2011 MST209 exam solutions

The references to the Handbook are given as section followed by page number e.g. (5 p26)

1. Multiply by $2x + y$ to give
$xy + 8 = (2x + y)(x - 3) = 2x^2 + xy - 6x - 3y$
which simplifies to $6x$
$$3y = 2x^2 - 5x + 8 = 2(x - 4)(x = 1)$$
So the answer is B.
$2(x-4)(x+1)$

2. The auxiliary equation is
$\lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0$ so $\lambda = -4$ (twice)
so the answer is D. (5 p26)

3. $\mathbf{b} \times \mathbf{c} = (\mathbf{i}+\mathbf{j}) \times (\mathbf{j} + 2\mathbf{k}) = \mathbf{k} - 2\mathbf{j} + 2\mathbf{i}$ (18 p29)
$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2 - 2 = 0$
(15 p29) so the answer is C.

4. $\mathbf{j}$-component $= |\mathbf{F}|\cos\left(\frac{\pi}{2} - \theta\right) = |\mathbf{F}| \sin\theta$ (8 p31)
so the answer is A.

5. As x is measured downwards
the gravitational PE $= -mgx$ and the PE in the springs is $\frac{2k}{2}(x - l_0)^2$ so the answer is D. (2 p35)

6. $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ giving an eigenvalue of 3 so the answer is B. (1 p41).

7. $f(x,y) = e^x \cos y$ so $f(0,0) = 1$
$f_x = e^x \cos y$ so $f_x(0,0) = 1$
$f_y = -e^x \sin y$ so $f_y(0,0) = 0$
$p_{1(x,y)} = 1 + x$ so the answer is A. (12 p45)

8. No longer examined (but the answer is C).

9. [force] = MLT^{-2} [area] = L^2
[stress] = $\frac{[\text{force}]}{[\text{area}]} = \mathrm{ML}^{-1}\mathrm{T}^{-2}$ so the answer is B. (5 p53)

10. Length of damper $l = y - x$
Rate of change of length $\dot{l} = \dot{y} - \dot{x}$ $\hat{\mathbf{s}} = -\mathbf{i}$ so
$\mathbf{R} = -r(\dot{y} - \dot{x})(-\mathbf{i})$ so the answer is D. (4 p54)

11. $[-1 \quad 1]^T$ is an eigenvector for $\lambda = -36$ or $\omega = \sqrt{-36} = 6$ and $[1 \quad -1]^T = -1[-1 \quad 1]^T$ so the answer is D. (9 p57)

12. Converting $2\mathbf{i} + 3\mathbf{j}$ to a unit vector we get $\frac{2\mathbf{i}+3\mathbf{j}}{\sqrt{4+9}}$ (7 p28) so the velocity of the particle is $v\frac{2\mathbf{i}+3\mathbf{j}}{\sqrt{13}}$ and the momentum is $2mv\frac{2\mathbf{i}+3\mathbf{j}}{\sqrt{4+9}}$ (9 p58) so the answer is C.

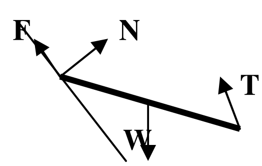
13. The function is even and has period 2 so the answer is A. (7 p61)

14. The $\mathbf{e}_\theta$-component of $\mathbf{grad}\, f = \frac{1}{\rho}\frac{\partial f}{\partial \theta} = \frac{1}{\rho}(\rho \sec^2\theta)$ so the answer is B. (15 p65)

15. No longer examined (but the answer is B).

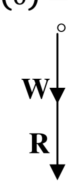
16. Using a trial solution (7 p27) $y = ax + b$ $\frac{dy}{dx} = a$ and $\frac{d^2y}{dx^2} = 0$. Substituting into the equation gives
$$7a + 12(ax + b) = 36x$$
Comparing coefficients we get
$12a = 36$ and $7a + 12b = 0$
So $a = 3$ and $b = -\frac{21}{12} = -\frac{7}{4}$
So the particular integral is $y = 3x - \frac{7}{4}$
When x becomes large the complementary function tends to zero and the solution tends to the particular integral which is essentially $y = 3x$ for large x.

17.



F is the force due to the friction between the plank and the slope
N is the normal reaction of the slope on the plank
W is the weight of the plank
T is the tension in the rope.

18. (a) $\mathbf{r}(0) = 2\mathbf{i}$ and $\dot{\mathbf{r}}(0) = 3\mathbf{i}$
(b)



W = weight $= -5g\mathbf{i}$
R = air resistance $= -c_2D^2|\mathbf{v}|\mathbf{v}$ where $\mathbf{v} = \dot{x}\mathbf{i}$ $c_2 = 0.2$ and $D = 0.1$ (13 p34) so $\mathbf{R} = -0.002\dot{x}^2\mathbf{i}$

18 (c) Using Newton's 2nd law (9 (c) p33)

$m\ddot{\mathbf{r}} = \mathbf{R} + \mathbf{W}$ where $m = 5$

Resolving in the $\mathbf{i}$-direction $5\ddot{x} = -0.002\dot{x}^2 - 5g$

or $\ddot{x} = -0.0004\dot{x}^2 - g$

19. The augmented matrix (11 p37) is

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 6 \\ 4 & 7 & 1 & 10 \\ 2 & 4 & -3 & 9 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 6 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & -2 & 3 \end{array}\right) \begin{array}{l} R_1 \\ R_{2a} = R_2 - R_1 \\ R_{3a} = R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 6 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -5 & 5 \end{array}\right) \begin{array}{l} R_1 \\ R_{2a} \\ R_{3b} = R_{3a} - R_{2a} \end{array}$$

so

$$2x + 3y - z = 6 \quad (1)$$
$$y + 3z = -2 \quad (2)$$
$$-5z = 5 \quad (3)$$

(3) gives $z = -1$

Substituting in (2) $y = 1$

Substituting in (3) $2x = 6 - 3 - 1 = 2$ so the solution is $x = 1, y = 1$ and $z = -1$.

20. $\mathbf{r} = (t^2 - 1)\mathbf{i} + \sqrt{2}(t - 8)\mathbf{j}$

a) velocity $= \dot{\mathbf{r}} = 2t\mathbf{i} + \sqrt{2}\mathbf{j}$ (3 p32)

b) $\dot{\mathbf{r}} \cdot \mathbf{r} = 2t(t^2 - 1) + 2(t - 8) = 2t^3 - 16$ (15 p29)

This is zero when $t^3 = 8$ or $t = 2$.

c) The velocity is perpendicular to the position vector (15 p29) and so we are at an extreme of the motion.

21. a) $\mathbf{r} = R\mathbf{e}_r$ (3 p59)

b) $\dot{\mathbf{r}} = R\dot{\theta}\mathbf{e}_\theta = R(t + \cos(2t))\mathbf{e}_\theta$ (3 p59)

c) Angular momentum

$\mathbf{l} = \mathbf{r} \times m\dot{\mathbf{r}} = mR^2(t + \cos(2t))\mathbf{k}$ as $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{k}$ (12 p60)

d) The torque law gives

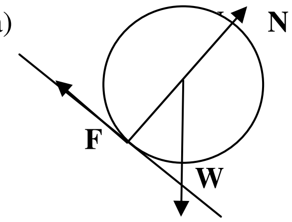
$\mathbf{\Gamma} = \dot{\mathbf{l}} = mR^2 (1 - 2\sin(2t))\,\mathbf{k}$ (13 p60)

22. Let $u = X(x)T(t)$ and substitute into the equation (9 p63) to give $X''T = \frac{1}{c^2}(T''X + T'X)$

Dividing by XT $\frac{X''}{X} = \frac{1}{c^2}\left(\frac{(T''+T')}{T}\right) = \mu$ (a constant)

So $X'' - \mu X = 0$ and $T'' + T' - \mu c^2 T = 0$.

23. a)

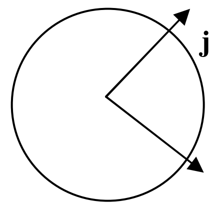


$\mathbf{W}$ = weight of cylinder

$\mathbf{F}$ = frictional force between the plane and the cylinder

$\mathbf{N}$ = normal reaction of the plane on the cylinder.

b)



As **N** and **W** acts through the centre of the cylinder (the origin) the only force that creates a torque is **F.**

$\mathbf{F} = -|\mathbf{F}|\mathbf{i}$ acts at position $-R\mathbf{j}$ and so the total torque is

$\mathbf{\Gamma} = -R\mathbf{j} \times \mathbf{F} = -R\mathbf{j} \times -|\mathbf{F}|\mathbf{i} = -R|\mathbf{F}|\mathbf{k}$ (13 p32)

c) The equation of rotational motion is

$I\ddot{\theta} = -R|\mathbf{F}|$ (16 p75)

where the angle θ is measured anticlockwise.

24. a) Separating the variables (9 p26)

$$\frac{1}{y^2}\frac{dy}{dx} = 5 + 6x^2$$

Integrating wrt x

$\int \frac{1}{y^2} dy = 5x + 2x^3 + C$ or $-y^{-1} = 5x + 2x^3 + C$

Giving $y = -\frac{1}{5x + 2x^3 + C}$

$y(0) = 1$ gives $C = -1$ so the solution of the equation is

$$y = -\frac{1}{5x + 2x^3 - 1}$$

b) (7 p25) $f(x, y) = x^3 - 4xy$

$x_0 = 0, \quad Y_0 = 1, \quad h = 0.1$ so $x_1 = 0.1$

$Y_1 = Y_0 + hf(x_0, Y_0) = 1 + 0.1(0) = 1 \quad x_2 = 0.2$

$Y_2 = Y_1 + hf(x_1, Y_1) = 1 + 0.1(0.001 - 0.4)$

$= 1 - 0.1(0.399) = 0.9601$

The approximate solution at $x = 0.2$ is 0.9601.

c) Using the integrating factor method (13 p26). The integrating factor $p(x) = \exp(\int 4x dx) = e^{2x^2}$

Multiplying the equation by the integrating factor gives

$$e^{2x^2}\frac{dy}{dx} + 4xe^{2x^2}y = \frac{d}{dx}\left(e^{2x^2}y\right) = x^3e^{2x^2}$$

24 c) cont Integrating using hint

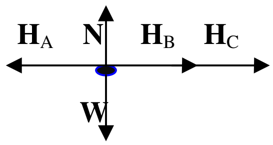
$e^{2x^2}y = \int x^3 e^{2x^2} dx = \frac{1}{4}x^2 e^{2x^2} - \frac{1}{8}e^{2x^2} + C$

Dividing by e^{2x^2} $y = \frac{1}{4}x^2 - \frac{1}{8} + Ce^{-2x^2}$

$y(0) = 1$ gives $1 = -\frac{1}{8} + C$ or $C = \frac{9}{8}$

The required solution is $y = \frac{1}{4}x^2 - \frac{1}{8} + \frac{9}{8}e^{-2x^2}$

25. a)



N is the normal reaction of the track on the particle
W is the weight of the particle
$\mathbf{H}_A$ is the force in spring AP
$\mathbf{H}_B$ is the force in spring PB
$\mathbf{H}_C$ is the force in spring PC

b) $\mathbf{W} = -3g\mathbf{j}$ $\mathbf{N} = |\mathbf{N}|\mathbf{j}$

The length of spring AP is x so using $\mathbf{H} = -k(l - l_0)\hat{\mathbf{s}}$ (1 p34) $\mathbf{H}_A = -6(x-1)\mathbf{i}$

The length of the spring PB is $3 - x$ so

$\mathbf{H}_B = -4(3 - x - 2)(-\mathbf{i}) = 4(1 - x)\mathbf{i}$

The length of spring PC is $4 - x$

$\mathbf{H}_C = -2(4 - x - 3)(-\mathbf{i}) = 2(1 - x)\mathbf{i}$

In equilibrium $\mathbf{N} + \mathbf{W} + \mathbf{H}_A + \mathbf{H}_B + \mathbf{H}_C = \mathbf{0}$ (3 p30)

Resolving in the **j** direction gives $|\mathbf{N}| = 3g$

c) Resolving in the **i** direction

$$-6(x-1) + 4(1-x) + 2(1-x) = 0$$

or $-12(x-1) = 0$

This is satisfied by $x = 1$ which gives the equilibrium position.

d) By Newton's second law (9(c) p33)

$3\mathbf{a} = \mathbf{N} + \mathbf{W} + \mathbf{H}_A + \mathbf{H}_B + \mathbf{H}_C$ where $\mathbf{a} = \ddot{x}\mathbf{i}$

Resolving in the **i** direction

$3\ddot{x} = -12(x-1)$ or $\ddot{x} = -4x + 4$

e) This can be written $\ddot{x} + 4x = 4$ which has solution

$$x = A\cos(2t) + B\sin(2t) + 1 \quad (4\ \text{p35})$$

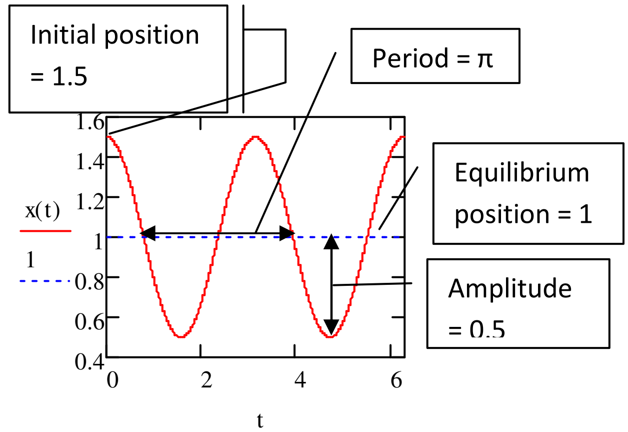
$$\dot{x} = -2A\sin(2t) + 2B\cos(2t)$$

If it is released from rest, $\dot{x} = 0$ when $t = 0$ so $B = 0$

Also $x = 1.5$ at $t = 0$ so $1.5 = A + 1$ or $A = 0.5$

Giving $x = 0.5\cos(2t) + 1$

f)



26. a) In matrix form the equations are

$\dot{\mathbf{x}} = \begin{bmatrix} 3 & -4 \\ -1 & 3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}e^{-t}$ (1 p42)

The matrix of coefficients has characteristic equation

$\begin{vmatrix} 3-\lambda & -4 \\ -1 & 3-\lambda \end{vmatrix} = 0$ (13 p41) or $(3-\lambda)^2 - 4 = 0$

Giving $\lambda^2 - 6\lambda + 5 = 0$ which factorises to

$(\lambda - 5)(\lambda - 1) = 0$ to give $\lambda = 5$ and $\lambda = 1$

The eigenvectors are given by

$$(3-\lambda)x - 4y = 0$$
$$-x + (3-\lambda)y = 0$$

When $\lambda = 5$ these reduce to $x = -2y$ so a typical eigenvector is $[-2\ 1]^T$

When $\lambda = 1$ these reduce to $x = 2y$ so a typical eigenvector is $[2\ 1]^T$

The complementary function is

$\mathbf{x}_c = C\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{5t} + D\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{t}$ (11, 12 p43)

For the particular integral try $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}e^{-t}$ (14 p43)

Substituting in the original equation we get

$$-\begin{bmatrix} a \\ b \end{bmatrix}e^{-t} = \begin{bmatrix} 3 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}e^{-t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}e^{-t}$$

Cancelling e^{-t} we get

$$-a = 3a - 4b + 4 \text{ or } b - a = 1 \ (1)$$

and $-b = -a + 3b + 2$ or $4b - a = -2$ (2)

Subtracting (1) from (2) we get $3b = -3$ or $b = -1$

Substituting in (1) $a = -2$ so $\mathbf{x}_p = \begin{bmatrix} -2 \\ -1 \end{bmatrix}e^{-t}$

And the general solutions (13 p43) is

$$\mathbf{x} = C\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{5t} + D\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{t} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}e^{-t}$$

When $t = 0$ $\mathbf{x} = \mathbf{0}$ so $0 = -2C + 2D - 2$ and

$0 = C + D - 1$ which give $D - C = 1$ and $D + C = 1$

So $D = 1$ and $C = 0$ giving

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{t} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}e^{-t}$$

b) When $t \to \infty$ $e^t \to 0$ and so **x** will be in the direction $[2 \quad 1]^T$ and increase rapidly.

27. a) For equilibrium $\frac{dx}{dt} = \frac{dy}{dt} = 0$ (6 p47) so

$$xy - 2x^2 = 0 \text{ or } x(y - 2x) = 0 \quad (1)$$
$$4 - 4x^2 - y^2 = 0 \quad (2)$$

From (1) $x = 0$ and substituting this into (2) gives

$$y^2 = 4 \text{ or } y = \pm 2$$

So $(0, 2)$ and $(0, -2)$ are equilibrium points.

From (1) $y = 2x$ and substituting this into (2) gives

$8x^2 = 4$ or $x^2 = \frac{1}{2}$ or $x = \pm\frac{1}{\sqrt{2}}$ with $y = \pm\sqrt{2}$.

We have two more equilibrium points $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$

b) For the linearised approximation we need the Jacobian for $u = xy - 2x^2$ and $v = 4 - 4x^2 - y^2$

So $\mathbf{J}(x, y) = \begin{bmatrix} y - 4x & x \\ -8x & -2y \end{bmatrix}$ (8,9 p47)

$\mathbf{J}(0,2) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$ and as it is a diagonal matrix the eigenvalues are 2 and -4 so they are real and distinct with opposite signs making (0, 2) a saddle point. (10 p47)

$\mathbf{J}(0, -2) = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$ and as it is a diagonal matrix the eigenvalues are -2 and 4 so they are real and distinct with opposite signs making $(0, -2)$ a saddle point.

$\mathbf{J}\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right) = \begin{bmatrix} -\sqrt{2} & \frac{1}{\sqrt{2}} \\ -4\sqrt{2} & -2\sqrt{2} \end{bmatrix}$ and its characteristic equation is $\left(-\lambda - \sqrt{2}\right)\left(-\lambda - 2\sqrt{2}\right) + 4 = 0$ or

$$\lambda^2 + 3\sqrt{2}\lambda + 8 = 0$$

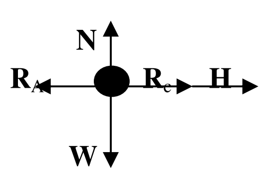
Giving $\lambda = \frac{-3\sqrt{2} \pm i\sqrt{24}}{2}$ so the eigenvalues are complex with negative real component so $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ is a spiral sink (10 p48)

$\mathbf{J}\left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} \\ 4\sqrt{2} & 2\sqrt{2} \end{bmatrix}$ and its characteristic equation is $\left(\sqrt{2} - \lambda\right)\left(2\sqrt{2} - \lambda\right) + 4 = 0$ or

$$\lambda^2 - 3\sqrt{2}\lambda + 8 = 0$$

Giving $\lambda = \frac{3\sqrt{2} \pm i\sqrt{24}}{2}$ so the eigenvalues are complex with positive real component so $\left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ is a spiral source. (10 p48)

28. a)



N is the normal reaction of the track on the particle
W is the weight of the particle
$\mathbf{R}_A$ is the force in damper *AP*
$\mathbf{R}_C$ is the force in damper *PC*
H is the force in spring *PB*

b) $\mathbf{N} = |\mathbf{N}|\mathbf{j}$ $\quad \mathbf{W} = -4g\mathbf{i}$ as there is no motion in the **j**-direction $\mathbf{N} + \mathbf{W} = \mathbf{0}$ so $\mathbf{N} = 4g\mathbf{j}$

Length of damper $AP = x - y$ so

$\mathbf{R}_A = -2(\dot{x} - \dot{y})\mathbf{i}$ (4 p54)

Length of damper $PC = 3 - x$ so

$\mathbf{R}_A = -4(-\dot{x})(-\mathbf{i}) = -4\dot{x}\mathbf{i}$

Length of spring $PB = 3 - x$ so

$\mathbf{H} - 9(3 - x - 1)(-\mathbf{i}) = 9(2 - x)\mathbf{i}$ (1 p34)

c) By Newton's second law

$4\mathbf{a} = \mathbf{N} + \mathbf{W} + \mathbf{R}_A + \mathbf{R}_C + \mathbf{H}$ where $\mathbf{a} = \ddot{x}\mathbf{i}$

Resolving in the **i**-direction

$$4\ddot{x} = -2\dot{x} + 2\dot{y} - 4\dot{x} + 18 - 9x$$

Giving $4\ddot{x} + 6\dot{x} + 9x = 18 + 2\dot{y}$ as required.

d) i) The natural frequency is $\sqrt{\frac{9}{4}} = \frac{3}{2}$ (8 p55)

ii) The damping ration is $\alpha = \frac{6}{2\sqrt{4 \times 9}} = \frac{1}{2}$

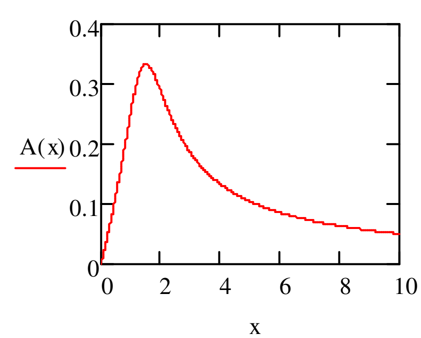
As $\alpha < 1$ the system is weakly damped (7 p55)

e) When $y = 1 + \cos(\Omega t)$ $\dot{y} = -\Omega\sin(\Omega t)$ so the equation becomes $4\ddot{x} + 6\dot{x} + 9x = 18 - 2\Omega\sin(\Omega t)$

As $-\sin(\Omega t) = \cos\left(\Omega t + \frac{\pi}{2}\right)$ (p15) we can use the same formula for the steady state amplitude as for $\cos(\Omega t)$ with $P = 2\Omega$ so using the formula in 14 p56 we have

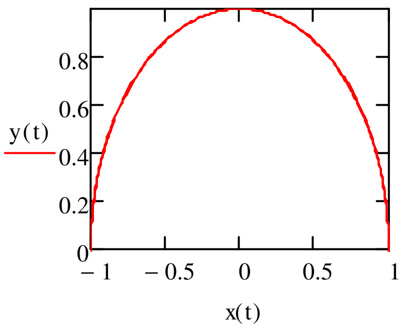
$$A = \frac{2\Omega}{\sqrt{(9 - 4\Omega^2)^2 + 36\Omega^2}}$$

f) $A = 0$ when $\Omega = 0$ and $A \to 0$ when $\Omega \to \infty$



The resonance occurs about the natural frequency as the system is weakly damped.

29. a) At $t = 0$ we start at the origin and at $t = 3$ we return to the origin. At $t = 1$ $x \to R$ and $y \to 0$ from both directions and at $t = 2$ $x \to -R$ and $y \to 0$ from both directions so the path is continuous and closed.



(The sketch is with $R = 1$ but shows the shape which is a semi-circle but includes the line $y = 0$.)

b) The scalar line integral is calculated in three parts.
For $0 \le t \le 1$ call the curve OA

$\mathbf{r} = Rt\mathbf{i}$ and $\frac{d\mathbf{r}}{dt} = R\mathbf{i}$ and $\mathbf{F} = 3R^2t^2\mathbf{i}$ so $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3R^3t^2$

and the line integral on this portion of the curve is (12 p67)

$$\int_{OA} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 3R^3t^2 dt = [R^3t^3]_0^1 = R^3$$

For $1 < t \le 2$ call the curve AB

$$\mathbf{r} = R\cos(\pi(t-1))\mathbf{i} + R\sin(\pi(t-1))\mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = -\pi R\sin(\pi(t-1))\mathbf{i} + \pi R\cos(\pi(t-1))$$

$$\mathbf{F} = \left(3R^2\cos^2(\pi(t-1)) + 3R^2\sin^2(\pi(t-1))\right)\mathbf{i} + 6R^2\cos(\pi(t-1))\sin(\pi(t-1))\,\mathbf{j}$$

$$= 3R^2\mathbf{i} + 6R^2\cos(\pi(t-1))\sin(\pi(t-1))\,\mathbf{j}$$

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -3\pi R^3\sin(\pi(t-1)) + 6\pi R^3\cos^2(\pi(t-1))\sin(\pi(t-1))$$

So $\int_{AB} \mathbf{F} \cdot d\mathbf{r}$

$$= \int_1^2 3\pi R^3\left(2\cos^2(\pi(t-1)) - 1\right)\sin(\pi(t-1))\,dt$$

$$= 3\pi R^3\left[-\frac{2}{3\pi}\cos^3(\pi(t-1)) + \frac{1}{\pi}\cos(\pi(t-1))\right]_1^2$$

$$= 3R^3\left[-\frac{2}{3}\cos^3\pi + \cos\pi + \frac{2}{3} - 1\right] = 3R^3\left[\frac{4}{3} - 2\right]$$

$$= -2R^3$$

For $2 < t \le 3$ call the curve BO $\mathbf{r} = R(t-3)\mathbf{i}$

$\frac{d\mathbf{r}}{dt} = R\mathbf{i}$ and $\mathbf{F} = 3R^2(t-3)^2\mathbf{i}$ so $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3R^3(t-3)^2$

So

$$\int_{BO} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 3R^3(t-3)^2 dt = [R^3(t-3)^3]_2^3 = R^3$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{OA} \mathbf{F} \cdot d\mathbf{r} + \int_{AB} \mathbf{F} \cdot d\mathbf{r} + \int_{BO} \mathbf{F} \cdot d\mathbf{r}$$
$$= R^3 - 2R^3 + R^3 = 0$$

c) $\mathbf{curl\ F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$ (7 p67)

$$= \left(\frac{\partial}{\partial x}(6xy) - \frac{\partial}{\partial y}(3x^2 + 3y^2)\right)\mathbf{k}$$

$$= (6y - 6y)\mathbf{k} = \mathbf{0}$$

d) The vector field is conservative as **curl F = 0** (20 p68)

30.a) Let B be the hemisphere then as it is symmetrical about the z-axis $-\pi < \phi \le \pi$. We are only considering the top half of the sphere so $0 \le \theta \le \frac{\pi}{2}$. The radius of the hemisphere is R so $0 \le r \le R$

M = $\int_B f\,dV$ where f is the density function. (14 p70)

$$= \int_0^R \int_0^{\frac{\pi}{2}} \int_{-\pi}^{\pi} cr(r^2\sin\theta)\,d\phi d\theta dr \quad \text{(13 p70)}$$

$$= \int_0^R \int_0^{\frac{\pi}{2}} 2\pi cr^3\sin\theta\,d\theta dr$$

$$= 2\pi c\int_0^R [-r^3\cos\theta]_0^{\frac{\pi}{2}} dr = 2\pi c\int_0^R r^3 dr$$

$$= 2\pi c\left[\frac{r^4}{4}\right]_0^R = \frac{\pi cR^4}{2}$$

b) M of I about the z-axis= $I = \int_B cr(r\sin\theta)^2 dV$ (16 p71)

$$I = \int_0^R \int_0^{\frac{\pi}{2}} \int_{-\pi}^{\pi} cr^5\sin^3\theta\,d\phi d\theta dr$$

$$= 2\pi c\int_0^R \int_0^{\frac{\pi}{2}} r^5(1 - \cos^2\theta)\sin\theta\,d\theta dr$$

$$= 2\pi c\int_0^R r^5\left[-\cos\theta + \frac{1}{3}(\cos^3\theta)\right]_0^{\frac{\pi}{2}} dr$$

$$= 2\pi c\int_0^R r^5\left(1 - \frac{1}{3}\right) dr = \frac{4}{3}\pi c\left[\frac{r^6}{6}\right]_0^R$$

$$= \frac{2\pi c}{9}R^6 = \frac{4}{9}MR^2$$

c) Using the parallel axis theorem (8 p74)
M of I about vertical axis through end of diameter

$$= I + MR^2 = \frac{4}{9}MR^2 + MR^2 = \frac{13}{9}MR^2$$