The Open University

MST209/B

Course Examination 2008

Mathematical Methods and Models

Monday 6 October 2008 **2.30 pm–5.30 pm**

Time allowed: 3 hours

Personal Identifier								
Examination No.								

You are **not** allowed to use a calculator in this examination.

There are three parts to this paper. In each part of the paper the questions are arranged in the order they appear in the course. There are 115 marks available, but scores greater than 100 will be rounded down to 100.

Part 1 consists of 15 questions each worth 2 marks. You are advised to spend no more than 1 hour on this part. Enter one option in each box provided on the question paper; use your answer book(s) for any rough work. Incorrect answers are not penalised. Cross out mistakes and write your answer next to the box provided.

Part 2 consists of 8 questions each worth 5 marks. You are advised to spend no more than $1\frac{1}{4}$ hours on this part.

Part 3 consists of 7 questions each worth 15 marks. Your best three marks will be added together to give a maximum of 45 marks.

In Parts 2 and 3: Write your answers in the answer book(s) provided. The marks allocated to each part of each question are given in square brackets in the margin. Unless you are directed otherwise in the question, you may use any formula or other information from the *Handbook* provided that you give a reference. Do **not** cross out any answers unless you have supplied a better alternative — everything not crossed out may receive credit.

At the end of the examination, **check that you have written your personal identifier and examination number on each answer book used (as well as in the boxes above)**. Failure to do so may mean that your work cannot be identified. Use the paper fastener provided to fix together all your answer books, and the question paper, with your signed desk record on top.

PART 1

Each question in this part of the paper is worth 2 marks. Fill in the appropriate response in the box alongside the question.

Question 1

Select the option that gives the largest possible domain of the function f of the real variable x defined by

$$f(x) = \ln(2 - x) + \ln(3 + x).$$

Options Answer:

A $x < 2$ **B** $x > -3$

C $-3 < x < 2$ **D** $-2 < x < 3$

Question 2

This question concerns the two following differential equations.

$$\frac{dy}{dx} - \frac{2}{x}y = 1 \tag{1}$$

and

$$\frac{dy}{dx} = (y^2 + 4)(x - 2). \tag{2}$$

Select the option that gives a correct statement.

Options Answer:

A Both Equation (1) and Equation (2) can be solved using an integrating factor.

B Equation (1) can be solved using separation of variables, Equation (2) can be solved using an integrating factor.

C Equation (1) can be solved using an integrating factor, Equation (2) can be solved using separation of variables.

D Both Equation (1) and Equation (2) can be solved using separation of variables.

Question 3

Select the option that gives the cross product

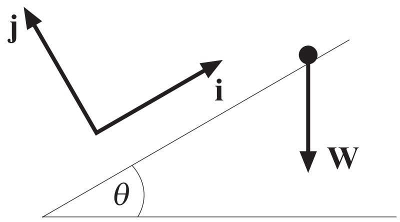
$$(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

Options Answer:

A $5\mathbf{i} - 5\mathbf{k}$ **B** $-5\mathbf{i} + 5\mathbf{k}$ **C** $-5\mathbf{i} - 5\mathbf{k}$ **D** $5\mathbf{i} + 5\mathbf{k}$

Question 4

Consider the vector $\mathbf{W}$ and the axes shown below.



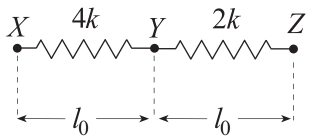
Select the option giving the $\mathbf{i}$-component of $\mathbf{W}$.

Options Answer:

A $|\mathbf{W}|\cos\theta$ **B** $|\mathbf{W}|\sin\theta$ **C** $-|\mathbf{W}|\cos\theta$ **D** $-|\mathbf{W}|\sin\theta$

Question 5

Two springs, each of natural length $2l_0$, with stiffnesses $4k$ and $2k$, respectively, are connected together as shown, with the two endpoints fixed at points X and Z. The point Y is then held so that each spring has length l_0.



Take the datum for potential energy in each spring at the point of zero extension. Select the option that gives the total potential energy stored in the springs.

Options Answer:

A kl_0^2 **B** $2kl_0^2$ **C** $3kl_0^2$ **D** $4kl_0^2$

Question 6

The matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ has eigenvalues 2 and 3. Select the option that gives the eigenvalues of $\mathbf{A}^{-3}$.

Options Answer:

A -1, 0 **B** -6, -9 **C** 8, 27 **D** 1/8, 1/27

Question 7

Consider the system of differential equations

$$\begin{cases} \dot{x} = x + 4y - \cos 3t - 4\sin 3t \\ \dot{y} = 6x - 3y + \sin 3t. \end{cases}$$

Select the option that gives a suitable candidate for a particular solution for this system.

Options Answer: $\square$

A $x = a\cos 3t + b\sin 3t$
$y = a\cos 3t + b\sin 3t$

B $x = a\cos 3t + b\sin 3t$
$y = c\cos 3t + d\sin 3t$

C $x = a\cos 3t + b\sin 3t$
$y = c\sin 3t$

D $x = a\cos 3t + a\sin 3t$
$y = b\cos 3t + b\sin 3t$

Question 8

Select the option that gives $\partial f/\partial\theta$ for the function $f(\rho,\theta,z) = \rho^2\sin^2\theta\,\cos z$.

Options Answer: $\square$

A $2\rho^2\sin\theta\cos\theta\,\cos z$ **B** $-2\rho^2\sin\theta\cos\theta\,\cos z$

C $-\rho^2\sin^2\theta\,\sin z$ **D** $2\rho\sin^2\theta\,\cos z$

Question 9

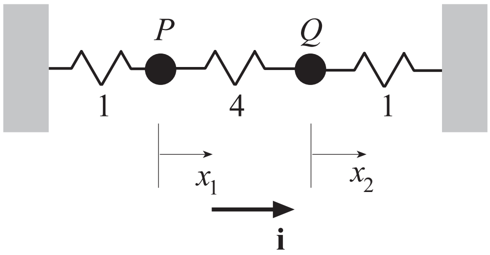
Power is defined as the rate of energy used per unit of time ($\mathrm{J\,s^{-1}}$). Select the option that gives the dimensions of power.

Options Answer: $\square$

A $\mathrm{ML^2}$ **B** $\mathrm{ML^2T^{-1}}$ **C** $\mathrm{ML^2T^{-2}}$ **D** $\mathrm{ML^2T^{-3}}$

Question 10

Two particles P and Q, each of mass 1 kg, are attached to three springs and the ends of the two outer springs are fixed as shown. The stiffness of each spring is shown next to it and the particles are constrained to move in a horizontal line.



x_1 and x_2 are the displacements of the particles from their respective equilibrium positions. The dynamic matrix of the system is

$$\mathbf{M} = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix}.$$

$\mathbf{M}$ has eigenvalues -1 and -9, with corresponding eigenvectors $[1 \quad 1]^T$ and $[1 \quad -1]^T$. The particles are initially displaced from equilibrium with $x_1 = -2$ and $x_2 = 2$ and released from rest. Select the option that describes the subsequent behaviour of the system.

Options Answer: ☐

A P and Q remain at rest.

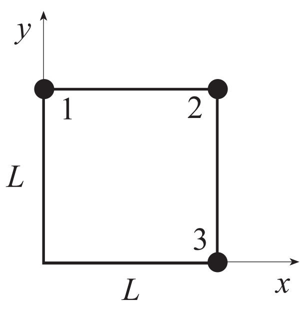
B P and Q exhibit in-phase normal mode motion.

C P and Q exhibit phase-opposed normal mode motion.

D P and Q do not exhibit normal mode motion.

Question 11

A light wire frame is formed as a square of side L, and is placed with one vertex at the origin. The coordinate axes are aligned with the square, as shown. Particles of mass 1 kg, 2 kg and 3 kg are fixed to the other three corners as shown.



Which option gives the coordinates of the centre of mass of the system?

Options Answer: ☐

A $(\frac{5}{6}L, \frac{1}{2}L)$ **B** $(\frac{1}{6}L, \frac{1}{2}L)$ **C** $(\frac{5}{6}L, \frac{1}{3}L)$ **D** $(\frac{1}{6}L, \frac{1}{3}L)$

Question 12

A particle of mass m has position $\mathbf{r}(t) = 2\,\mathbf{i} + t\,\mathbf{k}$. Select the option that gives the angular momentum of the particle about the origin at time t.

Options Answer:

A $\mathbf{0}$ **B** $2m\,\mathbf{j}$ **C** $-2m\,\mathbf{j}$ **D** $mt\,\mathbf{j}$

Question 13

Which option gives $\mathbf{grad}\,\phi$ where

$$\phi(x, y, z) = xyz.$$

Options Answer:

A $\mathbf{0}$ **B** $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ **C** $yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$ **D** $xyz(\mathbf{i}+\mathbf{j}+\mathbf{k})$

Question 14

A cylinder of radius R and height h lies on the x-y-plane, with its axis along the z-axis. The density of the cylinder varies with distance from the axis and is given at the point with cylindrical polar coordinates $(\rho,\ \theta,\ z)$ by the function $c\rho$, where c is a constant. Select the option that gives the moment of inertia of the cylinder about its axis.

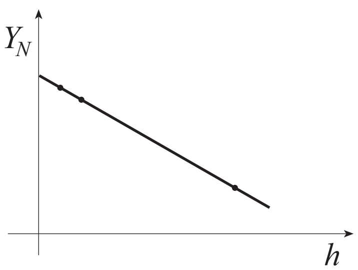
Options Answer:

A $\displaystyle\int_0^h \int_0^{\pi} \int_0^R c\rho^4\,d\rho\,d\theta\,dz$ **B** $\displaystyle\int_0^h \int_0^{\pi} \int_0^R c\rho^3\,d\rho\,d\theta\,dz$

C $\displaystyle\int_0^h \int_{-\pi}^{\pi} \int_0^R c\rho^4\,d\rho\,d\theta\,dz$ **D** $\displaystyle\int_0^h \int_{-\pi}^{\pi} \int_0^R c\rho^3\,d\rho\,d\theta\,dz$

Question 15

A numerical method is applied to a particular differential equation $dy/dx = f(x, y)$ over the range $x = 0$ to $x = 1$. Y_N is the resulting estimate of y at $x = 1$, and a graph of Y_N against the step-size h is shown below.



Select the option that gives the likely order of the numerical method.

Options Answer:

A 1 **B** 2 **C** 3 **D** 4

PART 2

Each question in this part of the paper is worth 5 marks.

Question 16

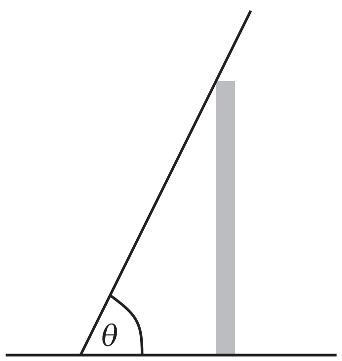
Solve the initial-value problem

$$\frac{dy}{dx} = x\cos x - y\tan x \quad (0 \le x < \pi/2), \quad y(0) = 1,$$

expressing your answer in the form $y = f(x)$. [5]

Question 17

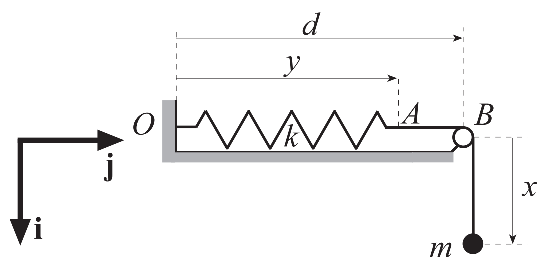
A model rod of mass m rests on a rough wall with its base on a rough horizontal floor. The rod makes an angle θ with the horizontal.



Draw a force diagram showing all the forces acting on the rod. Briefly define each force. [5]

Question 18

A particle of mass m is attached to a model string of length l. The string passes over a model pulley B, and the particle hangs a distance x vertically below. The other end of the string is attached to one end A of a horizontal model spring of natural length l_0 and stiffness k. The other end of the spring is attached to a fixed point O. The distance OB is d.



The origin is at O and at time t the length of the spring is y. You may assume that $y < d$ for all t.

(a) Express y in terms of x, l and d. [2]

(b) Determine the equation of motion of the particle in terms of m, d, x, l, l_0, k and g, the magnitude of the acceleration due to gravity. [3]

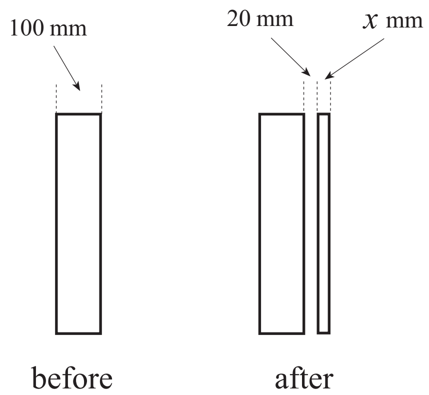
Question 19

Use the method of Gaussian elimination to determine the solution of the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 \qquad\quad &= 1 \\ 4x_1 + 9x_2 + 3x_3 &= 8 \\ -3x_1 - 4x_2 + 8x_3 &= 7 \end{aligned}$$ [5]

Question 20

A solid brick wall consists of a layer of brick, 100 mm thick, and of thermal conductivity $0.5\,\mathrm{Wm^{-1}K^{-1}}$. It is proposed to dry-line the wall using plasterboard of thickness x mm and thermal conductivity $0.2\,\mathrm{Wm^{-1}K^{-1}}$, with an air gap of 20 mm.



The convective heat transfer coefficients for the inside and outside surfaces, respectively, are $10\,\mathrm{Wm^{-2}K^{-1}}$ and $100\,\mathrm{Wm^{-2}K^{-1}}$, and the combined heat transfer coefficient of the air gap is $10\,\mathrm{Wm^{-2}K^{-1}}$.

The temperature of the room is 300 K, and the ambient outside temperature is 270 K.

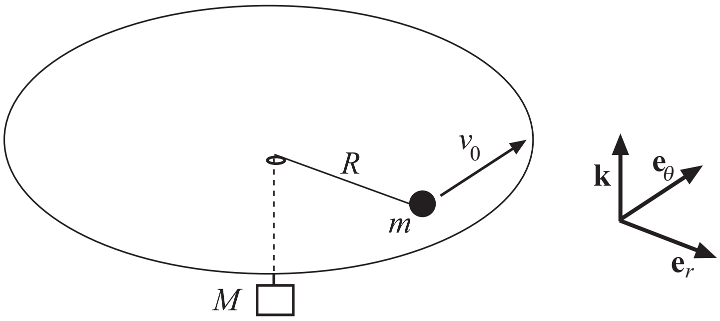
(a) Calculate the U-value of the wall before lining. [2]

(b) Calculate the U-value of the wall after lining. [2]

(c) Hence find the value of x if the rate of heat transfer is to be half its value prior to lining. [1]

Question 21

A toy car of mass m is on a frictionless horizontal table. The car is attached to a model string whose other end is attached to a mass M. The string passes through a smooth hole in the table so that the mass is suspended vertically, with half the string below the table. Model the car as a particle. A suitable coordinate system is shown below.



The car is given an initial speed v_0 perpendicular to the line of the string and in the plane of the table, and subsequently moves in a circle of radius R with constant speed.

(a) Express the acceleration of the car in vector form. *[2]*

(b) Determine v_0 in terms of m, M, R and g, the magnitude of the acceleration due to gravity. *[3]*

Question 22

A vector field is given in cylindrical polar coordinates by

$$\mathbf{V}(\rho,\, \theta,\, z) = -\rho \sin\theta\, \mathbf{e}_\rho + \rho\cos\theta\, \mathbf{e}_\theta + \mathbf{k}.$$

(a) Find $\operatorname{div} \mathbf{V}$. *[2]*

(b) Find $\mathbf{curl}\, \mathbf{V}$. *[3]*

Question 23

A paraboloid has equation $z = x^2 + y^2$.

(a) Show that the surface area of the portion of the paraboloid that lies over the disc $x^2 + y^2 \leq 2$ is given by

$$\int_{-\pi}^{\pi} \int_0^{\sqrt{2}} r\sqrt{1 + 4r^2}\, dr\, d\theta.$$ *[2]*

(b) Hence show that the surface area is $\dfrac{13\pi}{3}$. *[3]*

PART 3

Each question in this part of the paper is worth 15 marks. All of your answers will be marked and the marks from your best three answers will be added together. A maximum of 45 marks can be obtained from this part.

Question 24

Consider the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 25e^{-2t} + 27t.$$

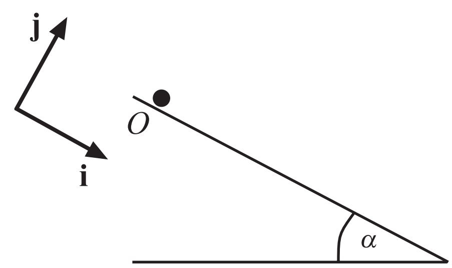
(a) Find the general solution of this equation. [9]

(b) Hence find the particular solution that satisfies $y(0) = 0$ and $\dot{y}(0) = 0$. [5]

(c) Describe briefly the long-term behaviour of y. [1]

Question 25

A toboggan of mass m is to slide down a slope inclined at an angle α to the horizontal. The toboggan starts from a point O, which is taken as origin, and is given an initial speed v_0 down the slope in the direction of the unit vector $\mathbf{i}$. Model the toboggan as a particle.



All resistive forces except air resistance may be ignored. The magnitude of the air resistance is kv, where v is the speed of the toboggan at time t after leaving O, and k is a positive constant.

(a) Draw a force diagram showing all the forces acting on the particle during its downward motion. Express each force in vector terms. [3]

(b) Write down the equation of motion of the particle during the downward motion and show that it may be expressed as

$$m\frac{dv}{dt} = mg\sin\alpha - kv.$$ [1]

(c) What is the terminal speed v_T of the toboggan? [2]

(d) Solve the differential equation in part (b) to obtain v as a function of time if the initial speed v_0 is less than v_T. [5]

(e) Hence find the distance d that the toboggan has moved along the slope at time t. [4]

Question 26

On mixing, a particular type of mortar contains a chemical M in each of two forms X and Y. In the presence of form Y, form X tends to change into form Y. Similarly, in the presence of form X, form Y tends to change into form X. During the period of transition, the rates at which these changes occur are

$$\dot{x} = 5x - 2y, \qquad \dot{y} = -10x + 4y,$$

where x and y are the respective masses of of X and Y present t hours after mixing the mortar.

(a) Express the equations in matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, clearly defining the vector $\mathbf{x}$ and the matrix $\mathbf{A}$. *[2]*

(b) Determine the general solutions for x and y as functions of t. *[6]*

Initially equal masses of 6 kg of X and Y are present. The mortar remains unstable until M is present in only one of the forms X and Y.

(c) Find a particular solution of the system that satisfies these initial conditions. *[3]*

(d) At what time does one chemical reduce to zero concentration, and which chemical is it? *[2]*

(e) Show that at this time the amount of the remaining chemical present has increased from its initial value by 50%. *[2]*

Question 27

Two populations x and y are interdependent, and a model of their coexistence is given as

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - \frac{x}{X} - \frac{y}{Y}\right) \\ \frac{dy}{dt} &= -y\left(1 - \frac{x}{2X} - \frac{2y}{Y}\right), \end{aligned}$$

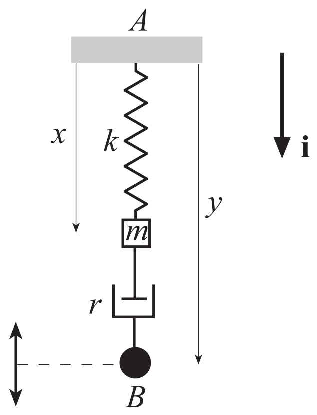
where X and Y are positive constants.

(a) Find all the equilibrium points of this system. *[5]*

(b) Classify each of the equilibrium points. *[10]*

Question 28

A particle of mass m is attached to a model spring of stiffness k and natural length l_0, whose other end is attached to a fixed point A. The particle hangs vertically, making a distance x from A, and is also attached to a model damper with damping constant r. The other end of the damper is the point B, which is forced to move so that its distance from A is y. The unit vector $\mathbf{i}$ points downward, as shown in the figure.



(a) Draw a force diagram showing all the forces acting on the particle. Express these forces in vector form. [4]

(b) Show that the equation of motion of the particle is given by

$$m\ddot{x} + r\dot{x} + kx = kl_0 + mg + r\dot{y}. \qquad [2]$$

The following values are now to be used.

m	k	l_0	r
1 kg	13 $\mathrm{Nm^{-1}}$	1 m	4 $\mathrm{Nsm^{-1}}$

(c) If the point B moves such that

$$y(t) = y_0 - \cos t \quad (y_0 > 5\,\mathrm{m}),$$

show that the particular integral for the equation of motion is given by

$$x(t) = \tfrac{3}{10}\sin t - \tfrac{1}{10}\cos t + 1 + \tfrac{1}{13}g,$$

and hence that the amplitude of the steady state oscillations is $1/\sqrt{10}$. [9]

Question 29

A model string is stretched to length L. One end is fixed at O, and the other end is attached to a smooth wire by a frictionless ring. The transverse displacement of the string at distance x from O at time t is given by $u(x,t)$.

The transverse oscillations of the string are described by the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad (0 < x < L, \quad t > 0),$$

subject to the boundary conditions

$$u(0,\,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,\,t) = 0.$$

Separation of variables, putting $u(x,t) = X(x)T(t)$, leads to the ordinary differential equation

$$\frac{d^2X}{dx^2} + \mu X = 0,$$

with boundary conditions $X(0) = \dfrac{dX}{dx}(L) = 0$, and you may assume that $\mu > 0$.

(a) Use the boundary conditions to determine the possible values that μ can take. Hence show that a solution of the above ordinary differential equation is given by

$$X(x) = A_n \sin\left((2n-1)\frac{\pi x}{2L}\right)$$

for any positive integer n. [3]

(b) Assume that the string is initially at rest. The differential equation satisfied by $T(t)$ is

$$\frac{d^2T}{dt^2} + \mu c^2 T = 0, \qquad \frac{dT}{dt}(0) = 0,$$

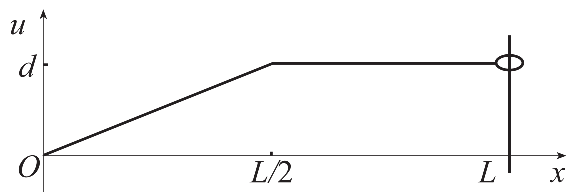
where

$$\mu = \frac{(2n-1)^2\pi^2}{4L^2}.$$

Solve this differential equation and hence show that the general solution of the partial differential equation is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left((2n-1)\frac{\pi x}{2L}\right)\cos\left((2n-1)\frac{\pi c t}{2L}\right).$$ [4]

(c) Initially the string is displaced by an amount d at its midpoint, as shown.



Show that $A_n = \dfrac{16d}{(2n-1)^2\pi^2} \sin\left((2n-1)\dfrac{\pi}{4}\right) \qquad (n = 1, 2, 3, \ldots)$.

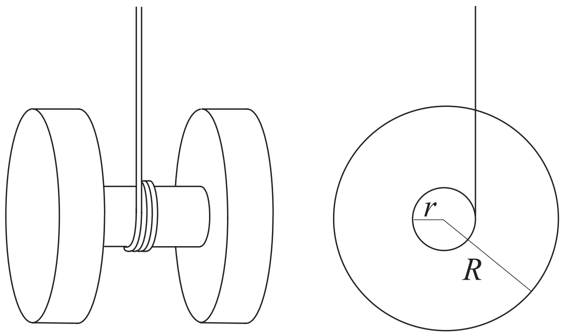
(Hint: You may find the integral

$$\int x \sin(\alpha x)\, dx = \frac{1}{\alpha^2} \sin(\alpha x) - \frac{x}{\alpha} \cos(\alpha x) + c$$

useful.) [8]

Question 30

A child's toy consists of two uniform discs, each of mass M and radius R joined by a uniform shaft of mass m and radius r ($r < R$). A model string is wound round the shaft and held vertically, leaving the toy to fall under gravity.



Assume that the tension in the string acts halfway along the shaft, the axis of the shaft remains horizontal and the string unwinds from the shaft without slipping. You may also ignore all resistive forces.

(a) Draw a force diagram showing all the forces acting on the toy, indicating the point of action of each force. [2]

(b) Give a suitable coordinate system to describe the linear motion of the centre of mass. [1]

(c) Write down an equation for the linear motion of the centre of mass. [2]

(d) Give a suitable coordinate system for the rotational motion about the centre of mass. [2]

(e) Write down an equation for the rotational motion about the centre of mass. [2]

(f) Write down an equation relating the angle of rotation to the distance fallen. [1]

(g) Show that the magnitude of the acceleration of the toy is given by

$$\frac{g(2M+m)r^2}{M(R^2+2r^2)+\frac{3}{2}mr^2}.$$

[5]

[END OF QUESTION PAPER]