Following [1], let

$$\lambda_t = \lambda_0\, e^{-\beta t} + \sum_{i=1}^{M_t} Y_t\, e^{-\beta(t-S_i)}$$

where

- λ_0 is the *initial value* of λ_t
- Y_i is a sequence of i.i.d. non-negative random variables distributed according to distribution function $G(y)\forall y>0$ on the positive half-line, being such that its expected value at any given instant is $E[Y_t]=\mu$
- S_i is a sequence of event times of a Poisson process M_t with constant intensity ρ
- β is the rate of exponential intensity decrease

the infinitesimal generator $A\cdot f(\Lambda,\lambda,t)$ of the process (Λ_t,λ_t,t) acting on a function $f(\Lambda,\lambda,t)\in \mathrm{Dom}(A)$ is given by

$$A\,f(\Lambda,\lambda,t) = \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial \Lambda} - \beta\lambda\frac{\partial f}{\partial \lambda} + \rho\left(\int_0^\infty f(\Lambda,\lambda+y,t)\mathrm{d}G(y) - f(\Lambda,\lambda,t)\right) \tag{1}$$

Sufficient conditions for f being in the domain of A is that f be differentiable with respect to its arguments Λ,λ,t for all values and that

$$\left|\int_0^\infty f(\Lambda,\lambda+y,t)\mathrm{d}G(y) - f(\Lambda,\lambda,t)\right| < \infty\forall\Lambda,\lambda,t \tag{2}$$

1 A Suitable Martingale for the Probability Generating Function

To find a probability generating function, let us first identify a suitable martingale. Let $W_t=\beta\Lambda_t+\lambda_t$ and $Z_t=\lambda_t\, e^{\beta t}$ then the infinitesimal generator $A\cdot f(W_t,Z_t,t)$ of the process (W_t,Z_t,t) acting on a function $f(w,z,t)\in\mathrm{Dom}(A)$ is given by

$$A\cdot f(w,z,t) = \frac{\partial f}{\partial t} + \rho\left[\int_0^\infty f(w+y,z+ye^{\beta t},t)\mathrm{d}G(y) - f(w,z,t)\right] \tag{3}$$

A necessary condition for $f(w,z,t)$ to be a martingale is that its corresponding infinitesimal generator must satisify $A\cdot f=0$. The equation

$$\dot{h}(t) - \rho(1-\hat{g}(a+b e^{\beta t}))h(t) = 0 \tag{4}$$

has the solution of the form

$$h(t) = K e^{\rho\int_0^t(1-\hat{g}(a+b\,e^{\beta s}))\mathrm{d}s} \tag{5}$$

where K is an arbitrary constant. Therefore, if a and b are such that $a\geqslant 0$ and $b\geqslant -a e^{-\beta u}$ where u is a fixed time to which Λ_t and λ_t evolve, then

$$e^{-a\beta\Lambda_t}\, e^{-(a+b\,e^{\beta t})\lambda_t}\, e^{\rho\int_0^t(1-\hat{g}(a+b\,e^{\beta s}))\mathrm{d}s} \tag{6}$$

is martingale where $\hat{g}(x)=\int_0^t e^{-x\,y}\mathrm{d}G(y)\forall t>0$.

Bibliography

[1] Angelos Dassios and Jiwook Jang. The distribution of the interval between events of a cox process with shot noise intensity. *Journal of Applied Mathematics and Stochastic Analysis*, 2008(367170):14, 2008.