When is one thing equal to some other thing?

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June 12, 2007

In memory of Saunders MacLane

1 The awkwardness of equality

One can't do mathematics for more than ten minutes without grappling, in some way or other, with the slippery notion of *equality*. Slippery, because the way in which objects are presented to us hardly ever, perhaps never, immediately tells us–without further commentary–when two of them are to be considered equal. We even see this, for example, if we try to define real numbers as decimals, and then have to mention aliases like $20 = 19.999\dots$, a fact not unknown to the merchants who price their items \$19.99.

The heart and soul of much mathematics consists of the fact that the "same" object can be presented to us in different ways. Even if we are faced with the simple-seeming task of "giving" a large number, there is no way of doing this without also, at the same time, "giving" a hefty amount of extra structure that comes as a result of the way we pin down—or the way we present—our large number. If we write our number as 1729 we are, sotto voce, offering a preferred way of "computing it" (add one thousand to seven hundreds to two tens to nine). If we present it as $1 + 12^3$ we are recommending another mode of computation, and if we pin it down—as Ramanujuan did— as the first number expressible as a sum of two cubes in two different ways, we are being less specific about how to compute our number, but have underscored a characterizing property of it within a subtle diophantine arena.

The issue of "presentation" sometimes comes up as a small pedagogical hurdle— no more than a pebble in the road, perhaps, but it is there– when one teaches young people the idea of *congruence mod N*. How should we think of $1, 2, 3, \dots$ mod 691? Are these ciphers just members of a *new* number system that happens to have similar notation as some of our integers? Are we to think of them as equivalence classes of integers, where the equivalence relation is congruence mod 691? Or are we happy to deal with them as the good old integers, but subjected to that equivalence relation? The eventual answer, of course, is: all three ways—having the flexibility to adjust our viewpoint to the needs of the moment is the key. But that may be too stiff a dose of flexibility to impose on our students all at once.

To define the mathematical objects we intend to study, we often—perhaps always—first make it understood, more often implicitly than explicitly, how we intend these objects to be presented to us, thereby delineating a kind of *super-object;* that is, a species of mathematical objects garnished with a repertoire of *modes of presentation*. Only once this is done do we try to erase the scaffolding of the presentation, to say when two of these super-objects—possibly presented to us in wildly different ways— are to be considered *equal*. In this oblique way, the objects that we truly want enter the scene only defined as *equivalence classes of explicitly presented objects.* That is, as specifically presented objects with the specific presentation ignored, in the spirit of "ham and eggs, but hold the ham."

This issue has been with us, of course, forever: the general question of *abstraction*, as separating what we want from what we are presented with. It is neatly packaged in the Greek verb *aphairein,* as interpreted by Aristotle[1] in the later books of the *Metaphysics* to mean simply *separation*: if it is *whiteness* we want to think about, we must somehow separate it from *white horse, white house, white hose,* and all the other white things that it invariably *must* come along with, in order for us to experience it at all.

The little trireme of possibilities we encounter in teaching congruence mod 691 (i.e., is 5 mod 691 to be thought of as a symbol, or a stand-in for any number that has remainder 5 when divided by 691, or should we take the tack that it (i.e., "5 mod 691") *is* the (equivalence) class of all integers that are congruent to 5 mod 691?) has its analogue elsewhere—perhaps everywhere—in mathematics. Familiarity with a concept will allow us to finesse, or ignore, this, as when we are happy to deal with a fraction a/b ambiguously as an equivalence class of pairs of integers (a, b) with $b \neq 0$, where the equivalence relationship is given by the rule $(a, b) \sim (a', b')$ if and only if $ab' = a'b$, or as a particular member of this class. Few mathematical concepts enter our repertoire in a manner other than ambiguously a *single object* and at the same time an *equivalence class of objects.* This is especially true for the concept of *natural number*, as we shall see in the next section where we examine the three possible ways we have of coming to terms with the number 5.

One of the templates of modern mathematics, category theory, offers its own formulation of *equivalence* as opposed to *equality*; the spirit of category theory allows us to be content to determine a mathematical object, as one says in the language

[1] Aristotle first uses this term in Book XI Chap 3 1061a line 29 of the *Metaphysics*; his discussion in Book XIII, Chap 2 begins to confront some of the puzzles the term poses.

of that theory, *up to canonical isomorphism.* The categorical viewpoint is, however, more than merely "content" with the inevitability that any particular mathematical object tends to come to us along with the contingent scaffolding of the specific way in which it is presented to us, but has this inevitability built in to its very vocabulary, and in an elegant way, makes profound use of this. It will allow itself the further flexibility of viewing any mathematical object "as" a *representation* of the theory in which the object is contained to the proto-theory of modern mathematics, namely, to *set theory.*

My aim in this article is to address a few points about *mathematical objects* and *equality of mathematical objects* following the line of thought of the preceding paragraph. I see these "points" borne out by the doings of working mathematicians as they go about their daily business thinking about, developing, and communicating mathematics, but I haven't found them specifically formulated anywhere[2]. I don't even see how questions about these issues can even be raised within the framework of vocabulary that people employ when they talk about *the foundations of mathematics* for so much of the literature on philosophy of mathematics continues to keep to certain staples: formal systems, consistency, undecidability, provability and unprovability, and rigor in its various manifestations.

To be sure, people have exciting things to talk about, when it comes to the list of topics I have just given. These issues have been the focus of dramatic encounters—famous "conversations," let us call them—that represent turning points in our understanding of what the very mission of mathematics should be. The ancient literature—notably, Plato's comment about how the mathematicians bring their analyses back to the *hypotheses* that they frame, *but no further*—already delineates this mission[3]. The early modern literature—epitomized by the riveting use that Kant made of his starkly phrased question "how is pure mathematics possible?"—offers a grounding for it. In the century just past, we have seen much drama regarding the grounds for mathematics: the Frege-Russell correspondence, the critique that L.E.J. Brouwer made of the modern dealings with infinity and with Cantor's set theory, Hilbert's response to this critique, leading to his magnificent

[2] The faintest resonance, though, might be seen in the discussion in Books 13 and 14 of Aristotle's *Metaphysics* which hits at the perplexity of whether the so-called *mathematicals* (that ostensibly play their role in the platonic theory of forms) occur *uniquely* for each mathematical concept, or *multiply.*

[3]

> I think you know that students of geometry, calculation, and the like hypothesize the odd and the even, the various figures, the three kinds of angles, and other things akin to these in each of their investigations, as if they know them. They make these their hypotheses and don't think it necessary to give any account of them, either to themselves or others, as if they were clear to everyone.

[Plato 1997] *Republic* Book VI 510c.

invention of formal systems, and the work of Gödel, itself an extraordinary comment on the relationship between the mission of mathematics and the manner in which it formulates its deductions.

Formal systems remain our lingua franca. The general expectation is that any particular work we happen to do in mathematics should be, or at least should be capable of being, packaged within some formal system or other. When we want to legitimize our modes of operation regarding anything, from *real numbers* to *set theory* to *cohomology*—we are in the habit of invoking *axiomatic systems* as standard-bearer. But when it comes to a crisis of rigorous argument, the open secret is that, for the most part, mathematicians who are not focussed on the architecture of formal systems per se, mathematicians who are *consumers* rather than *providers*, somehow achieve a sense of utterly firm conviction in their mathematical doings, without actually going through the exercise of translating their particular argumentation into a brand-name formal system.

If we are shaky in our convictions as to the rigor of an argument, an excursion into formal systems is rarely the thing that will shore up faith in the argument. To be sure, it is often very helpful for us to write down our demonstations very completely using pencil and paper or our all-efficient computers. In any event, no matter how wonderful and clarifying and comforting it may be for mathematician X to know that all of his or her proofs have, so far, found their expression within the framework of Zermelo-Frankel set theory, the chances are that mathematician X, if quizzed on what—exactly—those axioms are, might be at a loss to answer.

Of course, it is vitally important to understand, as fully as we can, what tools we need to assemble in order to justify our arguments. But to appreciate, and discuss, a grand view of the nature of mathematical objects that has taken root in mathematical culture during the past half-century, we must also become conversant with a language that has a thrust somewhat different from the standard fare of *foundations*. This newer vocabulary has phrases like *canonical isomorphism, "unique up to unique isomorphism", functor, equivalence of category* and has something to say about every part of mathematics, including the definition of the natural numbers.

2 Defining natural numbers

Consider natural numbers; for instance, the number 5. Here are three approaches to the task of defining the number 5.

- We could, in our effort to define the number 5, deposit five gold bars in, say, Gauss's observatory in Göttingen, and if ever anyone wants to determine whether or not their set has cardinality five, they would make a quick trip to Göttingen and try to put the elements of their set in one-one correspondence with the bullion deposited there. Of course, there are many drawbacks to this approach to defining the number "five," the least of which is that it has the smell of contingency. Let us call this kind of approach the *bureau of standards attitude towards definition:* one chooses a representative exemplar of the mathematical object one wishes to define, and then gives a criterion for any other mathematical object to be viewed as *equal* to the exemplar. There is, after all, something nice and crisp about having a single concrete exemplar for a mathematical concept.

- The extreme opposite approach to this is Frege's: define a **cardinality** (for example, five) as an equivalence class in the set of all sets, the equivalence relation $A \sim B$ being the existence of a one-one correspondence between A and B. The advantage, here, is that it is a criterion utterly devoid of subjectivity: no set is preferred and chosen to govern as benchmark for any other set; no choice (in the realm of sets) is made at all. The disadvantage, after Russell, is well known: the type of universal quantification required in Frege's definition, at least when the equivalence classes involved are considered to be sets, leads to paradox. The Frege-Russell correspondence makes it clear that one cannot, or at least one should not, be too greedy regarding unconditional quantification. To keep clear of immediate paradox, we introduce the word *class* into our discussion, amend the phrase *set of all sets* to *class of all sets*, and hope for the best.

- A fine compromise between the above two extremes is to do what we all, in fact, do: a strategy that captures the best features of both of the above approaches. What I mean here, by the way, is to indicate what we *do*, rather than what we *say we do* when quizzed about our foundations. I allow my notation $1, 2, 3, 4, 5, 6, \ldots$ to play the role of my personal bureau-of-standards within which I happily make my calculations. I think of the set $\{1, 2, 3, 4, 5\}$, for example, as a perfectly workable exemplar for quintuples. Meanwhile you use your notation $1', 2', 3', 4', 5', 6', \ldots$ (or whatever it is) to play a similar role with respect to your work and thoughts, the basic issue being whether there is a faithful translation of structure from the way in which you view natural numbers to the way I do.

Equivalence (of structure) in the above "compromise" is the primary issue, rather than *equality* of mathematical objects. Furthermore, it is the *structure* intrinsic to the whole gamut of natural numbers that plays a crucial role there. For only in terms of this structure (packaged, perhaps, as a version of Peano's axioms) do we have a criterion to determine when your understanding of "natural numbers," and mine, admit "faithful translations" one to another. A consequence of such an approach—which is the standard modus operandi of mathematics ever since Hilbert—is that any single mathematical object, say the number 5, is understood primarily in terms of the structural relationship it bears to the other natural numbers. Mathematical objects are determined by–and understood by—the network of relationships they enjoy with all the other objects of their species.

3 Objects versus structure

Mathematics thrives on going to extremes whenever it can. Since the "compromise" we sketched above has "mathematical objects determined by the network of relationships they enjoy with all the other objects of their species," perhaps we can go to extremes within this compromise, by taking the following further step. Subjugate the role of the *mathematical object* to the role of its network of relationships—or, a further extreme—simply *replace* the mathematical object by this network.

This may seem like an impossible balancing act. But one of the elegant–and surprising—accomplishments of category theory is that it performs this act, and does it with ease.

4 Category Theory as balancing act rather than "Foundations"

There are two great modern formulas—as I'll call them—for packaging entire mathematical theories. There is the concept of **formal system**, following David Hilbert, as discussed above. There is also the concept of **category**, the great innovation of Samuel Eilenberg and Saunders MacLane. Now, these two formulas have vastly different intents.

A formal system representing a mathematical theory has, within it, all of the mechanics and vocabulary necessary to discuss proofs, and the generation of proofs, in the mathematical theory; indeed, that is mainly what a formal system is all about.

In contrast, a **category** is quite sparse in its vocabulary: it can say nothing whatsoever about proofs; a category is a mathematical entity that, in the most succinct of languages, captures the essence of what a mathematical theory consists: *objects of the theory*, allowable *transformations* between these objects, and a *composition law* telling us how to compose two transformations when the range of the first transformation is the domain of the second.

It stands to reason, then, that the concept of category cannot provide us with anything that goes under the heading of "foundations." Nevertheless, in its effect on our view of mathematical objects it plays a fine balancing role: it extracts—as I hope you will see—the best elements from both a Fregean and a bureau-of-standards attitude towards the formulation of mathematical concepts.

5 Example: The category of sets.

Even before I describe *category* more formally, it pays to examine *the category of sets* as an example. The category of sets, though, is not just "an" example, it is the proto-type example; it is as much *an* example of a category as Odette is *un* amour de Swann.

The enormous complexity to set theory is one of the great facts of life of mathematics. I suppose most people before Cantor, if they ever had a flicker of a thought that *sets* could occur at all as mathematical objects, would have expected that a rather straightforward theoretical account of the notion would encompass everything that there was to say about those objects. As we all know, nothing of the sort has transpired.

The famous attitude of St. Augustine towards the notion of "time," (i.e., "What then is time? If no one asks me, I know what it is. If I wish to explain it to him who asks, I do not know.") mirrors my attitude—and I would suppose, most people's attitude—toward sets. If I retain my naive outlook on *sets,* all is, or at least seems to be, well; but once I embark on formulating the notion rigorously and specifically, I am either entangled, or else I am forced to make very contingent choices.

Keeping to the bare bones, a *set theory* will consist of

- the repertoire of **elements** of the theory, and if I wish to refer to one of them I will use a lower case symbol, e.g., $a, b, \ldots$
- the repertoire of **sets** of the theory, and for these I will use upper case symbols, e.g., $X, Y, \ldots$
- the relation of **containment** telling us when an element x is contained in a set X ($x \in X$); each set X is extensionally distinguished by the elements, $x \in X$, that are in it.
- the **mappings** $f : X \to Y$ between sets of the theory, each mapping f uniquely characterized by stipulating for every $x \in X$ the (unique) image, $f(x) \in Y$, of that element x.
- the guarantee that if $f : X \to Y$ and $g : Y \to Z$ are mappings in my theory, I can form the **composition** $g \cdot f : X \to Z$ by the rule that for any element $x \in X$ the value $(g \cdot f)(x) \in Z$ is just $g(f(x))$.

We neither lose nor gain anything by adding the requirement that, for any object X, the identity mapping $1_X : X \to X$ is a bona fide mapping in our set theory, so for convenience let us do that. Also, we see that our composition rule is *associative* in the standard sense of multiplication, i.e., $(h \cdot g) \cdot f = h \cdot (g \cdot f)$, when these compositions can be made, and that our identity mappings play the role of "unit."

Much has been omitted from this synopsis–all traces of quantifications, for example— and certain things have been hidden. The repeated use of the word *repertoire* is already a hint that something big is being hidden. It would be downright embarrassing, for example, to have replaced the words "repertoire" in the above description by "set," for besides the blatant circularity, we would worry, with Russell, about what arcane restrictions we would have then to make regarding our universal quantifier, once that is thrown into the picture. "Repertoire" is my personal neologism; the standard word is *class* and the notion behind it deserves much discussion; we will have some things to say about it in the next section. You may notice that I refrained from using the word "repertoire" when talking about mappings. A subtle issue, but an important one, is that we *may* boldly require that all the mappings from a given set X to a given set Y form a bona fide *set* in our theory, and not merely an airy repertoire. This is a source of power, and we adopt it as a requirement. Let us refer, then, to a theory such as we have just sketched as a **bare set theory**.

A *bare set theory* can be stripped down even further by forgetting about the *elements* and a fortiori the *containment relations*. What is left?

We still have the *objects* of the theory, i.e., the repertoire (synonymously: class) of its sets. For any two sets X and Y we have the *set* of mappings from X to Y; and for any three sets X, Y, Z and two mappings $f : X \to Y$ and $g : Y \to Z$ we have the mapping that is the composition of the two, $g \cdot f : X \to Z$, this composition rule admitting "units" and satisfying the associative law.

This *further-stripped-down bare set theory* is our first example of a category: it is the underlying **category** of the bare set theory.

The concept of *class* which will occur in the definition of category, and has already occurred in our proto-example, now deserves some discussion.

6 *Class* as a library with strict rules for taking out books

I'm certain that there are quite precise formulations of the notion of *class*, but here is a ridiculously informal user's-eye-view of it. Imagine a library with lots of books, administered by a somewhat stern librarian. You are allowed to take out *certain* subcollections of books in the library, but not all. You know, for example, that you are forbidden to take out, at one go, *all* the books of the library. You assume, then, that there are other subcollections of books that would be similarly restricted. But the full bylaws of this library are never to be made completely explicit. This doesn't bother you overly because, after all, you are interested in reading, and not the legalisms of libraries.

In observing how mathematicians tend to use the notion *class*, it has occurred to me that this notion seems really never to be put into play without some *background* version of set theory understood already. In short by a *class*, we mean a collection of objects, with some restrictions on which subcollections we, as mathematicians, can deem *sets* and thereby operate on with the resources of our set theory. I'm perfectly confident that this seeming circularity can be–and probably has been–ironed out. But there it is.

7 Category

A category $\mathcal{C}$ is intrinsically a relative notion for it depends upon having a set theory in mind; a *bare set theory* such as sketched above will do.

Fixing on a "bare set theory," a **category** $\mathcal{C}$ (*modeled* on this bare set theory) is given by the following[4]:

- a *class* of things called **the objects of** $\mathcal{C}$ and denoted $\mathrm{Ob}(\mathcal{C})$;

- given any two objects X, Y of $\mathcal{C}$, a *set* denoted $\mathrm{Mor}_{\mathcal{C}}(X, Y)$, which we think of as the set of *transformations* from the object X to the object Y; we refer to these transformations as **morphisms from** X **to** Y and usually denote such a morphism f as a labelled arrow $f : X \to Y$;

- given any three objects X, Y, Z of $\mathcal{C}$ and morphisms $f : X \to Y$ and $g : Y \to Z$ we are provided with a law that tells us how to "compose" these morphisms to get a morphisms
$$g \cdot f : X \to Z.$$
Intuitively, we are thinking of f and g as "transformations," and *composition* of them means that we imagine "first" applying f to get us from X to Y and "then" applying g to get us from Y to Z. The rule that associates to such a pair (f, g) the composition,
$$(f, g) \;\mapsto\; g \cdot f$$
we think of as a sort of "multiplication law."

  One also requires that morphisms playing the role of "identity elements" 1_X in $Mor_{\mathcal{C}}(X, X)$ with respect to this composition law exist; that is, for any morphism $f : X \to Y$ we have $f \cdot 1_X = f$; and similarly, for any morphism $e : V \to X$ we have $1_X \cdot e = e$. Finally the composition law is assumed to be associative, in the evident sense.

[4]Category-theorists will note that I am restricting my attention to what are called *locally small categories*.

As for the word *class* that enters into the definition, we will, at the very least, want, for any object X in our category, that the singleton set consisting of that object,$\{X\}$, be viewed as a bona fide set of our set theory.

This concept of *category* is an omni-purpose affair: we have our categories of sets, where the objects are sets, the morphisms are mappings of sets; we have the category of *topological spaces* whose objects are the eponymous ones, and whose morphisms are continuous maps. We have the algebraic categories: the category of *groups* where the morphisms are homomorphisms, the category of *rings with unit element* where the morphisms are ring homomorphisms (that preserve the unit element), etc. Every branch and sub-branch of mathematics can package their entities in this format. In fact, at this point in its career it is hard to say whether the role of category in the context of mathematical work is more descriptive, or more prescriptive. It frames a possible template for any mathematical theory: the theory should have *nouns* and *verbs*, i.e., objects, and morphisms, and there should be an explicit notion of composition related to the morphisms; the theory should, in brief, be packaged by a category. There is hardly any species of mathematical object that doesn't fit into this convenient, and often enlightening, template.

Template is a crucial feature of categories, for in its daily use, a category avoids any really detailed discussion of its underlying set theory. This clever manner in which category theory engages with set theory shows, in effect, that it has learned the Augustinian lesson. Category theory doesn't legislate *which* set theory we are to use, nor does it even give ground-rules for what "a" *set theory* should be. As I have already hinted, one of the beautiful aspects of category theory is that it is up to you, the category-theory-user to supply "a" set theory, a bare category of sets $\mathcal{S}$, for example. A category is a B.Y.O.S.T. party, i.e., you *bring your own set theory* to it. Or, you can adopt an even more curious stance: you can view $\mathcal{S}$ as something of a free variable, and consequently, end up by making no specific choice!

So, for example, "the" category of rings with unit element is, more exactly, a mold that you can impress on any bare set theory. To be sure, you want your set theory to be sufficiently rich so as to hold this impression: if there were no sets at all in your *set theory*, you wouldn't get much.

You might wonder why the framers of the notion of category bothered to use two difficult words *class* and *set* rather than only one, in their definition. One could, after all, simply require that there be a *set* of objects of the category, rather than bring in the airy word *class*. In fact, people do that, at times: it is standard to call a category whose objects form a *set* a **small category**, and these small categories do have their uses. Indeed, if you are worried about foundational issues, it is hardly a burden to restrict attention to small categories. But I think the reason that the notion of class is invoked has to do with the high ambition we have for categories: categories are meant to offer a fluid vocabulary for whole 'fields of mathematics' like group theory or topology, with a Fregean desire for freedom from the contingency implicit in subjective choices.

8 Equality versus isomorphism

The major concept that replaces *equality* in the context of categories is *isomorphism.* An **isomorphism** $f : A \to B$ between two objects A, B of the category $\mathcal{C}$ is a morphism in the category $\mathcal{C}$ that can be "undone," in the sense that there is another morphism $g : B \to A$ playing the role of the inverse of f; that is, the composition $gf : A \to A$ is the identity morphism 1_A and the composition $fg : B \to B$ is the identity morphism 1_B. The essential lesson taught by the categorical viewpoint is that it is usually either quixotic, or irrelevant, to ask if a certain object X in a category $\mathcal{C}$ is *equal* to an object Y. The query that is usually pertinent is to ask for a *specific isomorphism* from X to Y.

Note the insistence, though, on a *specific* isomorphism; although it may be useful to be merely assured of the *existence* of isomorphisms between X and Y, we are often in a much better position if we can pinpoint a specific isomorphism $f : X \to Y$ *characterized by an explicitly formulated property, or list of properties.* In some contexts, of course, we simply have to make do without being able to pinpoint a specific isomorphism. If, for example, I manage to construct an algebraic closure of the finite field $\mathbf{F}_2$ (i.e., of the field consisting of two elements), and am told that someone halfway around the world has also constructed such an algebraic closure, I *know* that there exists an isomorphism between the two algebraic closures but–without any further knowledge—I have no way of pinpointing a specific isomorphism. In contrast, desipte my ignorance of the manner in which my colleague at the opposite end of the world went about constructing her algebraic closure, I can, with utter confidence, put my finger on a *specific* isomorphism between the group of automorphisms of my algebraic closure and the group of automorphisms of the other algebraic closure[5]. The fact that the algebraic closures are not yoked together by a *specified* isomorphism is the source of some theoretical complications at times, while the fact that their automorphism groups are seen to be isomorphic via a cleanly specified isomorphism is the source of great theoretical clarity, and some profound number theory.

A uniquely specified isomorphism from some object X to an object Y characterized by a list of explicitly formulated properties—this list being sometimes, the truth be told, only implicitly understood—is usually dubbed a "canonical isomorphism." The "canonicality" here depends, of course, on the list. It is this brand of *equivalence*, then, that in category theory replaces *equality*: we wish to determine objects, as people say, *"up to canonical isomorphism."*

[5] for these automorphism groups are both topologically generated by the field automorphism consisting of squaring every element

9 An example of categorical vocabulary: Initial Objects

We also have at our immediate disposal, a broad range of concepts that can be defined purely in terms of the structure that we have already elucidated. For example, if we are given a category $\mathcal{C}$, an **initial object** Z of $\mathcal{C}$ is an object Z that has the property that given *any* object X of $\mathcal{C}$ there is a unique morphism of the category $i_X : Z \to X$ from Z to X; that is, the set $Mor_{\mathcal{C}}(Z, X)$ consists of the single element $\{i_X\}$.

Suppose that a category $\mathcal{C}$ has an *initial object* Z. There may, very well, be quite a number of objects vying for the role of *initial object of this category* $\mathcal{C}$. But given another contender, call it Z', there is a unique morphism $i_{Z'} : Z \to Z'$ since Z is an initial object, and a unique morphism $i'_Z : Z' \to Z$ since Z' is. Also, again since Z is an initial object, there can only be one morphism from Z to Z, and the identity morphism $\mathbf{1}_Z : Z \to Z$ fills this role just fine, so we must have that $i'_Z \cdot i_{Z'} = \mathbf{1}_Z$ and, for similar reasons, $i_Z \cdot i'_{Z'} = \mathbf{1}_{Z'}$. In summary, $i_{Z'}$ and i'_Z are (inverse) isomorphisms, and provide us with *canonical, in fact the only, isomorphisms* between Z and Z'. One way of citing this is to say, as people do, that the initial object of a category—if it exists—is *unique up to unique isomorphism.* To be sure it is not unique as "object," but rather, as "something else." It is this difference, what the "something else" consists in, that we are exploring.

10 Defining the natural numbers as an "initial object."

For this discussion, let us start by considering "the" initial object in the category of rings with unit. As we shall see, such an initial object does exist, given that the underlying bare set theory is not ridiculously impoverished. Such an initial object is "unique up to unique isomorphism," as all initial objects are. What is it?

Well, by the definition of initial object, it must be a ring Z (with a unit element) that admits a unique ring homomorphism (preserving unit elements) to any ring with unit. Since *the ring of ordinary integers* $\mathbf{Z}$ has precisely this property (there being one and only one ring homomorphism from $\mathbf{Z}$ to any ring with unit, the one that sends $1 \in \mathbf{Z}$ to the unit of the range ring) "the" initial object in the category of rings with unit is nothing more nor less than $\mathbf{Z}$ but, of course, only "up to unique isomorphism."

The previous paragraph situated $\mathbf{Z}$ among its fellow rings with unit element. Let us fashion a similar discussion for the Natural Numbers, highlighting the type of structure that Peano focussed on, when formulating his famous axioms.

For this, I want to define a category denoted $\mathcal{P}$ that I will call **the Peano category**.

The objects, $Ob(\mathcal{P})$, of the Peano category consists of triples (X, x, s) where X is a set, $x \in X$ is an element (call it a **base point**), and $s : X \to X$ is a mapping of X to itself (a "self-map" which we might call the *successor map*).

Given two objects $\mathcal{X} = (X, x, s)$ and $\mathcal{Y} = (Y, y, t)$ of $\mathcal{P}$, a morphism

$$F : (X, x, s) \to (Y, y, t)$$

in the category $\mathcal{P}$ is a mapping of sets $f : X \to Y$ with the property that

- f preserves base points; i.e., $f(x) = y$, and
- f respects the self-maps s and t, in the sense that $f \cdot s = t \cdot f$, i.e., we have for all elements $z \in X$, $f(s(z)) = t(f(z))$.

We will denote by $\mathrm{Mor}_{\mathcal{P}}(X, Y)$ the set of morphisms of the Peano category from X to Y, i.e., the set of such F's.

For any choice of bare set theory, we have thereby formed the category which we will call $\mathcal{P}$. If our bare set theory, on which the category is modeled, is at all decent—e.g., is one of the standard set theories containing the set of natural numbers $\mathcal{N} = \{1, 2, 3, \dots\}$, then $\mathcal{N}$ may be viewed as an object of $\mathcal{P}$, its *base point* being given by $1 \in \mathcal{N}$, and the self-map $s : \mathcal{N} \to \mathcal{N}$ being given by the rule that sends a natural number to its successor, i.e., $n \mapsto n + 1$.

Given any object $\mathcal{X} = (X, x, s)$ in Ob $\mathcal{P}$ *there is one and only one* morphism from $\mathcal{N}$ to $\mathcal{X}$ in the category $\mathcal{P}$; it is given by the mapping of sets sending $1 \in \mathcal{N}$ to the base point $x \in X$ (for, indeed, any morphism in $\mathcal{P}$ is required to send base point to base point) and the mapping is then "forced," from then on, by the formula $f(n + 1) = sf(n)$.

In summary, there is a unique morphsim in $\mathcal{P}$ from the natural numbers to any object in the category. That is, the natural numbers, $\mathcal{N}$, is an initial object of $\mathcal{P}$.

Moreover, as any initial object in any category is uniquely characterized, up to unique isomorphism, by its role *as initial object*, the natural numbers when viewed as *initial object of* $\mathcal{P}$ is similarly pinned down.

This strategy of defining the *Natural Numbers* as "an" initial object in a category of (what amounts to) discrete dynamical systems, as we have just done, is revealing, I think; it isolates, as Peano himself had done, the fundamental role of mere *succession* in the formulation of the natural numbers. It also follows the third of the three formats we listed for defining natural numbers; it is, in a sense that deserves to be understood, a compromise strategy between a bureau-of-standards kind of definition, and a Fregean universal quantification approach. Notice, though, its elegant shifting sands. At the very least, we have a definition that depends upon a selection of *a* set theory, as well as an agreement to deal with the object Z pinned down "up to unique isomorphism." We have even further to go, but first let us discuss how our definition differs in approach from the standard way of expressing Peano's axioms.

11 Light, shadow, dark

In elementary mathematics classes, we usually describe Peano's axioms that characterize the natural numbers roughly as follows.

> The natural numbers $\mathcal{N}$ is a set with a chosen element $1 \in \mathcal{N}$ and an injective ("successor") function $s : \mathcal{N} \to \mathcal{N}$ such that $1 \notin s(\mathcal{N})$ and such that **mathematical induction** holds, in the sense that if $P(n)$ is any proposition which may be formulated for all $n \in \mathcal{N}$, and for which $P(1)$ is true, and which has the further property that whenever $P(n)$ is true then $P(s(n))$ is true, then $P(n)$ is true for all $n \in \mathcal{N}$.

This, of course, has shock-value: it recruits the entire apparatus of propositional verification to its particular end. The fact that, especially when taken broadly, mathematical induction has extraordinary consequences, is amply illustrated by the ingenious work of Gentzen[6]. To formalize things, we tame these axioms by explicitly providing a setting in which the words *proposition* and *true* make sense.

The easiest way of comparing the *Peano axioms* with the *Peano category* as modes of defining natural Numbers, is to ask what each of these formats

- shines a *spotlight* on?
- keeps in the *shadows*?

  and

- keeps in the *dark*?

Both ways of pinpointing *natural numbers* are fastidiously explicit about the fact that a certain discrete dynamical system is involved: each shine their spotlight

[6]Gentzen developed a *normal form* for propositions and deductions in (Peano) arithmetic, and he noted that if it were permitted to employ—in one's demonstrations— a version of *mathematical induction* that ranges over *all* demonstrations in arithmetic (these demonstrations being organized according to their natural partial ordering) one can actually *prove* the *consistency of arithmetic*; see [Gentzen 1936] and [Gentzen 1938].

To be sure, there is an inherent circularity issue here, beyond the fact that one is calling forth an unusually powerful version of mathematical induction, but Gentzen's ideas are not the less interesting for all this. His tactic was to assume that the line

$$\text{“}0 = 1\text{”}$$

occurred in a demonstration expressed in normal form, and then to examine what the line immediately preceding $0 = 1$ in this putative demonstration could possibly be. From the structure of *normal form demonstrations* one sees that there could be no such line, and as a consequence, one could never deduce a contradiction in arithmetic by a demonstration that has been expressed in *normal form*.

on the essence of *iteration*, the successor function. The Peano axioms do this by focussing in somewhat more detail on the elementary properties of this successor function s, requiring as those axioms do, that 1 not be in the image of s, and that s be injective. The Peano category approach does this by simply considering the entire species of discrete dynamical systems with chosen base point.

Both modes of definition need a way of insisting on a certain "minimality" for the structure of natural numbers that they are developing. The Peano axioms formulate this "minimality" by dependence upon the domino effect of truth in a mathematically inductive context. The Peano category approach formulates "minimality" by considering the position of the natural numbers as a discrete dynamical system, among all discrete dynamical systems.

The Peano axiom approach calls up the full propositional apparatus of mathematics. But the details of the apparatus are kept in the shadows: you are required to "bring your own" propositional vocabulary if you wish to even begin to flesh out those axioms. The Peano category approach keeps all this in the dark: no mention whatsoever is made of propositional language.

The Peano axiom approach requires —at least explicitly—hardly any investment in some specific brand of set theory. At most one set is on the scene, the set of natural numbers itself. In contrast, the Peano category approach forces you to "bring your own set theory" to make sense of it.

When we gauge the differences in various mathematical viewpoints, it is a good thing to contrast them not only by what equipment these viewpoints ultimately invoke to establish their stance, for *ultimately* they may very well require exactly the same things, but also to pay attention to the order in which each piece of equipment is introduced and to the level of explicitness required for it to play its role.

12 Representing one theory in another

If categories package entire mathematical theories, it is natural to imagine that we might find the shadow of one mathematical theory (as packaged by a category $\mathcal{C}$) in another mathematical theory (as packaged by a category $\mathcal{D}$). We might do this by establishing a "mapping" of the entire category $\mathcal{C}$ to the category $\mathcal{D}$. Such a "mapping" should, of course, send basic features (i.e., objects, morphisms) of $\mathcal{C}$ to corresponding features of the category $\mathcal{D}$, and moreover, it must relate the composition law of morphisms in $\mathcal{C}$ to the corresponding law for morphisms of $\mathcal{D}$; we call such a "mapping" a **functor** from $\mathcal{C}$ to $\mathcal{D}$.

To give a functor F from $\mathcal{C}$ to $\mathcal{D}$, then, we must stipulate how we associate to any object X of $\mathcal{C}$ a well-defined object $F(X)$ of $\mathcal{D}$, and to any morphism between objects $f : X \to Y$ of $\mathcal{C}$ a well-defined morphism $F(f) : F(X) \to F(Y)$ between corresponding objects of $\mathcal{D}$; and, as mentioned, this relationship between objects and morphisms in $\mathcal{C}$ to objects and morphisms in $\mathcal{D}$ must respect identity morphisms, and the composition laws of these categories[7]. Let us denote such a functor F from $\mathcal{C}$ to $\mathcal{D}$ by a broken arrow

$$F : \mathcal{C} \dashrightarrow \mathcal{D}.$$

In this way, we have a vocabulary for establishing bridges between whole disciplines of mathematics; we have a way of representing grand aspects of, say, topology in algebra (or conversely) by establishing functors from the category of topological spaces to the category of groups (or conversely): construct the pertinent functors from the one category to the other!

The easiest thing to do, at least in mathematics, is to forget, and the forgetting process offers us some elementary functors, such as the functor from *topological spaces* to *sets* that passes from a topological space to its underlying set, thereby forgetting its topology. Of course one should also pay one's respects to the simplest of functors, the *identity functor*

$$\mathcal{C} \overset{\mathbf{1}_{\mathcal{C}}}{\dashrightarrow} \mathcal{C},$$

which when presented with any object U of $\mathcal{C}$ it gives it back to you intact, as it does each morphism.

The more profound bridges between fields of mathematics are achieved by more interesting constructions. But there is a ubiquitous type of functor, as easy to construct as one can imagine, and yet extraordinarily revealing. Given any object X in any category $\mathcal{C}$ we will construct (in section 14) an important functor (we will denote it F_X) from $\mathcal{C}$ to $\mathcal{S}$, the category of sets upon which $\mathcal{C}$ was modeled. This functor F_X will be enough to "reconstruct" X, but—as you might guess—only "up to canonical isomorphism."

But before we do this, we need to say what we mean by a morphism from one functor to another.

[7] By definition, then, a functor F from $\mathcal{C}$ to $\mathcal{D}$ associates to each object U of $\mathcal{C}$ an object of $\mathcal{D}$, call it $F(U)$; and to each morphism $h : U \to V$ of $\mathcal{C}$, a morphism of $\mathcal{D}$, call it $F(h) : F(U) \to F(V)$. If the morphism $\mathbf{1}_U : U \to U$ is the identity, then we require that $F(\mathbf{1}_U) = \mathbf{1}_{F(U)}$, i.e., that it be the identity morphism of the object $F(U)$ in $\mathcal{D}$. When we say that the functor F *respects* composition laws we mean that if $g : V \to W$ is a morphism of $\mathcal{C}$ (so that we can form the composition $g \cdot h : U \to W$ in the category $\mathcal{C}$) we have the law

$$F(g \cdot h) = F(g) \cdot F(h) : F(U) \to F(W).$$

13 Mapping one functor to another

If we are given two functors,

$$F, G : \mathcal{C} \dashrightarrow \mathcal{D},$$

by a **morphism of functors**

$$\mu : F \longrightarrow G$$

we mean that we are given, for each object U of $\mathcal{C}$, a morphism

$$\mu_U : F(U) \to G(U)$$

in the category $\mathcal{D}$ which respects the structures involved [8].

To offer a humble example, for any functor

$$F : \mathcal{C} \dashrightarrow \mathcal{D},$$

we have the *identity morphism of the functor F to itself,*

$$F \xrightarrow{\mathbf{1}_F} F$$

which associates, to an object U of $\mathcal{C}$, the identity morphism $\mathbf{1}_{F(U)} : F(U) \to F(U)$ (this being the identity morphism of the object $F(U)$ in the category $\mathcal{D}$). You might think that this example is not very enlightening, but it already holds its surprises; in any event, in the next section we shall visit an important large repertoire of morphisms of functors.

Once we have settled on the definition of *morphism* of functors, our way is clear to define **isomorphism of functors** for the definition of this notion follows the natural format for the definition of isomorphism related to absolutely *any* species of mathematical object. Namely, an *isomorphism of the functor $F : \mathcal{C} \dashrightarrow \mathcal{D}$ to the functor $G : \mathcal{C} \dashrightarrow \mathcal{D}$* is a morphism of functors $\mu : F \to G$ for which there is a morphism of functors going the other way, $\nu : G \to F$ such that $\nu \cdot \mu : F \to F$ and $\mu \cdot \nu : G \to G$ are equal to the respective identity morphisms (of functors).

To understand the notion of *isomorphism of functors* I find it particularly illuminating to consider, for the various categories of interest, what the *automorphisms of the identity functor* consist of. Note, to take a random example, that if $\mathcal{V}$ is the category of vector spaces over a field k, then multiplication by any nonzero scalar (i.e., element of k^*) is an automorphism of the identity functor. That is, let

[8] in the sense that for every pair of objects U, V of $\mathcal{C}$, and morphism $h : U \to V$ in $\mathrm{Mor}_{\mathcal{C}}(U, V)$ we have the equality

$$G(h) \cdot \mu_U = \mu_V \cdot F(h),$$

both left and right hand side of this equation being morphisms $F(U) \to G(V)$ in the category $\mathcal{D}$.

$1_{\mathcal{V}} : \mathcal{V} - - \to \mathcal{V}$ denote the identity functor; for any fixed nonzero scalar $\lambda \in k^*$ we can form (for all vector spaces U over k) the morphism in $\mathcal{V}$,

$$\lambda_U : U \to U$$

defined by $x \mapsto \lambda \cdot x$, and this data can be thought of as giving an isomorphism of functors

$$\lambda : 1_{\mathcal{V}} \cong 1_{\mathcal{V}}.$$

14 An object "as" a functor from the theory-in-which-it-lives to set theory

Given an object X of a category $\mathcal{C}$, we shall define a specific functor (that we will denote F_X) that encodes the essence of the object X. The functor F_X will, in fact, determine X up to canonical isomorphism.

This functor F_X maps the category $\mathcal{C}$ to the category $\mathcal{S}$ of sets (the same category of sets on which $\mathcal{C}$ is "modeled," as we've described in section 7 above).

Here is how it is defined. The functor F_X assigns to any object Y of $\mathcal{C}$ the set of morphisms from X to Y; that is,

$$F_X(Y) := \mathrm{Mor}_{\mathcal{C}}(X, Y).$$

Now, $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ is indeed a *set,* i.e., an object of $\mathcal{S}$, so we have described a mapping from *objects of* $\mathcal{C}$ to *sets*,

$$Y \ \mapsto \ F_X(Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y).$$

Moreover, to every morphism $g : Y \to Z$ of $\mathcal{C}$, our functor F_X assigns the mapping of sets

$$F_X(g) : F_X(Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y) \longrightarrow F_X(Z) = \mathrm{Mor}_{\mathcal{C}}(X, Z)$$

given simply by composition with g; i.e., if $f \in F_X(Y)$, the mapping $F_X(g)$ sends f to $g \cdot f \in F_X(Y)$:

$$f \ \mapsto g \cdot f.$$

In this way, every object X of any category $\mathcal{C}$ gives us a functor, F_X from $\mathcal{C}$ to $\mathcal{S}$.

Also any morphism $h : X' \to X$ in $\mathcal{C}$ gives rise to a morphism of functors $\eta : F_X \to F_{X'}$ by this simple formula: for an element Y of $\mathcal{C}$, the morphism h gives us a mapping of sets $\eta_Y : F_X(Y) \to F_{X'}(Y)$ by sending any $f : X \to Y$ in $F_X(Y)$ to $f \cdot h : X' \to Y$ in $F_{X'}(Y)$. The rule associating to an object Y the mapping of sets η_Y produces our morphism of functors $\eta : F_X \to F_{X'}$.

The fundamental, but miraculously easy to establish, fact is that the object X is entirely retrievable (however, only up to canonical isomorphism, of course) from knowledge of this functor F_X. This fact, a consequence of a result known as Yoneda's Lemma, can be expressed this way:

Theorem: *Let X, X' be objects in a category $\mathcal{C}$. Suppose we are given an isomorphism of their associated functors $\eta : F_X \cong F_{X'}$. Then there is a unique isomorphism of the objects themselves,*

$$h : X' \cong X$$

that gives rise—as in the process described above—to this isomorphism of functors.

The beauty of this result is that it has the following decidedly structuralist, or Wittgensteinian language-game, interpretation:

> *an object X of a category $\mathcal{C}$ is determined (always only up to canonical isomorphism, the recurrent theme of this article!) by the network of relationships that the object X has with all the other objects in $\mathcal{C}$.*

Yoneda's lemma, in its fuller expression, tells us that the set of morphisms (of the category $\mathcal{C}$) from an object X to an object Y is naturally in one-one correspondence with the set of morphisms of the functor F_Y to the functor F_X.

In brief, we have (or rather, Yoneda has) reconstructed the category $\mathcal{C}$, objects and morphisms alike, purely in terms of functors to sets; i.e., in terms of networks of relationships that deal with the entire category at once[9].

With all this, Yoneda's Lemma is one of the many examples of a mathematical result that is both extrardinarily consequential, and also extraordinarily easy to prove. [10]

15 Representable functors

The following definition (especially as it pervades the mathematical work of Alexander Grothendieck) marked the beginning of a significantly new viewpoint in our subject.

[9] The connection between Yoneda's lemma and structuralist and/or Wittgensteinian attitudes towards *meaning* was discussed in Michael Harris's review of *Mathematics and the Roots of Postmodern Theory* [Harris 2003]

[10] A full proof, for example, is given neatly and immediately via a single diagram in the "wikipedia entry" (http://en.wikipedia.org/wiki/Yoneda's_lemma). For an accessible introductory reference to the ideas of category theory, see the article by Daniel K. Biss [Biss 2003]. For a more technical, but still relatively gentle, account of category theory, see Saunders MacLane's *Categories for the working mathematician* [Mac Lane 1971].

> A functor $F : \mathcal{C} \dashrightarrow \mathcal{S}$, from a category $\mathcal{C}$ to the category of sets $\mathcal{S}$ on which it is modeled, is said to be **represented** by an object X of $\mathcal{C}$ if an isomorphism of functors $F \cong F_X$ is given. The functor F is said, simply, to be **representable** if it can be represented by some object X.

If you consult the theorem quoted at the end of the last section you see that Yoneda's lemma, then, guarantees that *if* a functor F is representable, then F determines the object X that represents it up to unique isomorphism.

One of the noteworthy lessons coming from subjects such as algebraic geometry is that often, when it is important for a theory to make a construction of a particular object that performs an important function, we have a ready description of the functor F that it would represent, if it exists. Often, indeed, the basic utility of the object X that represents this functor F comes exactly from that: that X represents the functor![11] Although a specific construction of X may tell us more about the particularities of X, there is no guarantee that all the added information a construction provides—or any of it—furthers our insight beyond guaranteeing representability of F.

Some of the important turning points in the history of mathematics can be thought of as moments when we achieve a fuller understanding of what it means for one "thing" to *represent* another "thing." The issue of *representation* is already implicit in the act of *counting,* as when we say that *these two mathematical units "represent" those two cows.* Leibniz dreamed of a scheme for a universal language that would reduce ideas "to a kind of alphabet of human thought" and the ciphers in his universal language would be manipulable *representations* of ideas.

Kant reserved the term *representation* (Vorstellung) for quite a different role. Here is the astonishing way in which this concept makes its first appearance in the *Critique of Pure Reason:*[12]

> There are only two possible ways in which synthetic representations and their objects ... can meet one another. Either the object (*Gegenstand*) alone must make the representation possible, or the representation alone must make the object possible.

It is this *either-or*, this dance between *object* and *representation*, that animates lots of what follows in Kant's *Critique of Pure Reason.* With meanings quite remote from Kant's, the same two terms, *object* and *representation*, each provide grounding for the other, in our present discussion.

[11] Students of algebra encounter this very early in their studies: the *tensor product* is (happily) nowadays first taught in terms of its functorial characterization, with its construction only coming afterwards; this is also the case for *fiber products*, for *localization in commutative algebra*; indeed this is the pattern of exposition for lots of notions in elementary mathematics, as it is for many of the grand constructions in modern algebraic geometry.

[12] [Kant 1961], p. 125.

Nowadays, whole subjects of mathematics are seen as represented in other subjects, the "represented" subject thereby becoming a powerful tool for the study of the "representing" subject, and vice versa.

It sometimes happens that the introduction of a term in a mathematical discussion is the signal that an important shift of viewpoint is taking place, or is about to take place. An emphasis on "representability" of functors in a branch of mathematics suggests an ever so slight, but ever so important, shift. The lights are dimmed on *mathematical objects* and beamed rather on the corresponding functors; that is, on the networks of relationships entailed by the objects. The functor has center stage, the object that it represents appears almost as an afterthought. The lights are dimmed on on *equality of mathematical objects* as well, and focussed, rather, on canonical isomorphisms, and equivalence.

16 The Natural Numbers as functor

Allow me to define, for *any* category, a particularly humble functor. If $\mathcal{C}$ is a category with underlying set theory $\mathcal{S}$ define a functor

$$I : \mathcal{C} - - \rightarrow \mathcal{S}$$

as follows:

If X is an object of $\mathcal{C}$, let

$$I(X) := \{X\};$$

that is, the set $I(X)$ is the singelton consisting in the set containing only one element: the object X. If $f : X \rightarrow Y$ is any morphism in $\mathcal{C}$, $I(f) : \{X\} \rightarrow \{Y\}$ is the unique mapping of singelton sets. We may think of our functor I as a *singleton functor*: it is a functor from $\mathcal{C}$ to the category of set that assigns to each object of the category $\mathcal{C}$ a singleton set. Any two "singleton functors" are (uniquely) isomorphic as functors. Is our functor I representable?

The answer here is clean. The functor I is representable if and only if our category $\mathcal{C}$ has an initial object. For if Z is an initial object, then F_Z, by the very meaning of *initial object*, is a singleton functor (there is a unique morphism from Z to any object X of the category). Therefore F_Z is isomorphic as a functor to I. Conversely, any object that represents I has the feature that it needs for us to deem it an initial object of $\mathcal{C}$.

This viewpoint gives us a way of pinning down the natural numbers from a different angle, which at first glance may seem quite strange.

> *The natural numbers is defined uniquely, up to unique isomorphism, as an object of the Peano category $\mathcal{P}$ that represents the singleton functor I.*

There is aspect to this definition that Frege might have liked: nothing "bureau-of-standards-like," nothing that smacks of a subjective choice of some particular exemplar, has entered this description. But where, in this definition, are the tangible, familiar, natural numbers? You may well ask this question; for—despite the crispness of the above definition—the concept embodied by the good old symbols $1, 2, 3, \ldots$ appears to have holographically smeared itself over the panoply of little "discrete dynamical systems" given by the objects of $\mathcal{P}$. And the category $\mathcal{P}$ itself, remember, is but a template, dependent upon an underlying set theory. But we have even further to go.

17 Equivalence of Categories

If the grand lesson is that *equivalence* has some claim to priority over *equality* in the mathematical theories packaged by categories, why are *categories* themselves untouched by this insight? The answer is that they are not. With this brief Q & A, to say nothing of the title of this section, you will not be surprised to find that what is next on the agenda is

Definition. *A functor $F : \mathcal{C} --\to \mathcal{D}$ from the category $\mathcal{C}$ to $\mathcal{D}$ is called an* **equivalence of categories** *if there is a functor going the other way, $G : \mathcal{D} --\to \mathcal{C}$ such that $G \cdot F$ is isomorphic to the identity functor from $\mathcal{C}$ to $\mathcal{C}$, and $F \cdot G$ is isomorphic to the identity functor from $\mathcal{D}$ to $\mathcal{D}$.*

and that we are specifically interested in the nature of many of our categories, *only up to equivalence.* So with this elementary vocabulary, entire theories are allowed to shift—up to equivalence.

18 Object and problem

Following Kronecker, we sometimes allow ourselves to think, say, of $\sqrt{2}$ as nothing more than a cipher that obeys the standard rules of arithmetic and about which *all we know* is that its square is 2. This characterization, to be sure, doesn't pin it down, for $-\sqrt{2}$ has precisely the same description. Nevertheless, there is no contradiction here, for having named our cipher $\sqrt{2}$ we have given birth to a specific creature of mathematics, and $-\sqrt{2}$ is just another creature with (evidently!) a different name. It is a clarifying move (in fact, the essence of algebra) to usher into the mathematical arena, and to name, certain mathematical objects that are unspecified beyond the sole fact that they are a *solution to a certain explicit problem*; in this case: a solution to the polynomial equation $X^2 = 2$.

When we do such a thing, what is sharply delineated is the *problem,* the object being a tag for (a solution to) the problem.

In the same spirit, any functor, explicitly given, from a category $\mathcal{C}$ to the category of sets $\mathcal{S}$ that the category is modeled on,

$$F : \mathcal{C} - - \to \mathcal{S}$$

may be construed as formulating an *explicit problem:*

> **PROBLEM:** Find "an" object X of the category C together with an isomorphism of functors
>
> $$\iota : F_X \cong F.$$

In a word, solve the above problem for the unknown X. To be sure, if we find two solutions,

$$\iota : F_X \cong F \text{ and } \iota' : F_{X'} \cong F,$$

then

$$\iota^{-1} \cdot \iota' : F_{X'} \cong F_X$$

is an isomorphism of representable functors and so, by Yoneda's Lemma, is induced from a unique isomorphism

$$X' \cong X;$$

i.e., the solution is unique, up to unique isomorphism.

The moral here, is that it is the *problem* that is explicit, while the *object* (that represents the solution of the problem) follows the theme of this essay: it is *unique up to unique isomorphism.*

19 Object and equality

The habitual format for discussions regarding the grounding of mathematics shines a bright light on modes of formulating assertions, organizing and justifying proofs of those assertions, and on setting up the substrate for it all—which is invariably a specific set theory. In doing this a battery of choices will be made. These choices smack of contingency, of viewing the clarity of mathematics through some *subjective lens* or other.

I imagine that all of us want to ignore—when possible—the contingent, and seize the essential, aspect of any idea. If we are of the make-up of Frege, who relentlessly strove to rid mathematical foundations of subjectivism (Frege excoriated the writings of Husserl—incorrectly, in my opinion—for ushering *psychologism* into mathematics), we look to *universal quantification* as a possible method of effacing the contingent— drowning it, one might say, in the sea of *all* contingencies. But this doesn't work.

A stark alternative—the viewpoint of categories— is precisely to *dim* the lights where standard mathematical foundations shines them brightest. Instead of focussing on the question of modes of justification, and instead of making any explicit choice of set theory, the genius of categories is to provide a vocabulary that keeps these issues at bay. It is a vocabulary that can say nothing whatsoever about proofs, and that works with any—even the barest–choice of a set theoretic language, and that captures the essential template nature of the mathematical concepts it studies, showing these concepts to be—indeed—separable from modes of justification, and from the substrate of ever-problematic set theory. *Separable* but not forever separated, effecting the kind of *aphairesis* that Aristotle might have wanted, for, as we have said, *you* must bring your own set theory, and *your* own mode of proof, to this party. With the other lights low, the mathematical concepts shine out in this new beam, as pinned down by the web of relations they have with all the other objects of their species. What has receded are set theoretic language and logical apparatus. What is now fully incorporated, center stage under bright lights, is the curious *class* of objects of the category, a template for the various manners in which a mathematical object of interest might be presented to us. The basic touchstone is that, in appropriate deference to the manifold ways an object can be presented to us, objects need only be given *up to unique isomorphism*, this being an enlightened view of what it means for one thing to be equal to some other thing.

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