac{(2n-1)\pi x}{2L}\right)\cos\left(\frac{(2n-1)\pi ct}{2L}\right)$$

c) Initially the string has profile

$$f(x)=\begin{cases}\frac{2dx}{L} & 0\leq x\leq\frac{L}{2}\\ d & \frac{L}{2}<x\leq L\end{cases}$$

When $t=0$ $u=f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{(2n-1)\pi x}{2L}\right)$

The right-hand side is a Fourier sine series (page 61) with some of the terms missing but we can use a similar formula for A_n.

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx$$

$$=\frac{2}{L}\left\{\int_0^{\frac{L}{2}}\frac{2dx}{L}\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx\right.$$

$$\left.+\int_{\frac{L}{2}}^{L}d\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx\right\}$$

$$A_n = \frac{4d}{L^2}\int_0^{l/2} x\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx + \frac{2d}{L}\left[-\frac{2L}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{2L}\right)\right]_{L/2}^{L}$$

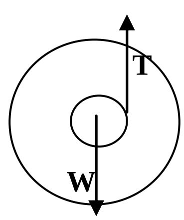
$$= \frac{4d}{L^2}\left[\frac{4L^2}{(2n-1)^2\pi^2}\sin\left(\frac{(2n-1)\pi x}{2L}\right) - \frac{2Lx}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{2L}\right)\right]_0^{\frac{L}{2}} + \frac{4d}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{4}\right)$$

(using the given integral)

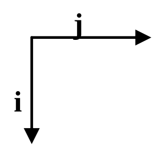
$$= \frac{16d}{(2n-1)^2\pi^2}\sin\left(\frac{(2n-1)\pi}{4}\right) - \frac{4d}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{4}\right) + \frac{4d}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi}{4}\right)$$

$$= \frac{16d}{(2n-1)^2\pi^2}\sin\left(\frac{(2n-1)\pi}{4}\right) \quad n = 1,2,3,\ldots$$

30 (a)



(b)



c) $\mathbf{W} = (2M+m)g\mathbf{i}$ $\mathbf{T} = -|\mathrm{T}|\mathbf{i}$

The equation of motion is

$$(2M+m)\mathbf{a} = \mathrm{W} + \mathrm{T}$$

Resolving in the **i**-direction gives

$$(2M+m)\ddot{x} = (2M+m)g - |\mathrm{T}|$$

(Write down means you down have to derive it)

d) Polar coordinates with the origin at the centre of the axle and θ measured anticlockwise from the horizontal.

e) The equation of rotational motion is

$I\ddot{\theta}\mathbf{k} = \mathbf{r}_\mathrm{T} \times \mathbf{T}$ where $\mathbf{r}_\mathrm{T} = r\mathbf{j}$ and I is the moment of inertia of the toy about the axle. (page 75)

Or $I\ddot{\theta} = (\mathbf{r}_\mathrm{T} \times \mathbf{T}).\mathbf{k}$

f) $x = r\theta$ (as the amount of string unravelled when the toy has rotated an angle θ is $r\theta$)

g) $\mathbf{r}_\mathrm{T} \times \mathrm{T} = r\mathbf{j} \times -|\mathbf{T}|\mathbf{i} = r|\mathbf{T}|\mathbf{k}$

From e) $I\ddot{\theta} = r|\mathbf{T}|$

From f) $\ddot{x} = r\ddot{\theta}$ so $|\mathbf{T}| = \frac{I\ddot{x}}{r^2}$

Substitute in c)

$$(2M+m)\ddot{x} = (2M+m)g - I\frac{\ddot{x}}{r^2} \quad (1)$$

I = Moment of inertia of discs + Moment of inertia of axle (page 74)

$$I = 2 \times \frac{I}{2}MR^2 + \frac{1}{2mr^2} = MR^2 + \frac{1}{2}mr^2$$

Substituting in (1) and rearranging

$$(2M+m)\ddot{x} + \frac{MR^2 + \frac{1}{2}mr^2}{r^2}\ddot{x} = (2M+m)g$$

$$\left(2Mr^2 + mr^2 + MR^2 + \frac{1}{2}mr^2\right)\ddot{x} = g(2M+m)r^2$$

$$\left(M(R^2+2r^2) + \frac{3}{2}mr^2\right)\ddot{x} = g(2M+m)r^2$$

$$\ddot{x} = \frac{g(2M+m)r^2}{\left(M(R^2+2r^2) + \frac{3}{2}mr^2\right)}$$