Let $N(t)$ represent the cumulative number of events of before time t [3]

$$\lambda(t) = \frac{E\{\mathrm{d}N(t)\}}{\mathrm{d}t} \tag{1}$$

and $\mu_N(t)$ me be the associated covariance density of the stochastic contiuous-time counting function $N(t)$ given by

$$\mu_N(\tau) = \frac{E\{\mathrm{d}N(t+\tau)\mathrm{d}N(t)\}}{\mathrm{d}t^2} - \frac{E\{\mathrm{d}N(t)^2\}}{\mathrm{d}t^2} = \frac{E\{\mathrm{d}N(t+\tau)\mathrm{d}N(t)\}}{\mathrm{d}t^2} - \lambda(t)^2 \tag{2}$$

which satifies the time-reflection relations

$$\mu(\tau) = \begin{cases} \mu(-\tau) & \tau < 0 \\ \infty & \tau = 0 \\ \mu(\tau) & \tau > 0 \end{cases} \tag{3}$$

with the additional natural condition that point process is simple

$$E\{\mathrm{d}N(t)^2\} = E\{\mathrm{d}N(t)\} \tag{4}$$

meaning that events cannot and do not occur multiply(concomitantly), so that the complete covariance density is

$$\mu^{(c)}(\tau) = \lambda\delta(\tau) + \mu(\tau) \tag{5}$$

where $\delta(t)$ is the Dirac delta function and $\mu(\tau)$ is continuous in around the neighborhod of the origin

$$\lim_{\tau\to 0^+}\mu(\tau) = \lim_{\tau\to 0^-}\mu(\tau) \tag{6}$$

The complete spectral density for $N(t)$ is defined by

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\tau w}\mu^{(c)}(\tau)\mathrm{d}\tau = \frac{1}{2\pi}\left\{\lambda + \int_{-\infty}^{\infty} e^{-i\tau\omega}\mu(\tau)\mathrm{d}\tau\right\} \tag{7}$$

The multi-dimensional version of the above 1-dimensional construction above neatly facilitaties the simultaneous study of an amalgamation of (stochastic point) processes considered in relation to one another and is denoted by $N_i(t)$ where $i, j \in \{1...k\}$ and whose covariance densities are related by

$$\mu_{N_{ij}}(t,\tau) = \frac{E\{\mathrm{d}N_i(t+\tau)\mathrm{d}N_j(t)\}}{\mathrm{d}t^2} - \lambda_i(t)\lambda_j(t) \tag{8}$$

where

$$\lambda_i(t) = \frac{E\{\mathrm{d}N_i(t)\}}{\mathrm{d}t} \tag{9}$$

which satisfies

$$\mu_{N_{rs}}(\tau) = \begin{cases} \mu_{\mathrm{sr}}(-\tau) & \tau < 0 \\ \infty & \tau = 0 \\ \mu_{rs}(\tau) & \tau > 0 \end{cases} \tag{10}$$

so that (5) transforms to

$$\mu^{(c)}_{N_{rs}}(\tau) = \delta(\tau)\mathrm{diag}(\Lambda) + \mu(\tau) \tag{11}$$

so that the its associated self-adjoint spectral density matrix can be expressed similiarly to (7) as

$$F_{N_{rs}}(\omega)=\frac{1}{2\pi}\left\{\mathrm{diag}(\Lambda)+\int_{-\infty}^{\infty}e^{-i\tau\omega}\mu_{N_{rs}}(\tau)\mathrm{d}\tau\right\} \tag{12}$$

By (5) and time-refelection symmetry it can be seen that $\mu(\tau)$ can be re-expressed as the combination of two integrals

$$\begin{aligned}\mu(\tau)\ &=\int_{-\infty}^{\tau}f(\tau-v)\mu^{(c)}(v)\mathrm{d}v\\ &=\lambda g(\tau)+\int_{-\infty}^{\tau}f(\tau-v)\mu(v)\mathrm{d}v\\ &=\lambda g(\tau)+\int_{0}^{\infty}f(\tau+v)\mu(v)\mathrm{d}v+\int_{0}^{\tau}f(\tau-v)\mu(v)\mathrm{d}v\end{aligned} \tag{13}$$

The process $\Lambda(t)$ when considered to be self-exciting can be expressed as

$$\Lambda(t)=\nu+\int_{-\infty}^{t}g(t-u)\mathrm{d}N(u) \tag{14}$$

then, conditional upon the process being stationary, that is, its largest eigenvector being no more than 1, the average intensity is

$$\lambda=E\{\Lambda(t)\}=\nu+\lambda\int_{-\infty}^{t}g(t-u)\mathrm{d}u=\frac{\nu}{1-\int_{0}^{\infty}g(v)\mathrm{d}v} \tag{15}$$

This is the part where the standard exponential recursions are derived. The task is to do the same, with the exponential powerlaw approximation kernel methods. The 'standard exponential kernel' is the one defined by

$$g(t)=\sum_{j=1}^{P}\alpha_j e^{-\beta_j t} \tag{16}$$

The Laplace transform of $g(t)$ is

$$\begin{aligned}\int_0^\infty g(t)e^{-ts}\,\mathrm{d}t\ &=\int_0^\infty\sum_{j=1}^{P}\alpha_j e^{-\beta_j t}e^{-ts}\,\mathrm{d}t\\ &=\sum_{j=1}^{P}\frac{\alpha_j}{s+\beta_j}\end{aligned} \tag{17}$$

It's probably worth mentioning the reference to Poisson measues in [4, 5.3] and the fact that in [2, 2.2.3] it is mentioned that the Fourier transform of the 'Gaussian' e^{-tx^2} is (essentially) is probably related to the fact that the intensity function of the Hawkes process having kernel (16) has an autocovariance function which is essential equal to itself, that is, it has the exact same functional form. TODO: See [1, 12.8] and use the notions of scale-invariance and self-similarity.

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