HOMEWORK 1. Probability

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1 Exercise 1

a) The marginal distributions $p(x)$ and $p(y)$

$p(X = x_1) = 0.1 + 0.05 + 0.01 = 0.16$
$p(X = x_2) = 0.02 + 0.1 + 0.05 = 0.17$
$p(X = x_3) = 0.03 + 0.05 + 0.03 = 0.11$
$p(X = x_4) = 0.1 + 0.07 + 0.05 = 0.22$
$p(X = x_5) = 0.1 + 0.2 + 0.04 = 0.34$

$p(Y = y_1) = 0.01 + 0.02 + 0.03 + 0.1 + 0.1 = 0.26$
$p(Y = y_2) = 0.05 + 0.1 + 0.05 + 0.07 + 0.2 = 0.47$
$p(Y = y_3) = 0.1 + 0.05 + 0.03 + 0.05 + 0.04 = 0.27$

b) The conditional distributions $p(x|Y = y_1)$ and $p(x|Y = y_3)$

$p(X = x_1|Y = y_1) = \frac{0.01}{0.26} = 0.038$
$p(X = x_2|Y = y_1) = \frac{0.02}{0.26} = 0.077$
$p(X = x_3|Y = y_1) = \frac{0.03}{0.26} = 0.115$
$p(X = x_4|Y = y_1) = \frac{0.1}{0.26} = 0.38$
$p(X = x_5|Y = y_1) = \frac{0.1}{0.26} = 0.38$

$p(X = x_1|Y = y_3) = \frac{0.1}{0.27} = 0.37$
$p(X = x_2|Y = y_3) = \frac{0.05}{0.27} = 0.185$
$p(X = x_3|Y = y_3) = \frac{0.03}{0.27} = 0.111$
$p(X = x_4|Y = y_3) = \frac{0.05}{0.27} = 0.185$
$p(X = x_5|Y = y_3) = \frac{0.04}{0.27} = 0.148$

2 Exercise 2

Show that $E_X[X] = E_Y[E_X[x|y]]$

Left side $E_X[X] = \sum_x x.P(X = x)$ (1)

Right side $E_Y[E_X[x|y]$

$= E_Y[\sum_x x.P(X = x|Y = y)]$

$= \sum_y \sum_x x.P(X = x|Y = y).P(Y = y)$

$= \sum_y \sum_x x.P(Y = y|X = x).P(X = x)$
$= \sum_x x.P(X = x) \sum_y P(Y = y|X = x)$
$= \sum_x x.P(X = x) \qquad (2)$
From (1) and (2), $E_X[X] = E_Y[E_X[x|y]]$.

3 Exercise 3

$P(X) = 0.207$
$P(Y) = 0.5$
$P(X|Y) = 0.365$

a) Use both X and y
$P(X \cap Y) = P(X|Y).P(Y) = 0.365.0.5 = 0.1825$

b) Use Y, given that they don't use X
$P(Y|\overline{X}) = \frac{P(\overline{X}|Y).P(Y)}{P(\overline{X})}$
$= \frac{[P(Y)-P(X|Y)].P(Y)}{1-P(X)}$
$= \frac{[0.5-0.365].0.5}{1-0.207}$
$= 0.085$

4 Exercise 4

Prove $V_X = E_X[x^2] - (E_X[x])^2$
$Var(X) = \sum(x_i - E[x])^2.p_i$
$= \sum[x_i^2 - 2x_i.E[x] + (E[x])^2].p_i$
$= \sum x_i^2.p_i - 2x_i.E[x].p_i + (E[x])^2.p_i$
$= \sum x_i^2.p_i - \sum 2x_i.p_i.E[x] + \sum(E[x])^2.p_i$
$= E[x].x_i - 2(E[x])^2 + (E[x])^2$
$= E[x^2] - (E[x])^2$

5 Exercise 5

There are 3 cases of place 1 car and 2 goats

	Door1	Door2	Door3
Case 1:	car	goat	goat
Case 2:	goat	car	goat
Case 3:	goat	goat	car

Suppose choosing Door1, P(A) be the probability of choosing correctly the car, in second chance:
- If not change the option:
$P_{notchange1}(A) = 1$
$P_{notchange2}(A) = 0$

$P_{notchange3}(A) = 0$
$\Rightarrow P_{notchange}(A) = \frac{1}{3}$

- If change the option:
$P_{change1}(A) = 0$
$P_{change2}(A) = 1$
$P_{change3}(A) = 1$
$\Rightarrow P_{change}(A) = \frac{2}{3}$

We have $P_{notchange} < P_{change}$.
Therefore, we should change the option in second chance.