Solutions to Assignment 4, 16 points

5.17 (6 points = 2 + 2 + 2) Construct the parity check matrix H and the generator matrix G for a linear code which can:

1. detect 1 error. Hint: this means that

   - the code distance should be $c_d = 2$,
   - i.e. every single column of H should be linearly independent,
   - i.e. none of the columns of H should be a 0-column. (*)

   For the data $k = 5$, the minimal codeword length for which it is possibl ensure that each of column H is non-zero is $n = 6$. The parity check mɛ H is of size $(n-k) \times n = 6 \times 1$:

$$H = [A^T I_1] = [111111]$$

   where the matrix $A^T = [11111]$, So, the matrix A is:

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

   and the generator matrix $G = [I_5 A]$ is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

2. correct 1 error. Hint: this means that

   - the code distance should be $c_d = 3$,
   - i.e. every pair of columns of H should be linearly independent,

- i.e. no two columns of H should be equal + condition (*) should hold. (**)

For the data $k = 5$, the minimal codeword length for which it is possible to ensure that the condition (**) holds is $n = 9$. For a smaller n, there is not enough different non-zero columns of length $n - 5$. For example, if we take $n = 8$, we need to have 8 different non-zero columns of length $8 - 5 = 3$, which is not possible. The total number of columns of length 3 is $2^3 = 8$, so one of the 8 columns is all-0.

For $n = 9$, a possible parity check matrix H is (there are many other possibilities):

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding generator matrix is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

3. correct 1 error + detect one more error. Hint: this means that

   - the code distance should be $c_d = 4$,
   - i.e. every 3 columns of H should be linearly independent,
   - i.e., no two columns of H should sum up to a third column + condition (**) should hold. (***)

For the data $k = 5$, the minimal codeword length for which it is possible to ensure that the condition (***) holds is $n = 10$. A possible parity check matrix H is (there are many other possibilities):

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding generator matrix is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

5.21 part 1 (2 points)

1. Construct the parity check matrix H and the generator matrix G of a Hamming code for 11-bit data.

   A possible parity check matrix H is (any other permutation of 15 non-zero columns will do):

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

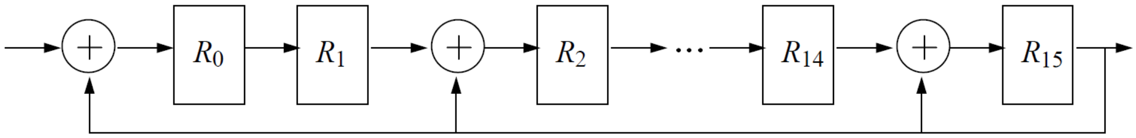
   The corresponding generator matrix is:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

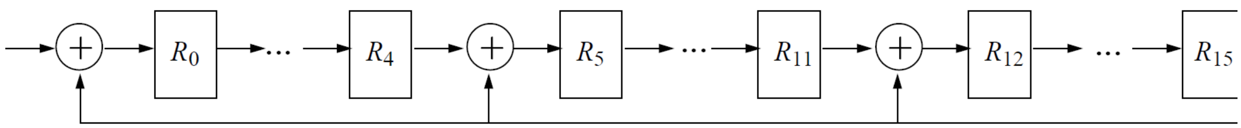
5.32 (3 points = 1 + 1 + 0.5 + 0.5)

a) Draw an LFSR decoding circuit for CRC codes with the following generator polynomials:

$$CRC-16: 1+x^2+x^{15}+x^{16}$$



$$CRC-CCITT: 1+x^5+x^{12}+x^{16}$$



b) Use the first generator polynomial for encoding the data $1+x^3+x^4$ (multiply two polynomials and write down the resulting polynomial).

$$(1+x^3+x^4)(1+x^2+x^{15}+x^{16}) = 1+x^2+x^3+x^4+x^5+x^6+x^{15}+x^{16}+x^{18}+x^{20}$$

c) Suppose that the error $1+x+x^2$ is added to the codeword you obtained in the previous task. Divide the resulting polynomial by the first generator polynomial and check whether this error will be detected or not.

$$(1+x^2+x^3+x^4+x^5+x^6+x^{15}+x^{16}+x^{18}+x^{20})+(1+x+x^2) = x+x^3+x^4+x^5+x^6+x^{15}+x^{16}+x^{18}+x^{20}$$

By dividing the above word by $1+x^2+x^{15}+x^{16}$, we get the reminder $1+x+x^2$, so the error will be detected.

5.34 (1 point) The code distance of the resulting code is 2.

Berger code for 3-bit data:

data bits			check bits	
d_0	d_1	d_2	c_0	c_1
0	0	0	1	1
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	0	0

5.36 for the case when 4-bit data never include the word [1111] (2 points)

There are two ways to reduce the number of check bits of a Berger code for 4-bit data from 3 to 2. First, if the original data never include the word 1111. Then, the checkbits are obtained in the usual way:

data bits				check bits	
d_0	d_1	d_2	d_3	c_0	c_1
0	0	0	0	1	1
0	0	0	1	1	0
0	0	1	0	1	0
0	0	1	1	0	1
0	1	0	0	1	0
0	1	0	1	0	1
0	1	1	0	0	1
0	1	1	1	0	0
1	0	0	0	1	0
1	0	0	1	0	1
1	0	1	0	0	1
1	0	1	1	0	0
1	1	0	0	0	1
1	1	0	1	0	0
1	1	1	0	0	0

5.37 (2 points = 1 + 1)

(a) $3N$ arithmetic code for 3-bit data:

dataword			codeword				
d_0	d_1	d_2	c_0	c_1	c_2	c_3	c_4
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	1
0	1	0	0	0	1	1	0
0	1	1	0	1	0	0	1
1	0	0	0	1	1	0	0
1	0	1	0	1	1	1	1
1	1	0	1	0	0	1	0
1	1	1	1	0	1	0	1

(b) Any single-bit error, and any multiple-bit error which changes the codeword to a word which is not a multiple of 3 is detected. For example 00000 $\rightarrow$ 00010.

Multiple-bit errors which change the codeword to a word which is a multiple of 3 are not detected. For example 00000 $\rightarrow$ 00011.