M337/A

The Open University

Third Level Course Examination 2001

Complex Analysis

Monday, 8 October, 2001 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I (64% of the marks) you should attempt as many questions as you can.

In Part II (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach all your answer books together using the fastener provided.

The use of calculators is **NOT** permitted in this examination.

PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

Let $\alpha = -2 + 2i$.

(a) Plot α and $\overline{\alpha}$ in the complex plane. [1]

(b) Write down $|\alpha|$ and $\operatorname{Arg}\alpha$. [2]

(c) Determine $\alpha^{\frac{1}{3}}$, the principal cube root of α. Write your answer in Cartesian form. [2]

(d) On your sketch indicate the approximate positions of the three cube roots of α. [2]

(e) Write down the smallest positive integer k such that α^k is real. [1]

Question 2

Let $A = \{z : 1 < |z| < 2\}$, $B = \{z : -\pi < \operatorname{Arg} z < \pi\}$ and f be the function $f(z) = 1/z$.

(a) Make separate sketches of the sets A and B. [2]

(b) For which of the sets A and B is/are the following statements true. (No justification need be given.)

(i) The set is a region.

(ii) The function f is analytic on the set.

(iii) For any closed contour Γ in the set

$$\int_{\Gamma} f(z)\, dz = 0.$$

(iv) The function f is bounded on the set. [6]

Question 3

(a) (i) Determine the standard parametrization for the line segment Γ_1 from -1 to i.

(ii) Evaluate

$$\int_{\Gamma_1} \operatorname{Re} z\, dz.$$ [3]

(b) Determine an upper estimate for the modulus of

$$\int_{\Gamma_2} \frac{\operatorname{Log} z}{5 + z^2}\, dz$$

where Γ_2 is the line segment from $1 - i$ to $1 + i$. [5]

Question 4

(a) Let $f(z) = e^{\sinh z}$.

(i) Find the Taylor series about 0 for the function f up to the term in z^3.

(ii) For what values of z does the Taylor series represent f? Justify your answer. [4]

(b) Find the Laurent series for the function

$$g(z) = \frac{1}{z^2+1}$$

about the point 0 on the region $\{z : |z| > 1\}$. State the general term of the series. [4]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{1}{z^3-1}$$

at each of the poles of f. [4]

(b) Hence evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^3-1}\, dt.$$ [4]

Question 6

(a) Show that $|\sinh z| \le e^{|\operatorname{Re} z|}$. [2]

(b) Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sinh z}{n^2+1}$$

converges uniformly on $E = \{z : |\operatorname{Re} z| \le 3\}$. [4]

(c) Evaluate $\Gamma\left(-\frac{1}{2}\right)$, where Γ is the gamma function. [2]

Question 7

Let $q(z) = \overline{z} - i$ be a velocity function.

(a) Explain why q represents a model fluid flow on $\mathbb{C}$. [1]

(b) Determine a stream function for this flow and hence find the equations for the streamline through the point 1 and the streamline through the point $-1-i$. [4]

(c) Sketch the streamlines found in part (b), showing the direction of flow, and also indicate any degenerate streamlines. [3]

Question 8

(a) Show that i is a periodic point of the function $f(z) = z^3 + i$, and determine whether it is (super-) attracting, repelling or indifferent. [3]

(b) Determine which of the following points c lie in the Mandlebrot set.

(i) $c = -\frac{1}{2} + \frac{1}{2}i$.

(ii) $c = -1 - i$.

Justify your answer in each case. [5]

PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Use the Cauchy-Riemann Theorem and its converse to determine the set of points of $\mathbb{C}$ on which the function

$$f(z) = z^2 + 3(\operatorname{Im} z)^2 + i6(\operatorname{Re} z)^2$$

is differentiable. [8]

(b) Let g be the function $g(z) = z^2 + 2$.

(i) Show that g is conformal on $\mathbb{C} - \{0\}$.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = i + e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = it \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point $2i$, and sketch Γ_1 and Γ_2 on the same diagram.

(iii) Describe the effect of g on a small disc centred at $2i$ and hence make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(2i)$.

(iv) Make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(0)$. [10]

Question 10

Let f be the function

$$f(z) = \exp\left(\frac{1}{z+1}\right),$$

and let $C_1 = \{z : |z| = \frac{1}{2}\}$, $C_2 = \{z : |z| = 2\}$.

(a) Show that f has just one singularity, and that it is an essential singularity. [2]

(b) Evaluate the following integrals naming any standard results that you use and checking that their required conditions hold.

(i) $\displaystyle\int_{C_1} f(z)\,dz$

(ii) $\displaystyle\int_{C_1} \frac{f(z)}{(4z+1)^2}\,dz$

(iii) $\displaystyle\int_{C_2} f(z)\,dz$

(iv) $\displaystyle\int_{C_2} \frac{f(z)}{z}\,dz$ [16]

Question 11

(a) Let f be the function $f(z) = -z + z^3$.

(i) Show that f is one-one near 0.

(ii) Invert the Taylor series for f about 0, giving the first three non-vanishing terms. [7]

(b) For the function $g(z) = z^2 + i$, determine

$$\max\{|g(z)| : |z| \leq 1\}$$

and find the point or points at which this maximum is obtained. [7]

(c) Let h be analytic and one-one on $D = \{z : |z| < 1\}$. For each of the statements in (i) and (ii) below, write down whether or not it is necessarily true. If you decide that it is true, then prove it; if you decide that it is false, then write down a counter-example.

(i) $h(D)$ is a region.

(ii) $h(D)$ is bounded. [4]

Question 12

(a) Determine the extended Möbius transformation $\hat{f}_1$ which maps i to 0, $\frac{1}{2}(1+i)$ to 1 and 1 to ∞. [3]

(b) Let $R = \{z : |z| < 1,\ \operatorname{Re} z + \operatorname{Im} z > 1\}$ and $S = \{w : \operatorname{Im} w > 0\}$.

(i) Sketch the regions R and S, indicating the position of $\frac{1}{2}(1+i)$ in your sketch of R.

(ii) Determine the image R_1 of R under $\hat{f}_1$ of part (a).

(iii) Hence determine a conformal mapping f from R onto S.

(iv) Write down the rule of the inverse function f^{-1}. [15]

[END OF QUESTION PAPER]