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Temporal disaggregation of economic time series: towards a dynamic extension

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Keywords: temporal disaggregation, related series, dynamic model

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Temporal disaggregation of economic time series: towards a dynamic extension[1]

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1. Statement of the problem

A traditional problem often faced by National Statistical Institutes (NSIs) and more generally by economic researchers is the interpolation or distribution of economic time series observed at low frequency into compatible higher frequency data. While interpolation refers to estimation of missing observations of stock variables, a distribution (or temporal disaggregation) problem occurs for flows aggregates and time averages of stock variables.

The need for temporal disaggregation can stem from a number of reasons. For example NSIs, due to the high costs involved in collecting the statistical information needed for estimating national accounts, could decide to conduct large sample surveys only annually. Consequently, quarterly (or even monthly) national accounts could be obtained through an indirect approach, that is by using related quarterly (or monthly) time series as indicators of the short-term dynamics of the annual aggregates. As another example, econometric modelling often implies the use of a number of time series, some of which could be available only at lower frequencies, and therefore it could be convenient to disaggregate these data instead of estimating, with a significant loss of information, the complete model at lower frequency level.

Temporal disaggregation[2] has been extensively considered by previous econometric and statistical literature and many different solutions have been proposed so far. Broadly speaking, two alternative approaches have been followed[3]:

1) methods which do not involve the use of related series but rely upon purely mathematical criteria or time series models to derive a smooth path for the unobserved series;
2) methods which make use of the information obtained from related indicators observed at the desired higher frequency.

The first approach comprises purely mathematical methods, as those proposed by Boot *et al.* (1967) and Jacobs (1994), and more theoretically founded model-based methods (Wei and Stram, 1990) relying on the Arima representation of the series to be disaggregated. The latter approach, more interesting for our purposes, includes, amongst others, the adjustment procedure due to Denton (1971) and the related Ginsburgh's (1973) approach, the method proposed by Chow and Lin (1971)

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[1] Lavoro svolto nell'ambito del Task-Project di Eurostat "Construction of higher frequency data when only lower ones are available". Ringrazio Marco Marini, Gian Luigi Mazzi e Fabio Sartori per i commenti su precedenti versioni del paper. Resto ovviamente il solo responsabile di eventuali errori.

[2] In this report we deal exclusively with issues of temporal disaggregation of univariate time series. For a discussion on the multivariate case, see Di Fonzo (1994).

[3] Eurostat (1999) contains a survey, taxonomy and description of the main temporal disaggregation methods proposed by literature, and in Bloem *et al.* (2001) two chapters analyze the relevant techniques to be used for benchmarking, extrapolation and related problems occurring in the compilation of quarterly national accounts.

and further developed by Bournay and Laroque (1979), Fernández (1981) and Litterman (1983). Moreover, Al-Osh (1989), Wei and Stram (1990), Guerrero (1990) and Guerrero and Martinez (1995) combine an Arima-based approach with the use of high frequency related series in a regression model to overcome some arbitrariness in the choice of the stochastic structure of the high frequency disturbances.

However, in these very last years other papers on temporal aggregation and disaggregation of time series have been produced, and the resurgence of interest has been so sudden that many of them are still unpublished (sometimes under work) or are going to appear soon in the specialized journals.

Though the behaviour of NSIs could sometimes appear conservative enough in the introduction of new techniques, it should be stressed that their introduction in a routine process (such as that of the estimation of quarterly national accounts) requires that at least the following requirements are fulfilled:

1) the techniques should be flexible enough to allow for a variety of time series to be treated easily, rapidly and without too much intervention by the producer;
2) they should be accepted by the international specialized community;
3) the techniques should give reliable and meaningful results;
4) the statistical procedures involved should be run in an accessible and well known, possibly user friendly, and well sounded software program, interfacing with other relevant instruments typically used by data producers (i.e. seasonal adjustment, forecasting, identification of regression models,…).

The difficulties in achieving these goals have led NSIs using indirect approaches to quarterly disaggregation to rely mainly on techniques developed some thirty years ago (as Denton, 1971 or Chow and Lin, 1971). For obvious reasons, these techniques do not consider the more recent developments of econometric literature (typically, the introduction of the dynamic specifications, the co-integration and common component analyses and the use of an unobserved time series models environment) and therefore sometimes they demonstrate obsolete and not responding to the increasing demand for more sophisticated and/or theoretically well founded statistical and mathematical methods in estimating national accounts figures.

Focusing for the moment on temporal disaggregation, broadly speaking two research lines seem to be pre-eminent and will be discussed in this survey:

a) techniques using dynamic regression models (and possibly transformed data) in the identification of the relationship linking the series to be estimated and the (set of) related time series;
b) techniques using formulations in terms of unobserved component models/structural time series (either in levels or transformed) and the Kalman filter to get optimal estimates of missing observations by a smoothing algorithm (Gudmundsson, 1999, Hotta and Vasconcellos, 1999, Proietti, 1999, Gómez, 2000).

In this paper we discuss and extend the main methods in category a), namely those by Gregoir (1995), Salazar *et al.* (1994, 1997, 1998) and Santos Silva and Cardoso (2001), while as for techniques under b) we refer to Moauro and Savio (2001). This choice can be explained by the fact that, at the present state of art, methods based on a structural time series modelling approach appear very promising, for they bring into the estimation problem a more thorough consideration of the (possibly multivariate, Harvey, 1989, Harvey and Koopmans, 1997) data generating process[4]. On the other hand, from our point of view this approach doesn't still fulfil the minimal requirements 1)-4) for NSIs we quoted above.

[4] Another advantage of this approach lies in the fact that both unadjusted and seasonally adjusted series can be simultaneously estimated.

The work is organized as follows: in the next section the terms of the problem are rapidly presented, along with the notation that will be used in the rest of the paper. In section 3 we review the classical approaches to the disaggregation of a single time series based on both a constrained minimization of a quadratic loss function and a static regression model and discuss some fundamental identification issues. The links between the two approaches will be made clear, for they will be useful for the results presented in the next sections. In section 4 we deal with the management of the logarithmic transformation and discuss the model in first differences of the variables, by this way giving a first enlargement of the classical temporal disaggregation approach considered so far. The reasons suggesting an enlargement towards dynamic structure of the underlying regression model on which the temporal disaggregation is based are discussed in section 5. A disaggregation procedure founded on the simple dynamic regression model worked out by Gregoir (1995) and Salazar *et al.* (1997, 1998) is then presented (section 6) along with a closed form expression for the estimated high-frequency values. Section 7 is devoted to an almost complete review of the method by Santos Silva and Cardoso (2001), whose proposal permits to deal with a simple dynamic framework without conditioning on the initial observations, allowing by this way for a straightforward formulation of the disaggregation formulae. To make clear the procedures and in order to understand and empirically evaluate the way the methods work, in section 8 we consider two practical applications: the former replicates the application originally developed by Santos Silva and Cardoso (2001), that is the estimation of quarterly US personal consumption (1953q1-1984q4, seasonally adjusted) by temporal disaggregating the relevant annual series using US personal disposable income as related series[5]. The latter concerns the estimation of monthly Italian industrial value added (1970q1-2001q3, seasonally adjusted) using the relevant monthly industrial production index as related series.

2. Notation

According to Salazar *et al.* (1994), we adopt the following notation convention. Single observations of low-frequency (LF) data are denoted by a single subscript, i.e. y_t, and are observed in T consistently spaced periods, which is a key assumption for the methods outlined in the following sections.

Our aim is to derive an estimate of the underlying high-frequency (HF) series, whose unknown values are denoted by a double-subscript, so that $y_{t,u}$ denotes the HF value of Y in sub-period u of period $t = 1,\ldots,T$, which is assumed to have periodicity s. For example, s=3 if we require monthly estimates of a quarterly observed series, s=4 if we want quarterly estimates for yearly data and s=12 if monthly data are required for an annually observed series.

The $(T \times 1)$ vector of LF data is denoted by

$$\mathbf{y}_l = (y_1,\ldots,y_t,\ldots,y_T)'$$

while the $(n \times 1)$ vector of HF data is denoted by $\mathbf{y}_h$. Notice that we must have $n \geq sT$. If $n = sT$, then $\mathbf{y}_h = (y_{1,1},\ldots,y_{T,s})'$ and we face a problem of distribution or interpolation. If $n > sT$, also an extrapolation issue has to be considered, with the difference $n - sT$ being the number of HF sub-periods not subject to temporal aggregation constraints.

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[5] The data have been published by Greene (1997, tab. 17.5).

Similarly, any $(T \times K)$ matrix of LF data is denoted by $\mathbf{X}_l$ and its $(n \times K)$ HF counterpart is written as $\mathbf{X}_h$. The columns of the LF matrix $\mathbf{X}_l$ are denoted by $\mathbf{x}_{l,k}$ and those in the HF matrix $\mathbf{X}_h$ by $\mathbf{x}_{h,k}$, where $k = 1,\ldots,K$ denotes the relevant variable of the matrix $\mathbf{X}_l$ and $\mathbf{X}_h$, respectively. Accordingly, $\mathbf{x}_t$ and $\mathbf{x}_{t,u}$ are $(K \times 1)$ vectors containing the LF and HF observations on K related series in period t and (t,u), respectively.

We assume, in general, that there exists a (time-invariant) constraint linking the LF and HF data given by

$$y_t = \sum_{u=1}^{s} c_u y_{t,u}, \qquad t = 1,\ldots,T, \tag{2.1}$$

where the weights $\{c_u\}_{u=1}^{s}$ are known *a priori* and are typically 1 if the LF data are simply aggregates of the high-frequency data or $\frac{1}{s}$ if the LF data are averages of the HF data. If the LF data have been obtained by systematically sampling the HF ones, then the weights are all zeroes but either the last one (stock variable observed at the end of the LF period) or the first one (stock variable observed at the beginning of the LF period).

We express (2.1) in matrix terms by defining a $(T \times n)$ matrix $\mathbf{C}$ that links the (observed) LF vector $\mathbf{y}_l$ to that of the corresponding (unknown) HF series $\mathbf{y}_h$. If $n = sT$ the aggregation matrix $\mathbf{C}$ has a block-diagonal structure: $\mathbf{C} = \mathbf{I}_T \otimes \mathbf{c}'$, where $\mathbf{c} = (c_1,\ldots,c_u,\ldots,c_s)'$ and $\otimes$ denotes the Kronecker product. Hence, we have

$$\mathbf{y}_l = \mathbf{C}\mathbf{y}_h. \tag{2.2}$$

In the case of a pure distribution problem it is

$$\underset{(T \times sT)}{\mathbf{C}} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix} = \mathbf{I}_T \otimes \mathbf{c}',$$

where $\mathbf{c} = (1,1,\ldots,1)'$. Alternatively, in the case of a pure interpolation situation, with end of period values, we have:

$$\underset{(T \times sT)}{\mathbf{C}} = \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} = \mathbf{I}_T \otimes \mathbf{c}',$$

where $\mathbf{c} = (0,0,\ldots,1)'$. When an extrapolation problem is also present, n-sT columns of zeroes must be added to the previous matrices.

3. A general formulation of the disaggregation problem

Without any loss of generality assume that we want to estimate a column vector $\mathbf{y}_h$ of n HF values of a given variable Y. For this purpose we dispose of a vector $\mathbf{y}_l$ of T LF observations of the same variable. The first element of $\mathbf{y}_h$ will correspond to the first sub-period of the first LF period in $\mathbf{y}_l$ (e.g., the first month of the first quarter). Also assume we observe K related HF series whose observations are grouped in a $(n \times K)$ matrix $\mathbf{X}_h$.

Define the estimator of $\boldsymbol{\beta}$ and $\mathbf{y}_h$ as the optimal solution of the following problem:

$$\min_{\mathbf{y}_h, \boldsymbol{\beta}} L(\mathbf{y}_h, \boldsymbol{\beta}) = (\mathbf{y}_h - \mathbf{X}_h\boldsymbol{\beta})'\mathbf{W}(\mathbf{y}_h - \mathbf{X}_h\boldsymbol{\beta}) \quad \text{subject to } \mathbf{C}\mathbf{y}_h = \mathbf{y}_l, \tag{3.1}$$

where $\mathbf{W}$ is a $(n \times n)$ positive definite metric matrix and $\boldsymbol{\beta}$ is a $(K \times 1)$ column vector of coefficients. $L(\mathbf{y}_h, \boldsymbol{\beta})$ is a quadratic loss function in the differences between the series to be estimated $\mathbf{y}_h$ and a linear combination $\mathbf{X}_h\boldsymbol{\beta}$ of the related HF series.

The optimal solutions $\hat{\mathbf{y}}_h$ and $\hat{\boldsymbol{\beta}}$ are easily obtained from (3.1) (Pinheiro and Coimbra, 1992):

$$\hat{\mathbf{y}} = \mathbf{X}_h\hat{\boldsymbol{\beta}} + \mathbf{W}^{-1}\mathbf{C}'\left(\mathbf{C}\mathbf{W}^{-1}\mathbf{C}'\right)^{-1}\left(\mathbf{y}_l - \mathbf{X}_l\hat{\boldsymbol{\beta}}\right), \tag{3.2}$$

$$\hat{\boldsymbol{\beta}} = \left[\mathbf{X}_l'\left(\mathbf{C}\mathbf{W}^{-1}\mathbf{C}'\right)^{-1}\mathbf{X}_l\right]^{-1}\mathbf{X}_l'\left(\mathbf{C}\mathbf{W}^{-1}\mathbf{C}'\right)^{-1}\mathbf{y}_l. \tag{3.3}$$

Solutions (3.2)-(3.3) obtained according a quadratic-linear approach may be derived in a different way (Chow and Lin, 1971). Assume that we have the HF regression model

$$\mathbf{y}_h = \mathbf{X}_h\boldsymbol{\beta} + \mathbf{u}_h, \qquad E(\mathbf{u}_h \mid \mathbf{X}_h) = \mathbf{0}, \qquad E\left(\mathbf{u}_h\mathbf{u}_h' \mid \mathbf{X}_h\right) = \mathbf{V}_h. \tag{3.4}$$

Pre-multiplying equation (3.4) by $\mathbf{C}$, we obtain the LF regression

$$\mathbf{y}_l = \mathbf{X}_l\boldsymbol{\beta} + \mathbf{u}_l, \tag{3.5}$$

with $\mathbf{u}_l = \mathbf{C}\mathbf{u}_h$, $E(\mathbf{u}_l \mid \mathbf{X}_l) = \mathbf{0}$, $E\left(\mathbf{u}_l\mathbf{u}_l' \mid \mathbf{X}_l\right) = \mathbf{C}\mathbf{V}_h\mathbf{C}' = \mathbf{V}_l$. It is interesting to notice that $\hat{\boldsymbol{\beta}}$ in (3.3) is the Generalized Least Squares (GLS) estimator of $\boldsymbol{\beta}$ in the LF regression (3.5) if we let $\mathbf{V}_h = \mathbf{W}^{-1}$.

From the Gauss-Markov theorem, it is obvious that (3.3) is the Best Linear (in $\mathbf{y}_l$) Unbiased Estimator (BLUE) for $\boldsymbol{\beta}$, conditional to $\mathbf{X}_h$. The estimator $\hat{\mathbf{y}}_h$ is obtained by correcting the linear combination of the HF series $\mathbf{X}_h\hat{\boldsymbol{\beta}}$ by distributing the estimated LF residuals $\hat{\mathbf{u}}_l = \mathbf{y}_l - \mathbf{X}_l\hat{\boldsymbol{\beta}}$ among the HF sub-periods. The covariance matrix of $\hat{\mathbf{y}}_h$ is given by (Bournay and Laroque, 1979)

$$E(\hat{\mathbf{y}}_h - \mathbf{y}_h)(\hat{\mathbf{y}}_h - \mathbf{y}_h)' = (\mathbf{I}_n - \mathbf{L}\mathbf{C})\mathbf{V}_h + (\mathbf{X}_h - \mathbf{L}\mathbf{X}_l)\left(\mathbf{X}_l'\mathbf{V}_l^{-1}\mathbf{X}_l\right)^{-1}(\mathbf{X}_h - \mathbf{L}\mathbf{X}_l)', \tag{3.6}$$

where $\mathbf{L} = \mathbf{V}_h \mathbf{C}' \mathbf{V}_l^{-1}$. From (3.6) standard errors of the estimated HF series may be calculated and used to produce, for example, reliability indicators of the estimates (van der Ploeg, 1985) as

$$\frac{\hat{y}_{t,u}}{\hat{\sigma}_{\hat{y}_{t,u}}}, \quad t=1,\ldots,T, \quad u=1,\ldots,s,$$

or, assuming $u_{t,u}$ normal, confidence intervals.

3.1. Alternative choices of $\mathbf{V}_h$ and the fundamental identification issue

From a practical point of view, when one is interested in applying one of the procedures presented so far, the key issue lies in identifying the covariance matrix $\mathbf{V}_h$ from $\mathbf{V}_l$. This latter can be estimated from the data, possibly by imposing an *ARIMA* structure on the aggregate but, in general, the covariance matrix of the HF disturbances cannot be uniquely identified from the relationship $\mathbf{V}_l = \mathbf{C}\mathbf{V}_h\mathbf{C}'$.

For the case when indicators are available several restrictions on the DGP of $u_{t,u} | \mathbf{X}_h$ have been proposed in order to simplify the problem (Eurostat, 1999):

- Chow and Lin (1971): $u_{t,u} | \mathbf{X}_h \quad AR(1)$;
- Fernàndez (1981): $u_{t,u} | \mathbf{X}_h$ Random Walk ;
- Litterman (1983): $u_{t,u} | \mathbf{X}_h \quad ARIMA(1,1,0)$;

Wei and Stram (1990) consider the general case where $u_{t,u} | \mathbf{X}_h \quad ARIMA(p,d,q)$. They deal with sufficient condition under which the derivation of $\mathbf{V}_h$ from $\mathbf{V}_l$ is possible for the *ARIMA* model class when $\mathbf{c} = \mathbf{i}_s$; Barcellan and Di Fonzo (1994) provide the extension for general $\mathbf{c}$.

Despite the proposal of Wei and Stram (1990) encompasses the three models above, statistical agencies often limit their considerations to the Chow and Lin procedure (or to suitable extensions) because of its computational simplicity. As a consequence, little or no attention is posed on the data generating mechanism and it is often the case that the residuals of the LF regression distributed across the periods are realizations of integrated processes. Also, the procedure is applied twice using both raw and seasonally adjusted indicators, in order to get a raw and seasonally adjusted disaggregated or interpolated series. Moreover, as noted by the authors themselves in their concluding remarks, “The usefulness of this method in practice (…) surely depends on the validity of the regression model assumed”.

The approach of Wei and Stram has the merit of reducing the arbitrariness of the parametrization of $\mathbf{V}_h$ that characterizes the ‘classical’ solutions. On the other hand, given the (usually) reduced number of aggregated observations on which the identification procedure must be based, its application is not straightforward. In fact, as Proietti (1999) points out, the Wei and Stram approach relies heavily on model identification for the aggregate series according to the Box-Jenkins strategy, which fundamentally hinges upon the correlograms. These have poor sample properties for the typical sample sizes occurring in economics. Furthermore, due to the increase in the standard errors (Rossana and Seater, 1995), the autocorrelations may become insignificant and low order models are systematically chosen. The Monte Carlo evidence presented in Chan (1993) shows that the approach is likely to perform comparatively bad when $T<40$ (which is not an infrequent size in economics).

4. A first enlargement of the classic approach: temporal disaggregation of a log or log-differenced variable

4.1. Preliminary remark. A regression model in first differences: another look at Fernàndez (1981)

Consider the following representation of the underlying HF data:

$$\Delta y_{t,u} = \Delta \mathbf{x}'_{t,u} \boldsymbol{\beta} + \varepsilon_{t,u} \tag{4.1}$$

where $\Delta = 1 - L$, L being the lag operator, $\boldsymbol{\beta}$ is a $(K \times 1)$ vector of fixed, unknown parameters and the process $\{\varepsilon_{t,u}\}$ is *white noise* with zero mean and variance σ_ε^2. Notice that implicit assumptions of model (4.1) are that:

- the constant is not part of vector $\mathbf{x}_{t,u}$;
- a constant term appears in $\Delta \mathbf{x}_{t,u}$ only if $\mathbf{x}_{t,u}$ contains a linear deterministic trend.

We will turn on these two points later.

Representation (4.1) implies

$$y_{t,u} = \mathbf{x}'_{t,u} \boldsymbol{\beta} + u_{t,u} \tag{4.2}$$

where the process $\{u_{t,u}\}$, defined by $\Delta u_{t,u} = \varepsilon_{t,u}$, follows a *random walk without drift.* Implicitly therefore it is assumed that the two series are not cointegrated (or, if they are, that the cointegrating vector differs from $\boldsymbol{\beta}$).

Now, let us consider the $(n \times n)$ matrix $\mathbf{D} = \{d_{ij}\}$ such that $d_{ii} = 1$, $d_{i,i-1} = -1$ and zero elsewhere:

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}, \tag{4.3}$$

whose inverse is the lower triangular matrix:

$$\mathbf{D}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Model (4.2) can thus be re-written in matrix form as

$$\mathbf{y}_h = \mathbf{X}_h \boldsymbol{\beta} + \boldsymbol{\xi}_h \tag{4.4}$$

where $\boldsymbol{\xi}_h = \mathbf{D}^{-1}\boldsymbol{\varepsilon}_h$ is a zero-mean stochastic vector with covariance matrix given by $E\left(\boldsymbol{\xi}_h \boldsymbol{\xi}_h^{'}\right) = \sigma_\varepsilon^2 \mathbf{D}^{-1}\left(\mathbf{D}^{-1}\right)' = \sigma_\varepsilon^2 \left(\mathbf{D}'\mathbf{D}\right)^{-1}$, where

$$\left(\mathbf{D}'\mathbf{D}\right)^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}.$$

If the HF model contains an intercept, that is

$$\Delta y_{t,u} = \alpha + \Delta \mathbf{x}_{t,u}^{'} \boldsymbol{\beta} + \varepsilon_{t,u}, \tag{4.5}$$

the process $\left\{u_{t,u}\right\}$ follows a *random walk with drift*, that is $\Delta u_{t,u} = \alpha + \varepsilon_{t,u}$. Notice that this case is equivalent to augment the original (not differenced) indicator matrix by a linear deterministic trend variable, such that

$$\mathbf{X}_h^* = \left[\mathbf{j} \quad \mathbf{X}_h\right],$$

where, without loss of generality, $\mathbf{j}$ is a $(n \times 1)$ vector with generic element $j_{t,u} = (t-1)s + u$. The HF model becomes

$$\mathbf{y}_h = \mathbf{X}_h^* \boldsymbol{\beta}^* + \boldsymbol{\xi}_h, \tag{4.6}$$

where $\boldsymbol{\beta}^*$ is a $\left((k+1)\times 1\right)$ vector of parameters whose first element is the drift α.

In the light of what we have discussed so far, the user should not put deterministic variables as either a constant or a linear deterministic trend (or both) in vector $\mathbf{x}_{t,u}$, because, as shown in tab. 4.1, they are either incompatible or encompassed by models (4.1) and (4.5).

Model (4.4) has been considered by Fernàndez (1981) first (see also Di Fonzo, 1987, pp. 51-52) and later by Pinheiro and Coimbra (1992), Salazar *et al.* (1994)[6] and Eurostat (1999). Following the classical result by Chow and Lin (1971, 1976), the MMSE estimated HF vector in the case of random walk without drift is given by:

$$\hat{\mathbf{y}}_h = \mathbf{X}_h \hat{\boldsymbol{\beta}} + \left(\mathbf{D}'\mathbf{D}\right)^{-1} \mathbf{C}' \left[\mathbf{C}\left(\mathbf{D}'\mathbf{D}\right)^{-1}\mathbf{C}'\right]^{-1} \left(\mathbf{y}_l - \mathbf{X}_l \hat{\boldsymbol{\beta}}\right),$$

[6] However, contrary to what Salazar *et al.* (1994, p. 6) state, this procedure *is not* an amended version of Ginsburgh's (1973) method, which is a well known two-step adjustment procedure, but rather the optimal (one step) solution to the temporal disaggregation problem given either model (4.4) or (4.6).

where

$$\hat{\boldsymbol{\beta}} = \left\{ \mathbf{X}_l^{'} \left[\mathbf{C}\left(\mathbf{D}'\mathbf{D}\right)^{-1} \mathbf{C}' \right]^{-1} \mathbf{X}_l \right\}^{-1} \mathbf{X}_l^{'} \left[\mathbf{C}\left(\mathbf{D}'\mathbf{D}\right)^{-1} \mathbf{C}' \right]^{-1} \mathbf{y}_l .$$

The solution for the case in which a drift is present is simply obtained by substituting $\boldsymbol{\beta}$, $\mathbf{X}_h$ and $\mathbf{X}_l$ with $\boldsymbol{\beta}^*$, $\mathbf{X}_h^*$ and $\mathbf{X}_l^* = \mathbf{C}\mathbf{X}_h^*$, respectively.

Tab. 4.1: Constant and linear trend in the regression model in first differences

Deterministic regressor in $\mathbf{x}_{t,u}$		**Stochastic disturbances in the high frequency regression model $y_{t,u} = \mathbf{x}_{t,u}^{'}\boldsymbol{\beta} + u_{t,u}$**	
Constant	**Linear trend**	**Random walk without drift**	**Random walk with drift**
NO	**NO**	$(1-L)y_{t,u} = (1-L)\mathbf{x}_{t,u}^{'}\boldsymbol{\beta} + \varepsilon_{t,u}$	$(1-L)y_{t,u} = \alpha + (1-L)\mathbf{x}_{t,u}^{'}\boldsymbol{\beta} + \varepsilon_{t,u}$
YES	**NO**	----- *	----- *
YES	**YES**	----- *	---- **
NO	**YES**	$(1-L)y_{t,u} = \alpha + (1-L)\mathbf{w}_{t,u}^{'}\boldsymbol{\beta} + \varepsilon_{t,u}$ ***	--- ****

* $(1-L)\mathbf{x}_{t,u}$ contains a systematically null element: the estimation cannot be performed.

** $(1-L)\mathbf{x}_{t,u}$ contains both a systematically null element and a systematically constant (not zero) element: the estimation cannot be performed.

*** We write $\mathbf{x}_{t,u} = \left(j, \mathbf{w}_{t,u}^{'}\right)'$, $j = (t-1)s + u$. This case is thus equivalent to use only the 'true indicators' $\mathbf{w}_{t,u}$ in a model which assumes a random walk with drift for the high-frequency disturbances.

**** $(1-L)\mathbf{x}_{t,u}$ contains a systematically constant (not zero) element: the estimation cannot be performed.

As Salazar *et al.* (1994) note, the GLS procedure above, and in general the estimates obtained according to Chow and Lin's approach, are satisfactory in the case where the temporal aggregation constraint is linear and there are no lagged dependent variables in the regression. We next discuss how the approach can be extended to deal with logarithmic and lagged dependent variables.

4.2. Log-transformed variable in a static model

Empirical studies on macroeconomic time series show that in many circumstances it is strongly recommended to use logarithms of the original data to achieve better modelizations of time series. In particular, most macroeconomic aggregate time series become stationary after applying first differences to their logarithms. Unfortunately, logarithmic transformation being not additive, for a distribution problem the standard disaggregation results can not be directly applied.

The problem of dealing with log-transformed variables in a disaggregation framework has been considered by Pinheiro and Coimbra (1992), Salazar *et al.* (1994, 1997), Proietti (1999) and Aadland (2000). The approach followed here is strictly related to the last reference, which is rather comprehensive on this issue and whose results regarding the temporal disaggregation of flows variables confirm those found by Salazar *et al.* (1994, 1997). Moreover, Salazar *et al.* (1997) and Proietti (1999) tackle the related problem of the adjustment of the estimated values to fulfil the temporal aggregation constraints.

Let us consider first the case of temporal disaggregation of a flow variable *Y*, whose high-frequency values $y_{t,u}$ are unknown, $t = 1,\dots,T$ being the low-frequency index and $u = 1,\dots,s$ the high-frequency one (*s=3* in the quarterly/monthly, *s=4* in the annual/quarterly and *s=12* in the annual/monthly cases, respectively).

As for *Y*, (i) we observe the temporal aggregate $y_t = \sum_{u=1}^{s} y_{t,u}$ and (ii) we suppose that the model at disaggregated level be

$$\ln y_{t,u} = \mathbf{x}_{t,u}^{'}\boldsymbol{\beta} + u_{t,u}, \qquad t = 1,\dots,T; \quad u = 1,\dots,s,$$

that is

$$z_{t,u} = \mathbf{x}_{t,u}^{'}\boldsymbol{\beta} + u_{t,u}, \qquad t = 1,\dots,T; \quad u = 1,\dots,s, \tag{4.7}$$

where $z_{t,u} \equiv \ln y_{t,u}$ and $u_{t,u}$ is a zero-mean random disturbance.

Now, let us consider the first order Taylor series expansion of $\ln y_{t,u}$ around the period-level average of the variable to be estimated, $\bar{y}_t = \frac{1}{s}\sum_{u=1}^{s} y_{t,u} = \frac{y_t}{s}$:

$$\ln y_{t,u} = z_{t,u} \quad \ln \bar{y}_t + \frac{1}{\bar{y}_t}\left(y_{t,u} - \bar{y}_t\right) = \ln y_t - \ln s + \frac{s y_{t,u}}{y_t} - 1. \tag{4.8}$$

Summing up (4.8) over the high-frequency periods we have

$$\sum_{u=1}^{s} z_{t,u} \quad s \ln y_t - s \ln s + \frac{s \sum_{u=1}^{s} y_{t,u}}{y_t} - s = s \ln y_t - s \ln s.$$

So, discarding the approximation and summing up (4.7) over *u*, we have the observable aggregated model

$$z_t = \mathbf{x}_t^{'}\boldsymbol{\beta} + u_t, \qquad t = 1,\dots,T, \tag{4.9}$$

where $z_t = s \ln y_t - s \ln s$, $\mathbf{x}_t = \sum_{u=1}^{s} \mathbf{x}_{t,u}$ and $u_t = \sum_{u=1}^{s} u_{t,u}$.

Depending on the assumptions made on the disturbance term in model (4.7), the classic Chow and Lin's disaggregation approach, according to the original AR(1) formulation or in one of its well-known variants (Fernàndez, 1981, Litterman, 1983, see section 3.1), can thus be applied using model (4.9)[7].

[7] In the next section we shall see that producing log-transformed HF data following the procedure by Fernàndez (1981) has a straightforward economic interpretation in terms of modelling the rate of change of the variable.

Let us re-write (4.7) in matrix form as

$$\mathbf{z}_h = \mathbf{X}_h \boldsymbol{\beta} + \mathbf{u}_h ,$$

with $E\left(\mathbf{u}_h \mid \mathbf{X}_h\right) = \mathbf{0}$ and $E\left(\mathbf{u}_h \mathbf{u}_h' \mid \mathbf{X}_h\right) = \mathbf{V}_h$, assumed known. The BLU solution to the problem of obtaining estimates of $\mathbf{z}_h$ coherent with the aggregate $\mathbf{z}_l$ is given by (see section 3):

$$\hat{\mathbf{z}}_h = \mathbf{X}_h \hat{\boldsymbol{\beta}} + \mathbf{V}_h \mathbf{C}' \mathbf{V}_l^{-1} \left(\mathbf{z}_l - \mathbf{X}_l \hat{\boldsymbol{\beta}}\right),$$
$$\hat{\boldsymbol{\beta}} = \left(\mathbf{X}_l' \mathbf{V}_l^{-1} \mathbf{X}_l\right)^{-1} \mathbf{X}_l' \mathbf{V}_l^{-1} \mathbf{z}_l ,$$

where $\mathbf{V}_l = \mathbf{C}\mathbf{V}_h\mathbf{C}'$.

As a consequence, we estimate disaggregated values $\hat{z}_{t,u}$ such that

$$\sum_{u=1}^{s} \hat{z}_{t,u} = z_t = s \ln y_t - s \ln s , \qquad t = 1,\ldots,T .$$

A natural estimate for $y_{t,u}$ is thus given by

$$\hat{y}_{t,u} = \exp\left(\hat{z}_{t,u}\right), \qquad t = 1,\ldots,T; \quad u = 1,\ldots,s . \tag{4.10}$$

Due to the approximation, the estimated values in (4.10) will generally violate the aggregation constraints:

$$\sum_{u=1}^{s} \hat{y}_{t,u} \neq y_t \quad \Leftrightarrow \quad y_t - \sum_{u=1}^{s} \hat{y}_{t,u} = r_t \neq 0 , \quad t = 1,\ldots,T .$$

As suggested by Proietti (1999), the simplest solution is to adopt the Denton (1971) algorithm so as to distribute the residuals across the HF periods. The vector stacking the final estimates, $\tilde{\mathbf{y}}$, is obtained adding to the preliminary estimates in $\hat{\mathbf{y}}$ a term resulting from the distribution of the LF residuals in the vector **r**:

$$\tilde{\mathbf{y}} = \hat{\mathbf{y}} + \left(\mathbf{D}'\mathbf{D}\right)^{-1} \mathbf{C}' \left[\mathbf{C}\left(\mathbf{D}'\mathbf{D}\right)^{-1} \mathbf{C}'\right]^{-1} \mathbf{r} , \tag{4.11}$$

where $\mathbf{C} = \mathbf{I}_T \otimes \mathbf{i}_s'$ is the aggregation matrix for flows, $\mathbf{i}_s = [1,1,\ldots,1]'$ is a ($s \times 1$) vector and $\mathbf{D}$ is the $n \times n$ matrix in (4.3).

When the LF data are the temporal averages of the HF ones (index variable), we have

$$y_t = \frac{1}{s} \sum_{u=1}^{S} y_{t,u} , \qquad t = 1,\ldots,T .$$

After some algebra[8], we find that

$$\sum_{u=1}^{s} \ln y_{t,u} \quad \ln y_t \,. \tag{4.12}$$

For an index variable the observable aggregated variable z_t to be used in model (4.9) is thus $z_t = \ln y_t$, the other results remaining unchanged, apart in Denton's formula (4.11) the definition of the aggregation matrix **C**, which is $\mathbf{C} = \mathbf{I}_T \otimes \mathbf{i}_s^{'} / s$ for the problem in hand.

Finally, when the temporal aggregate takes the form of end-of-stock period,

$$y_t = y_{t,s}\,, \qquad\qquad t = 1,\ldots,T\,,$$

the dependent variable in model (4.9) is simply $z_t = \ln y_t$. Similar conclusion holds also for beginning-of-period stock variables. As it's obvious, in both cases no further adjustment to the estimates $\hat{y}_{t,u} = \exp\left(\hat{z}_{t,u}\right)$ is needed. A synthesis of the results presented so far is shown in tab. 4.2.

Tab. 4.2: Approximation and adjustment needed to deal with a logarithmic transformation of the variable to be disaggregated*

Type of		*Temporal aggregates*		*Adjustment needed for*
variable	*aggregation*	y_t	z_t	$\hat{y}_{t,u} = \exp\left(\hat{z}_{t,u}\right)$
Flow	Sum	$\sum_{u=1}^{s} y_{t,u}$	$s \ln y_t - s \ln s$	Yes
Index	Average	$\frac{1}{s}\sum_{u=1}^{s} y_{t,u}$	$\ln y_t$	Yes
Stock (eop)	Systematic sampling (eop)	$y_{t,s}$	$\ln y_t$	No
Stock (bop)	Systematic sampling (bop)	$y_{t,1}$	$\ln y_t$	No

* eop and bop stand for 'end of period' and 'beginning of period', respectively

To conclude on this issue, when dealing with a log-transformed variable in a static regression model, the researcher can use one of the commonly available (for example, in Ecotrim) standard disaggregation procedures, with the only caution to give the procedure as input the transformed aggregated variable z_t, whose form will depend on the type of variable/aggregation in hand. The estimated values $\hat{z}_{t,u}$ have then to be transformed as $\hat{y}_{t,u} = \exp\left(\hat{z}_{t,u}\right)$: for stock variables the procedure so far produces estimates coherent with the original aggregated series. For both flow and index variables the low-frequency residuals r_t must be evaluated and, if the discrepancies are relatively large, adjusted estimates according to Denton's procedure should be calculated.

---

[8] The first order Taylor expansion is now around y_t: $\ln y_{t,u} = z_{t,u} \quad \ln y_t + \frac{1}{y_t}\left(y_{t,u} - y_t\right)$. Summing up over *u* we find expression (4.12).

4.3. The *deltalog* model

Many economic models are expressed in terms of rate of change of the variable of interest, i.e. logarithmic difference,

$$\Delta \ln y_{t,u} = \ln \frac{y_{t,u}}{y_{t,u-1}} = \Delta z_{t,u},$$

where $z_{t,u} = \ln y_{t,u}$ and, with obvious notation, $y_{t,0} = y_{t-1,s}$. In this case the HF regression model in first differences (4.1) becomes[9]

$$\Delta z_{t,u} = \Delta \mathbf{x}_{t,u}^{'} \boldsymbol{\beta} + \varepsilon_{t,u}. \tag{4.13}$$

Using the results of the previous sections, estimates of $\mathbf{y}_h$ coherent with $\mathbf{y}_l$ and obtained in the framework of the *deltalog* model (4.13) are simply obtained using the procedure by Fernàndez (1981, see section 4.1)[10]:

$$\hat{\mathbf{y}}_h = \exp\left(\hat{\mathbf{z}}_h\right),$$

where

$$\hat{\mathbf{z}}_h = \mathbf{X}_h \hat{\boldsymbol{\beta}} + \left(\mathbf{D'D}\right)^{-1} \mathbf{C'} \left[\mathbf{C}\left(\mathbf{D'D}\right)^{-1} \mathbf{C'}\right]^{-1} \left(\mathbf{z}_l - \mathbf{X}_l \hat{\boldsymbol{\beta}}\right),$$

$$\hat{\boldsymbol{\beta}} = \left\{\mathbf{X}_l^{'} \left[\mathbf{C}\left(\mathbf{D'D}\right)^{-1} \mathbf{C'}\right]^{-1} \mathbf{X}_l\right\}^{-1} \mathbf{X}_l^{'} \left[\mathbf{C}\left(\mathbf{D'D}\right)^{-1} \mathbf{C'}\right]^{-1} \mathbf{z}_l .$$

Now, in order to get an interesting interpretation of the way this disaggregation procedure works, let us consider the case of a temporally aggregated flow variable. Pre-multiplying (4.13) by the polynomial $\left(1 + L + \ldots + L^{s-1}\right)$ and using the approximation relationship $\sum_{u=1}^{s} \ln y_{t,u} \quad z_t = s \ln y_t - s \ln s$, we find

$$\sum_{u=1}^{s} \ln \frac{y_{t,u}}{y_{t-1,u}} \quad \Delta z_t = s \Delta \ln y_t, \tag{4.14}$$

where, in this case, $\Delta = 1 - \mathrm{B}$, B being the lag operator operating at low-frequency. In other words, the (logarithmic) growth rate of the temporally aggregated variable, $\Delta \ln y_t = \ln \frac{y_t}{y_{t-1}}$, can be viewed as an approximation of a *s*-period average of past and current growth rates of $y_{t,u}$ (see Aadland, 2000).

---

[9] In model (4.13) the related series are not log-transformed in order to safe notation. The results remain of course valid in this case too.

[10] For either flow or index variable, a further adjustment according to Denton's formula (4.11) should be performed to exactly fulfil the temporal aggregation constraints.

As a matter of fact, $\ln y_{t,u}$ contains a unit root (at the higher frequency) and, after application of the transformation polynomials, $\ln y_t$ displays a unit root at the lower frequency too. In fact, by temporally aggregating equation (4.13), and taking into account relationship (4.14), we obtain a model expressed in terms of the low-frequency rate of change:

$$\sum_{u=1}^{s} \Delta \ln y_{t,u} = \mathbf{x}_t^{'}\boldsymbol{\beta} + \varepsilon_t \qquad \Leftrightarrow \qquad s\Delta \ln y_t = \mathbf{x}_t^{'}\boldsymbol{\beta} + \varepsilon_t . \tag{4.15}$$

Similar results hold also when the other types of aggregation (index and stock variables) are considered. For, in these cases the observable, aggregated model is simply

$$\Delta \ln y_t = \mathbf{x}_t^{'}\boldsymbol{\beta} + \varepsilon_t .$$

It's thus possible to say that, by using the *deltalog* model, the variable of interest is disaggregated in such a way that its estimated HF rates of change are coherent (approximately for flow and index variables) with the LF counterparts.

4.4. An example: the estimation of monthly Italian Value Added

To get more insights about the effect of working with log-transformations in a disaggregation framework, and only with illustrative purposes[11], monthly Italian industrial value added (1970:01-2001:09) has been estimated by temporal disaggregating the available quarterly seasonally adjusted data (y_t , source: Istat) using the monthly seasonally adjusted series of Italian industrial production index ($x_{t,u}$, source: Bank of Italy) and according to the *deltalog* model

$$\Delta \ln y_{t,u} = \beta_0 + \beta_1 \Delta \ln x_{t,u} + \varepsilon_{t,u} , \quad t\text{=1970q1,…,2001q3}, \quad u\text{=1,2,3,} \tag{4.16}$$

where $y_{t,0} = y_{t-1,3}$ and $\varepsilon_{t,u}$ is a white noise.

Estimated parameters[12] are all significant (see tab. 4.3), while the determination coefficient of the auxiliary quarterly GLS regression (0.354) is rather low.[13]

Tab. 4.3: Estimates of parameters of the auxiliary quarterly model

Coefficient	Estimate	Std. error	*t*-statistic	*p*-value
β_0	6.586	0.395	16.664	0.000
β_1	0.807	0.097	8.280	0.000

[11] A preliminary study of the dynamic properties of the available series is of course a necessary pre-requisite for the application of any model.

[12] The estimates have been obtained using the Gauss routine `DYNCHOW` developed on the behalf of Eurostat.

[13] On the meaning of the determination coefficient in models in differences, and on comparing regression models in levels and first differences, see Harvey (1980) and the discussion in Maddala (2001, pp. 230-234). It should be noted that in this case the auxiliary quarterly regression model has not spherical disturbances, for it is estimated by GLS, making the comparison even more difficult.

Fig. 4.1 shows the percentage discrepancies between the original, quarterly Italian data, y_t, and the preliminary quarterly sum of the monthly estimates, $\hat{y}_t = \sum_{u=1}^{3} \hat{y}_{t,u}$, obtained according to the *deltalog* model.

It's worth noting that the discrepancies are all negative (the quarterly sums of the estimated monthly values overestimate the 'true' figures) but they are with no doubt negligible, as they are practically always less than 0.04 percentage points.

To fulfil the temporal aggregation constraints, adjusted estimates according to Denton's first order procedure have been calculated. As figg. 4.2 and 4.3 clearly show, the amount of 'correction' to the preliminary estimates is very small and by no way changes the dynamic profile of the estimated series.

Fig. 4.1: Quarterly percentage discrepancies of the aggregated estimated Italian monthly value added: $r_t = \frac{y_t - \hat{y}_t}{y_t} \times 100$, $t = 1970q1 - 2001q3$

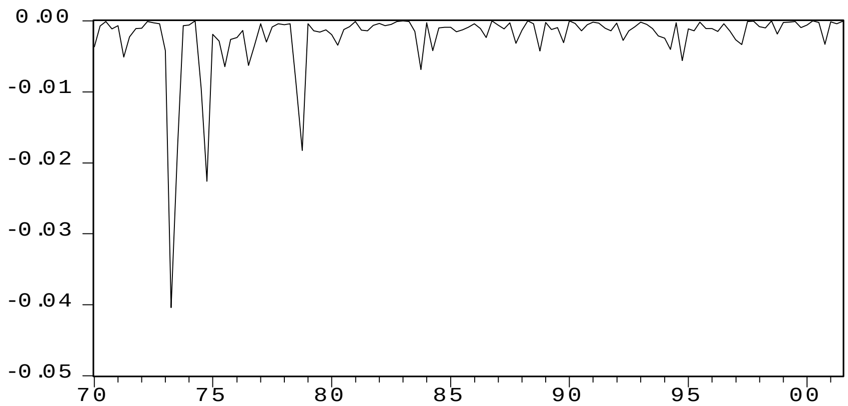


Fig. 4.2: Monthly percentage discrepancies of the estimated Italian monthly value added:
$r_{t,u} = \frac{\hat{y}_{t,u} - \tilde{y}_{t,u}}{\hat{y}_{t,u}} \times 100$, $t = 1970q1 - 2001q3$, $u = 1,2,3$

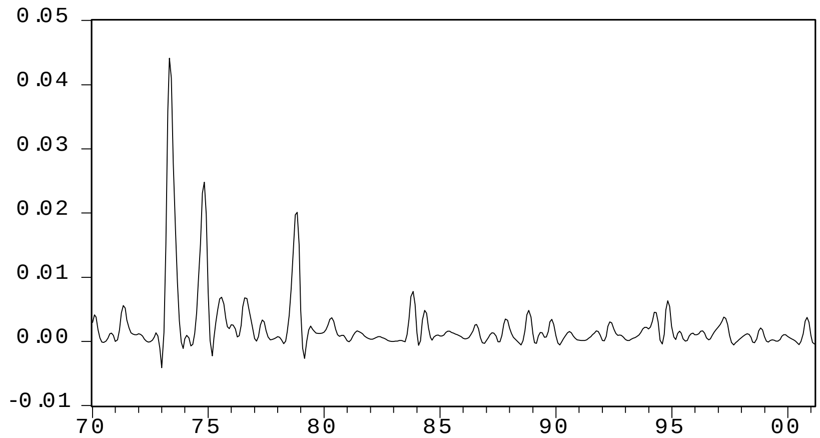
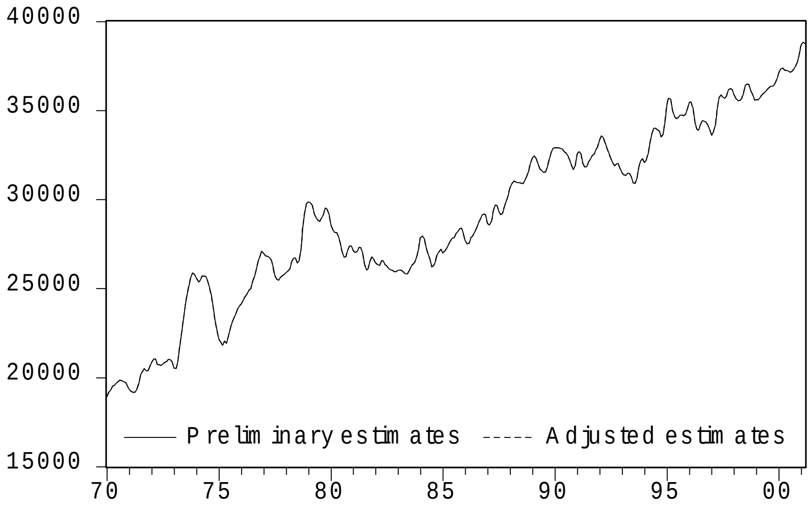


Fig. 4.3: Preliminary and adjusted estimates of the Italian monthly value added



5. Towards a dynamic framework for disaggregation

One of the major drawback of the Chow and Lin's approach to temporal disaggregation of an economic time series is that the HF model on which the method is based is static, giving rise to some doubt about its capability to model more sophisticated dynamic, as those usually encountered in applied econometrics work. Strictly linked to this issue is the observation that a successful implementation of the HF regression model (3.4) requires $u_{t,u}$ to be generated by a stationary process. This means that (3.4) must form a cointegrating regression if $y_{t,u}$ and the related series are integrated series.

On the other side, the classical extensions of the Chow and Lin's approach available in literature (Fernández, 1981, Litterman, 1983) 'expand' the original AR(1) disturbance structure to simple integrated models (random walk and random walk-Markov model for Fernández, 1981, and Litterman, 1983, respectively), which implicitly means that $y_{t,u}$ and the related series are not cointegrated and are thus to be modelled in differenced form.

In general, according to the standard Chow and Lin's approach, the dynamic path of the unobserved variable is derived only from the information given by the HF related series. In practice, a more reliable description of the system under study may be given by simply adding lagged values of the variable of interest to the basic high-frequency regression equation.

Gregoir (1995) and Salazar *et al.* (1997, 1998) considered a simple dynamic regression model as a basis to perform the temporal disaggregation of a variable of interest, deriving the estimates in a framework of constrained optimization of a quadratic loss function (see section 3). In both cases, however, the algorithms needed to calculate the estimates and their standard errors seem rather complicated and not straightforward to be implemented in a computer program.

In a recent paper Santos Silva and Cardoso (2001, hereafter SSC) developed an extension of the Chow and Lin's temporal disaggregation method based on the same linear dynamic model considered by Gregoir (1995) and Salazar *et al.* (1997, 1998). By means of a well-known transformation developed to deal with distributed lag model (Klein, 1958, Harvey, 1990), a closed form solution is derived according to the standard BLU approach of Chow and Lin (1971, 1976), along with a straightforward expression of the covariance matrix of the disaggregated series.

These results seem to encourage in using the method by SSC, for both reasonability (and simplicity) of the model, which is able to deal with simple dynamic structures as compared with the essentially static nature of the original Chow and Lin's approach, and for the simplicity of calculations needed to get the estimated high frequency values, which is particularly interesting for the subjects (i.e., statistical institutes) which are typically in charge of temporal disaggregation activities as part of their current work of producing high-frequency series from (and coherent with) the available source data.

We present both approaches to temporal disaggregation according to a linear dynamic regression model, to establish the practical equivalence from a modelling point of view and to point out that the only real difference between the two approaches is the way the first observation is treated by each of them.

However, it should be noted that Salazar *et al.* (1997, 1998) formulate the link between interpoland (the variable to be disaggregated) and related series in terms of a more general, possibly non linear, dynamic model, which encompasses the Gregoir's and SSC formulation (see appendix A). From a theoretical point of view this sounds well, but practically the estimation problems deriving from a number of lags of the interpoland variable greater than one (or, eventually, greater than the seasonal order when dealing with raw series) are emphasized by the fact that we have to move from an observable model which is aggregated, with consequent loss of information. As for non-linearity, in the previous section we discussed the way to conveniently transform the data in order to preserve the quadratic-linear approach even in presence of a logarithmic transformation, which is probably the most widely used non linear transformation. We have seen that the disaggregated estimates present only negligible discrepancies with the observed aggregated values.

For these reasons, according to Gregoir (1995), in what follows the attention is focused on the first order linear dynamic regression model:

$$(1-\phi L) y_{t,u} = \mathbf{x}'_{t,u}\boldsymbol{\beta} + \varepsilon_{t,u}, \quad t = 1,\dots,T \quad u = 1,\dots,s, \tag{5.1}$$

where $|\phi| < 1$, $\mathbf{x}_{t,u}$ is a $(K \times 1)$ vector of (weakly) exogenous variables, eventually comprehensive of lagged independent variables, $\boldsymbol{\beta}$ is a $(K \times 1)$ vector of parameters and $\varepsilon_{t,u} \quad WN(0, \sigma^2_\varepsilon)$. If needed, the model can be enlarged to take into account further lags of the dependent variable in a straightforward manner (Salazar *et al.*, 1998, see appendix A).

6. The solution of Salazar *et al.*

Salazar *et. al.* (1997, 1998) consider a slightly different representation of the underlying high frequency data as compared to model (5.1), which explicitly takes into account the presence of the lagged dependent variable:

$$(1-\phi L) y_{t,u} = \mathbf{x}'_{t,u}\boldsymbol{\beta} + \varepsilon_{t,u}, \quad \begin{cases} t = 1 & u = 2,\dots,s \\ t = 2,\dots,T & u = 1,2,\dots,s \end{cases}. \tag{6.1}$$

Let us pre-multiply model (6.1) by the polynomial $\left(1+\phi L+\ldots+\phi^{s-1}L^{s-1}\right)$. Given that $\left(1+\phi L+\ldots+\phi^{s-1}L^{s-1}\right)\left(1-\phi L\right)=\left(1-\phi^{s}L^{s}\right)$, it is

$$\left(1-\phi^{s}L^{s}\right)y_{t,u}=\left(1+\phi L+\ldots+\phi^{s-1}L^{s-1}\right)\mathbf{x}_{t,u}^{'}\boldsymbol{\beta}+\left(1+\phi L+\ldots+\phi^{s-1}L^{s-1}\right)\varepsilon_{t,u}. \tag{6.2}$$

Pre-multiplying (6.2) by the polynomial $\left(c_1+c_2L+\ldots+c_{s-1}L^{s-1}\right)$, where the weights $\left\{c_u\right\}_{u=1}^{s}$ have been previously defined, we obtain the observable aggregated model, that is:

$$\left(1-\phi^{s}L^{s}\right)y_{t}=\left(1+\phi L+\ldots+\phi^{s-1}L^{s-1}\right)\mathbf{x}_{t}^{'}\boldsymbol{\beta}+\left(1+\phi L+\ldots+\phi^{s-1}L^{s-1}\right)\varepsilon_{t},\quad t=2,\ldots,T, \tag{6.3}$$

where $\mathbf{x}_t=\sum_{u=1}^{s}c_u\mathbf{x}_{t,u}$ and $\varepsilon_t=\sum_{u=1}^{s}c_u\varepsilon_{t,u}$, t=2,…,T.

6.1. The effect of temporal aggregation on a simple first order dynamic model

To make the effects of temporal aggregation on model (6.1) clear, without loss of generality let us consider the simple dynamic regression model

$$\left(1-\phi L\right)y_{t,u}=\alpha+\beta x_{t,u}+\varepsilon_{t,u},\qquad \begin{array}{ll} t=1 & u=2,\ldots,s \\ t=2,\ldots,T & u=1,\ldots,s \end{array}.$$

If Y is a flow variable and s=3, the LF observable quarterly aggregated model can be written as

$$\left(1-\phi^{3}B\right)y_{t}=3\alpha\left(1+\phi+\phi^{2}\right)+\beta x_{t}+\beta\phi z_{1,t}+\beta\phi^{2}z_{2,t}+v_{t},\quad t=2,\ldots,T, \tag{6.4}$$

where B is the LF lag operator such that $By_t=y_{t-1}$ and

$$z_{1,t}=x_{t-1,3}+x_{t,1}+x_{t,2},$$

$$z_{2,t}=x_{t-1,2}+x_{t-1,3}+x_{t,1},$$

$$v_t=\left(1+\phi L+\phi^{2}L^{2}\right)\left(1+L+L^{2}\right)\varepsilon_{t,3}.$$

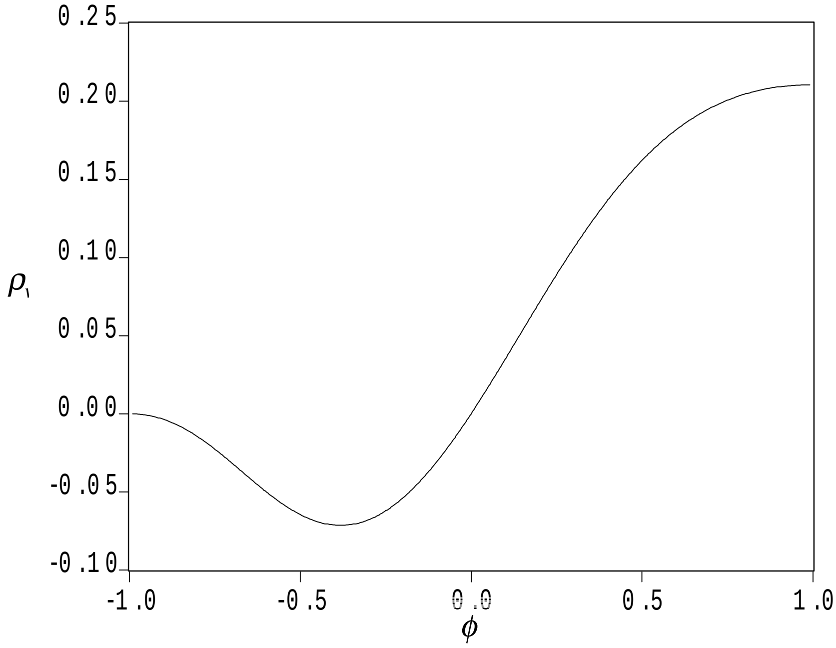
The LF stochastic disturbance follows a zero-mean MA(1) process (Abeysinghe, 2000) with

$$Var\left(v_t\right)=\sigma_v^2=\sigma_\varepsilon^2\left(3+4\phi+5\phi^2+4\phi^3+3\phi^4\right),$$

$$Corr\left(v_t,v_{t-1}\right)=\rho_{v,1}=\frac{\phi\left(1+\phi\right)^2}{3+4\phi+5\phi^2+4\phi^3+3\phi^4}.$$

The graph of $\rho_{v,1}$ is represented in fig. 6.1, which shows that, for $\phi>0$, the autocorrelation ranges from 0 to about 0.21.

Fig. 6.1: First order autocorrelation of the stochastic disturbance in model (6.4)



When s=4 (i.e, annual aggregation of a quarterly flows variable), the LF observable annual aggregated model can be written as

$$\left(1-\phi^4 B\right) y_t = 4\alpha\left(1+\phi+\phi^2+\phi^3\right)+\beta x_t+\beta\phi z_{1,t}+\beta\phi^2 z_{2,t}+\beta\phi^3 z_{3,t}+v_t, \qquad t=2,\ldots,T, \tag{6.5}$$

where

$$z_{1,t} = x_{t-1,4}+x_{t,1}+x_{t,2}+x_{t,3},$$

$$z_{2,t} = x_{t-1,3}+x_{t-1,4}+x_{t,1}+x_{t,2},$$

$$z_{3,t} = x_{t-1,2}+x_{t-1,3}+x_{t-1,4}+x_{t,1},$$

$$v_t = \left(1+\phi L+\phi^2 L^2+\phi^3 L^3\right)\left(1+L+L^2+L^3\right)\varepsilon_{t,4}.$$

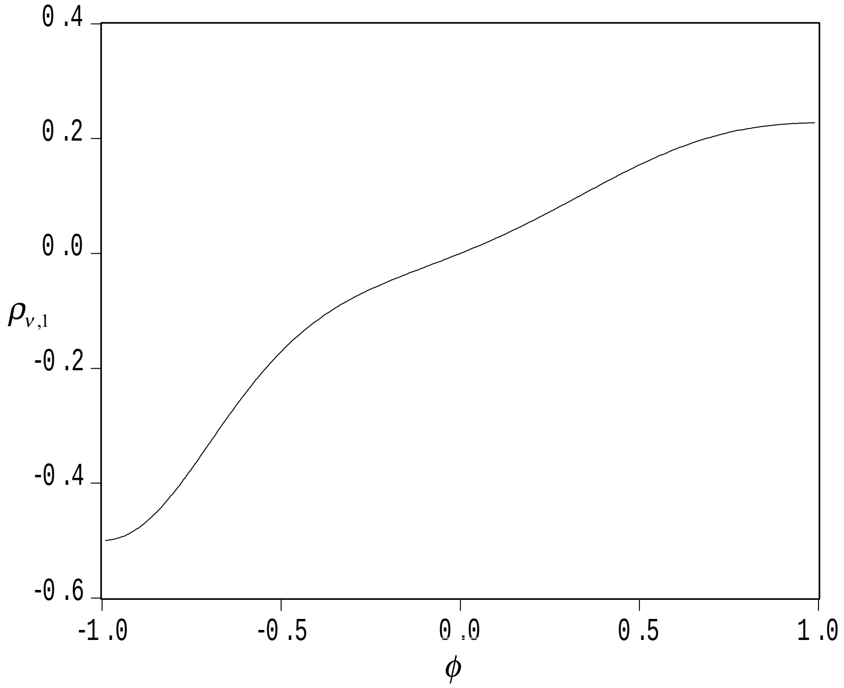
The LF stochastic disturbance follows a zero-mean MA(1) process with

$$\sigma_v^2 = \sigma_\varepsilon^2\left(4+6\phi+8\phi^2+8\phi^3+8\phi^4+6\phi^5+4\phi^6\right),$$

$$\rho_{v,1} = \frac{\phi\left(1+2\phi+4\phi^2+2\phi^3+\phi^4\right)}{4+6\phi+8\phi^2+8\phi^3+8\phi^4+6\phi^5+4\phi^6}.$$

The graph of $\rho_{v,1}$ is represented in fig. 6.2, which shows that, overall, the autocorrelation ranges from -0.5 to about 0.23.

Fig. 6.2: First order autocorrelation of the stochastic disturbance in model (6.5)



Finally, if a stock variable is involved, the quarterly and annual observable aggregated models are, respectively[14],

Quarterly model

$$\left(1-\phi^3 B\right) y_t = \alpha\left(1+\phi+\phi^2\right)+\beta x_{t,3}+\beta\phi x_{t,2}+\beta\phi^2 x_{t,1}+v_t\,, \qquad t=1,\dots,T$$

$$v_t = \left(1+\phi L+\phi^2 L^2\right)\varepsilon_{t,3}$$

Annual model

$$\left(1-\phi^4 B\right) y_t = \alpha\left(1+\phi+\phi^2+\phi^3\right)+\beta x_{t,4}+\beta\phi x_{t,3}+\beta\phi^2 x_{t,2}+\beta\phi^3 x_{t,1}+v_t\,, \qquad t=1,\dots,T$$

$$v_t = \left(1+\phi L+\phi^2 L^2+\phi^3 L^3\right)\varepsilon_{t,4}\,.$$

It should be noted that in both cases v_t follows a white noise with $\sigma_v^2 = \sigma_\varepsilon^2\left(1+\sum_{j=1}^{s-1}\phi^j\right)^2$.

6.2. GLS estimation of the observable aggregated model

In order to get closed form expressions, suitable to be implemented in a computer program, it is convenient to express model (6.3) in matrix form. Let us consider the $\left((n-1)\times n\right)$ matrix $\mathbf{D}_\phi^*$ such that

[14] We consider a stock variable observed at the end of the HF period.

$$\mathbf{D}_{\phi}^{*} = \begin{bmatrix} -\phi & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\phi & 1 & \cdots & 0 & 0 \\ 0 & 0 & -\phi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi & 1 \end{bmatrix}. \tag{6.6}$$

Model (6.1) can thus be written as

$$\mathbf{D}_{\phi}^{*}\mathbf{y}_h = \mathbf{X}_h^{*}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_h^{*}, \tag{6.7}$$

where

$$\mathbf{X}_h^{*} = \{\mathbf{x}_{t,u}\}, \begin{cases} t=1 & u=2,\ldots,s \\ t=2,\ldots,T & u=1,2,\ldots,s \end{cases}$$

is the $((n-1)\times K)$ matrix of HF related series bereaved of $\mathbf{x}_{1,1}$, and $\boldsymbol{\varepsilon}_h^{*}$ is a zero mean stochastic vector with $E\left(\boldsymbol{\varepsilon}_h^{*}\boldsymbol{\varepsilon}_h^{*\prime}\right) = \sigma_{\varepsilon}^{2}\mathbf{I}_{n-1}$.

Now, let $\mathbf{K}_{\phi}$ be the $((n-s)\times(n-1))$ matrix given by

$$\mathbf{K}_{\phi} = \begin{bmatrix} \phi^{s-1} & \phi^{s-2} & \phi^{s-3} & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & \phi^{s-1} & \phi^{s-2} & \cdots & \phi & 1 & \cdots & 0 & 0 \\ 0 & 0 & \phi^{s-1} & \cdots & \phi^{2} & \phi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \phi & 1 \end{bmatrix} \tag{6.8}$$

and partition the aggregation matrix $\mathbf{C}$ previously defined as follows:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0}' \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix}, \tag{6.9}$$

where $\mathbf{C}_1 = \mathbf{c}'$ and $\mathbf{C}_2 = \mathbf{I}_{T-1} \otimes \mathbf{c}'$. Notice that $\mathbf{C}_2\mathbf{K}_{\phi}\mathbf{D}_{\phi}^{*} = \mathbf{D}_{\phi^s}^{*}\mathbf{C}$, where

$$\mathbf{D}_{\phi^s}^{*} = \begin{bmatrix} -\phi^{s} & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\phi^{s} & 1 & \cdots & 0 & 0 \\ 0 & 0 & -\phi^{s} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi^{s} & 1 \end{bmatrix}$$

is a $((T-1)\times T)$ matrix. Thus, pre-multiplying model (6.7) by $\mathbf{C}_2\mathbf{K}_\phi$ gives the observable aggregated model (6.3) expressed in matrix form:

$$\mathbf{D}^*_{\phi^s}\mathbf{y}_l = \mathbf{C}_2\mathbf{K}_\phi\mathbf{X}^*_h\boldsymbol{\beta} + \mathbf{C}_2\mathbf{K}_\phi\boldsymbol{\varepsilon}^*_h,$$

that is

$$\mathbf{y}^*_{l,\phi} = \mathbf{X}^*_{l,\phi}\boldsymbol{\beta} + \boldsymbol{\varepsilon}^*_{l,\phi}, \qquad (6.10)$$

where $\mathbf{y}^*_{l,\phi} = \mathbf{D}^*_{\phi^s}\mathbf{y}_l$, $\mathbf{X}^*_{l,\phi} = \mathbf{C}_2\mathbf{K}_\phi\mathbf{X}^*_h$ and $\boldsymbol{\varepsilon}^*_{l,\phi} = \mathbf{C}_2\mathbf{K}_\phi\boldsymbol{\varepsilon}^*_h$, with $E\left(\boldsymbol{\varepsilon}^*_{l,\phi}\boldsymbol{\varepsilon}^{*'}_{l,\phi}\right) = \sigma^2_\varepsilon\left(\mathbf{C}_2\mathbf{K}_\phi\mathbf{K}'_\phi\mathbf{C}'_2\right) = \mathbf{V}^*_{l,\phi}$.

For any fixed ϕ, the BLU estimator of $\boldsymbol{\beta}$ in model (6.10) is obtained through Generalized Least Squares estimation:

$$\hat{\boldsymbol{\beta}}_\phi = \left[\mathbf{X}^{*'}_{l,\phi}\left(\mathbf{V}^*_{l,\phi}\right)^{-1}\mathbf{X}^{*'}_{l,\phi}\right]^{-1}\mathbf{X}^{*'}_{l,\phi}\left(\mathbf{V}^*_{l,\phi}\right)^{-1}\mathbf{y}^*_{l,\phi}.$$

The parameter ϕ being generally unknown, $\boldsymbol{\beta}$ and ϕ can be estimated either by minimization of the weighted sum of squared residual,

$$\left(\mathbf{y}^*_{l,\phi} - \mathbf{X}^*_{l,\phi}\boldsymbol{\beta}\right)'\left(\mathbf{V}^*_{l,\phi}\right)^{-1}\left(\mathbf{y}^*_{l,\phi} - \mathbf{X}^*_{l,\phi}\boldsymbol{\beta}\right),$$

or, assuming the gaussianity of $\varepsilon_{t,u}$, by maximizing the log-likelihood function

$$\ell^*(\phi,\boldsymbol{\beta}) = -\frac{T-1}{2}\ln 2\pi - \frac{1}{2}\ln\left|\mathbf{V}^*_{l,\phi}\right| - \frac{1}{2}\left(\mathbf{y}^*_{l,\phi} - \mathbf{X}^*_{l,\phi}\boldsymbol{\beta}\right)'\left(\mathbf{V}^*_{l,\phi}\right)^{-1}\left(\mathbf{y}^*_{l,\phi} - \mathbf{X}^*_{l,\phi}\boldsymbol{\beta}\right). \qquad (6.11)$$

In both cases, given an estimate $\hat{\phi}$, the estimated regression coefficients are calculated as

$$\hat{\boldsymbol{\beta}}_{\hat{\phi}} = \left[\mathbf{X}^{*'}_{l,\hat{\phi}}\left(\mathbf{V}^*_{l,\hat{\phi}}\right)^{-1}\mathbf{X}^*_{l,\hat{\phi}}\right]^{-1}\mathbf{X}^{*'}_{l,\hat{\phi}}\left(\mathbf{V}^*_{l,\hat{\phi}}\right)^{-1}\mathbf{y}^*_{l,\hat{\phi}}. \qquad (6.12)$$

6.3. Estimates of $\mathbf{y}_h$ as solution of a constrained optimization problem

The estimates of $\boldsymbol{\beta}$ and ϕ obtained so far can be used to get a 'preliminary' estimate of $\mathbf{D}^*_\phi\mathbf{y}_h$, say $\mathbf{z}^*_h = \mathbf{X}^*_h\hat{\boldsymbol{\beta}}$**,** to be used in the following constrained optimization problem:

$$\min_{\mathbf{y}_h}\left(\mathbf{D}^*_\phi\mathbf{y}_h - \mathbf{z}^*_h\right)'\left(\mathbf{D}^*_\phi\mathbf{y}_h - \mathbf{z}^*_h\right) \qquad \text{subject to } \mathbf{y}_l = \mathbf{C}\mathbf{y}_h. \qquad (6.13)$$

In order to conveniently deal with the first sub-period value, let us re-write the loss function to be minimized in (6.13) as follows:

$$\left(\mathbf{A}_1\mathbf{y}^1_h + \mathbf{A}_2\mathbf{y}^2_h - \mathbf{z}^*_h\right)'\left(\mathbf{A}_1\mathbf{y}^1_h + \mathbf{A}_2\mathbf{y}^2_h - \mathbf{z}^*_h\right),$$

where the HF vector $\mathbf{y}_h$ has been partitioned as $\mathbf{y}_h = \begin{bmatrix} \mathbf{y}_h^1 \\ \mathbf{y}_h^2 \end{bmatrix}$, $\mathbf{y}_h^1$ and $\mathbf{y}_h^2$ being $(s \times 1)$ and $((n-s)\times 1)$, respectively, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are $((n-1)\times s)$ and $((n-1)\times(n-s))$ matrices, respectively, such that $\mathbf{D}_\phi^* = [\mathbf{A}_1 \vdots \mathbf{A}_2]$:

$$\mathbf{A}_1 = \begin{bmatrix} -\phi & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\phi & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\phi \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ -\phi & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi & 1 \end{bmatrix}.$$

Let us consider the lagrangean function

$$L\left(\mathbf{y}_h^1, \mathbf{y}_h^2, \boldsymbol{\lambda}\right) = \left(\mathbf{A}_1\mathbf{y}_h^1 + \mathbf{A}_2\mathbf{y}_h^2 - \mathbf{z}_h^*\right)'\left(\mathbf{A}_1\mathbf{y}_h^1 + \mathbf{A}_2\mathbf{y}_h^2 - \mathbf{z}_h^*\right) - 2\boldsymbol{\lambda}'\left(\mathbf{C}\mathbf{y}_h - \mathbf{y}_l\right),$$

that is, equivalently,

$$L\left(\mathbf{y}_h^1, \mathbf{y}_h^2, \lambda_1, \boldsymbol{\lambda}_2\right) = \left(\mathbf{A}_1\mathbf{y}_h^1 + \mathbf{A}_2\mathbf{y}_h^2 - \mathbf{z}_h^*\right)'\left(\mathbf{A}_1\mathbf{y}_h^1 + \mathbf{A}_2\mathbf{y}_h^2 - \mathbf{z}_h^*\right) - 2\lambda_1\left(\mathbf{C}_1\mathbf{y}_h^1 - y_l^1\right) - 2\boldsymbol{\lambda}_2'\left(\mathbf{C}_2\mathbf{y}_h^2 - \mathbf{y}_l^2\right) \qquad (6.14)$$

where the aggregation matrix $\mathbf{C}$ has been partitioned as in (6.7) and we partitioned the aggregated vector as $\mathbf{y}_l = \begin{bmatrix} y_l^1 \\ \mathbf{y}_l^2 \end{bmatrix}$, y_l^1 being the (scalar) LF values of the first period, whereas $\mathbf{y}_l^2$ has dimension $((T-1)\times 1)$, and the Lagrange multiplier vector as $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \boldsymbol{\lambda}_2 \end{bmatrix}$.

An estimate of $\mathbf{y}_h$ coherent with the temporal aggregate $\mathbf{y}_l$ can be obtained by minimizing the lagrangean function (6.14). Given the first order condition,

$$\begin{cases} \dfrac{\partial L}{\partial \mathbf{y}_h^1} & = & \mathbf{0} \\ \dfrac{\partial L}{\partial \mathbf{y}_h^2} & = & \mathbf{0} \\ \dfrac{\partial L}{\partial \lambda_1} & = & 0 \\ \dfrac{\partial L}{\partial \boldsymbol{\lambda}_2} & = & \mathbf{0} \end{cases}$$

the solution for $\mathbf{y}_h$ is given by (see the appendix B)

$$\begin{bmatrix} \hat{\mathbf{y}}_h^1 \\ \hat{\mathbf{y}}_h^2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{'}\mathbf{A}_1 & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{A}_2^{'}\mathbf{A}_2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_1\mathbf{z}_h^* + \mathbf{C}_1^{'}\left[\mathbf{C}_1\left(\mathbf{A}_1^{'}\mathbf{A}_1\right)^{-1}\mathbf{C}_1^{'}\right]^{-1}\left[y_l^1 - \mathbf{C}_1\left(\mathbf{A}_1^{'}\mathbf{A}_1\right)^{-1}\mathbf{A}_1^{'}\mathbf{z}_h^*\right] \\ \mathbf{A}_2\mathbf{z}_h^* + \mathbf{C}_2^{'}\left[\mathbf{C}_2\left(\mathbf{A}_2^{'}\mathbf{A}_2\right)^{-1}\mathbf{C}_2^{'}\right]^{-1}\left[\mathbf{y}_l^2 - \mathbf{C}_2\left(\mathbf{A}_2^{'}\mathbf{A}_2\right)^{-1}\mathbf{A}_2^{'}\mathbf{z}_h^*\right] \end{bmatrix}, \tag{6.15}$$

where

$$\begin{aligned} \mathbf{T}_{12} &= \mathbf{A}_1^{'}\mathbf{A}_2 - \mathbf{C}_1^{'}\left[\mathbf{C}_1\left(\mathbf{A}_1^{'}\mathbf{A}_1\right)^{-1}\mathbf{C}_1^{'}\right]^{-1}\mathbf{C}_1\left(\mathbf{A}_1^{'}\mathbf{A}_1\right)^{-1}\mathbf{A}_1^{'}\mathbf{A}_2, \\ \mathbf{T}_{21} &= \mathbf{A}_2^{'}\mathbf{A}_1 - \mathbf{C}_2^{'}\left[\mathbf{C}_2\left(\mathbf{A}_2^{'}\mathbf{A}_2\right)^{-1}\mathbf{C}_2^{'}\right]^{-1}\mathbf{C}_2\left(\mathbf{A}_2^{'}\mathbf{A}_2\right)^{-1}\mathbf{A}_2^{'}\mathbf{A}_1. \end{aligned} \tag{6.16}$$

7. The method of Santos Silva and Cardoso

For notation convenience, let $\tau = s(t-1)+u$ be the index running on the HF periods and re-write model (5.1) as follows:

$$y_\tau = \phi y_{\tau-1} + \mathbf{x}_\tau^{'}\boldsymbol{\beta} + \varepsilon_\tau, \qquad \tau = 1,\ldots,n. \tag{7.1}$$

Starting from an idea originally formulated by Tserkezos (1991, 1993), SSC suggest to recursively substitute (Klein, 1958) in (7.1), thus obtaining

$$y_\tau = \left(\sum_{i=0}^{\tau-1}\phi^i\mathbf{x}_{\tau-i}^{'}\right)\boldsymbol{\beta} + \phi^\tau y_0 + \left(\sum_{i=0}^{\tau-1}\phi^i\varepsilon_{\tau-i}\right). \tag{7.2}$$

Now, observe that

$$y_0 = \left(\sum_{i=0}^{+\infty}\phi^i\mathbf{x}_{-i}^{'}\right)\boldsymbol{\beta} + \left(\sum_{i=0}^{\infty}\phi^i\varepsilon_{-i}\right),$$

so that its expected value conditionally to the infinite past is

$$\eta = E\left(y_0 \mid \mathbf{x}_0, \mathbf{x}_{-1}, \ldots\right) = \left(\sum_{i=0}^{+\infty}\phi^i\mathbf{x}_{-i}^{'}\right)\boldsymbol{\beta}.$$

Then model (7.2) can be written as

$$y_\tau = \left(\sum_{i=0}^{\tau-1}\phi^i\mathbf{x}_{\tau-i}^{'}\right)\boldsymbol{\beta} + \phi^\tau\eta + \left(\sum_{i=0}^{\infty}\phi^i\varepsilon_{\tau-i}\right), \qquad \tau = 1,\ldots,n. \tag{7.3}$$

According to Harvey (1990), we treat the so-called *truncation remainder* η as a fixed (and unknown) parameter to be estimated. In other words, the HF relationship on which the disaggregation procedure will be based is:

$$y_\tau = \mathbf{x}_{\phi,\tau}^{'}\boldsymbol{\beta} + \phi^\tau \eta + u_\tau , \quad u_\tau = \phi u_{\tau-1} + \varepsilon_\tau , \quad \tau = 1,\dots,n , \quad u_0 = 0 , \tag{7.4}$$

where $\mathbf{x}_{\phi,\tau} = \left(\sum_{i=0}^{\tau-1} \phi^i \mathbf{x}_{\tau-i} \right)$ is a $(K \times 1)$ vector containing the weighted sum of current and past values of regressors and u_τ is an error term generated by an AR(1) stationary process.

Model (7.4) can be written in matrix form as

$$\mathbf{y}_h = \mathbf{X}_{h,\phi}\boldsymbol{\beta} + \mathbf{q}_\phi \eta + \mathbf{u}_h , \tag{7.5}$$

where $\mathbf{q}_\phi$ is a $(n \times 1)$ vector given by

$$\mathbf{q}_\phi = \begin{bmatrix} \phi \\ \phi^2 \\ \vdots \\ \phi^n \end{bmatrix}$$

and $\mathbf{D}_\phi \mathbf{u}_h = \boldsymbol{\varepsilon}_h$, $\mathbf{D}_\phi$ being the $(n \times n)$ matrix

$$\mathbf{D}_\phi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\phi & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\phi & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi & 1 \end{bmatrix}.$$

Notice that model (7.5) can be obtained by transforming the matrix counterpart of model (7.1), that is

$$\mathbf{D}_\phi \mathbf{y}_h = \mathbf{X}_h \boldsymbol{\beta} + \mathbf{q}\eta + \boldsymbol{\varepsilon}_h = \mathbf{Z}_h \boldsymbol{\gamma} + \boldsymbol{\varepsilon}_h , \tag{7.6}$$

where $\mathbf{q} = (\phi, 0, 0, \dots, 0)'$ is a $(n \times 1)$ vector whose unique not zero (and equal to ϕ) element is the first one, $\mathbf{Z} = [\mathbf{X} \vdots \mathbf{q}]$ is a $(n \times (K+1))$ matrix and $\boldsymbol{\gamma} = [\boldsymbol{\beta}' \vdots \eta]'$ is a $((K+1) \times 1)$ vector of parameters.

Given that the inverse of $\mathbf{D}_\phi$ is the lower triangular matrix

$$\mathbf{D}_\phi^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \phi & 1 & 0 & \cdots & 0 & 0 \\ \phi^2 & \phi & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & \cdots & \phi & 1 \end{bmatrix},$$

model (7.5) is obtained by pre-multiplication of (7.6) by $\mathbf{D}_{\phi}^{-1}$, that is:

$$\mathbf{y}_h = \mathbf{D}_{\phi}^{-1}\mathbf{Z}_h\boldsymbol{\gamma} + \mathbf{D}_{\phi}^{-1}\boldsymbol{\varepsilon}_h = \mathbf{Z}_{h,\phi}\boldsymbol{\gamma} + \mathbf{u}_h,$$

where $E\left(\mathbf{u}_h \mid \mathbf{Z}_{h,\phi}\right) = \mathbf{0}$ and $E\left(\mathbf{u}_h\mathbf{u}_h' \mid \mathbf{Z}_{h,\phi}\right) = \sigma_{\varepsilon}^2\mathbf{V}_h$, $\mathbf{V}_h$ being the Toeplitz matrix

$$\mathbf{V}_h = \left(\mathbf{D}_{\phi}'\mathbf{D}_{\phi}\right)^{-1} = \begin{bmatrix} 1 & \phi & \phi^2 & \cdots & \phi^{n-2} & \phi^{n-1} \\ \phi & 1 & \phi^2 & \cdots & \phi^{n-3} & \phi^{n-2} \\ \phi^2 & \phi & 1 & \cdots & \phi^{n-4} & \phi^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi^{n-1} & \phi^{n-2} & \phi^{n-3} & \cdots & \phi & 1 \end{bmatrix}.$$

The aggregated model is given by

$$\mathbf{y}_l = \mathbf{C}\mathbf{Z}_{h,\phi}\boldsymbol{\gamma} + \mathbf{C}\mathbf{u}_h. \tag{7.7}$$

SSC suggest to estimate ϕ by maximizing the log-likelihood function

$$\ell(\phi,\boldsymbol{\beta}) = -\frac{T}{2}\ln 2\pi - \frac{1}{2}\ln\left|\mathbf{V}_{l,\phi}\right| - \frac{1}{2}\left(\mathbf{y}_{l,\phi} - \mathbf{Z}_{l,\phi}\boldsymbol{\beta}\right)'\mathbf{V}_{l,\phi}^{-1}\left(\mathbf{y}_{l,\phi} - \mathbf{Z}_{l,\phi}\boldsymbol{\beta}\right), \tag{7.8}$$

where $\mathbf{Z}_{l,\phi} = \mathbf{C}\mathbf{Z}_{h,\phi}$ and $\mathbf{V}_{l,\phi} = \mathbf{C}\mathbf{V}_{h,\phi}\mathbf{C}'$, via a scanning procedure on ϕ variable into the stationarity region (-1,+1).

With respect to other temporal disaggregation procedures based on dynamic models (Salazar *et al.*, 1998, Gregoir, 1995), the method by SSC has the advantage of not having any trouble for the estimation of HF values for the first LF period and, most of all, of producing disaggregated estimates which fulfil the temporal aggregation constraints along with their standard errors in a straightforward way. In fact, the estimated values can be obtained according to the classical Chow and Lin's procedure:

$$\hat{\mathbf{y}}_h = \mathbf{Z}_{h,\hat{\phi}}\hat{\boldsymbol{\gamma}}_{\hat{\phi}} + \mathbf{V}_{h,\hat{\phi}}\mathbf{C}'\mathbf{V}_{l,\hat{\phi}}^{-1}\left[\mathbf{y}_l - \mathbf{Z}_{l,\hat{\phi}}\hat{\boldsymbol{\gamma}}_{\hat{\phi}}\right],$$

$$\hat{\boldsymbol{\gamma}}_{\hat{\phi}} = \left(\mathbf{Z}_{l,\hat{\phi}}'\mathbf{V}_{l,\hat{\phi}}^{-1}\mathbf{Z}_{l,\hat{\phi}}\right)^{-1}\mathbf{Z}_{l,\hat{\phi}}'\mathbf{V}_{l,\hat{\phi}}^{-1}\mathbf{y}_l,$$

while an estimate of the covariance matrix of $\hat{\mathbf{y}}_h$ is readily available as

$$E\left(\hat{\mathbf{y}}_h - \mathbf{y}\right)\left(\hat{\mathbf{y}}_h - \mathbf{y}\right)' = \left(\mathbf{I}_n - \mathbf{L}_{\hat{\phi}}\mathbf{C}\right)\mathbf{V}_{h,\hat{\phi}} + \left(\mathbf{X}_h - \mathbf{L}_{\hat{\phi}}\mathbf{X}_l\right)\left(\mathbf{X}_l'\mathbf{V}_{l,\hat{\phi}}^{-1}\mathbf{X}_l\right)^{-1}\left(\mathbf{X}_h - \mathbf{L}_{\hat{\phi}}\mathbf{X}_l\right)',$$

with $\mathbf{L}_{\hat{\phi}} = \mathbf{V}_{h,\hat{\phi}}\mathbf{C}'\mathbf{V}_{l,\hat{\phi}}^{-1}$.

8. Dynamic Chow and Lin extension at work: Two empirical applications

To illustrate the use of the dynamic extension of the classical Chow and Lin's approach described so far, in this section we consider two empirical applications. In the first one we replicate (and extend) the application of SSC, that is we disaggregate annual US personal consumption, obtained by summation of quarterly data from 1954q1 to 1983q4, using US quarterly personal disposable income (seasonally adjusted, 1954q1-1983q4) as related indicator. The data have been taken by Table 17.5 of Greene's (1997) textbook. In the second example we perform the estimation of monthly Italian industrial value added by temporal disaggregating the available quarterly series (1970q1-2001q3, seasonally adjusted, at constant prices, source: Istat) using the monthly Italian industrial production index (1970:01-2001:09, seasonally adjusted, source: Bank of Italy) as related series.

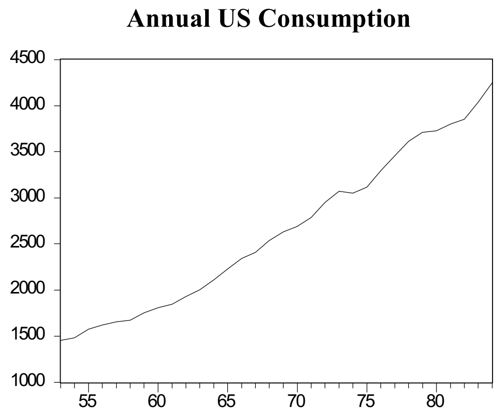
While in the former case the evaluation of the results can be made with respect to the 'true' HF data, in the latter the results are to be judged only in terms of goodness-of-fit and 'economic reasonableness' of the estimated quarterly regression model and, eventually, by comparing different estimates.

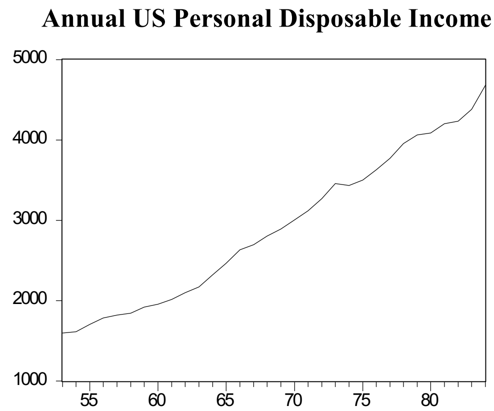
In both cases a preliminary analysis of the dynamical properties of the available series is performed, in order to assess the use of a dynamic model. A number of estimates have been then calculated according to different specifications, both in the original data and in log-transformed form.

8.1. Quarterly disaggregation of US personal consumption

The series of interest (at annual frequency, see fig. 8.1) show clear upward trends. As regards their dynamic properties, in short we find (tab. 8.1 and 8.2) that the individual series, in both levels and logs, are I(1) but the residual based ADF tests (tab. 8.3) do not clearly support the cointegration hypothesis.

Fig. 8.1: Annual time series used in the first temporal disaggregation example





Tab. 8.1: Unit roots tests for annual US consumption (1953-1984)

Statistic	*Intercept and trend*		
	None	*Only intercept*	*Both*
Levels			
ADF(1)	3.536563	1.539602	1.736423
	(-1.9526)	(-2.9627)	(-3.5670)
PP	10.68784	2.677142	-1.507800
	(-1.9521)	(-2.9591)	(-3.5614)
First differences			
ADF(1)	-0.844724	-3.477077	-4.626932
	(-1.9530)	(-2.9665)	(-3.5731)
PP	-0.464600	-3.021454	-3.490603
	(-1.9526)	(-2.9627)	(-3.5670)
Log-levels			
ADF(1)	4.063573	-0.647084	-2.342295
	(-1.9526)	(-2.9627)	(-3.5670)
PP	11.43835	-0.400788	-1.930294
	(-1.9521)	(-2.9591)	(-3.5614)
First differences in logs			
ADF(1)	-1.307424	-4.532890	-4.358359
	(-1.9530)	(-2.9665)	(-3.5731)
PP	-.904924	-4.416819	-4.370993
	(-1.9526)	(-2.9627)	(-3.5670)

MacKinnon 5% critical values for rejection of hypothesis of a unit root in parentheses
PP test statistics have been calculated using 3 lags truncation for Bartlett kernel (Newey and West, 1994).

Tab. 8.2: Unit roots tests for annual US disposable income (1953-1984)

Statistic	*Intercept and trend*		
	None	*Only intercept*	*Both*
Levels			
ADF(1)	3.770442	1.562176	-1.814597
	(-1.9526)	(-2.9627)	(-3.5670)
PP	9.425927	2.279405	-1.852807
	(-1.9521)	(-2.9591)	(-3.5614)
First differences			
ADF(1)	-0.420614	-2.794364	-3.502259
	(-1.9530)	(-2.9665)	(-3.5731)
PP	-0.421621	-3.315762	-3.733314
	(-1.9526)	(-2.9627)	(-3.5670)
Log-levels			
ADF(1)	4.062705	-0.799427	-1.888553
	(-1.9526)	(-2.9627)	(-3.5670)
PP	9.496625	-0.444275	-1.717144
	(-1.9521)	(-2.9591)	(-3.5614)
First differences in logs			
ADF(1)	-0.982749	-3.484757	-3.337831
	(-1.9530)	(-2.9665)	(-3.5731)
PP	-0.914404	-4.457921	-4.437147
	(-1.9526)	(-2.9627)	(-3.5670)

MacKinnon 5% critical values for rejection of hypothesis of a unit root in parentheses
PP test statistics have been calculated using 3 lags truncation for Bartlett kernel (Newey and West, 1994).

Tab. 8.3: Residual-based cointegration tests: ADF(1) on Greene's data

	τ_c	τ_{ct}
Levels	-2.632536	-2.553634
Log-levels	-2.700094	-2.644761
5% asymptotic critical values*	-3.34	-3.78

* Davidson and MacKinnon (1993), Table 20.2, p. 722.

For illustrative purposes we pursued estimates according to the following variants of a first order dynamic model[15] using the method by SSC:

Variant	Model
1	$(1-\phi L)y_{t,u} = \alpha + \beta x_{t,u} + \varepsilon_{t,u}$
2	$(1-\phi L)y_{t,u} = \beta x_{t,u} + \varepsilon_{t,u}$
3	$(1-\phi L)\ln y_{t,u} = \alpha + \beta \ln x_{t,u} + \varepsilon_{t,u}$
4	$(1-\phi L)\ln y_{t,u} = \beta \ln x_{t,u} + \varepsilon_{t,u}$

It should be noted that variant 1 has been considered by SSC in their original paper. However, the non-significant intercepts in both variants 1 and 3 (see tab. 8.4) suggest to adopt either variant 2 (dynamic model in levels without intercept) or variant 4 (dynamic model in logarithms without intercept). As shown in tab. 8.5, where also the results obtained *via* the classical approaches by Chow and Lin, Fernández and Litterman are considered[16], variant 4 of the dynamic model offers the best performance in terms of errors in levels and quarterly percentage changes, while it is outperformed by estimates according to Litterman if annual percentage changes are considered.

Tab. 8.4: Estimates of parameters of the auxiliary annual regression on Greene's data

Variant	α	β	η	ϕ
1	1.095 (0.289)	0.672 (137.0)	357.951 (3.810)	0.259 (0.855)
2	---	0.680 (468.9)	365.7 (3.958)	0.253 (0.830)
3	-0.015 (-0.521)	0.652 (144.9)	5.846 (64.27)	0.341 (1.295)
4	---	0.618 (2906)	5.839 (77.06)	0.373 (1.543)
4*	---	0.616 (2903)	---	0.375 (1.506)

In parentheses are reported *t*-statistics.
* According to Salazar *et al.*

[15] A feasible model selection strategy could be based on post-sample comparison of forecasts obtained according to different LF observable models (Abeysinghe, 1998).
[16] The reported estimates for Chow and Lin and Litterman are different from (and correct) those in SSC.

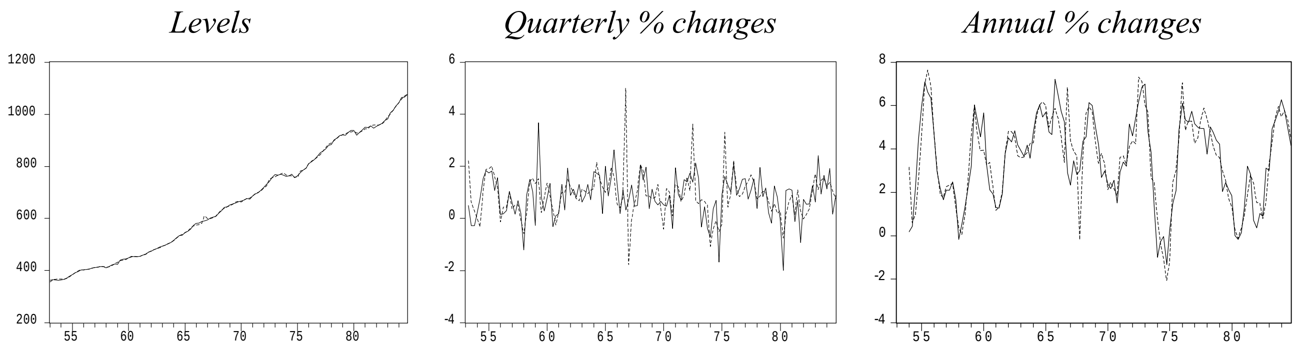
Tab. 8.5: Performance indicators of the disaggregated estimates

	Static model			*Dynamic model*			
	Chow-Lin	**Fernandez**	**Litterman**	**Variant 1**	**Variant 2**	**Variant 4**	**Variant 4***
Levels							
Correlation	0.9997	0.99973	0.99981	0.99978	0.99982	0.99982	0.99982
Std. dev.	0.80	0.75	0.62	0.68	0.68	0.62	0.62
Min.	-5.56	-5.15	-4.14	-4.21	-4.21	-3.35	-3.34
Max	2.04	1.89	1.52	1.50	1.50	1.85	1.84
Quarterly % changes							
Correlation	0.39895	0.4151	0.46564	0.45279	0.45296	0.48945	0.49019
Std. dev.	1.22	1.14	0.96	0.95	0.95	0.84	0.84
Min.	-7.78	-7.20	-5.77	-5.81	-5.81	-4.67	-4.66
Max	5.67	5.28	4.32	3.38	3.38	2.51	2.49
Annual % changes							
Correlation	0.84156	0.85834	0.89943	0.87241	0.87251	0.89041	0.89080
Std. dev.	1.19	1.11	0.91	1.03	1.03	0.94	0.93
Min.	-6.28	-5.90	-4.91	-4.68	-4.68	-3.94	-3.93
Max	4.73	4.34	3.39	3.97	3.97	3.18	3.17

* According to Salazar *et al.*

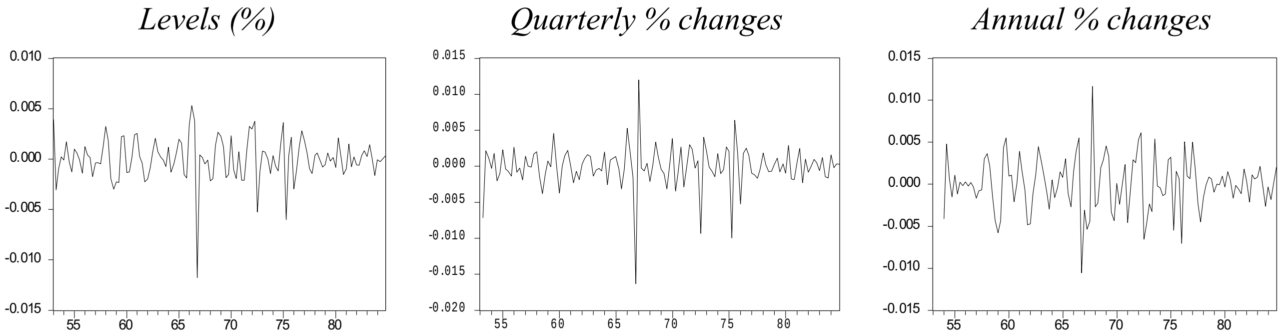
The visual inspection of actual and estimated (through variant 4) levels and percentage changes (fig. 8.2) confirms that the results are surely very good in terms of levels and annual changes, while the ability of the estimated quarterly changes to capture the 'true' dynamics of the series is less pronounced.

Fig. 8.2: Actual (continuous line) and estimated (through variant 4 dynamic model, dotted line) levels and percentage changes of US personal consumption



Finally, it should be noted that the results obtained using the method by Salazar *et al.* (last row of tab. 8.4 and last column of tab. 8.5) are practically the same as those obtained according to SSC. As a consequence, the discrepancies between the two estimates according variant 4 dynamic model (fig. 8.3) are practically negligible.

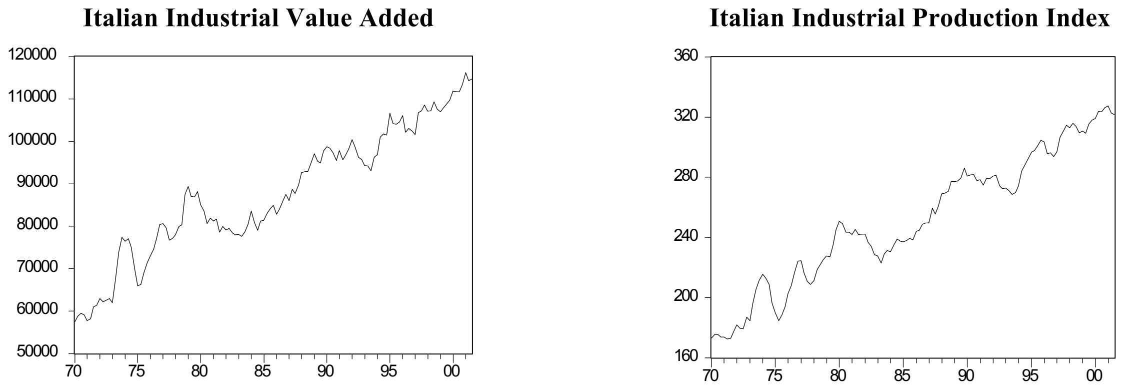
Fig. 8.3: Discrepancies between two estimates (obtained through variant 4 dynamic model according to either SSC or Salazar *et al.*) of US personal consumption



8.2. Monthly disaggregation of quarterly Italian industrial value added

The series to be disaggregated and the chosen indicator are represented in fig. 8.4. As confirmed by the unit roots tests (tab. 8.6 and 8.7), both series are I(1). Moreover, the residual based ADF test τ_c (tab. 8.8) is coherent with the hypothesis of cointegration.

Fig. 8.4: Quarterly time series used in the second temporal disaggregation example



Tab. 8.9 contains parameters' estimates for dynamic models in both levels and logarithms, and precisely according to variants 1, 2 and 3 (that is, model in levels with or without intercept, and model in logs with intercept, which in this last case turns out to be significant). Concentrating on the estimates obtained through variants 2 and 3, we find that the HF estimated values are very similar, as the discrepancies reported in fig. 8.5 clearly show.

To conclude this practical example, the estimated monthly series of Italian industrial value added is represented in fig. 8.6. Estimates obtained following SSC and Salazar *et al*. are practically the same in this case too, and have not been reported.

Tab. 8.6: Unit roots tests for quarterly Italian industrial value added (1970q1-2001q3)

Statistic	*Intercept and trend*		
	None	*Only intercept*	*Both*
Levels			
ADF(4)	1.982030	-1.203263	-3.684817
	(-1.9426)	(-2.8851)	(-3.4469)
PP	2.098253	-1.060944	-3.321493
	(-1.9425)	(-2.8844)	(-3.4458)
First differences			
ADF(4)	-5.680731	-6.450299	-6.439057
	(-1.9426)	(-2.8853)	(-3.4472)
PP	-9.522318	-9.821026	-9.781038
	(-1.9425)	(-2.8845)	(-3.4461)
Log-levels			
ADF(4)	2.131530	-1.869253	-3.875026
	(-1.9426)	(-2.8851)	(-3.4469)
PP	2.178485	-1.635426	-3.289813
	(-1.9425)	(-2.8844)	(-3.4458)
First differences in logs			
ADF(4)	-5.875567	-6.603323	-6.682620
	(-1.9426)	(-2.8853)	(-3.4472)
PP	-.760612	-8.990481	-8.971272
	(-1.9425)	(-2.8845)	(-3.4461)

MacKinnon 5% critical values for rejection of hypothesis of a unit root in parentheses
PP test statistics have been calculated using 4 lags truncation for Bartlett kernel (Newey and West, 1994).

Tab. 8.7: Unit roots tests for quarterly Italian industrial production index (1970q1-2001q3)

Statistic	*Intercept and trend*		
	None	*Only intercept*	*Both*
Levels			
ADF(4)	2.429368	-0.994235	-3.386788
	(-1.9426)	(-2.8851)	(-3.4469)
PP	2.161043	-0.930934	-3.354105
	(-1.9425)	(-2.8844)	(-3.4458)
First differences			
ADF(4)	-5.412656	-6.265364	-6.251745
	(-1.9426)	(-2.8853)	(-3.4472)
PP	-7.624805	-7.911961	-7.882067
	(-1.9425)	(-2.8845)	(-3.4461)
Log-levels			
ADF(4)	2.527806	-1.550708	-3.285249
	(-1.9426)	(-2.8851)	(-3.4469)
PP	2.175486	-1.365844	-3.191530
	(-1.9425)	(-2.8844)	(-3.4458)
First differences in logs			
ADF(4)	-5.69054	-6.519446	-6.593100
	(-1.9426)	(-2.8853)	(-3.4472)
PP	-7.314959	-7.552667	-7.538137
	(-1.9425)	(-2.8845)	(-3.4461)

MacKinnon 5% critical values for rejection of hypothesis of a unit root in parentheses
PP test statistics have been calculated using 4 lags truncation for Bartlett kernel (Newey and West, 1994).

Tab. 8.8: Residual-based cointegration tests: ADF(4) on Italian data

	τ_c	τ_{ct}
Levels	-3.363091	-3.364926
Log-levels	-3.450114	-3.483024
5% asymptotic critical values*	-3.34	-3.78

* Davidson and MacKinnon (1993), Table 20.2, p. 722.

Tab. 8.9: Estimates of the auxiliary quarterly regression on Italian data

Variant	α	β	η	ϕ
1	185.342 (1.197)	60.178 (33.068)	18524.1 (15.397)	0.823 (23.793)
2	---	60.973 (197.2)	18653.7 (15.543)	0.826 (24.080)
3	0.982 (42.127)	0.157 (29.712)	9.832 (225.5)	0.837 (25.300)
3*	0.966 (41.583)	0.154 (29.309)	---	0.840 (25.226)

In parentheses are reported *t*-statistics.
* According to Salazar *et al.*

Fig. 8.5: Discrepancies between two estimates (obtained through variants 2 and 3 of a dynamic model) of monthly Italian industrial value added

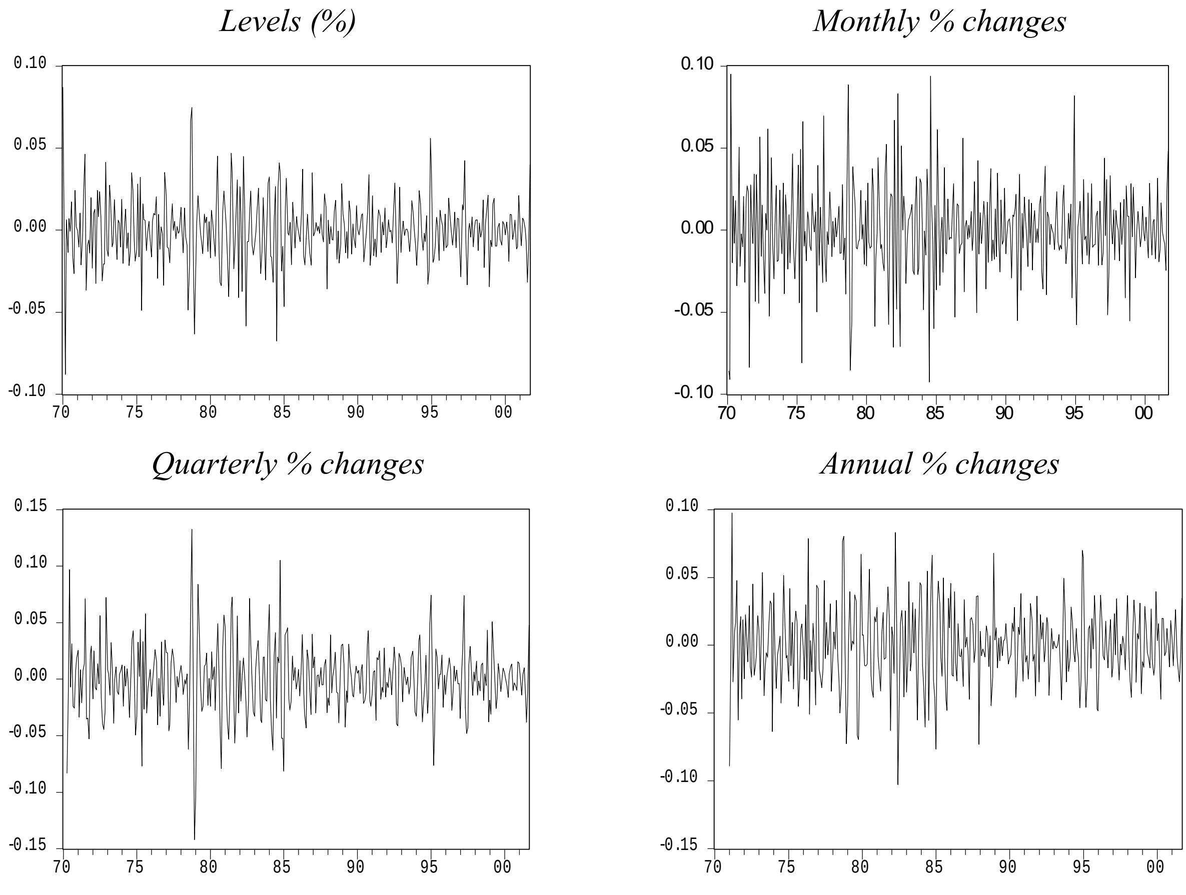
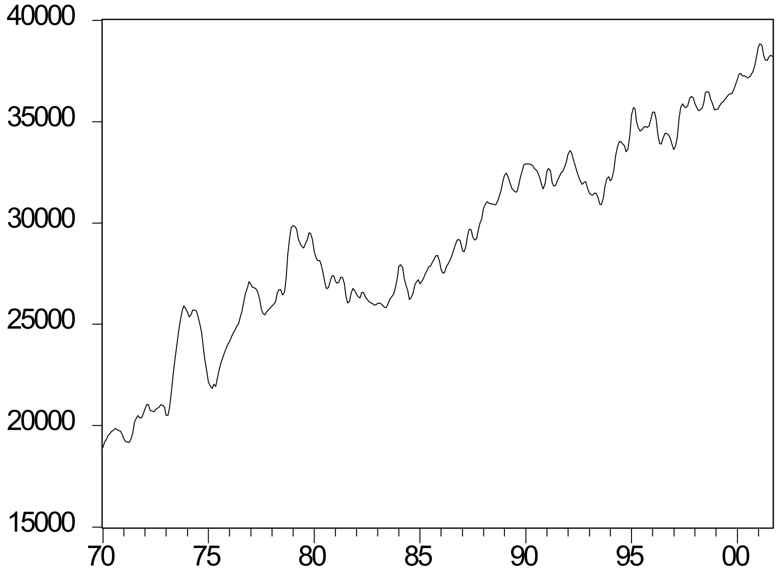


Fig. 8.6: Monthly Italian industrial value added estimated through variant 3 dynamic model



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Appendix A
The general dynamic formulation of Salazar *et al.*

Salazar *et al.* (1997, 1998) consider the following non linear dynamic regression model linking the K observed indicator variables $x_{t,u}^{j}$, $j=1,\ldots,K$, to the unobserved HF interpoland $y_{t,u}$:

$$\alpha(L)f(y_{t,u})=\beta_0+\sum_{j=1}^{K}\beta_j(L)x_{t,u}^{j}+\varepsilon_{t,u},\quad u=1,\ldots,s,\quad t=0,1,\ldots,T, \tag{A.1}$$

where $\alpha(L)=1-\sum_{i=1}^{p}\alpha_i L^i$ and $\beta_j(L)=1-\sum_{k=0}^{q_j}\beta_{j,k}L^j$ are scalar lag polynomials of orders p and q_j respectively operating on the transformed unobserved HF dependent variable, $f(y_{t,u})$, and the observed HF indicator variables $x_{t,u}^{j}$, $j=1,\ldots,K$. The functional form $f(\cdot)$ used in constructing the interpoland in (A.1) is assumed known.

The possibility that the dependent variable $f(y_{t,u})$ in (A.1) is a non-linear function of the interpoland $y_{t,u}$ reflects a frequent occurrence in applied macro-econometric research; for example, a logarithmic transformation is often employed. Of course, the exogenous indicator variables $\{x_{t,u}^{j}\}$ may themselves also be transformations of other underlying variables. It is assumed that the lag lengths p and q_j, $j=1,\ldots,K$, are chosen sufficiently large so that the error terms $\{\varepsilon_{t,u}\}$ may be assumed to possess zero mean, constant variance, and to be serially uncorrelated and uncorrelated with lagged values of $f(y_{t,u})$ and current and lagged values of $\{x_{t,u}^{j}\}$.

The regression equation (A.1) is quite general. For example, if $\alpha_i=0$ for all $i>0$, then the model is essentially static in level of $f(y_{t,u})$. If $\alpha_1=-1$ and $\alpha_i=0$ or all $i>1$, then the model involves the HF first difference of $f(y_{t,u})$. Other values for the parameters $\{\alpha_i\}$ allow a general specification of the dynamics in (A.1). In the special case in which the sum of the coefficients on the dependent variable is unity, $\sum_i\alpha_i=1$, the left hand side of (A.1) may be re-expressed as a scalar lag polynomial of order p-1 operating on the first difference of the dependent variable $f(y_{t,u})$. When $\sum_{i=1}^{p}\alpha_i\neq 1$, there is a long-run relationship linking $f(y_{t,u})$ and $\{x_{t,u}^{j}\}$, $j=1,\ldots,K$; in particular, if $f(y_{t,u})$ and $\{x_{t,u}^{j}\}$, $j=1,\ldots,K$, are difference stationary, there exists a co-integrating relationship between $f(y_{t,u})$ and $\{x_{t,u}^{j}\}$, $j=1,\ldots,K$. Furthermore, in this case, a test of the restriction $\sum_{i=1}^{p}\alpha_i=1$ corresponds to a test of the null hypothesis that there is no co-integrating relationship (Engle and Granger, 1987).

A straightforward application of lag polynomials can transform the dynamic model so that only observed frequencies appear. For, it is possible to transform (A.1) into a regression equation involving only s-order lags of $f(y_{t,u})$ by pre-multiplying $\alpha(L)$ by a suitable polynomial function of the HF lag operator L whose coefficients depend on $\{\alpha_i\}$. Salazar *et al.* (1998) consider only the case in which the maximum lag length is p=1, but they claim that a more general solution for p>1 can be found by factoring $\alpha(L)$ in terms of its roots and treating each factor using the method valid for p=1 (see Astolfi *et al.*, 2000).

Now, let $\lambda(L)=1+\lambda_1 L+\ldots+\lambda_{(s-1)p}L^{(s-1)p}$ be a lag polynomial of order $(s-1)p$ such that

$$\lambda(L)\alpha(L)=\pi(L),$$

where $\pi(L)$ is the lag polynomial of order ps

$$\pi(L)=1-\pi_1 L-\ldots-\pi_{ps}L^{ps}$$

such that $\pi_j=0$ if $k = s(j\text{-}1)+1,\ldots,sj\text{-}1$, for some j = 1,2,…,p, that is

$$\pi(L)=1-\pi_s L^s-\pi_{2s}L^{2s}-\ldots-\pi_{ps}L^{ps}. \tag{A.2}$$

Notice that pre-multiplying model (A.1) by the transformation polynomial $\lambda(L)$, the HF unobserved model is converted into the LF observed model

$$\pi(L)f(y_\tau)=\lambda(1)\beta_0+\lambda(L)\sum_{j=1}^{K}\beta_j(L)x_\tau^j+\lambda(L)\varepsilon_\tau,\quad \tau=(p+1)s,(p+2)s,\ldots,Ts, \tag{A.1}$$

where, as in section 7, for notation convenience we let $\tau=s(t-1)+u$ be the index running on the HF periods. For example, if p = 1 the transformation polynomial is

$$\lambda(L)=1+\alpha_1 L+\ldots+\alpha_1^{s-1}L^{s-1},$$

as we showed in section 6, where $\alpha_1=\phi$. As a consequence, $\pi_s=\alpha_1^s$ and $\pi_j=0$, $j\neq s$.

Abeysinghe and Tay (2000)[17] provide the general solution to the problem of finding the coefficients of the lag transformation polynomial as functions of the autoregressive parameters in the original model (A.1).

---

[17] Abeysinghe and Tay (2000) consider a slightly different notation, using $\alpha(L)=1+\alpha_1 L+\ldots+\alpha_p L^p$ instead of $\alpha(L)=1-\alpha_1 L-\ldots-\alpha_p L^p$.

Appendix B
Derivation of the solution to the constrained optimization problem (6.13)

Partial derivatives of the lagrangean function (6.14) with respect to $\mathbf{y}_h^1$ and $\mathbf{y}_h^2$ are, respectively,

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{y}_h^1} &= 2\mathbf{A}_1'\mathbf{A}_1\mathbf{y}_h^1 + 2\mathbf{A}_1'\mathbf{A}_2\mathbf{y}_h^2 - 2\mathbf{A}_1'\mathbf{z}_h^* - 2\lambda_1\mathbf{C}_1' \\ \frac{\partial L}{\partial \mathbf{y}_h^2} &= 2\mathbf{A}_2'\mathbf{A}_1\mathbf{y}_h^1 + 2\mathbf{A}_2'\mathbf{A}_2\mathbf{y}_h^2 - 2\mathbf{A}_2'\mathbf{z}_h^* - 2\mathbf{C}_2'\boldsymbol{\lambda}_2\end{aligned} . \tag{B.1}$$

Equating (B.1) to zero and solving for λ_1 and $\boldsymbol{\lambda}_2$ gives:

$$\begin{aligned}\lambda_1\mathbf{C}_1' &= \mathbf{A}_1'\mathbf{A}_1\mathbf{y}_h^1 + \mathbf{A}_1'\mathbf{A}_2\mathbf{y}_h^2 - \mathbf{A}_1'\mathbf{z}_h^* \\ \mathbf{C}_2'\boldsymbol{\lambda}_2 &= \mathbf{A}_2'\mathbf{A}_1\mathbf{y}_h^1 + \mathbf{A}_2'\mathbf{A}_2\mathbf{y}_h^2 - \mathbf{A}_2'\mathbf{z}_h^*\end{aligned} . \tag{B.2}$$

Pre-multiplying (B.2) by the block-diagonal matrix

$$\begin{bmatrix}\mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1} & \mathbf{0}' \\ \mathbf{0} & \mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\end{bmatrix}$$

and using the constraints $\mathbf{C}_1\mathbf{y}_h^1 = y_l^1$ and $\mathbf{C}_2\mathbf{y}_h^2 = \mathbf{y}_l^2$, we obtain

$$\begin{cases}\lambda_1\mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1}\mathbf{C}_1' &= y_l^1 + \mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1}\mathbf{A}_1'\mathbf{A}_2\mathbf{y}_h^2 - \mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1}\mathbf{A}_1'\mathbf{z}_h^* \\ \mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\mathbf{C}_2'\boldsymbol{\lambda}_2 &= \mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\mathbf{A}_2'\mathbf{A}_1\mathbf{y}_h^1 + \mathbf{y}_l^2 - \mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\mathbf{A}_2'\mathbf{z}_h^*\end{cases} .$$

The Lagrange multipliers are thus given by

$$\begin{cases}\lambda_1 &= \left[\mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1}\mathbf{C}_1'\right]^{-1}\left[y_l^1 + \mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1}\mathbf{A}_1'\mathbf{A}_2\mathbf{y}_h^2 - \mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1}\mathbf{A}_1'\mathbf{z}_h^*\right] \\ \boldsymbol{\lambda}_2 &= \left[\mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\mathbf{C}_2'\right]^{-1}\left[\mathbf{y}_l^2 + \mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\mathbf{A}_2'\mathbf{A}_1\mathbf{y}_h^1 - \mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\mathbf{A}_2'\mathbf{z}_h^*\right]\end{cases} . \tag{B.3}$$

Equating (B.1) to zero and substituting (B.3) gives the following system

$$\begin{bmatrix}\mathbf{A}_1'\mathbf{A}_1 & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{A}_2'\mathbf{A}_2\end{bmatrix}\begin{bmatrix}\mathbf{y}_h^1 \\ \mathbf{y}_h^2\end{bmatrix} = \begin{bmatrix}\mathbf{A}_1\mathbf{z}_h^* + \mathbf{C}_1'\left[\mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1}\mathbf{C}_1'\right]^{-1}\left[y_l^1 - \mathbf{C}_1\left(\mathbf{A}_1'\mathbf{A}_1\right)^{-1}\mathbf{A}_1'\mathbf{z}_h^*\right] \\ \mathbf{A}_2\mathbf{z}_h^* + \mathbf{C}_2'\left[\mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\mathbf{C}_2'\right]^{-1}\left[\mathbf{y}_l^2 - \mathbf{C}_2\left(\mathbf{A}_2'\mathbf{A}_2\right)^{-1}\mathbf{A}_2'\mathbf{z}_h^*\right]\end{bmatrix}, \tag{B.4}$$

where

$$\mathbf{T}_{12} = \mathbf{A}_1^{'}\mathbf{A}_2 - \mathbf{C}_1^{'}\left[\mathbf{C}_1\left(\mathbf{A}_1^{'}\mathbf{A}_1\right)^{-1}\mathbf{C}_1^{'}\right]^{-1}\mathbf{C}_1\left(\mathbf{A}_1^{'}\mathbf{A}_1\right)^{-1}\mathbf{A}_1^{'}\mathbf{A}_2\,,$$

$$\mathbf{T}_{21} = \mathbf{A}_2^{'}\mathbf{A}_1 - \mathbf{C}_2^{'}\left[\mathbf{C}_2\left(\mathbf{A}_2^{'}\mathbf{A}_2\right)^{-1}\mathbf{C}_2^{'}\right]^{-1}\mathbf{C}_2\left(\mathbf{A}_2^{'}\mathbf{A}_2\right)^{-1}\mathbf{A}_2^{'}\mathbf{A}_1\,.$$

Expression (6.15) is just the solution of system (B.4), that is

$$\begin{bmatrix}\hat{\mathbf{y}}_h^1\\ \hat{\mathbf{y}}_h^2\end{bmatrix} = \begin{bmatrix}\mathbf{A}_1^{'}\mathbf{A}_1 & \mathbf{T}_{12}\\ \mathbf{T}_{21} & \mathbf{A}_2^{'}\mathbf{A}_2\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{A}_1\mathbf{z}_h^* + \mathbf{C}_1^{'}\left[\mathbf{C}_1\left(\mathbf{A}_1^{'}\mathbf{A}_1\right)^{-1}\mathbf{C}_1^{'}\right]^{-1}\left[y_l^1 - \mathbf{C}_1\left(\mathbf{A}_1^{'}\mathbf{A}_1\right)^{-1}\mathbf{A}_1^{'}\mathbf{z}_h^*\right]\\ \mathbf{A}_2\mathbf{z}_h^* + \mathbf{C}_2^{'}\left[\mathbf{C}_2\left(\mathbf{A}_2^{'}\mathbf{A}_2\right)^{-1}\mathbf{C}_2^{'}\right]^{-1}\left[\mathbf{y}_l^2 - \mathbf{C}_2\left(\mathbf{A}_2^{'}\mathbf{A}_2\right)^{-1}\mathbf{A}_2^{'}\mathbf{z}_h^*\right]\end{bmatrix}.$$