M337 Complex Analysis Exam - 1995

There are some 1995 exam solutions available which were produced by I.R van de Stadt, BSc. Hons. This document has been produced to help you if you are unable to find a copy of the 1995 exam paper. The exam paper given below has been reconstructed from the solutions mentioned above.

Most of the questions are very similar to those set in subsequent years so I think this reconstruction is fairly accurate. I have doubts about the following reconstructed questions

> Question 3. This seems shorter than those on later papers. Is there a missing part?
>
> Question 12 (b)(i). The format of this is different from subsequent years.

Answers

I have not checked these. I have noticed the following typos.

Question 8
(a) Add subscript n at end of 2nd line.

Question 9
(a) (i) Replace sinh z by sinh y, cosh x by cosh y, sinh x by sinh y.
(a) (ii) Replace sinh x by sinh y.

PART I

(i) *You should attempt as many questions as you can in this part.*
(ii) *Each question in this part carries 8 marks.*

Question 1

(a) Let $w = \frac{1}{i-1}$.

(i) Determine Arg w. **[2]**
(ii) Determine the principal fourth root of w in polar form. **[3]**

(b) Give the Cartesian form of i^i, simplifying your answer as far as possible. **[3]**

Question 2

Let $A = \{ z : -\pi/2 < \mathrm{Arg}(z - 1) < \pi/2 \}$ and $B = \{ z : |z| \le 2\}$.

(a) Make separate sketches of the sets ∂A and $(B - \overline{A}) \cup A$. **[3]**
(b) Write down which of the sets, if any, is not a region :
A, $A \cup \{1\}$, $A \cap B$, and $A - B$.
(c) State whether or not the following sets are compact :
$B - A$, and $\overline{A}$. **[5]**

Question 3

Let $f(z) = \mathrm{Im}\,(z^2)$.

(a) What is the domain of the function f. Is f continuous? **[2]**
(b) Determine the standard parameterization for the line segment Γ for the line segment from 0 to $1 + i$.

Evaluate $\int_\Gamma f(z)\,dz$. **[6]**

Question 4

Let $f(z) = e^z \cos z$.

(a) Find the Taylor series about 0 for the function f up to the term in z^4. For what values of z does the Taylor series represent f? Justify your answer. **[4]**

(b) (i) Evaluate $\int_C \frac{f(z)}{z^2}\,dz$, where $C = \{z : |z| = 1\}$ **[2]**

(b) (ii) Determine the first 3 non-zero terms of the Taylor series about 0 for
$g(z) = e^z (\cos z - \sin z)$. **[2]**

Question 5

(a) Find the residue of the function

$$f(z) = \frac{e^{3iz}}{z^2 + 4}$$

at each of the poles of f. **[4]**

(b) Hence evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{\cos 3t}{t^2 + 4}\, dt\,.$$ **[4]**

Question 6

(a) Show that if $|z| = 1$ then $|e^z| > 1/3$. **[2]**

Let the function $f(z) = e^z - \frac{1}{3} z^4$.

(b) Find the number of zeros of f in $\{ z : |z| \le 1 \}$. **[4]**

(c) Hence evaluate the integral

$$\int_\Gamma \frac{1}{f(z)}\, dz$$

where $\Gamma = \{ z : |z| = 1 \}$ **[2]**

Question 7

Let $q(z) = 2/\bar{z}$ be a velocity function on $\mathbb{C} - \{0\}$.

(a) Explain why q represents a model fluid flow on $\mathbb{C} - \{0\}$. **[1]**

(b) Determine a stream function for this flow. Hence find the equations of the streamlines through the points i and $1 + i$, and sketch these streamlines, indicating the directions of flow in each case. **[5]**

(c) Evaluate the flux of q across the unit circle $\{ z : |z| = 1\}$. **[2]**

Question 8

(a) Prove that the iteration sequence

$$z_{n+1} = 2z_n(1 - z_n), \qquad n = 0, 1, 2, \ldots,$$

with $z_0 = -1$ is conjugate to the iteration sequence

$$w_{n+1} = w_n^2$$

with $w_0 = 3$. **[3]**

(b) Which of the following points c lie in the Mandelbrot :

(i) $c = -1 - i$;

(ii) $c = - 1/4$.

Justify your answer in each case. **[5]**

PART II

(iii) *You should attempt no more than* **TWO** *questions in this part.*
(iv) *Each question in this part carries 18 marks.*

Question 9

(a) (i) Show that

$$| \sin z |^2 = \sin^2 x + \sinh^2 y, \text{ and}$$
$$| \cos z |^2 = \cos^2 x + \sinh^2 y,$$

where $z = x + iy$. **[6]**

(ii) Hence, or otherwise, determine the values of z in the region $\{ z : -\pi < \operatorname{Re} z < \pi \}$ for which $|\tan z|^2 \le 1$. **[5]**

(c) Determine for which values of the real numbers a and b the function

$$f(z) = x^2 + 2axy\, i + by^2, \qquad \text{where } z = x + iy$$

is analytic on $\mathbb{C}$. **[7]**

Question 10

Let $f(z) = \dfrac{\sin z}{z(z^2+9)}$.

(a) Classify the singularities of f and find the residues at each of the poles. **[5]**

(b) Evaluate the integral $\int_\Gamma f(z)\,dz$ where

(i) $\Gamma = \{ z : |z| = 1 \}$.
(ii) $\Gamma = \{ z : |z| = \pi \}$. **[5]**

(c) Evaluate the integral $\int_\Gamma f(z)\,dz$, where Γ is a circle of radius 1 about the point 3i. **[3]**

(d) Find the residues of the function $g(z) = \dfrac{1}{zf(z)}$ at each pole inside the circle $\{ z : |z| = \pi \}$. **[5]**

Question 11

(a) (i) Show that

$$|\exp(e^{-z})| = \exp(e^{-x}\cos y), \quad \text{for } z = x + iy \in \mathbb{C}.$$

(ii) Determine

$$\max\{\, |\exp(e^{-z})| : -1 \le \operatorname{Re} z \le 1, -\pi \le \operatorname{Im} z \le \pi \,\}$$

and find the point or points at which this maximum is attained. **[9]**

(c) For $r > 0$, show that the functions

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{r}\right)^n \quad (|z| < r)$$

and

$$g(z) = -\sum_{n=0}^{\infty} \left(\frac{r}{z}\right)^n \qquad (|z| > r)$$

are indirect analytic continuations of each other. **[9]**

Question 12

(a) (i) Determine the extended Möbius transformation $\hat{f}$ which maps 0 to $-i$, 1 to 1, and ∞ to i. **[3]**

(ii) Find the image of the extended imaginary axis under $\hat{f}$. **[3]**

Let $h(z) = z^{1/2}$.

(b) (i) Prove that $h(z)$ is a one-one conformal mapping on $\{ z : \operatorname{Re} z > 0\}$.
Hence show that the composite function $g = h \circ f$ is a one-one conformal mapping.
Determine the image of $\{ z : \operatorname{Re} z > 0\}$ under g.

(b) (ii) Write down a formula for the corresponding inverse function g^{-1}. **[12]**

[END OF QUESTION PAPER]