

# An Expression For The Argument of $\zeta$ at Zeros on the Critical Line

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## Abstract

It is conjectured that when  $t = t_n$  is the imaginary part of the  $n$ -th zero of  $\zeta$  on the critical line, the normalised argument  $S(t) = \pi^{-1} \arg \zeta\left(\frac{1}{2} + it\right)$  is equal to  $S(t) = S_n(t_n) = n - \frac{3}{2} - \frac{\vartheta(t_n)}{\pi}$  where  $\vartheta(t)$  is the Riemann-Siegel  $\vartheta$  function. If  $S(t_n) = S_n(t_n) \forall n \in \mathbb{Z}^+$  then the exact transcendental equation for the Riemann zeros has a solution for each positive integer  $n$  which proves that Riemann's hypothesis is true since the counting function for zeros on the critical line is equal to the counting function for zeros on the critical strip in that case.

## 1 Introduction

### 1.1 Motivation

The goal of this paper is to show that a solution  $t_n$  exists for each  $n$  of the exact transcendental equation for the Riemann zeros

**Definition 1.** *The exact equation for the imaginary part of the  $n$ -th zero of  $\zeta$  on the critical line*

$$\frac{\vartheta(t_n)}{\pi} + S(t_n) + 1 = n - \frac{3}{8} \quad (1)$$

which is the same equation as in [1, Equation 20] but with 1 added to both the left and right sides of the equation.  $\zeta\left(\frac{1}{2} + it_n\right) = 0$

#### 1.1.1 The Riemann-Siegel $\vartheta(t)$ Function

The Riemann-Siegel  $\vartheta$  function is defined by

$$\begin{aligned} \vartheta(t) &= -\frac{i}{2} \left( \ln \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \ln \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \right) - \frac{\ln(\pi) t}{2} \\ &= \arg\left(\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right) - \frac{\ln(\pi) t}{2} \end{aligned} \quad (2)$$

Let

$$\tilde{\vartheta}(t) = \frac{t \ln\left(\frac{t}{2\pi e}\right)}{2} - \frac{\pi}{8}$$

be the approximate  $\vartheta$  function where the  $\Gamma$  function has been replaced with its Stirling approximation.

$$\Gamma(s) \simeq \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} \quad (3)$$

The  $\vartheta(t)$  function is not invertible but the inverse of its approximation  $\tilde{\vartheta}(t)$  is defined by a linear function of the Lambert W function given by

$$\tilde{\vartheta}^{-1}(t) = \frac{\pi + 8t}{4W\left(\frac{\pi + 8t}{8\pi e}\right)} \quad (4)$$

Let  $\text{frac}(x) = \begin{cases} x - \lfloor x \rfloor & x \geq 0 \\ x - \lceil x \rceil & x < 0 \end{cases} \forall x \in \mathbb{R}$  be the function which gives the fractional part of a real number by subtracting either the floor  $\lfloor x \rfloor$  or the ceiling  $\lceil x \rceil$  of  $x$  from  $x$ , depending upon its sign. Furthermore, let

$$S(t) = \pi^{-1} \arg \left( \zeta \left( \frac{1}{2} + it \right) \right) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} ((S(t + i\varepsilon) - S(t - i\varepsilon))) \quad (5)$$

be the argument of  $\zeta$  on the critical line.

**Definition 2.** The Riemann-von-Mangoldt formula makes use of Cauchy's argument principle to count the number of zeros **inside the critical strip**  $0 < \text{Im}(\rho_n) < t$  where  $\zeta(\sigma + i\rho_n)$  with  $0 < \sigma < 1$

$$\tilde{N}_s(t) = \frac{t}{2\pi} \ln \left( \frac{t}{2\pi e} \right) + \frac{7}{8} + S(t) + O(t^{-1}) \quad (6)$$

and this definition does not depend on the Riemann hypothesis (Conjecture 16). This equation has exactly the same form as the asymptotic Equation 7. [1, Equation 15]

**Definition 3.** The asymptotic equation for the  $n$ -th zero of the Hardy  $Z$  function

$$\frac{t_n}{2\pi} \ln \left( \frac{t_n}{2\pi} \right) + S(t_n) + O \left( \frac{1}{t_n} \right) + 1 = n - \frac{3}{8} \quad (7)$$

[1, Equation 20]

**Corollary 4.** The number of solutions of the asymptotic equation (7) for the zeros of the Hardy  $Z(t)$  function, over the interval  $[0, t]$  is given by

$$\tilde{N}_l(t) = \frac{t}{2\pi} \ln \left( \frac{t}{2\pi e} \right) + \frac{7}{8} + S(t) + O(t^{-1}) \quad (8)$$

which counts the number of zeros on the critical line.

**Conjecture 5.** The exact equation for the imaginary part of the  $n$ -th zero of  $\zeta(\frac{1}{2} + it)$  [1, Equation 20]

$$\frac{\vartheta(t_n)}{\pi} + S(t_n) - \left( n - \frac{3}{2} \right) \pi = 0 = \frac{\vartheta(t_n)}{\pi} + \frac{\arg \left( \zeta \left( \frac{1}{2} + it \right) \right)}{\pi} - \left( n - \frac{3}{2} \right) \pi \quad (9)$$

has a solution for each integer  $n \geq 1$  where  $t_n$  enumerate the zeros of  $Z$  on the real line and the zeros of  $\zeta$  on the critical line

$$\zeta \left( \frac{1}{2} + it_n \right) = 0 \forall n \in \mathbb{Z}^+ \quad (10)$$

where  $\mathbb{Z}^+$  denotes the positive integers. [1, Equation 14]

## 1.2 The Gram Points

The  $n$ -th Gram point is defined to be the solution of the equation

$$g(n) = \{\vartheta(t) = (n-1)\pi\} \quad (11)$$

A very accurate approximation  $\tilde{g}(n)$  to the Gram points  $g(n)$  is found by inverting  $\tilde{\vartheta}(t)$  to get the exact solution

$$\begin{aligned} \tilde{g}(n) &= \{t: \tilde{\vartheta}(t) - (n-1)\pi = 0\} \\ &= \left\{ t: \left( \frac{t \ln \left( \frac{t}{2\pi e} \right)}{2} - \frac{\pi}{8} \right) - (n-1)\pi = 0 \right\} \\ &= \frac{(8n-7)\pi}{4W \left( \frac{8n-7}{8e} \right)} \\ &= g(n) + O(\delta_n) \end{aligned} \quad (12)$$

where  $W$  is the Lambert  $W$  function, and the approximation bounds  $\delta_n$  when  $n = 1$  is  $\delta_1 = 0.00223698\dots$ , followed by  $\delta_2 = 0.00137812\dots$  and decreases monotonically with increasing  $n$ , that is,  $\delta_{n+1} < \delta_n$ . The inverse of  $\tilde{g}(n)$  is given by

$$\begin{aligned}\tilde{g}^{-1}(n) &= \{t: \tilde{g}(n) = 0\} \\ &= \frac{t \ln\left(\frac{t}{2\pi e}\right)}{2\pi} + \frac{7}{8} \\ &= \tilde{N}(t) - S(t) \\ &= \tilde{N}(t) - S(t)\end{aligned}\tag{13}$$

where  $\tilde{N}(t) = \tilde{N}_s(t) = \tilde{N}_l(t)$ .

### 1.2.1 An Expression for $S(t)$ at its Discontinuous Points

Define the infinite sequence of functions indexed by  $n \in \mathbb{Z}^+$

$$\begin{aligned}T_n(t) &= 1 + \lfloor \tilde{g}^{-1}(n) \rfloor - n \\ &= 1 + \left\lfloor \frac{t \ln\left(\frac{t}{2\pi e}\right)}{2\pi} + \frac{7}{8} \right\rfloor - n\end{aligned}\tag{14}$$

**Proposition 6.** *For each “bad” Gram point  $n$  there will be, within  $n \pm 2$ , a corresponding zero on the critical line which has an argument not on the principal branch. That is, if  $n$  is a “bad” Gram point then  $T_m(t_m) \neq 0$  for some  $m$  where  $|m - n| \leq 2$ .*

The function  $T_n = T_n(t_n)$  determines how many multiples of  $\pi$  to add or subtract to  $-\frac{1}{2} - \lfloor \frac{\vartheta(t_n)}{\pi} \rfloor$  so that it agrees with the argument of  $\zeta$  at a zero on the critical line where it is discontinuous, jumping by the multiplicity of the corresponding root, and having the value  $\lim_{\varepsilon \rightarrow 0} \frac{1}{2}((S(\rho + i\varepsilon) - S(\rho - i\varepsilon)))$  when  $\zeta(\rho) = 0$ .

**Definition 7.** *Let*

$$\begin{aligned}S_n(t) &= \frac{1}{2} - \text{frac}\left(\frac{\vartheta(t)}{\pi}\right) - T_n(t) \\ &= \frac{1}{2} - \text{frac}\left(\frac{\vartheta(t)}{\pi}\right) - (1 + \lfloor \tilde{g}^{-1}(n) \rfloor - n) \\ &= \frac{1}{2} - \text{frac}\left(\frac{\vartheta(t)}{\pi}\right) - 1 - \lfloor \tilde{g}^{-1}(n) \rfloor + n \\ &= -\frac{1}{2} - \text{frac}\left(\frac{\vartheta(t)}{\pi}\right) - \lfloor \tilde{g}^{-1}(n) \rfloor + n \\ &= n - \frac{1}{2} - \text{frac}\left(\frac{\vartheta(t)}{\pi}\right) - \lfloor \tilde{g}^{-1}(n) \rfloor \\ &= n - \frac{1}{2} - \text{frac}\left(\frac{\vartheta(t)}{\pi}\right) + \lfloor \frac{\vartheta(t)}{\pi} + 1 \rfloor \\ &= n - \frac{1}{2} - \left(\frac{\vartheta(t)}{\pi} + 1\right) \\ &= n - \frac{1}{2} - \frac{\vartheta(t)}{\pi} - 1 \\ &= n - \frac{3}{2} - \frac{\vartheta(t)}{\pi}\end{aligned}\tag{15}$$

Where we see that

$$S_n(t_n) = n - \frac{3}{2} - \frac{\vartheta(t_n)}{\pi} = S(t_n)\tag{16}$$

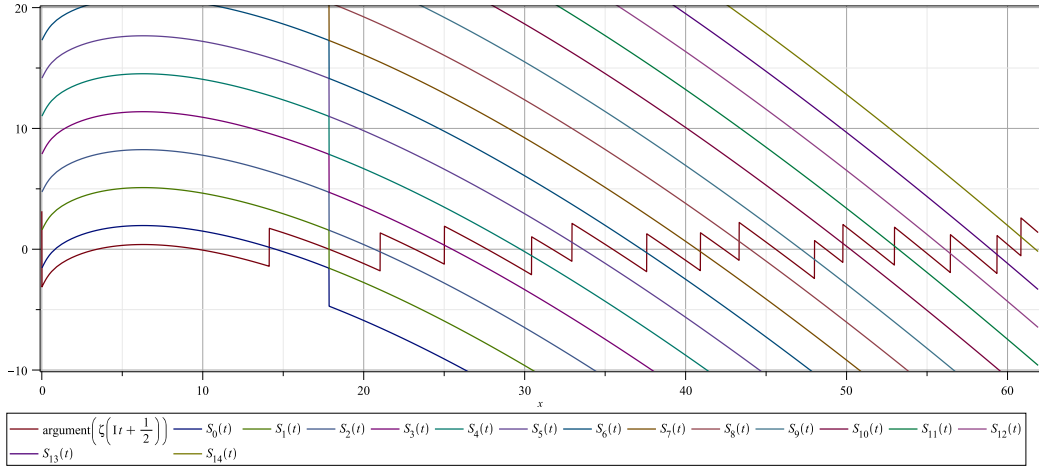
when  $t_n$  is the imaginary part of a zero of  $\zeta$  on the critical line.

**Conjecture 8.** The normalized argument  $S(t) = \pi^{-1} \arg \zeta\left(\frac{1}{2} + it\right)$  at the  $n$ -th non-trivial zero  $\zeta\left(\frac{1}{2} + it_n\right) = 0 \forall n \geq 1$  on the critical line is equal to  $S_n(t)$ , that is

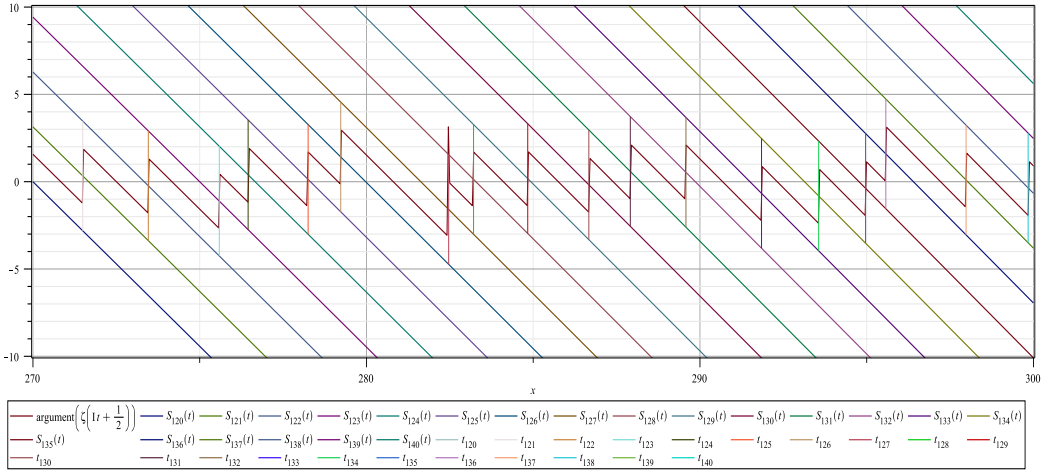
$$\begin{aligned} S(t_n) &= S_n(t_n) = \frac{1}{2} \left( \lim_{t \rightarrow t_n^-} S(t) + \lim_{t \rightarrow t_n^+} S(t) \right) \\ &= n - \frac{3}{2} - \frac{\vartheta(t_n)}{\pi} \end{aligned} \quad (17)$$

**Theorem 9.** If Conjecture 8 is true then Conjecture 5 is true and, due to Lemma 17, so is Conjecture 16, the Riemann hypothesis.

**Proof.** If  $s_\vartheta(t)S_n(t_n) = S(t_n)$  then  $S(t_n)$  is well-defined  $\forall n \geq 1$  since  $s_\vartheta(t)S_n(t_n)$  is well-defined  $\forall n \geq 1$ .  $\square$



**Figure 1.**  $S_n(t)$  and the imaginary parts of the roots of  $\zeta\left(\frac{1}{2} + it_n\right)$  marked with verticle lines just touching its touching its corresponding pair of neighboring curves  $S_n(t)$  for  $n = 0 \dots 14$

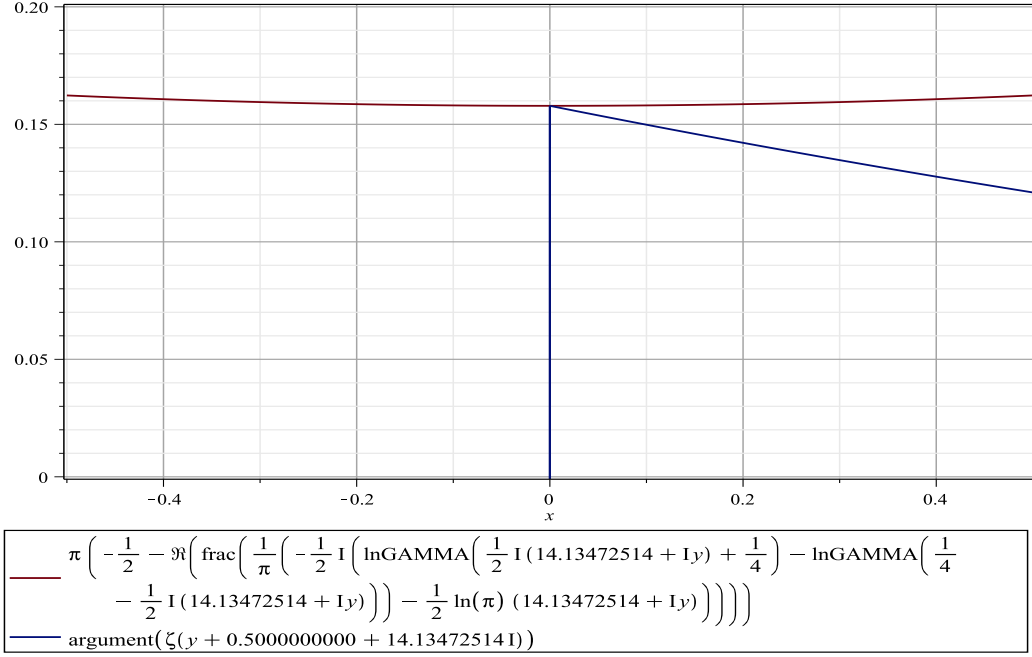


**Figure 2.**  $S_n(t)$  and the imaginary parts of the roots of  $\zeta\left(\frac{1}{2} + it_n\right)$  marked with verticle lines just touching its touching its corresponding pair of neighboring curves  $S_n(t)$  for  $n = 120 \dots 140$  which includes two “bad” Gram points at  $n = 126$  and  $n = 134$

**Theorem 10.**  $\arg\left(\zeta\left(\frac{1}{2} + ig(n)\right)\right) = 0 \forall n \in \mathbb{Z}^+$

**Proof.** The argument of any positive number  $x$  with  $\text{Im}(x) = 0$  is equal to 0 and  $\text{Im}\left(\zeta\left(\frac{1}{2} + ig(n)\right)\right) = 0$  since by definition the Gram points are the points where the imaginary part of  $\zeta$  on the critical line vanishes.  $\square$

**Conjecture 11.**  $\text{frac}\left(\frac{\vartheta(t)}{\pi}\right) + S(t) \in \{-1, 0, 1\} \forall 0 < t \in \mathbb{R}$



**Figure 3.** Illustration of convergence around the first zero

### 1.3 An Equivalent Expression for $S(t)$

Let

$$B(t) = N(t) + 2 = \frac{\vartheta(t)}{\pi} + S(t) + 3 \quad (18)$$

which is Backlund's  $\zeta$  zero counting function on the critical strip, plus 2, which is valid without assuming the Riemann hypothesis.

**Theorem 12.** *An equivalent expression for  $S(t) \forall 0 < t \in \mathbb{R}$  is*

$$\begin{aligned} \mathcal{S}(t) &= S_{B(t)}(t) - \frac{3}{2} \\ &= B(t) - \frac{3}{2} - \frac{\vartheta(t)}{\pi} - \frac{3}{2} \\ &= \frac{\vartheta(t)}{\pi} + S(t) + 3 - \frac{3}{2} - \frac{\vartheta(t)}{\pi} - \frac{3}{2} \\ &= \frac{\vartheta(t)}{\pi} + S(t) + 3 - 3 - \frac{\vartheta(t)}{\pi} \\ &= \frac{\vartheta(t)}{\pi} + S(t) - \frac{\vartheta(t)}{\pi} \\ &= S(t) \end{aligned} \quad (19)$$

**Proof.** Since  $S(t)$  appears on the left side and right sides of the equation, if we subtract  $S(t)$  from both sides we get

$$\begin{aligned} 0 &= \frac{\vartheta(t)}{\pi} + 3 - \frac{3}{2} - \frac{\vartheta(t)}{\pi} - \frac{3}{2} \\ &= 3 - \frac{3}{2} - \frac{3}{2} \\ &= 3 - 3 \\ &= 0 \end{aligned} \quad (20) \quad \square$$

### 1.3.1 A Bad/Good Gram Point Indicator

Let

$$g(n) = 2\pi e^{1+W\left(\frac{8n+1}{8e}\right)} \quad (21)$$

be the approximate value of the  $n$ -th Gram point. Now define

$$G(n) = \frac{Z(g(n))}{|Z(g(n))|} + \frac{Z(g(n+1))}{|Z(g(n+1))|} \quad (22)$$

then the function defined by

$$B(n) = \frac{1}{4}G(n-1)G(n) \quad (23)$$

takes on the value 1 when  $n$  is a “bad” Gram point for which  $(-1)^{n+1}Z(g(n)) > 0$  and the value 0 when it is a “good” Gram point  $(-1)^{n+1}Z(g(n)) < 0$ .

## 2 Appendix

### 2.1 Transcendental Equations Satisfied By The Nontrivial Riemann Zeros

**Definition 13.** *The critical line is the line in the complex plane defined by  $\text{Re}(t) = \frac{1}{2}$ .*

**Definition 14.** *The critical strip is the strip in the complex plane defined by  $0 < \text{Re}(t) < 1$ .*

**Theorem 15.** *If the limit*

$$\lim_{\delta \rightarrow 0^+} \arg\left(\zeta\left(\frac{1}{2} + \delta + it\right)\right) \quad (24)$$

*is exists and is well-defined  $\forall t$  then the left-hand side of Equation (7) is well-defined  $\forall t$ , and due to monotonicity, there must be a unique solution for every  $n \in \mathbb{Z}^+$ . [1, II.A]*

**Conjecture 16.** *(The Riemann hypothesis) All solutions  $t$  of the equation*

$$\zeta(t) = 0 \quad (25)$$

*besides the trivial solutions  $t = -2n$  with  $n \in \mathbb{Z}^+$  have real-part  $\frac{1}{2}$ , that is,  $\text{Re}(t) = \frac{1}{2}$  when  $\zeta(t) = 0$  and  $t \neq -2n$ .*

**Lemma 17.** *If the exact Equation (9) has a unique solution for each  $n \in \mathbb{Z}^+$  then Conjecture 16, the Riemann hypothesis, follows.*

**Proof.** If the exact equation has a unique solution for each  $n$ , then the zeros obtained from its solutions on the **critical line** can be counted since they are enumerated by the integer  $n$ , leading to the counting function  $N_0(t)$  in Equation (8). The number of solutions obtained on the **critical line** would saturate the counting function of the number of solutions on the **critical strip** so that  $N(t) = N_0(t)$  and thus all of the non-trivial zeros of  $\zeta$  would be enumerated in this manner. If there are zeros off of the critical line, or zeros with multiplicity  $m \geq 2$ , then the exact Equation (9) would fail to capture all the zeros on the critical strip which would mean  $N_0(t) < N(t)$ . [1, IX]  $\square$

## Bibliography

- [1] Guilherme França and André LeClair. Transcendental equations satisfied by the individual zeros of riemann zeta, dirichlet and modular l-functions. *Communications in Number Theory and Physics*, 2015.