Problem 1: A university has the following academic ranks for tenured/tenure-track faculty: assistant professor, associate professor, and professor. The first two ranks are tenure-track, so only professors have tenure. A tenure-track faculty member may be discharged, remain at his or her present rank, or be promoted to a higher rank. Discharged faculty are never rehired. Only tenure-track faculty members can be promoted or discharged. A professor may remain as his or her rank or retire. All changes happen at the end of an academic year.

a. The career path of a faculty member can be modeled using a Markov chain with 5 states. Define the state variable, the state space, and give the index set for the Markov chain.
    a. 1.Discharged
    b. 2.Assistant Professor
    c. 3.Associate Professor
    d. 4.Professor
    e. 5.Retired

State Space {1,2,3,4,5}

T{0,1,2,3,4…} at the end of each year

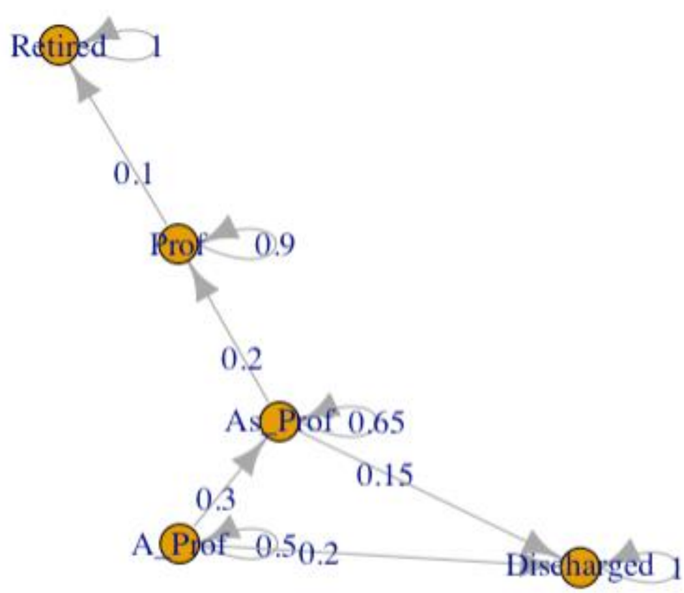
b. The promotion and discharge probabilities for assistant professors are 0.3 and 0.2, respectively. The promotion and discharge probabilities for associate professors are 0.2 and 0.15, respectively. On average, 10% of professors retire each year. Construct a transition matrix and transition diagram for the Markov chain.

```
library(diagram)

## Loading required package: shape

library(markovchain)

## Package:   markovchain
## Version:   0.8.5-4
## Date:      2021-01-07
## BugReport: https://github.com/spedygiorgio/markovchain/issues

Discharged <- c(1,0,0,0,0)
A_Prof<-c(0.2,0.5,0.3,0,0)
As_Prof<- c(0.15,0,0.65,0.2,0)
Prof<-c(0,0,0,0.9,0.1)
Retired <- c(0,0,0,0,1)

x <- cbind(Discharged,A_Prof,As_Prof,Prof,Retired)
colnames(x)<- c("Discharged","A_Prof","As_Prof","Prof","Retired")
transition_matrix<- matrix(x, nrow =5 ,byrow=TRUE)
```

```r
dtmcA <-
new("markovchain",transitionMatrix=transition_matrix,states=c("Discharged","A_Prof","As_Prof","Prof","Retired"), name="MarkovChain A")
dtmcA
```

```
## MarkovChain A
##  A  5 - dimensional discrete Markov Chain defined by the following states:
##  Discharged, A_Prof, As_Prof, Prof, Retired
##  The transition matrix  (by rows)  is defined as follows:
##             Discharged A_Prof As_Prof Prof Retired
## Discharged       1.00    0.0    0.00  0.0     0.0
## A_Prof           0.20    0.5    0.30  0.0     0.0
## As_Prof          0.15    0.0    0.65  0.2     0.0
## Prof             0.00    0.0    0.00  0.9     0.1
## Retired          0.00    0.0    0.00  0.0     1.0
```

```r
plot(dtmcA)
```



c. If the university's faculty consists of 45% assistant professors, 30% associate professors, and 25% professors at the end of this academic year, what should the distribution of faculty be at the end of the next academic year? Express answers to the nearest 0.001.

```
#current distribution
c <- c(0,0.45,0.3,0.25,0)
#distribution by year end
y<- c%*%transition_matrix
colnames(y) <-
c("Discharged","Assistant_Professor","Associate_Professor","Professor","Retir
ed")
y
```

```
##      Discharged Assistant_Professor Associate_Professor Professor Retired
## [1,]      0.135               0.225                0.33     0.285   0.025
```

Problem 2: Consider a timber growth model used in forest management. The states are size categories, measured by tree diameter. Every three years, all the trees in a stand were measured, and from that data the following transition matrix was determined. Units are in inches.

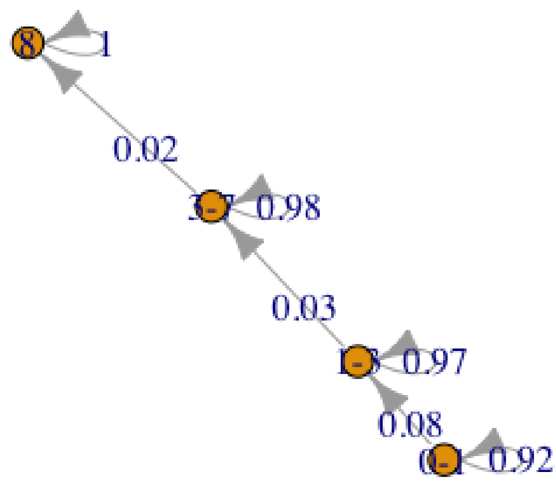
a. Define the state space for the Markov chain and draw the transition diagram. State Space

1. 0-1
2. 1-3
3. 3-7
4. >8

State Space {1,2,3,4}

```
library(diagram)
library(markovchain)

size_1 <- c(0.92,0.08,0,0)
size_2 <- c(0,0.97,0.03,0)
size_3 <- c(0,0,0.98,0.02)
size_4 <- c(0,0,0,1)

x <- cbind(size_1,size_2,size_3,size_4)
colnames(x)<- c("0-1","1-3","3-7","8")
rownames(x)<- c("0-1","1-3","3-7","8")
transition_matrix_2<- matrix(x, nrow =4 ,byrow=TRUE)
dtmcA <- new("markovchain",transitionMatrix=transition_matrix_2,states=c("0-
1","1-3","3-7","8"), name="MarkovChain A")
dtma



## MarkovChain A
##  A  4 - dimensional discrete Markov Chain defined by the following states:
##  0-1, 1-3, 3-7, 8
##  The transition matrix  (by rows)  is defined as follows:
##      0-1  1-3  3-7    8
## 0-1 0.92 0.08 0.00 0.00
## 1-3 0.00 0.97 0.03 0.00
## 3-7 0.00 0.00 0.98 0.02
## 8   0.00 0.00 0.00 1.00
```

```r
plot(dtmcA)
```



b. Note that the time for each transition is 3 years. This means that the index set is T = {0;1;2;3;4; : }and each number represents a 3-year period. So X1 is the state variable representing the size of a tree after 1 transition which is 3 years. Similarly, X2 takes on the number of the state representing tree size after 6 years, or two transitions.

c. Find the percentage of the current trees in the stand that are currently 0-1 inches in diameters which, after 15 years will be:

   a. i. 1-3 inches in diameter - 31.9%

   b. ii. 3-8 inches in diameter -2.1%

   c. iii. more than 8 inches in diameter -0.0449%

```
library(matrixcalc)
#Each transition is three years, so to get to 15,is 5 steps
P_5 <- matrix.power(transition_matrix_2,5)
P_5

##           [,1]     [,2]       [,3]         [,4]
## [1,] 0.6590815 0.319444 0.02102508 0.0004493904
## [2,] 0.0000000 0.858734 0.13556031 0.0057056610
## [3,] 0.0000000 0.000000 0.90392080 0.0960792032
## [4,] 0.0000000 0.000000 0.00000000 1.0000000000

inch_1_3 <-P_5[1,2]
inch_1_3

## [1] 0.319444

inch_3_8 <-P_5[1,3]
inch_3_8

## [1] 0.02102508

inch_8  <- P_5[1,4]
inch_8

## [1] 0.0004493904
```

d. Consider the current population of 0-1 inch diameter trees. Using the Markov model, find the percentage that will be less than 1 inch in diameter after 30 years.- 43.4%

```r
#Each transition is three years, so to get to 30,is 10 steps
P_10 <- matrix.power(transition_matrix_2,10)
P_10

##           [,1]      [,2]       [,3]        [,4]
## [1,] 0.4343885 0.4848571 0.07616618 0.004588288
## [2,] 0.0000000 0.7374241 0.23894604 0.023629833
## [3,] 0.0000000 0.0000000 0.81707281 0.182927193
## [4,] 0.0000000 0.0000000 0.00000000 1.000000000

inch_0_1 <-P_10[1,1]
inch_0_1

## [1] 0.4343885
```

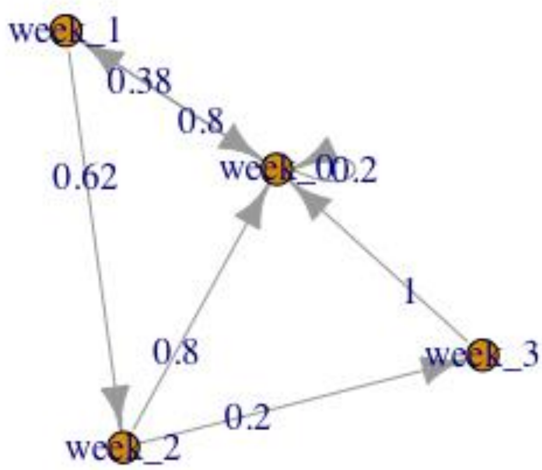
e. Suppose that of the trees currently in the stand, 50% have less than 1 inch diameter, 28% have 1-3 inch diameter, 13% have 3-8 inch diameter, and the rest have over 8 inch diameter. What will the distribution of diameters be in 12 years?

```r
inch_0_1 <- 0.50
inch_1_3 <- 0.28
inch_3_8 <- 0.13
inch_8 <- 1-(inch_0_1+inch_1_3+inch_3_8)
current <- c(inch_0_1,inch_1_3,inch_3_8,inch_8)

P_12 <- matrix.power(transition_matrix_2,4)
current%*%P_12

##           [,1]      [,2]      [,3]      [,4]
## [1,] 0.3581965 0.3830019 0.1576419 0.1011597
```

Problem 3: An electronic device contains a number of interchangeable parts that operate independently. Each component is subject to random failure which renders it completely inoperable. All components are inspected at the end of every week.A component which is found to have failed is replaced with an identical new component. Based on historical data, a new component has 0.2 probability of failing in the first week. Components that are 1 week old have a 0.375 probability of failingin the second week. Components that are 2 weeks old have a 0.8 probability of failing in the third week. No components lastonger than 4 weeks. Let the process be represented by the following state variable: 0:0.2 probability of failing 1:0.375 probability of failing 2:0.8 probability of failing 3:0 probability of failing

a. What is the transition matrix for this process?

```
week_0<- c(0.2,0.8,0,0)
week_1<- c(0.375,0,0.625,0)
week_2<- c(0.8,0,0,0.2)
week_3<- c(1,0,0,0)

x<- cbind(week_0,week_1,week_2,week_3)
colnames(x)<- c("week_0","week_1","week_2","week_3")
rownames(x)<- c("week_0","week_1","week_2","week_3")
transition_matrix_3<- matrix(x, nrow =4 ,byrow=TRUE)
dtmcA <-
new("markovchain",transitionMatrix=transition_matrix_3,states=c("week_0","wee
k_1","week_2","week_3"), name="MarkovChain A")
dtmcA

## MarkovChain A
##  A  4 - dimensional discrete Markov Chain defined by the following states:
##  week_0, week_1, week_2, week_3
##  The transition matrix  (by rows)  is defined as follows:
##        week_0 week_1 week_2 week_3
## week_0  0.200    0.8  0.000    0.0
## week_1  0.375    0.0  0.625    0.0
## week_2  0.800    0.0  0.000    0.2
## week_3  1.000    0.0  0.000    0.0

plot(dtmcA)
```



b. Determine E(3). What is the value of e(3)02 and what does it mean in the context of this problem?

   a. The expected number of visits from probability of failure in this first week probability of failure in the third week is 60%

```
P_2 <- matrix.power(transition_matrix_3,2)
P_3 <-  matrix.power(transition_matrix_3,3)

E <- diag(4) + transition_matrix_3 + P_2 + P_3
E[1,3]

## [1] 0.6
```

b. If a device contains a component that is two weeks old, what is the probability that the next time that it has a component that is two weeks old is 5 weeks from now?

```
P_5 <- matrix.power(transition_matrix_3,5)
```

```
P_5[3,3]
```

```
## [1] 0.156
```

c. What is the mean recurrence time for components being new? Find your answer by taking the reciprocal of the steady state probability.

```
library(MASS)
SS <- t(transition_matrix_3) - diag(4)
SS<- rbind(SS[1:3,],c(1,1,1,1))
ginv(SS)%*%(c(0,0,0,1))
```

```
##            [,1]
## [1,] 0.41666667
## [2,] 0.33333333
## [3,] 0.20833333
## [4,] 0.04166667
```