1

(i) $\phi_1 : \mathbb{C} \to \mathbb{C}$
$z \mapsto |z|$

Adding moduli of complex numbers does not look as though it would lead to a homomorphism. We try simple complex numbers to investigate.

Let $z_1 = 1$ and $z_2 = i$.

$\phi_1(z_1 + z_2) = \phi_1(1+i) = |1+i| = \sqrt{2}$

$\phi_1(z_1) + \phi_1(z_2) = |1| + |i| = 1 + 1 = 2 \neq \phi_1(z_1 + z_2)$

ϕ_1 is not a homomorphism.

(ii) $\phi_2 : \mathbb{C}^* \to \mathbb{C}^*$
$z \mapsto \mathrm{Re}(z)$

Let $z_1 = z_2 = 1+i$. $z_1 z_2 = (1+i)(1+i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$

$\phi_2(z_1 z_2) = \phi_2(2i) = \mathrm{Re}(2i) = 0$

$\phi_2(z_1)\phi_2(z_2) = \mathrm{Re}(1+i)\mathrm{Re}(1+i) = 1 \times 1 = 1 \neq \phi_2(z_1 z_2)$

ϕ_2 is not a homomorphism.

(iii) $\phi_3 : \mathbb{C} \to \mathbb{C}$
$z \mapsto \mathrm{Re}(z)$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. $z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$

$$\begin{aligned}\phi_3(z_1 + z_2) &= \mathrm{Re}((x_1 + x_2) + i(y_1 + y_2)) \\ &= x_1 + x_2 \\ &= \mathrm{Re}(z_1) + \mathrm{Re}(z_2) \\ &= \phi_3(z_1) + \phi_3(z_2)\end{aligned}$$

It follows that ϕ_3 is a homomorphism.

2 (a) (i) $\phi_1 : (\mathbb{R}, +) \to (\mathbb{Z}, +)$
$r \mapsto [r]$, ie the greatest integer less than or equal to r

(*Since this does not look like a homomorphism we try to find a counter-example immediately*)

Consider $\phi_1(0.6 + 0.6) = \phi_1(1.2) = 1$
$\phi_1(0.6) + \phi_1(0.6) = 0 + 0 = 0 \neq \phi_1(0.6 + 0.6)$

Hence ϕ_1 is not a homomorphism.

(ii) $\phi_2 : (\mathbb{R}^2, +) \to (\mathbb{R}, +)$
$(x, y) \mapsto 2x + y$

(*Since this is not obvious, we start by looking at the general case.).*

Consider $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$$\begin{aligned}
\phi_2((x_1, y_1) + (x_2, y_2)) &= \phi_2((x_1 + x_2, y_1 + y_2)) \\
&= 2(x_1 + x_2) + (y_1 + y_2) \\
&= 2x_1 + y_1 + 2x_2 + y_2 \\
&= (2x_1 + y_1) + (2x_2 + y_2) \\
&= \phi_2((x_1, y_1)) + \phi_2((x_2, y_2))
\end{aligned}$$

ϕ_2 is a homomorphism.

(iii) $\phi_3 : V \to (\mathbb{R}^*, \times)$
$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto a + b$

where $V = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R}, a^2 \neq b^2 \right\}$ and the operation is matrix multiplication.

(Since I have no particular intuition on this, I will look at the general case. It does not look too hard.)

$$\begin{aligned}
\phi_3\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} c & d \\ d & c \end{pmatrix}\right) &= \phi_3\left(\begin{pmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{pmatrix}\right) \\
&= ac + bd + ad + bc \\
&= (a+b)(c+d) \\
&= \phi_3\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right)\phi_3\left(\begin{pmatrix} c & d \\ d & c \end{pmatrix}\right)
\end{aligned}$$

Hence ϕ_3 is a homomorphism.

(b) (ii) $\text{Ker}(\phi_2) = \{(x, y) \in \mathbb{R}^2 : \phi_2(x, y) = 0\}$
$= \{(x, y) \in \mathbb{R}^2 : 2x - y = 0\}$
$= \{(x, y) \in \mathbb{R}^2 : 2x = y\}$
$= \{(x, 2x) \in \mathbb{R}^2\}$

$\text{Im}(\phi_2) = \{x \in \mathbb{R} : x = \phi_2(a, b) \text{ for some } (a, b) \in \mathbb{R}^2\}$

Consider any $x \in \mathbb{R}$. $x = \phi_2(0, -x)$. Hence $\text{Im}(\phi_2) = \mathbb{R}$.

(iii) $$\operatorname{Ker}(\phi_3) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 \neq b^2,\ \phi_3 \begin{pmatrix} a & b \\ b & a \end{pmatrix} = 1 \right\}$$
$$= \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 \neq b^2,\ a + b = 1 \right\}$$
$$= \left\{ \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix} : a \in \mathbb{R},\ a \neq \frac{1}{2} \right\}$$

Let $x \in \mathbb{R}^*$. We have $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in V$ and $\phi_3\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right) = x + 0 = x$.

ie $\operatorname{Im}(\phi_3) = \mathbb{R}^*$.