The Open University

MST209/J

Course Examination 2009

Mathematical Methods and Models

Friday 16 October 2009 **10.00 am – 1.00 pm**

Time allowed: 3 hours

Personal Identifier								
Examination No.								

You are **not** allowed to use a calculator in this examination.

There are three parts to this paper. In each part of the paper the questions are arranged in the order they appear in the course. There are 115 marks available, but scores greater than 100 will be rounded down to 100.

Part 1 consists of 15 questions each worth 2 marks. You are advised to spend no more than 1 hour on this part. Enter one option in each box provided on the question paper; use your answer book(s) for any rough work. Incorrect answers are not penalised. Cross out mistakes and write your answer next to the box provided.

Part 2 consists of 8 questions each worth 5 marks. You are advised to spend no more than $1\frac{1}{4}$ hours on this part.

Part 3 consists of 7 questions each worth 15 marks. Your best three marks will be added together to give a maximum of 45 marks.

In Parts 2 and 3: Write your answers in the answer book(s) provided. The marks allocated to each part of each question are given in square brackets in the margin. Unless you are directed otherwise in the question, you may use any formula or other information from the *Handbook* provided that you give a reference. Do **not** cross out any answers unless you have supplied a better alternative — everything not crossed out may receive credit.

At the end of the examination, **check that you have written your personal identifier and examination number on each answer book used (as well as in the boxes above)**. Failure to do so may mean that your work cannot be identified. Use the paper fastener provided to fix together all your answer books, and the question paper, with your signed desk record on top.

PART 1

Each question in this part of the paper is worth 2 marks. Fill in the appropriate response in the box alongside the question.

Question 1

Given that

$$2\ln y + \ln(3x+5) = 2\ln 2 + \ln(x-2) + \ln(x+2),$$

and that $x > 2$ and $y > 0$, select the option that gives y as a function of x.

Options Answer: $\square$

A $y = \dfrac{4(x^2-4)}{3x+5}$ **B** $y = \sqrt{\dfrac{4(x^2-4)}{3x+5}}$

C $y = \dfrac{3x+5}{4(x^2-4)}$ **D** $y = \sqrt{\dfrac{3x+5}{4(x^2-4)}}$

Question 2

This question concerns the differential equation

$$\frac{dy}{dx} - 2y = 4x + 3 \quad (x > 0, \quad y > 0).$$

Which of the following options is correct?

Options Answer: $\square$

A The differential equation may be solved using the separable variables method but not the integrating factor method.

B The differential equation may be solved using integrating factor method but not the the separable variables method.

C The differential equation may be solved using either the separable variables method or the integrating factor method.

D The differential equation cannot be solved using either the separable variables method or the integrating factor method.

Question 3

Select the option that gives the result of evaluating the expression

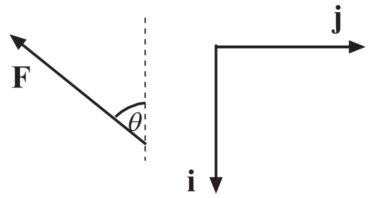
$$\mathbf{i} \cdot ((\mathbf{i} + \mathbf{j}) \times (\mathbf{j} - 2\mathbf{k})).$$

Options Answer: $\square$

A 0 **B** 1 **C** 2 **D** -2

Question 4

Consider the vector $\mathbf{F}$ and the axes shown below.



Select the option giving the $\mathbf{j}$-component of $\mathbf{F}$.

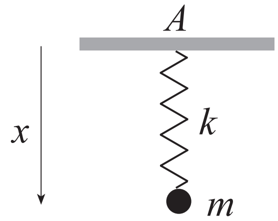
Options

Answer:

A $|\mathbf{F}|\cos\theta$ **B** $|\mathbf{F}|\sin\theta$ **C** $-|\mathbf{F}|\cos\theta$ **D** $-|\mathbf{F}|\sin\theta$

Question 5

A particle of mass m is attached to a model spring of stiffness k and natural length l_0. The other end of the spring is fixed at a point A and the spring is vertical. Take A as the datum for gravitational potential energy, and the datum for potential energy in the spring at the point of zero extension. Select the option that gives the total potential energy stored in the system when the mass is a distance x from A.



Options

Answer:

A $-mgx + \frac{1}{2}k(x - l_0)^2$ **B** $-mgx - \frac{1}{2}k(x - l_0)^2$

C $mgx - \frac{1}{2}k(x - l_0)^2$ **D** $mgx + \frac{1}{2}k(x - l_0)^2$

Question 6

Zero is one of the eigenvalues of the matrix

$$\begin{bmatrix} 3 & 0 & 6 \\ 2 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}.$$

Select the option that gives a corresponding eigenvector.

Options

Answer:

A $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T$ **B** $\begin{bmatrix} -2 & 0 & -1 \end{bmatrix}^T$

C $\begin{bmatrix} -2 & 0 & 1 \end{bmatrix}^T$ **D** $\begin{bmatrix} 1 & 0 & -2 \end{bmatrix}^T$

Question 7

Select the option that gives $\partial f/\partial\phi$ for the function $f(r,\theta,\phi) = r^2\cos^2\theta\sin^2\phi$.

Options

A $2r\sin^2\theta\cos^2\phi$ **B** $-2r^2\sin\theta\cos\theta\sin^2\phi$

C $2r^2\cos^2\theta\cos\phi\sin\phi$ **D** $-2r^2\sin^2\theta\cos\phi\sin\phi$

Answer: $\square$

Question 8

Two populations x and y are modelled by the system of differential equations

$$\dot{x} = rx^2 + 2sxy$$
$$\dot{y} = -qx + py^2,$$

where p, q, r and s are constants. Select the option that gives the Jacobian matrix for this system at the point (a, b).

Options

A $\begin{bmatrix} 2pb - qa & -qb \\ 2sb & 2ra \end{bmatrix}$ **B** $\begin{bmatrix} 2sb & 2ra \\ 2pb - qa & -qb \end{bmatrix}$

C $\begin{bmatrix} 2ra + 2sb & 2sa \\ -q & 2pb \end{bmatrix}$ **D** $\begin{bmatrix} -qb & 2pb - qa \\ 2ra & 2sb \end{bmatrix}$

Answer: $\square$

Question 9

When an aerofoil of area A is moving with speed v through a fluid of density ρ, the magnitude of the lift force F is given by

$$F = \tfrac{1}{2}\rho v^2 AC,$$

where C is the lift coefficient. Select the option that gives the dimensions of the constant C.

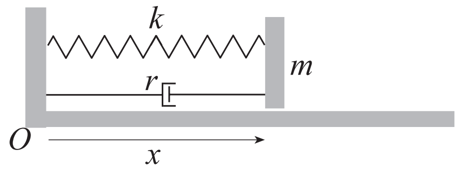
Options

A dimensionless **B** L **C** L^2 **D** $\mathrm{L}^2\mathrm{T}$

Answer: $\square$

Question 10

A particle of mass m slides in a straight line on a smooth horizontal table. It is connected to a fixed point O by a model spring of stiffness k and natural length l_0 and a model damper of damping constant r. The displacement of the particle from O at time t is x.



Which option gives the equation of motion of the particle?

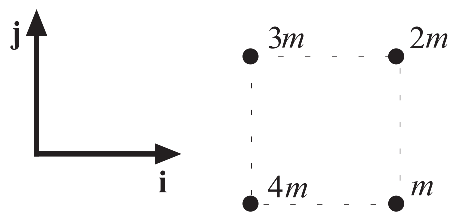
Options Answer: ☐

A $m\ddot{x} + r\dot{x} + kx = kl_0$ **B** $m\ddot{x} + r\dot{x} + kx = (k + r)l_0$

C $m\ddot{x} + (k + r)\dot{x} + kx = (k + r)l_0$ **D** $m\ddot{x} - r\dot{x} + kx = kl_0$

Question 11

Particles of mass m, $2m$, $3m$ and $4m$ are placed at the corners of a square of unit length, as shown.



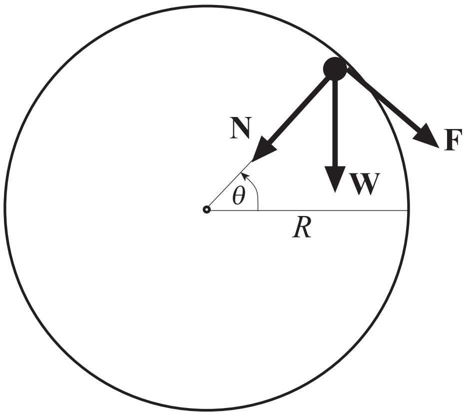
Taking the origin at the bottom left corner, and unit vectors as shown, select the option that gives the centre of mass of the four particles.

Options Answer: ☐

A $\mathbf{0}$ **B** $3\,\mathbf{i} + 5\,\mathbf{j}$ **C** $\frac{1}{2}\,\mathbf{i} + \frac{5}{6}\,\mathbf{j}$ **D** $\frac{3}{10}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$

Question 12

A particle of mass m is moving on the inside of a circular cylinder of radius R. The particle is moving in an anti-clockwise direction in a vertical plane. The unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ take their usual meanings.



Which option gives the tangential equation of motion of the bead?

Options

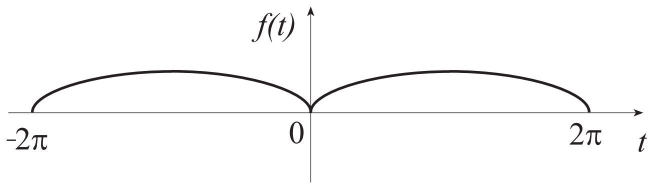
Answer: [ ]

A $mR\ddot{\theta} = -mg\cos\theta - |\mathbf{F}|$ **B** $mR\ddot{\theta} = -mg\cos\theta + |\mathbf{F}|$

C $mR\ddot{\theta} = mg\cos\theta - |\mathbf{F}|$ **D** $mR\ddot{\theta} = mg\cos\theta + |\mathbf{F}|$

Question 13

Consider the function $f(t)$ whose graph appears below.



Select the option that gives a possible Fourier series for the periodic extension of $f(t)$, where A_r and B_r are non-zero constants.

Options

Answer: [ ]

A $A_0 + \sum_{r=1}^{\infty} A_r \cos(rt)$ **B** $\sum_{r=1}^{\infty} B_r \sin(rt)$

C $1 + \sum_{r=1}^{\infty} B_r \sin(rt)$ **D** $A_0 + \sum_{r=1}^{\infty} A_r \cos(rt) + \sum_{r=1}^{\infty} B_r \sin(rt)$

Question 14

A scalar field is given in spherical polar coordinates by $h(r,\,\theta,\,\phi) = r\phi\sin\theta$. Which option gives **grad** h?

Options

Answer:

A $\mathbf{0}$

B $\phi\cos\theta\,\mathbf{e}_r + \phi\sin\theta\,\mathbf{e}_\theta + \mathbf{e}_\phi$

C $\phi\sin\theta\,\mathbf{e}_r + \phi\cos\theta\,\mathbf{e}_\theta + \mathbf{e}_\phi$

D $\phi\sin\theta\,\mathbf{e}_r + \phi\cos\theta\,\mathbf{e}_\theta - \mathbf{e}_\phi$

Question 15

A spherical shell has inner radius R_1 and outer radius R_2 and is made of material of uniform density κ. Select the option that gives the moment of inertia of the shell about a diameter.

Options

Answer:

A $\displaystyle\int_{R_1}^{R_2}\int_0^{\pi}\int_{-\pi}^{\pi} \kappa r^2 \sin\theta\, d\phi\, d\theta\, dr$

B $\displaystyle\int_{R_1}^{R_2}\int_0^{\pi}\int_{-\pi}^{\pi} \kappa r^2 \sin^2\theta\, d\phi\, d\theta\, dr$

C $\displaystyle\int_{R_1}^{R_2}\int_0^{\pi}\int_{-\pi}^{\pi} \kappa r^3 \sin^3\theta\, d\phi\, d\theta\, dr$

D $\displaystyle\int_{R_1}^{R_2}\int_0^{\pi}\int_{-\pi}^{\pi} \kappa r^4 \sin^3\theta\, d\phi\, d\theta\, dr$

PART 2

Each question in this part of the paper is worth 5 marks.

Question 16

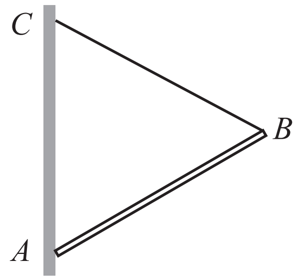
Solve the initial-value problem

$$x^2\frac{dy}{dx} + xy = 1 + x^2 \quad (x \geq 1), \quad y(1) = 1,$$

expressing your answer in the form $y = f(x)$. [5]

Question 17

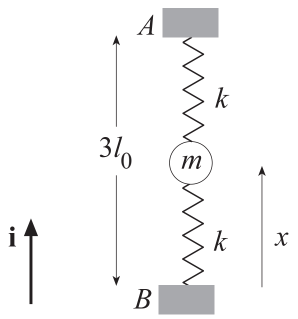
A heavy rod of mass M is smoothly hinged to a vertical wall at the point A. The rod is supported by means of a rope attached to the other end of the rod at B and to a point C on the wall vertically above A.



Model the rope as a model string, and the rod as a model rod. Draw a force diagram showing all the forces acting on the rod. Briefly define each force. [5]

Question 18

A particle of mass m is connected to two vertical model springs, each of stiffness k and natural length l_0. The other end of the first spring is attached to a fixed point A. The other end of the second spring is attached to a fixed point B that is a distance $3l_0$ below A. The particle is constrained to move in a vertical line between A and B, and there are no resistive forces.



(a) Draw a force diagram showing all the forces acting on the particle when it is a distance x from B. [2]

(b) Express each of these forces in vector form in terms of the given parameters. [2]

(c) Hence derive the equation of motion of the particle. [1]

Question 19

Consider the system of equations

$$\begin{aligned} x_1 \qquad\ \ - x_3 &= 3 \\ 2x_1 + x_2 \ \ - x_3 &= 8 \\ 3x_1 + x_2 - 2x_3 &= 11. \end{aligned}$$

Write these equations in augmented matrix form, and hence use Gaussian elimination to obtain the general solution of this system. *[5]*

Question 20

A part of the trajectory of a fairground car of mass m is described by the parametrization

$$\mathbf{r}(t) = 2\cos(2\pi t)\,\mathbf{i} - 3\sin(2\pi t)\,\mathbf{j} + (1 - t^2)\,\mathbf{k}, \qquad (-1 \leq t \leq 1),$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are a right-handed set of unit vectors, with $\mathbf{k}$ pointing vertically upwards. Model the car as a particle of mass m.

(a) Write down the velocity of the car at time t. *[1]*

(b) Write down the acceleration of the car at time t. *[2]*

(c) What is the vertical component of the force acting on the car at time t? *[2]*

Question 21

A ball of mass $2m$ moves along the $\mathbf{i}$-direction with speed u. It collides with a second ball, of identical size but mass m. After the collision the two balls move in the $\mathbf{i}$-direction, the first ball with speed v and the initially stationary ball with speed w. The coefficient of restitution between the two balls is e.

(a) What is the momentum of the ball of mass $2m$ before and after the collision? *[1]*

(b) Find v and w in terms of u. *[2]*

(c) Show that the energy lost in the collision is $\frac{1}{3}mu^2\,(1 - e^2)$. *[2]*

Question 22

A vector field is given in cylindrical polar coordinates by

$$\mathbf{V}(\rho, \theta, z) = \rho^2 \cos\theta\,\mathbf{e}_\theta.$$

(a) Find div $\mathbf{V}$. *[2]*

(b) Find **curl** $\mathbf{V}$. *[3]*

Question 23

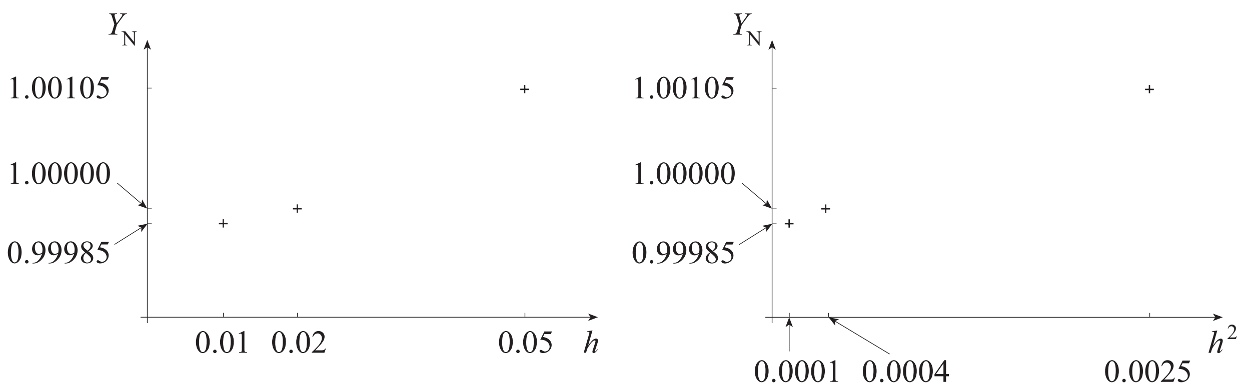
A certain numerical method is applied to a differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(0) = y_0,$$

in order to determine an approximation Y_N to the value $y(1)$. Successive approximations are obtained using step sizes $h = 0.05$, $h = 0.02$ and $h = 0.01$. The following values were obtained.

step size	Y_N
0.05	1.00105
0.02	1.00000
0.01	0.99985

The resulting plots of Y_N against h and against h^2 are shown below.



(a) What is the likely order of the numerical method? [1]

(b) Find an upper bound for the step size that should guarantee an accuracy of 6 decimal places in the required approximation to $y(1)$. [4]

PART 3

Each question in this part of the paper is worth 15 marks. All of your answers will be marked and the marks from your best three answers will be added together. A maximum of 45 marks can be obtained from this part.

Question 24

Consider the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 29\cos(3t) - 26\sin(3t).$$

(a) Find the general solution to this equation. [*9*]

(b) Hence find the particular solution that satisfies $y(0) = 7$ and $\dot{y}(0) = 4$. [*6*]

Question 25

This question concerns the modelling of an aeroplane take-off. The plane, which has mass m, stands at rest at a point at the end of a straight, horizontal runway. In order to take to the air, it must reach a speed v_1 relative to the ground. When turned to full power, the engines produce a constant force of magnitude P in the forward direction. The resistive forces at first increase with the speed, but then reach a maximum value k when the plane is moving at speed v_0. According to the model the magnitude of the resistive force is given by

$$R(v) = \begin{cases} \dfrac{k}{v_0} v & 0 \leq v \leq v_0 \\ k & v_0 < v. \end{cases}$$

(a) Draw a force diagram, showing all the forces acting on the plane while it is still in contact with the runway. Express each force in vector terms. [2]

(b) Write down the equation of motion for the plane for the period up until its speed reaches v_0 and show that it may be expressed as

$$\frac{mv_0\, v}{Pv_0 - kv} \frac{dv}{dx} = 1,$$

where x is the distance travelled by the plane from rest. [2]

(c) Solve this differential equation to obtain $v(x)$, and show that the distance x_1 travelled by the plane at the end of this period is

$$x_1 = \frac{mv_0^2 P}{k^2} \ln\left(\frac{P}{P-k}\right) - \frac{mv_0^2}{k}.$$

(Hint: You may find the integral

$$\int \frac{v}{a+bv}\, dv = \frac{v}{b} - \frac{a}{b^2} \ln(a+bv) \quad (a+bv > 0)$$

useful.) [7]

(d) Now consider the next period of motion. When the plane's speed has increased from v_0 to v show that the distance x travelled from rest may be expressed as

$$x = x_1 + \frac{m(v^2 - v_0^2)}{2(P-k)}.$$ [3]

(e) Hence show that the total distance x_2 travelled on the runway before becoming airborne is

$$x_2 = \frac{mv_0^2 P}{k^2} \ln\left(\frac{P}{P-k}\right) - \frac{mv_0^2}{k} + \frac{m(v_1^2 - v_0^2)}{2(P-k)}.$$ [1]

Question 26

Consider the following inhomogeneous system of first-order differential equations

$$\begin{aligned}\dot{x} &= 4x + 2y - 6t + 5\\ \dot{y} &= 4x - 3y - t + 5.\end{aligned}$$

(a) Express the system of equations in matrix form. [1]

(b) Find the eigenvalues of the matrix of coefficients and an eigenvector corresponding to each eigenvalue. Hence write down the complementary function for the system of equations. [5]

(c) Determine the particular integral to the inhomogeneous system and hence write down the general solution of the system given above. [4]

(d) If initially $x(0) = 0$ and $y(0) = -13$ find the particular solution at time t. [3]

(e) Describe the long term behaviour of this solution. [2]

Question 27

A cylindrical pipe is made of material with thermal conductivity κ. It has inner radius r, external radius R and length L. The pipe is fitted with a uniform layer of lagging of thickness b which has thermal conductivity κ_{lag}. There is no air gap between the outer surface of the pipe and the lagging.

(a) Let h_{in} and h_{out} be the convective heat transfer coefficients at the inside and outside surfaces of the lagged pipe, Θ_{in} and Θ_{out} the corresponding temperatures ($\Theta_{\text{in}} > \Theta_{\text{out}}$) and q the steady-state rate of heat transfer through the walls of the pipe. Starting from Fourier's Law, and justifying each step of your working, derive the expression

$$q = 2\pi L(\Theta_{\text{in}} - \Theta_{\text{out}})\left(\frac{1}{rh_{\text{in}}} + \frac{1}{\kappa}\ln\left(\frac{R}{r}\right) + \frac{1}{\kappa_{\text{lag}}}\ln\left(\frac{R+b}{R}\right) + \frac{1}{(R+b)h_{\text{out}}}\right)^{-1}$$

for the convective rate of heat transfer q from the interior of the pipe to the outside.

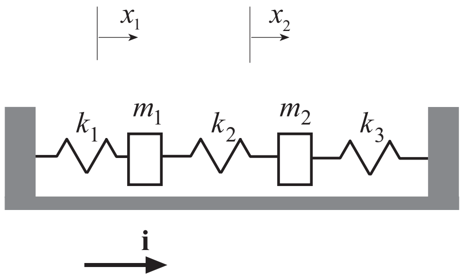
(Hint: For a pipe, Fourier's law is $q = -\kappa A\dfrac{d\theta}{dr}$, where A is the surface area, r is the radius and κ is the thermal conductivity. The rate of heat transfer at the surface of a cylindrical pipe is given by $q = 2\pi Lrh(\Theta_1 - \Theta_2)$, where r is the radius of the surface of the pipe, L is the length, Θ_1 is the temperature of the surface of the pipe, Θ_2 is the ambient temperature, and h is the convective heat transfer coefficient.) [10]

(b) Determine the thickness b of lagging in terms of R, h_{out} and κ_{lag} for which the heat loss from the lagged pipe is a maximum.

(Hint: You do not need to show that your result *is* a maximum rather than a mimimum, and you may assume that $\dfrac{\kappa_{\text{lag}}}{h_{\text{out}}} > R$.) [5]

Question 28

Two particles, of mass m_1 and m_2, are attached to three springs, of stiffness k_1, k_2 and k_3, respectively, as shown.



The free ends of two of the springs are attached to fixed points, and the particles are constrained to move in a horizontal line. The displacements of the two particles from their equilibrium positions are denoted x_1, x_2, respectively, and the unit vector **i** points horizontally as shown. You may ignore air resistance and any other frictional forces.

(a) Draw a force diagram showing the forces acting on each particle. *[2]*

(b) Express the changes in the spring forces (from their equilibrium values) acting on each particle in terms of the variables and parameters given above. *[2]*

(c) Write down the equation of motion of each particle, and hence show that the dynamic matrix of the system is

$$\begin{bmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} \end{bmatrix}.$$

[3]

(d) You are now given the additional information.

m_1	m_2	k_1	k_2	k_3
1 kg	1 kg	$2\,\mathrm{N\,m^{-1}}$	$1\,\mathrm{N\,m^{-1}}$	$2\,\mathrm{N\,m^{-1}}$

Find the normal mode angular frequencies of the system. *[3]*

(e) Find an eigenvector corresponding to each angular frequency, and hence write down the displacement of each particle from its equilibrium position as a function of time. *[3]*

(f) If the particles are released from rest with

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$

give a brief qualitative description of the subsequent motion. *[2]*

Question 29

The temperature u at a point on a insulated rod of length L is given by the partial differential equation

$$k\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

where k is a constant, t represents time, and x is the distance of the point from one end of the rod. Both ends of the rod are maintained at a constant temperature of 0°C.

(a) Use the method of separation of variables $u(x,t) = X(x)T(t)$ to show that the function $X(x)$ satisfies the differential equation

$$X'' - \mu X = 0,$$

for some constant μ, with boundary conditions $X(0) = X(L) = 0$. [3]

(b) Show that the only non-trivial solutions occur when $\mu < 0$. Hence write down the solution of the differential equation, clearly stating what values μ is allowed to take. [4]

(c) Write down the differential equation that T satisfies. Solve this, and hence determine an expression for the general solution $u(x,t)$ in terms of arbitrary constants. [5]

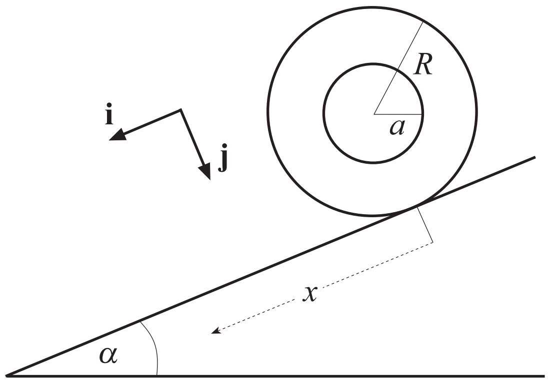
(d) Given the initial condition that at time $t = 0$ the temperature is given as a function of x by

$$u(x,0) = x(L-x), \quad (0 \leq x \leq L),$$

write down an expression that will determine the arbitrary constants. (There is no need to evaluate these constants.) [3]

Question 30

In this question you may ignore air resistance. A wheel consists of a metal cylinder of radius a and mass M on which is mounted a solid tyre of inner radius a, outer radius R and mass m. The wheel starts from rest and rolls without slipping down a rough plane inclined to the horizontal at an angle α.



(a) Determine the moment of inertia I of the wheel about its axis. [2]

(b) Draw a force diagram showing all the forces acting on the wheel during its descent. [2]

(c) Write down the equation of linear motion of the centre of mass. [2]

(d) At time t the wheel has rolled a distance x down the plane, and has angular speed $\dot{\theta}$. Show that the kinetic energy T of the wheel is given by

$$T = \tfrac{1}{2}\left((M+m)\dot{x}^2 + \tfrac{1}{2}Ma^2\dot{\theta}^2 + \tfrac{1}{2}m(R^2+a^2)\dot{\theta}^2\right). \qquad [3]$$

(e) Write down the rolling condition for the wheel. [1]

(f) Taking the datum for gravitational potential energy at the initial position of the centre of mass of the wheel, show that the total mechanical energy E of the system is

$$E = \tfrac{1}{2}\dot{x}^2\left(M + \tfrac{3}{2}m + \frac{(M+m)a^2}{2R^2}\right) - (M+m)gx\sin\alpha. \qquad [3]$$

(g) By differentiating with respect to t find the magnitude of the linear acceleration of the centre of mass of the wheel. [2]

[END OF QUESTION PAPER]