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Tiivistelmä — Referat — Abstract

In this master's thesis we develop homological algebra using category theory. We develop basic properties of abelian categories, triangulated categories, derived categories, derived functors, and t-structures. At the end of most of the chapters there is a short section for notes which guide the reader to further results in the literature.

Chapter 1 consists of a brief introduction to category theory. We define categories, functors, natural transformations, limits, colimits, pullbacks, pushouts, products, coproducts, equalizers, coequalizers, and adjoints, and prove a few basic results about categories like Yoneda's lemma, criterion for a functor to be an equivalence, and criterion for adjunction.

In chapter 2 we develop basics about additive and abelian categories. Examples of abelian categories are the category of abelian groups and the category of R-modules over any commutative ring R. Every abelian category is additive, but an additive category does not need to be abelian. In this chapter we also introduce complexes over an additive category, some basic diagram chasing results, and the homotopy category. Some well known results that are proven in this chapter are the five lemma, the snake lemma and functoriality of the long exact sequence associated to a short exact sequence of complexes over an abelian category.

In chapter 3 we introduce a method, called localization of categories, to invert a class of morphisms in a category. We give a universal property which characterizes the localization up to unique isomorphism. If the class of morphisms one wants to localize is a localizing class, then we can use the formalism of roofs and coroofs to represent the morphisms in the localization. Using this formalism we prove that the localization of an additive category with respect to a localizing class is an additive category.

In chapter 4 we develop basic properties of triangulated categories, which are also additive categories. We prove basic properties of triangulated categories in this chapter and show that the homotopy category of an abelian category is a triangulated category.

Chapter 5 consists of an introduction to derived categories. Derived categories are special kind of triangulated categories which can be constructed from abelian categories. If $\mathcal{A}$ is an abelian category and $C(\mathcal{A})$ is the category of complexes over $\mathcal{A}$, then the derived category of $\mathcal{A}$ is the category $C(\mathcal{A})[S^{-1}]$, where S is the class consisting of quasi-isomorphisms in $C(\mathcal{A})$. In this chapter we prove that this category is a triangulated category.

In chapter 6 we introduce right and left derived functors, which are functors between derived categories obtained from functors between abelian categories. We show existence of right derived functors and state the results needed to show existence of left derived functors. At the end of the chapter we give examples of right and left derived functors.

In chapter 7 we introduce t-structures. T-structures allow one to do cohomology on triangulated categories with values in the core of a t-structure. At the end of the chapter we give an example of a t-structure on the bounded derived category of an abelian category.

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An Introduction to Homological Algebra

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Preface

In this master's thesis we develop homological algebra from category theory point of view. At the end of most of the chapters there are some notes which guide the reader to further results in the literature. We develop the basics of abelian categories, triangulated categories and derived categories. Main results are that derived category of an abelian category is a triangulated category, left exact functors between abelian categories induce right derived functors between derived categories, and that one can do cohomology on triangulated category by using t-structures. For category theory we follow [Bor94a, Bor94b], for homological algebra [GM03], and for t-structures we follow both [HTT08] and [GM03].

The author originally intended to do his master's thesis on ℓ-adic sheaves to gain some understanding of the technical machinery used to prove the Weil conjecture. While sketching the results needed to prove the Weil conjectures, it became clear that this topic was too difficult for the author. Therefore the plan changed to write about homological algebra and the derived category of coherent sheaves on a curve. Due to lack of time and knowledge about algebraic geometry, the part about coherent sheaves on a curve was too much. Hence this thesis is only about homological algebra. I hope that the amount of details in this thesis would be valuable for a reader who wishes to understand basics of homological algebra.

Introduction

In this master's thesis we develop homological algebra by using category theory. Here is a short summary of the results of each chapter.

Chapter 1 gives a short introduction to category theory. It is shown how categories naturally arise when one considers collections of all various well-known mathematical objects. We prove well known results like the Yoneda's lemma, characterize when a functor is an equivalence of categories, and prove some results about limits and adjoints that we will need in the later sections. For a comprehensive introduction to category theory see the books [Bor94a] and [Bor94b].

Chapter 2 follows the book [Bor94b] to develop basics of the theory of abelian categories. First, in section 2.1 we introduce additive categories and in section 2.2 we define abelian categories, which turn out to be also additive categories, as shown in section 2.3, theorem 2.3.3. Then in section 2.4 we introduce the formalism of pseudo-elements (In [ML78] these are called members). This formalism allows one to use element style arguments in abelian categories to prove properties about morphisms. In section 2.5 we define the category of complexes, prove basic results about cohomology of a complex, and prove that the category of complexes over an additive category is an additive category, lemma 2.5.6, and that the category of complexes over an abelian category is an abelian category, theorem 2.5.7. Section 2.6 is devoted for important results about diagrams like 5-lemma, lemma 2.6.1, Snake lemma, corollary 2.6.4, and Functorial long exact sequence, theorem 2.6.6, in abelian categories and in the category of complexes over an abelian category. The last section 2.7 of this chapter studies the homotopy category of an additive category and an abelian category. This category is obtained from the category of complexes by using an equivalence relation on morphisms. Homotopy category will be important in the study of the derived category of an abelian category.

In chapter 3 we give a method to invert a class of morphisms in a category. This method, called localization of a category, is given in section 3.1 togerher with a universal property theorem 3.1.3 which characterizes the localization up to unique isomorphism of categories. Then in section 3.2 we introduce the formalism of roofs and coroofs, which are used to describe morphisms in the localized category when the class of inverted morphisms is a localizing class 3.2.1. In particular, localization of a category with respect to a localizing class preserves additive categories, by proposition 3.2.10, so that the localizing functor is additive, and in some cases the localizing functor preserves full subcategories, see proposition 3.2.7. In general, localization of a category is not well-defined, as shown by the example 5.2.2, because the collection of morphisms between two objects in the resulting "category" is not a set but a class. To justify the use of localization in the construction of derived category, lemma 5.1.9 shows that the derived category of R-modules is well-defined. Further results about the existence of the localization of a category may be found for example at [Wei95] and [Nee01].

Chapter 4 is concerned on triangulated categories. In section 4.1 we define triangulated categories and prove basic properties about triangulated categories, like corollary 4.1.6, an analog of 5-lemma for triangulated categories. The main result of section 4.2 is that the homotopy category of an abelian category is a triangulated category, see theorem 4.2.5. Then in section 4.3 we show that localization of a triangulated category is a triangulated category when the class of inverted morphisms is a localizing class compatible with triangulation.

Chapter 5 is devoted for derived categories. By definition 5.1.1 the derived category is the localization of the category of complexes over an abelian category along the class consisting of quasi-isomorphisms. The main result of section 5.1 is that the derived category is isomorphic to the localization of the homotopy category of the underlying abelian category with respect to the quasi-isomorphisms, and thus is a triangulated category. See theorem 5.1.8. To give an example of a derived category in section 5.2 we compute the derived category of finite dimensional vector spaces over a field.

In chapter 6 we develop the theory of right derived functors. We state the similar results for left derived functors. A right derived functor is an exact functor, in the sense of triangulated categories, between derived categories obtained from a left exact functor between the underlying abelian categories. The existence of this functor is proved in theorem 6.1.14 when we are given an adapted class of objects for a left exact functor. To give an example of a right derived functor, in section 6.2 we construct the derived functor $R\,\mathrm{Mor}$.

In the last chapter 7 we develop t-structures on triangulated categories. This structure gives one a way to obtain an abelian category from a triangulated category. Indeed, the core of a t-structure is an abelian category by theorem 7.3.2. Theorem 7.3.4 shows that one can do cohomology on a triangulated category with a t-structure, with values in the core. To give an example of a t-structure, in section 7.4 we define a standard t-structure on the bounded derived category of an abelian category.

Chapter 1

Introduction to categories

In this chapter we give a short introduction to basics of category theory. One could use category theory as foundations of mathematics, as shown in [MLM92, VI.10], but we use the Neumann-Bernays-Gödel (NBD) axiom system as foundation [Jec13, p.70]. The reason is that we want to be able to define the category of all sets. It is well-known that Russel's paradox implies that these do not form a set. The way to avoid this is to use classes, offered by the chosen axiom system. Alternatively one can use the axiom of universes + ZFC to overcome the same problem. See [Bor94a, 1.1] for a comparison of these approaches. In particular, NBD is an extension of Zermelo-Fraenkel set theory with the axiom of choice so that the set theory in the NBD axiom system is the one the reader is hopefully used to. For a comprehensive treatment of set theory see [Jec13].

1.1 Definitions and notation

Let us begin with the definition of a category.

Definition 1.1.1 (Category)**.** A *category* $\mathcal{C}$ consists of a class of objects $\operatorname{Ob}\mathcal{C}$, a class of morphisms $\operatorname{Mor}\mathcal{C}$ which associates to every pair of elements $X, Y \in \operatorname{Ob}\mathcal{C}$ a set $\operatorname{Mor}_{\mathcal{C}}(X,Y)$, also denoted by $\mathcal{C}(X,Y)$, and three maps $d, c : \operatorname{Mor}\mathcal{C} \to \operatorname{Mor}\mathcal{C}$, and $\circ$ called domain, codomain, and composition, such that the following conditions hold

C 1 For any $f \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$, we define c and d by

$$d(f) = X \qquad \text{and} \qquad c(f) = Y.$$

The composition $\circ$ is a map defined from the class

$$\bigcup_{X,Y,Z\in\operatorname{Ob}\mathcal{C}} \operatorname{Mor}_{\mathcal{C}}(Y,Z) \times \operatorname{Mor}_{\mathcal{C}}(X,Y)$$

to $\operatorname{Mor}\mathcal{C}$, such that for any morphisms $f \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ and $g \in \operatorname{Mor}_{\mathcal{C}}(Y,Z)$ the image of (g,f) is contained in $\operatorname{Mor}_{\mathcal{C}}(X,Z)$. For any two morphisms $f \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$ and $g \in \operatorname{Mor}_{\mathcal{C}}(Y,Z)$ we write $g \circ f$, or gf, for $\circ(g,f)$.

C 2 For any $X, Y, Z, W \in \operatorname{Ob}\mathcal{C}$, $f \in \operatorname{Mor}_{\mathcal{C}}(X,Y)$, $g \in \operatorname{Mor}_{\mathcal{C}}(Y,Z)$, and $h \in \operatorname{Mor}_{\mathcal{C}}(Z,W)$ the composition map satisfies

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

For this reason we usually omit brackets for composition of morphisms.

C 3 For any object $X \in \mathrm{Ob}\,\mathcal{C}$ there exist a morphism $\mathrm{Id}_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ such that for any $Y \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ we have

$$f \circ \mathrm{Id}_X = f \qquad \text{and} \qquad \mathrm{Id}_Y \circ f = f.$$

If $\mathrm{Ob}\,\mathcal{C}$ is a set, then the category $\mathcal{C}$ is said to be *small*.

The elements of $\mathrm{Ob}\,\mathcal{C}$ are called the objects of the category $\mathcal{C}$ and the elements of $\mathrm{Mor}\,\mathcal{C}$ are called the morphisms of $\mathcal{C}$. One can easily verify that for any category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{op}$, obtained by defining $\mathrm{Ob}\,\mathcal{C}^{op} = \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}_{\mathcal{C}^{op}}(X, Y) := \mathrm{Mor}_{\mathcal{C}}(Y, X)$ for all objects X and Y, is a category.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that $\mathcal{C}$ is a *subcategory* of $\mathcal{D}$ if $\mathrm{Ob}\,\mathcal{C} \subset \mathrm{Ob}\,\mathcal{D}$ and for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$ we have $\mathrm{Mor}_{\mathcal{C}}(X, Y) \subset \mathrm{Mor}_{\mathcal{D}}(X, Y)$.

Example 1.1.2. Here are some examples of categories. We leave it for the reader to verify that these are categories.

(i) The category of sets **Set** consists of all the sets as objects and for any sets X and Y we let $\mathrm{Mor}_{\mathbf{Set}}(X, Y)$ to consist of all functions from X to Y.

(ii) The category of abelian groups **Ab** consists of all abelian groups and for any abelian groups A and B the set $\mathrm{Mor}_{\mathbf{Ab}}(A, B)$ consists of all the group homomorphisms from A to B.

(iii) The category of commutative rings **CRing** is the category where the objects are commutative rings and $\mathrm{Mor}_{\mathbf{CRing}}(R, S)$ is the set of all ring homomorphisms from R to S for any commutative rings R and S.

(iv) Fix a commutative ring R. The category of R-modules **RMod** consists of all R-modules and the set $\mathrm{Mor}_{\mathbf{RMod}}(M, N)$ consists of all R-module homomorphisms from M to N for any R-modules M and N.

(v) The category of topological spaces **Top** consists of topological spaces and continuous maps between them.

(vi) Fix a topological space X. Denote by $\mathbf{Top}(X)$ the category where objects are the open subsets of X and morphisms are inclusions, that is, if U and V are open subsets such that $U \subset V$, then the set $\mathrm{Mor}_{\mathbf{Top}(X)}(U, V)$ consists of one element and otherwise $\mathrm{Mor}_{\mathbf{Top}(X)}(U, V)$ is an empty set. It is easy to see that this is a subcategory of **Top**.

For any mathematical objects one usually wants to consider maps which preserve the structure of the object. For categories such a map is called a functor.

Definition 1.1.3 (Functor)**.** A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of two maps $\mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and $\mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$, both denoted by F, such that for any object $X \in \mathrm{Ob}\,\mathcal{C}$, $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ and for any two morphisms $f, g \in \mathrm{Mor}\,\mathcal{C}$, such that gf is defined, we have $F(gf) = F(g)F(f)$. One can easily check that composition of two functors is a functor.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If for any objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$ the map $F : \mathrm{Mor}_{\mathcal{C}}(X, Y) \to \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$ is injective (resp. surjective, resp. bijective), then F is called *faithful* (resp. *full*, resp. *fully faithful*). A subcategory $\mathcal{C}$ of $\mathcal{D}$ is called *full* if the inclusion functor is full.

For a category $\mathcal{C}$ we denote by $\mathrm{Id}_{\mathcal{C}}$ the functor which is an identity both on objects and morphisms.

Example 1.1.4. Here are some examples of functors. The verification of these being functors is left to the reader.

(i) The category of small categories **Cat** consists of all small categories and the set of morphisms between two small categories consists of all functors between the categories.

(ii) Let $\mathcal{C}$ be a category. For any object $X \in \mathrm{Ob}\,\mathcal{C}$ we define a functor $\mathrm{Mor}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathbf{Set}$, called the *representable functor* of X, which maps an object Y of $\mathcal{C}$ to $\mathrm{Mor}_{\mathcal{C}}(X,Y)$. If $f : Y \to Z$ is a morphism in $\mathcal{C}$, then $\mathrm{Mor}_{\mathcal{C}}(X,-)(f) : \mathrm{Mor}_{\mathcal{C}}(X,Y) \to \mathrm{Mor}_{\mathcal{C}}(X,Z)$ is given by composition with f, i.e., $\psi \mapsto f \circ \psi$. One can check that this defines a functor.

An object X in a category $\mathcal{C}$ is *initial* if for any object Y of $\mathcal{C}$ there exists a unique morphism from X to Y. If for any object Y there exists a unique morphism from Y to X, then the object X is *terminal.* An object both initial and terminal is a *zero* object. If a category has a zero object, we call the composite $X \to 0 \to Y$ the *zero* morphism from X to Y.

A morphism $f : X \to Y$ in a category $\mathcal{C}$ is a *monomorphism* if for any two morphisms $g_1, g_2 : Z \to X$ such that $fg_1 = fg_2$ we have $g_1 = g_2$. The morphism f is an *epimorphism* if for any two morphism $g_1, g_2 : Y \to Z$ with $g_1 f = g_2 f$ we have $g_1 = g_2$. A morphism $f : X \to Y$ is an *isomorphism* if there exists a morphism $g : Y \to X$ such that $gf = \mathrm{Id}_X$ and $fg = \mathrm{Id}_Y$. One can easily show that an isomorphism is both a monomorphism and an epimorphism.

Functors can be viewed as morphisms of categories. To understand morphisms of categories better we define morphisms of morphisms of categories which can be thought of as some kind of homotopies between morphisms. In category theory such morphisms are called natural transformations.

Definition 1.1.5 (Natural transformation)**.** A *natural transformation* $\tau : F \to G$ of functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of a morphism $\tau(D) : F(D) \to G(D)$ for any object D of $\mathcal{D}$ such that for any morphism $f : X \to Y$ of $\mathcal{C}$ the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\tau(X)"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\tau(Y)"] & G(Y) \end{tikzcd}$$

commutes.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. We say that the functors F and G are *isomorphic*, written $F \cong G$, if there exists a natural transformation $\tau : F \to G$ such that for every object $X \in \mathcal{C}$ the morphism $\tau(X) : F(X) \to G(X)$ is an isomorphism in $\mathcal{D}$.

Example 1.1.6. Here are some examples of natural transformations.

(i) Let C and D be objects in a category $\mathcal{C}$ and let $f : C \to D$ be a morphism in $\mathcal{C}$. Then the morphism f induces a natural transformation between the representable functors $\mathrm{Mor}_{\mathcal{C}}(D,-)$ and $\mathrm{Mor}_{\mathcal{C}}(C,-)$, see example 1.1.4 (ii), denoted by $- \circ f$, which maps a morphism $\phi : D \to X$ to $\phi \circ f : C \to X$. Indeed, for any morphism $g : X \to Y$ we have

$$(\mathrm{Mor}_{\mathcal{C}}(C,-)(g) \circ (- \circ f)(X))(\phi) = g \circ \phi \circ f = ((- \circ f)(Y) \circ \mathrm{Mor}_{\mathcal{C}}(D,-)(g))(\phi),$$

so $- \circ f$ is a natural transformation.

(ii) For any two categories $\mathcal{C}$ and $\mathcal{D}$, we can define the category of functors $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$, also denoted $\mathcal{C}^{\mathcal{D}}$, from $\mathcal{C}$ to $\mathcal{D}$. Morphisms in this category are natural transformations of functors.

(iii) Let X be a topological space and $\mathbf{Top}(X)$ the category defined in example 1.1.2 (vi). The category $\mathbf{Fun}(\mathbf{Top}(X)^{op}, \mathbf{Set})$ (resp. $\mathbf{Fun}(\mathbf{Top}(X)^{op}, \mathbf{Ab})$, resp. $\mathbf{Fun}(\mathbf{Top}(X)^{op}, \mathbf{CRing})$, resp. $\mathbf{Fun}(\mathbf{Top}(X)^{op}, \mathbf{RMod})$) is called the category of *presheaves of sets* (resp. abelian groups, resp. commutative rings, resp. R-modules) on X.

Next we prove a well-known result which identifies natural transformations from a representable functor to a functor F with a set defined by F.

Theorem 1.1.7 (Yoneda's lemma)**.** *Let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$, and $\operatorname{Mor}_{\mathcal{C}}(X,-)$ the representable functor (ii). For any functor $F : \mathcal{C} \to$ **Set** we have a bijection*

$$\theta_{F,X} : Nat(\mathcal{C}(X,-),F) \to F(X),$$

where $Nat(\mathcal{C}(X,-),F)$ denotes the class of natural transformations from $\mathcal{C}(X,-)$ to F.

Proof. For any natural transformation $\sigma : \mathcal{C}(X,-) \to F$, define $\theta_{F,X}(\sigma) = \sigma(X)(\mathrm{Id}_X)$.

For any $x \in F(X)$ we define a natural transformation $\tau(x) : \mathcal{C}(X,-) \to F$ as follows. For any object $Y \in \mathcal{C}$, let $\tau(x)(Y) : \mathcal{C}(X,Y) \to F(Y)$, be the map $f \mapsto F(f)(x)$. Then for any morphism $g : Y \to Z \in \operatorname{Mor}\mathcal{C}$ the following diagram

$$\begin{tikzcd}
\mathcal{C}(X,Y) \arrow[r, "\tau(x)(Y)"] \arrow[d, "g\circ"] & F(Y) \arrow[d, "F(g)"] \\
\mathcal{C}(X,Z) \arrow[r, "\tau(x)(Z)"] & F(Z)
\end{tikzcd}$$

commutes. This shows that $\tau(x)$ is a natural transformation.

It suffices to show that $\tau(x)$ is the inverse of $\theta_{F,X}$. For any $x \in F(X)$ we have

$$\theta_{F,X}(\tau(x)) = \tau(x)(X)(\mathrm{Id}_X) = F(\mathrm{Id}_X)(x) = x$$

and

$$\tau(\theta_{F,X}(\sigma))(Y)(f) = \tau(\sigma(X)(\mathrm{Id}_X))(Y)(f) = (F(f)\sigma(X))(\mathrm{Id}_X) = \sigma(Y)(f),$$

where the last equality follows from the following commutative diagram

$$\begin{tikzcd}
\mathcal{C}(X,X) \arrow[r, "\sigma(X)"] \arrow[d, "f\circ"] & F(X) \arrow[d, "F(f)"] \\
\mathcal{C}(X,Y) \arrow[r, "\sigma(Y)"] & F(Y)
\end{tikzcd}$$

applied to Id_X. Hence $\theta_{F,X}$ is bijective. $\square$

By using Yoneda's lemma we can easily identify isomorphic representable functors.

Corollary 1.1.8. *Let $\mathcal{C}$ be a category and $X, Y \in \operatorname{Ob}\mathcal{C}$. Then $F : \operatorname{Mor}_{\mathcal{C}}(X,-) \to \operatorname{Mor}_{\mathcal{C}}(Y,-)$, given by some morphism $f : Y \to X$, by theorem 1.1.7, is an isomorphism if and only if f is an isomorphism.*

Proof. $\Rightarrow$: Suppose that F is an isomorphism. Let $g = (F(Y))^{-1}(\mathrm{Id}_Y) : X \to Y$ be a morphism in $\mathcal{C}$. Then $g \circ f = \mathrm{Id}_Y$. Let $G : \operatorname{Mor}_{\mathcal{C}}(Y,-) \to \operatorname{Mor}_{\mathcal{C}}(X,-)$ be the natural transformation induced by the morphism g like in theorem 1.1.7. Now, let $\phi = (G(X))^{-1}(\mathrm{Id}_X) : Y \to X$. We have $\phi \circ g = \mathrm{Id}_X$. Now $\phi = \phi \circ \mathrm{Id}_Y = \phi \circ g \circ f = \mathrm{Id}_X \circ f = f$. Hence $f \circ g = \mathrm{Id}_X$ and f is an isomorphism.

$\Leftarrow$: Suppose that $f : Y \to X$ is an isomorphism. Then for any object Z of $\mathcal{C}$, the map $F(Z)$ is injective because for any morphism $h : X \to Z$, $h \circ f \circ f^{-1} = h$. To see that $F(Z)$ is surjective, let $\psi : Y \to Z$ be any morphism of $\mathcal{C}$. Then the morphism ψf^{-1} is mapped to ψ by $F(Z)$. This completes the proof. $\square$

Definition 1.1.9 (Equivalence)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* of categories if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathrm{Id}_{\mathcal{C}}$ and $FG \cong \mathrm{Id}_{\mathcal{C}}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *essentially surjective* if for any object X of $\mathcal{D}$ there exists some object Y of $\mathcal{C}$ such that $F(Y) \cong X$.

Theorem 1.1.10 (Criteria for equivalence)**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is fully faithful and essentially surjective.*

Proof. $\Rightarrow$: Suppose F is an equivalence of categories. By definition there exist a functor $G : \mathcal{D} \to \mathcal{C}$ such that $\tau_1 : GF \cong \mathrm{Id}_{\mathcal{C}}$ and $\tau_2 : FG \cong \mathrm{Id}_{\mathcal{D}}$. For any object $X \in \mathcal{D}$, the morphism $\tau_2(X) : FG(X) \to X$ is an isomorphisms, so F is essentially surjective.

To show that F is faithful, let $f, g : X \to Y$ be two morphisms in $\mathcal{C}$ such that $F(f) = F(g)$. We have the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\tau_1(X)"] \arrow[d, "f"] & (GF)(X) \arrow[d, "(GF)(f)"] & X \arrow[l, "\tau_1(X)"'] \arrow[d, "g"] \\
Y \arrow[r, "\tau_1(Y)"] & (GF)(Y) & Y \arrow[l, "\tau_1(Y)"']
\end{tikzcd}$$

so $f = \tau_1(Y)^{-1} \circ (GF)(f) \circ \tau_1(X) = \tau_1(Y)^{-1} \circ (GF)(g) \circ \tau_1(X) = g$. Hence F is faithful. Similarly one shows that G is faithful.

Let $g : F(X) \to F(Y)$ be a morphism in $\mathcal{D}$. The following diagram commutes

$$\begin{tikzcd}
GF(X) \arrow[r, "\tau_1(X)"] \arrow[d, "G(g)"] & X \arrow[d, "f"] & GF(X) \arrow[l, "\tau_1(X)"'] \arrow[d, "GF(f)"] \\
GF(Y) \arrow[r, "\tau_1(Y)"] & Y & GF(Y) \arrow[l, "\tau_1(Y)"']
\end{tikzcd}$$

where $f = \tau_1(Y) \circ G(g) \circ \tau_1(X)^{-1}$. Since $\tau(X)$ and $\tau(Y)$ are isomorphisms, we have $G(g) = GF(f)$. Because G is faithful, we obtain $g = F(f)$. This shows that F is full.

$\Leftarrow$: Let F be fully faithful and essentially surjective. We define a functor $G : \mathcal{D} \to \mathcal{C}$ as follows: for any object Y of $\mathcal{D}$ fix an isomorphism $\epsilon_Y : Y \to FX$ and an object X of $\mathcal{C}$. Define $G(Y) = X$. For any morphism $g : Y_1 \to Y_2$ of $\mathcal{D}$ there exists the unique morphism $f_g : X_1 \to X_2$ in $\mathcal{C}$, where the objects X_1 and X_2 are the associated fixed objects of Y_1 and Y_2, which satisfies the equation $\epsilon_{Y_2}^{-1} \circ F(f_g) \circ \epsilon_{Y_1} = g$. We define $G(g) = f_g$.

Clearly $G(\mathrm{Id}_Y) = G(\epsilon_Y^{-1} \circ F(\mathrm{Id}_X) \circ \epsilon_Y) = \mathrm{Id}_X$, so G preserves identity morphisms. Let $g : Y_1 \to Y_2$ and $h : Y_2 \to Y_3$ be morphisms in $\mathcal{D}$ and X_1, X_2, and X_3 the corresponding fixed objects of $\mathcal{C}$, respectively. Then

$$\begin{aligned}
G(h \circ g) &= G(\epsilon_{Y_3}^{-1} \circ F(f_h \circ f_g) \circ \epsilon_{Y_1}^{-1}) \\
&= f_h \circ f_g \\
&= G(\epsilon_{Y_3}^{-1} \circ F(f_h) \circ \epsilon_{Y_2}) \circ G(\epsilon_{Y_2}^{-1} \circ F(f_g) \circ \epsilon_{Y_1}) \\
&= G(h) \circ G(g).
\end{aligned}$$

This shows that G respects composition.

It remains to show that $FG \cong \mathrm{Id}_{\mathcal{D}}$ and $GF \cong \mathrm{Id}_{\mathcal{C}}$. Commutativity of

$$\begin{tikzcd} FG(Y_1) \arrow[r, "\epsilon_{Y_1}^{-1}"] \arrow[d, "FG(g)"] & Y_1 \arrow[d, "g"] \\ FG(Y_2) \arrow[r, "\epsilon_{Y_2}^{-1}"] & Y_2 \end{tikzcd} \tag{1.1}$$

is clear from the equation $\epsilon_{Y_2} \circ FG(g) \circ \epsilon_{Y_1}^{-1} = g$, which follows from the definition of the functor G. This shows that $FG \cong \mathrm{Id}_{\mathcal{D}}$. To show that $GF \cong \mathrm{Id}_{\mathcal{C}}$, note that $\mathrm{Mor}_{\mathcal{C}}(GF(X), X) \cong \mathrm{Mor}_{\mathcal{D}}(FGF(X), F(X))$ for any object X because F is fully faithful. Since $FG \cong \mathrm{Id}_{\mathcal{D}}$ the objects $GF(X)$ and X are isomorphic. Let $\eta(X) : GF(X) \to X$ be the unique morphism such that $F(\eta(X)) = \epsilon_{F(X)}^{-1}$. Since F is fully faithful, commutativity of

$$\begin{tikzcd} GF(X_1) \arrow[r, "\eta(X_1)"] \arrow[d, "GF(f)"] & X_1 \arrow[d, "f"] \\ GF(X_2) \arrow[r, "\eta(X_2)"] & X_2 \end{tikzcd}$$

follows from commutativity of

$$\begin{tikzcd} FGF(X_1) \arrow[r, "\epsilon_{F(X_1)}^{-1}"] \arrow[d, "FGF(f)"] & F(X_1) \arrow[d, "F(f)"] \\ FGF(X_2) \arrow[r, "\epsilon_{F(X_2)}^{-1}"] & F(X_2) \end{tikzcd}$$

Commutativity of the latter diagram follows from commutativity of (1.1). This shows that $GF \cong \mathrm{Id}_{\mathcal{C}}$. □

Example 1.1.11 (Example of equivalence)**.** The above theorem shows that the inclusion functor of a full subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is an equivalence of categories if and only if every object of $\mathcal{C}$ is isomorphic to some object of $\mathcal{D}$.

1.2 Limits

In the study of categories, the most important constructions are limits and colimits. To define limits and colimits we need universal objects.

Definition 1.2.1 (Universal object)**.** Let $F : J \to \mathcal{C}$ be a functor and X an object of $\mathcal{C}$. A *universal object* from X to F, if it exists, is a pair $(Y \in \mathrm{Ob}\, J, f : X \to F(Y) \in \mathrm{Mor}_{\mathcal{C}}(X, F(Y)))$ such that for any other pair $(Z \in \mathrm{Ob}\, J, g : X \to F(Z) \in \mathrm{Mor}_{\mathcal{C}}(X, F(Z)))$ there exists a unique morphism $h : Y \to Z$ in J such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & F(Y) \arrow[d, "F(h)"] \\ & F(Z) \end{tikzcd}$$

Dually, a *universal object* from F to X, if it exists, is a pair $(Y \in \mathrm{Ob}\, J, f : F(Y) \to X)$ such that for any other pair $(Z \in \mathrm{Ob}\, J, g : F(Z) \to X)$ there exists a unique morphism $h : Z \to Y$ in J such that the following diagram commutes

$$\begin{tikzcd} F(Y) \arrow[r, "f"] & X \\ F(Z) \arrow[u, "F(h)"] \arrow[ur, "g"'] & \end{tikzcd}$$

The important thing about universal objects is that they are unique up to unique isomorphism. This means the following. Suppose $F : J \to \mathcal{C}$ is a functor, $X \in \operatorname{Ob}\mathcal{C}$, and $(Y, f : X \to F(Y))$ and $(Z, g : X \to F(Z))$ are universal objects from X to F. Then by definition there exist unique morphisms $h_1 : Y \to Z$ and $h_2 : Z \to Y$ making the following diagrams commutative

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & F(Y) \arrow[d, "F(h_1)"] \\ & F(Z) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & F(Z) \arrow[d, "F(h_2)"] \\ & F(Y) \end{tikzcd}$$

By the uniqueness property $F(h_2) \circ F(h_1) = \mathrm{Id}_{F(Y)}$ and $F(h_1) \circ F(h_2) = \mathrm{Id}_{F(Z)}$. In particular, the morphisms $F(h_1)$ and $F(h_2)$ are unique. Thus we say that the universal objects from X to F are unique up to unique isomorphism. Similarly for universal objects from F to X. The following lemma shows that the universal objects are unique up to unique isomorphism.

Lemma 1.2.2. *Let $F : J \to \mathcal{C}$ be a faithful functor, X an object of $\mathcal{C}$, and $(Y, f : X \to F(Y))$ and $(Y', f' : X \to F(Y'))$ universal objects from X to F. Then there exists a unique isomorphism $g : F(Y) \to F(Y')$ such that $gf = f'$. Similarly, the universal object from F to X is unique up to unique isomorphism.*

Proof. By definition of universal object we can find unique morphisms $h : Y \to Y'$ and $h' : Y' \to Y$ such that $fF(h) = f'$ and $f'F(h') = f$. Now $f'F(hh') = f' = f'F(\mathrm{Id}_X)$ and $fF(h'h) = f = fF(\mathrm{Id}_X)$. By uniqueness of the factorization, we have $hh' = \mathrm{Id}_{Y'}$ and $h'h = \mathrm{Id}_Y$. This shows that the universal objects from X to F are unique up to unique isomorphism.

Let $(Y, f : F(Y) \to X)$ and $(Y', f' : F(Y') \to X)$ be two universal objects from F to X. By definition of universal object there exist unique morphisms $h : Y \to Y'$ and $h' : Y' \to Y$ such that $fF(h') = f'$ and $f'F(h) = f$. Again by uniqueness of factorization $h'h = \mathrm{Id}_Y$ and $hh' = \mathrm{Id}_{Y'}$. Hence the universal objects from F to X are unique up to unique isomorphism. $\square$

Let $\mathcal{C}$ and J be categories. The functor

$$\Delta_J : \mathcal{C} \to \mathcal{C}^J$$

which sends an object C of $\mathcal{C}$ to the constant functor $\Delta_J(C) : J \to \mathcal{C}$, $j \mapsto C$, $(j \to j') \mapsto \mathrm{Id}_C$, is called the *diagonal functor*. Denote by Δ_J the full subcategory of $\mathcal{C}^J$ consisting of all diagonal functors. Let $F : J \to \mathcal{C}$ be a functor and $I : \Delta_J \to \mathcal{C}^J$ the inclusion functor. A natural transformation $\tau : \Delta_J(C) \to F$ is a *cone* on the diagram F with vertex C. A *universal cone* on F is a universal object from I to F. In other words, the pair $(\Delta_J(C), \tau : \Delta_J(C) \to F)$ is a universal cone if for any other pair $(\Delta_J(C'), \tau' : \Delta_J(C') \to F)$ there exists a unique morphism $C' \to C$ such that for any morphism $\phi : j \to j'$ of J the following diagram is commutative

$$\begin{tikzcd}
& & F(j) \arrow[dd, "F(\phi)"] \\
C' \arrow[r] \arrow[urr, "\tau'(j)"] \arrow[drr, "\tau'(j')"'] & C \arrow[ur, "\tau(j)"'] \arrow[dr, "\tau(j')"] & \\
& & F(j')
\end{tikzcd}$$

A *cocone* on F with vertex C, if it exists, is a natural transformation $\tau : F \to \Delta_J(C)$ and a *universal cocone* is a universal object from F to I. That is, the pair $(\Delta_J(C), \tau : F \to \Delta_J(C))$ is a universal cocone if for any other pair $(\Delta_J(C'), \tau' : F \to \Delta_J(C'))$ there exists a unique morphism $C \to C'$ such that for any morphism $\phi : j \to j'$ of J the following diagram is commutative

$$\begin{tikzcd}
F(j) \arrow[dd, "F(\phi)"'] \arrow[dr, "\tau(j)"'] \arrow[drr, "\tau'(j)"] & & \\
& C \arrow[r] & C' \\
F(j') \arrow[ur, "\tau(j')"] \arrow[urr, "\tau'(j')"'] & &
\end{tikzcd}$$

Definition 1.2.3 (Limit, Colimit)**.** Let $\mathcal{C}$ and J be a categories and $F : J \to \mathcal{C}$ a functor. If the universal cone (resp. universal cocone) on the diagram F exists, it is the *limit* (resp. *colimit*) of F and denoted $\varprojlim_{j \in J} F$ (resp. $\varinjlim_{j \in J} F$). If the category J is small, the universal cone (resp. universal cocone) is the *small limit* (resp. *small colimit*) of F.

In particular, since the universal object is unique up to isomorphism by 1.2.2 limits and colimits are unique up to unique isomorphism when they exist.

Example 1.2.4 (Constructions by limits and colimits)**.** Here are some examples of important constructions created by limits and colimits.

Product and coproduct Let $\mathcal{C}$ ba a category, J a discrete category, that is, $\mathrm{Mor}_{\mathcal{J}}(j, j') = \varnothing$ for $j \neq j'$, and $F : J \to \mathcal{C}$ a functor. The limit (resp. colimit) object of this functor is called the *product* (resp. *coproduct*) of the objects $F(j)$, $j \in J$, and is written $\prod_{j \in J} F(j)$ (resp. $\coprod_{j \in J} F(j)$).

Pushout and pullback Let $\mathcal{C}$ be a category, J a category consisting of three objects 1, 2 and 3 and two nontrivial morphisms $a : 1 \to 2$ and $b : 3 \to 2$, and let $F : J \to \mathcal{C}$ be a functor. A limit object together with the morphisms to $F1$ and $F3$ is the *pullback* of $F(a)$ and $F(b)$. Dually, the limit of $F : I^{op} \to \mathcal{C}$ is called the *pushout* of $F(a)$ and $F(b)$.

Equalizer and coequalizer Let $\mathcal{C}$ be a category, J be a category consisting of two objects 1 and 2 and two nontrivial morphisms $a, b : 1 \to 2$, and $F : J \to \mathcal{C}$. The *equalizer* (resp. the *coequalizer*) of the morphisms $F(a)$ and $F(b)$ is the limit (resp. colimit) object of the functor F.

In particular, equalizers are monomorphisms and dually coequalizers are epimorphisms. Indeed, if $e : E \to X$ is the equalizer of the pair $f, g : X \to Y$ and $ex = ey$, then $fex = gey$ and by universal property there exists a unique morphism u such that $eu = ex = ey$. Thus, we must have $x = y$. Similarly for coequalizers.

Kernel and cokernel Let $\mathcal{C}$ be a category, suppose $\mathcal{C}$ has a zero object, and let f be a morphism in $\mathcal{C}$. We define the *kernel* (resp. *cokernel*) of f to be the equalizer (resp. coequalizer) of the morphisms f and 0. In particular, kernels are monomorphisms and cokernels are epimorphisms, because equalizers are monomorphisms and coequalizers are epimorphisms.

The definition for equalizers allows us to define sheaves.

Example 1.2.5 (Sheaves)**.** Let X be a topological space. A functor $F : \mathbf{Top}(X)^{op} \to \mathbf{Set}$ is called a presheaf of sets. By changing the target category, one obtains presheaves of abelian groups, R-modules, and commutative rings. The presheaf F is said to be a sheaf if for any open subset U of X and for any covering $\{U_i\}_{i\in I}$ of U by open subsets, the following diagram

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \overset{\phi_i}{\underset{\psi_j}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \cap U_j)$$

is an equalizer.

The notion of a sheaf and presheaf can be generalized to Grothendieck topologies. See [MLM92, III] for Grothendieck topologies.

The following proposition gives a criterion for existence of finite products.

Proposition 1.2.6. *A category $\mathcal{C}$ which has the terminal object and products for all pairs of objects, has all finite products.*

Proof. We prove the statement by induction. A product of one object c of $\mathcal{C}$ is given by the identity morphism $\mathrm{Id}_c : c \to c$. Indeed, let $c \in \mathrm{Ob}\,\mathcal{C}$, J be the subcategory of $\mathcal{C}$ consisting of just c and Id_c, and let $I : J \to \mathcal{C}$ be the inclusion functor. Then for any morphism $f : d \to c$ the following diagram is commutative

$$\begin{tikzcd}
& & c \arrow[dd, "\mathrm{Id}_c"] \\
d \arrow[urr, "f"] \arrow[r, "f"] \arrow[drr, "f"'] & c \arrow[ur, "\mathrm{Id}_c"'] \arrow[dr, "\mathrm{Id}_c"] & \\
& & c
\end{tikzcd}$$

This shows that (c, Id_c) is the product of c.

Let $c_1, \ldots, c_n \in \mathrm{Ob}\,\mathcal{C}$ and suppose $\mathcal{C}$ has the product $c_1 \prod \ldots \prod c_{n-1}$. By assumption, the product $(c_1 \prod \ldots \prod c_{n-1}) \prod c_n$ exists. Let $\phi_i : d \to c_i$ be a family of morphisms in $\mathcal{C}$. Then there exists a unique morphism $\psi : d \to c_1 \prod \ldots \prod c_{n-1}$ such that $p_i\psi = \phi_i$. Hence, there exists a unique morphism $\epsilon : d \to (c_1 \prod \ldots \prod c_{n-1}) \prod c_n$ such that $p_1\epsilon = \psi$ and $p_2\epsilon = \phi_n$. This shows that $\mathcal{C}$ has the product $(c_1 \prod \ldots \prod c_{n-1}) \prod c_n$ where the projections are given by

$$\begin{cases} p_i = p_i p_1 & 1 \leqslant i \leqslant n-1 \\ p_n = p_2 & \text{otherwise} \end{cases}$$

$\square$

The following proposition will be used in the theory of abelian categories. We say that a category is *finite* if the set of all objects is finite and the set of morphisms between any two object is also finite.

Proposition 1.2.7. *Let $\mathcal{C}$ be a category. The following conditions are equivalent*

(i) $\mathcal{C}$ is finitely complete, that is, every functor $I \to \mathcal{C}$, from a finite category I, has a limit.

(ii) $\mathcal{C}$ has a terminal object, equalizers of all pairs of morphisms with the same domain and codomain, and all products between any pair of objects.

(iii) $\mathcal{C}$ has a terminal object and pullbacks.

Proof. $(i) \Rightarrow (ii)$: Terminal object is the limit of an empty category. Products and equalizers are limits by definition.

$(ii) \Rightarrow (i)$: Consider the following finite products

$$\left(\prod_{j\in J} F(j), (p_j)_{j\in J}\right) \qquad \text{and} \qquad \left(\prod_{\substack{j,j'\in \mathrm{Ob}\, J \\ f:j\to j'\in \mathrm{Mor}_J(j,j')}} F(c(f)), (p'_{c(f)})\right). \tag{1.2}$$

We let $\alpha, \beta : \prod_{j\in J} F(j) \to \prod_{j\to j'\in J} F(j')$ to be the unique morphisms such that

$$p'_{c(f)}\alpha = p_{c(f)} \qquad \text{and} \qquad p'_{c(f)}\beta = F(f)p_j,$$

for every $j, j' \in \mathrm{Ob}\, J$ and $f \in \mathrm{Mor}_J(j, j')$. We show that $(L, (p_j l)_{j\in J})$ defines the limit of F, where (L, l) is the equalizer of the pair (α, β).

For any morphisms $f : j \to j'$ of J, the equalities

$$F(f)p_j l = p'_{c(f)}\beta l = p'_{c(f)}\alpha l = p_{j'} l$$

show that $(L, (p_j l)_{j\in J})$ defines a cone of F. To show that it is the universal cone, let $(M, (q_j)_{j\in J})$ be another pair which defines cone of F. By the universal property of products there exists a unique morphism q' such that $p_j q' = q_j$ for all $j \in \mathrm{Ob}\, J$. For any morphism $f : j \to j'$ in J we have

$$p'_{c(f)}\alpha q' = p_{j'}q' = q_{j'} = F(f)q_j = F(f)p_j q' = p'_{c(f)}\beta q'.$$

This shows that $\alpha q' = \beta q'$ by the uniquness of the second product of (1.2). Hence there exists a unique factorization $q : M \to L$ such that $lq = q'$ and we have $p_j lq = p_j q' = q_j$. It remains to show that the morphism q is unique with this property. Suppose $p_j\bar{q} = q_j$ for some morphism $\bar{q}$ and all objects j of J. Hence

$$p'_j lq = p_j q' = q_j = p_j\bar{q} = p'_j l\bar{q},$$

so $lq = l\bar{q}$. The equality $\bar{q} = q$ follows from the fact that l is a monomorphism.

$(ii) \Rightarrow (iii)$: Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathcal{C}$. Take the product $X \prod Y$ and consider the equalizer

$$E \xrightarrow{\ e\ } X\prod Y \underset{gp_2}{\overset{fp_1}{\rightrightarrows}} Z\ .$$

We show that $(E, p_1 e, p_2 e)$ is the pullback of f and g. Suppose $\phi_1 : D \to X$ and $\phi_2 : D \to Y$ are morphisms such that $f\phi_1 = g\phi_2$. By universal property of products, there is a unique morphism $\psi : D \to X \prod Y$ such that $p_1\psi = \phi_1$ and $p_2\psi = \phi_2$. Hence $fp_1\psi = gp_2\psi$, so by the universal property of the equalizer there exists a unique morphism $\delta : D \to E$ with $\psi = e\delta$. This shows that $\mathcal{C}$ has pullbacks.

(iii) $\Rightarrow$ *(ii)* : Let X, Y be any objects of $\mathcal{C}$. The pullback of $X \to T$, $Y \to T$ gives the product of X and Y, where T is the terminal object of $\mathcal{C}$. Indeed, let $(Z, Z \to X, Z \to Y)$ be the pullback of $X \to T$ and $Y \to T$. For any object W and morphisms $W \to X$ and $W \to Y$ we have

$$W \to X \to T = W \to Y \to T,$$

because morphisms to terminal objects are unique. Thus, by definition of pullback, there exists a unique morphism $W \to Z$ such that

$$W \to X = W \to Z \to X \qquad \text{and} \qquad W \to Y = W \to Z \to Y.$$

This shows that Z is the product of X and Y.

To show that $\mathcal{C}$ has equalizers of all pairs of morphisms, let $f, g : X \to Y$ be morphisms in $\mathcal{C}$. Let $(E, e : E \to X, E \to Y)$ be the pullback of the morphisms $(f, g) : X \to Y \prod Y$ and $\Delta_Y : Y \to Y \prod Y$. Here (f, g) is the unique morphisms given by the definition of product associated to the morphisms f and g and Δ_Y is the unique morphism given by the definition of product associated to the morphisms Id_Y and Id_Y. We show that the morphism $e : E \to X$ is the equalizer of f and g. Let $h : Z \to X$ be any morphism such that $fh = gh$. Now

$$\begin{cases} p_1 \Delta_Y fh = fh = p_1(f, g)h \\ p_2 \Delta_Y fh = fh = gh = p_2(f, g)h \end{cases}$$

implies, by uniqueness of morphisms from Z to $Y \prod Y$, that $\Delta_Y fh = (f, g)h$. Hence by the definition of pullback there exists a unique morphism $\phi : Z \to E$ such that $e\phi = x$. This shows that $(E, e : E \to X)$ is the equalizer of f and g. $\square$

Proposition 1.2.8. *Let $\mathcal{C}$ be a category. The pullback of a monomorphism is a monomorphism. Dually, the pushout of an epimorphism is an epimorphism.*

Proof. Consider the following pullback diagram in $\mathcal{C}$

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

and suppose that g is a monomorphism. Let $u, v : Q \to W$ be morphisms such that $g'u = g'v$. Put $g'' = g'u$ and $f'' = f'u$. Now $fg'' = gf''$ and

$$\begin{cases} g'' = g'u \\ f'' = g'u \end{cases} \qquad \begin{cases} g'' = g'v \\ gf'' = fg'' = fg'u = fg'v = gf'v \end{cases}$$

so by the uniqueness property of pullback, $u = v$. Hence g' is a monomorphism.

The dual follows from this argument applied to $\mathcal{C}^{op}$. $\square$

1.3 Adjoints

For the rest of this chapter we study adjunctions. In particular, we show that adjunctions are unique up to unique isomorphism, and we give a criterion to prove that two functors are adjunctions, which means that one functor is left adjoint to other functor and the other is right adjoint to the first functor. We follow [Bor94a, 3.1]. Let us start by a definition of adjoints.

Definition 1.3.1 (Adjoint)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is *left adjoint* to a functor $G : \mathcal{D} \to \mathcal{C}$ if there exists a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ such that for all objects C of $\mathcal{C}$ the pair $(F(C), \eta(C))$ is the universal object from C to G. This means that for any morphism $f : C \to G(D)$ in $\mathcal{C}$ there exists a unique morphism $h : F(C) \to D$ in $\mathcal{D}$ such that the following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "\eta(C)"] \arrow[dr, "f"'] & GF(C) \arrow[d, "G(h)"] \\ & G(D) \end{tikzcd}$$

A functor $G : \mathcal{D} \to \mathcal{C}$ is *right adjoint* to a functor $F : \mathcal{C} \to \mathcal{D}$ if there exists a natural transformation $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ such that for all $D \in \mathrm{Ob}\,\mathcal{D}$ the pair $(G(D), \epsilon(D))$ is the universal object from F to D. That is, for any morphism $f : F(C) \to D$ in $\mathcal{D}$ there exists a unique morphism $h : C \to G(D)$ in $\mathcal{C}$ such that the following diagram is commutative

$$\begin{tikzcd} FG(D) \arrow[r, "\epsilon(D)"] & D \\ F(C) \arrow[u, "F(h)"] \arrow[ur, "f"] & \end{tikzcd}$$

The following lemma shows that the left and right adjoints are unique up to unique isomorphism.

Lemma 1.3.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $G, G' : \mathcal{D} \to \mathcal{C}$ right adjoints to F with ϕ, ϕ' the isomorphisms on the set of morphisms. Then $G \cong G'$. Similarly, if $F, F' : \mathcal{C} \to \mathcal{D}$ are left adjoints to G, then $F \cong F'$.*

Proof. Let $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ and $\epsilon' : FG' \to \mathrm{Id}_{\mathcal{D}}$ be natural transformations such that for any object $D \in \mathcal{D}$ the pairs $(G(D), \epsilon(D))$ and $(G'(D), \epsilon'(D))$ are universal objects from F to D. By definition of universal object, there exist unique morphisms $h : G(D) \to G'(D)$ and $h' : G'(D) \to G(D)$ such that $hh' = \mathrm{Id}_{G'(D)}$ and $h'h = \mathrm{Id}_{G(D)}$. These morphisms show that $G \cong G'$.

Similarly, let $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\eta' : \mathrm{Id}_{\mathcal{C}} \to GF'$ be natural transformations such that for any object C of $\mathcal{C}$ $(F(C), \eta_C)$ and $(F'(C), \eta'(C))$ are universal objects from C to G. By definition there exists unique morphisms $h : F(C) \to F'(C)$ and $h' : F'(C) \to F(C)$ such that $hh' = \mathrm{Id}_{F(C)}$ and $h'h = \mathrm{Id}_{F'(C)}$. These morphisms show that the functors F and F' are isomorphic. $\square$

The following lemma gives a relation of left and right adjoint of categories and their opposite categories. We denote by f^{op}, F^{op}, and τ^{op} the natural morphisms, functors, and natural transformations, respectively, in the corresponding opposite categories. Note that $(f^{op})^{op} = f$, $(F^{op})^{op} = F$, and $(\tau^{op})^{op} = \tau$.

Lemma 1.3.3. *A functor $F : \mathcal{C} \to \mathcal{D}$ is the left adjoint of $G : \mathcal{D} \to \mathcal{C}$ if and only if $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is the right adjoint of $G^{op} : \mathcal{D}^{op} \to \mathcal{C}^{op}$.*

Proof. $\Rightarrow$: Suppose that F is the left adjoint of G. Let $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ be the natural transformation such that for any object C of $\mathcal{C}$ the pair $(F(C), \eta(C))$ is the universal object from C to G. Thus for any morphism $f : C \to G(D)$ there exists a unique morphism $h : GF(C) \to G(D)$ such that the following diagram is commutative in the dual category

$$\begin{tikzcd} G^{op}F^{op}(C) \arrow[r, "\eta^{op}(C)"] & C \\ G^{op}(D) \arrow[u, "h^{op}"] \arrow[ur, "f^{op}"] & \end{tikzcd}$$

This shows that $(F^{op}(C), \eta^{op}(C))$ is the universal object from G^{op} to C. Hence F^{op} is the right adjoint of G^{op}.

$\Leftarrow$: Let F^{op} be the right adjoint of G^{op}. Then there exists a natural transformation $\epsilon^{op} : F^{op}G^{op} \to \mathrm{Id}_{\mathcal{C}}$ such that for all objects C of $\mathcal{C}$ the pair $(F^{op}(C), \epsilon^{op}(C))$ is a universal object from G^{op} to C. Thus for any morphism $f^{op} : G^{op}(D) \to C$ there exists a unique morphism $h^{op} : D \to F^{op}(C)$ such that the following diagram is commutative in the dual category

$$\begin{tikzcd} C \arrow[r, "\epsilon(C)"] \arrow[dr, "f"] & GF(C) \arrow[d, "G(h)"] \\ & G(D) \end{tikzcd}$$

This shows that the pair $F(C), \epsilon(C))$ is a universal object from C to G. Therefore F is the left adjoint to G. $\square$

We will use later the following theorem in chapter 7 to the theory of t-structures to prove that abstract truncations are adjoints to inclusion functors.

Theorem 1.3.4. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Then the following conditions are equivalent.*

(i) F is left adjoint to G.

(ii) There exists natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, called the counit *and* unit*, such that the following diagrams are commutative*

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, equal, "\mathrm{Id}_G"] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, equal, "\mathrm{Id}_F"] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \tag{1.3}$$

(iii) For any objects $C \in \mathrm{Ob}\,\mathcal{C}$ and $D \in \mathcal{D}$ there exists a bijection $\phi_{C,D} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \to \mathrm{Mor}_{\mathcal{C}}(C, G(D))$ such that for any morphisms $f : C' \to C$ of $\mathcal{C}$ and any morphism $g : D \to D'$ of $\mathcal{D}$ the following diagram is commutative

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{D}}(F(C), D) \arrow[r, "\phi_{C,D}"] \arrow[d, "g\circ - \circ F(f)"] & \mathrm{Mor}_{\mathcal{C}}(C, G(D)) \arrow[d, "G(g)\circ - \circ f"] \\ \mathrm{Mor}_{\mathcal{D}}(F(C'), D') \arrow[r, "\phi_{C',D'}"] & \mathrm{Mor}_{\mathcal{C}}(C', G(D')) \end{tikzcd} \tag{1.4}$$

Here $(g \circ - \circ F(f))(h) = ghF(f)$ and $(G(g) \circ - \circ f)(h) = G(g)hf$.

(iv) G is right adjoint to F.

Proof. $(i) \Rightarrow (ii)$: The natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ is given by the definition of left adjoint, so let us construct the natural transformation ϵ. Consider the universal object $(FG(D), \eta(G(D)))$ from $G(D)$ to G. Let $\epsilon(D) : FG(D) \to D$ be the unique morphism given by the definition of universal object such that the following diagram is commutative

$$\begin{tikzcd} G(D) \arrow[r, "\eta(G(D))"] \arrow[dr, "\mathrm{Id}_{G(D)}"] & GFG(D) \arrow[d, "G(\epsilon(D))"] \\ & G(D) \end{tikzcd}$$

To show that ϵ is a natural transformation, let $d : D \to D'$ be a morphism in $\mathcal{D}$. Then

$$G(\epsilon(D') \circ FG(d)) \circ \eta(G(D)) = G(\epsilon(D')) \circ GFG(d) \circ \eta(G(D)) = G(\epsilon(D')) \circ \eta(G(D')) \circ G(d) = G(d),$$

and

$$G(d \circ \epsilon(D)) \circ \eta(G(D)) = G(d) \circ G(\epsilon(D)) \circ \eta(G(D)) = D(d).$$

By uniqueness of the factorization of the universal object $\epsilon(D') \circ FG(d) = d \circ \epsilon(D)$. This shows that ϵ is a natural transformation.

To show the commutativity of the second triangle of (1.3), let $C \in \operatorname{Ob}\mathcal{C}$ and $(GF(C), \eta(C))$ the universal object from $F(C)$ to F. Then

$$\begin{aligned} G(\epsilon(F(C)) \circ F\eta(C)) \circ \eta(C) &= G\epsilon(F(C)) \circ GF\eta(C) \circ \eta(C) \\ &= \eta(C) = G(\mathrm{Id}_{F(C)}) \circ \eta(C). \end{aligned}$$

By uniquness of the factorization of universal object, $\epsilon(G(D)) \circ G\eta(D) = \mathrm{Id}_{F(C)}$. Hence the second triangle is commutative.

$(ii) \Rightarrow (iii)$: Given a morphism $d : F(C) \to D$, we define $\phi_{C,D}(d)$ to be the composite $G(d) \circ \eta(C)$. For a morphism $c : C \to G(D)$ we define $\tau_{C,D}(c)$ to be the composite $\epsilon(D) \circ F(c)$. We have

$$\begin{aligned} (\tau_{C,D} \circ \phi_{C,D})(d) &= \tau_{C,D}(G(d) \circ \eta(C)) = \epsilon(D') \circ F(G(d) \circ \eta(C)) = \epsilon(D') \circ FG(d) \circ F(\eta(C)) \\ &= d \circ \epsilon(F(C)) \circ F(\eta(C)) = d, \end{aligned}$$

and

$$\begin{aligned} (\phi_{C,D} \circ \tau_{C,D})(c) &= \phi_{C,D}(\epsilon(D) \circ F(c)) = G(\epsilon(D) \circ F(c)) \circ \eta(C) = G(\epsilon(D)) \circ GF(c) \circ \eta(C) \\ &= G(\epsilon(D)) \circ \eta(C') \circ c = c, \end{aligned}$$

so the maps $\tau_{C,D}$ and $\phi_{C,D}$ are mutual inverses. This shows that $\phi_{C,D}$ is bijective for all $C \in \operatorname{Ob}\mathcal{C}$ and $D \in \operatorname{Ob}\mathcal{D}$. To show that the diagram (1.4) is commutative, let $f : C' \to C$ and $g : D \to D'$ be morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively. Let $d : F(C) \to D \in \operatorname{Mor}_{\mathcal{D}}(F(C), D)$. Then

$$\begin{aligned} ((G(g) \circ - \circ f) \circ \phi_{C,D})(d) &= (G(g) \circ - \circ f)(G(d) \circ \eta(C)) = G(g) \circ G(d) \circ \eta(C) \circ f \\ &= G(g) \circ G(d) \circ GF(f) \circ \eta(C'), \end{aligned}$$

and

$$\begin{aligned} (\phi_{C',D'} \circ (g \circ - \circ F(f)))(d) &= \phi_{D',C'}(g \circ d \circ F(f)) = G(g \circ d \circ F(f)) \circ \eta(C') \\ &= G(g) \circ G(d) \circ GF(f) \circ \eta(C'). \end{aligned}$$

This shows the commutativity of the diagram.

$(iii) \Rightarrow (i)$: First note that for any object $C \in \operatorname{Ob}\mathcal{C}$ the morphism $\phi_{C,F(C)}(\mathrm{Id}_C) : C \to GF(C)$ defines a natural transformation by commutativity of the diagram (1.4). It suffices to show that for any object C of $\mathcal{C}$ the pair $(F(C), \phi_{C,C}(\mathrm{Id}_C))$ is a universal object from C to G. Let $f : C \to G(D)$ be a morphism in $\mathcal{D}$ and let $g : F(C) \to D$ be the morphism $\tau_{C,D}(f)$. Now

$$\begin{aligned} ((G(g) \circ - \circ \mathrm{Id}_C) \circ \phi_{C,F(C)})(\mathrm{Id}_{F(C)}) &= (G(g) \circ - \circ \mathrm{Id}_C) \circ \phi_{C,F(C)}(\mathrm{Id}_{F(C)}) = (G(g) \circ \phi_{C,F(C)})(\mathrm{Id}_{F(C)}) \\ &= (\phi_{C,D} \circ g)(\mathrm{Id}_{F(C)}) = (\phi_{C,D} \circ (g \circ - \circ \mathrm{Id}_{F(C)}))(\mathrm{Id}_{F(C)}), \end{aligned}$$

so the pair is the universal object. To show uniqueness of g, let $g' : F(C) \to D$ be a morphism such that $(G(g') \circ \phi_{C,F(C)})(\mathrm{Id}_{F(C)}) = f$. Then

$$\begin{aligned}
\phi_{C,D}(g') &= \phi_{C,D}((g' \circ - \circ \mathrm{Id}_{F(C)})(\mathrm{Id}_{F(C)})) \\
&= (G(g') \circ - \circ \mathrm{Id}_C)(\phi_{C,F(C)}(\mathrm{Id}_{F(C)}) \\
&= G(g') \circ \phi_{C,F(C)}(\mathrm{Id}_{F(C)}) \\
&= f = \phi_{C,D}(g),
\end{aligned}$$

where the second equality follows from commutativity of the diagram (1.4). Since $\phi_{C,D}$ is bijection, this shows that $g = g'$.

$(iv) \Leftrightarrow (iii)$: Suppose G is the right adjoint of F. By lemma 1.3.3 G^{op} is the left adjoint of F^{op}. By (iii) the following diagram is commutative for all morphisms $g^{op} : D' \to D$ and $f^{op} : C \to C'$

$$\begin{array}{ccc}
\mathcal{D}^{op}(G^{op}(D), C) & \xrightarrow{\phi_{D,C}} & \mathcal{C}^{op}(D, F^{op}(C)) \\
\Big\downarrow{\scriptstyle f \circ - \circ G^{op}(g^{op})} & & \Big\downarrow{\scriptstyle F^{op}(f^{op}) \circ - \circ g^{op}} \\
\mathcal{D}^{op}(G^{op}(D'), C') & \xrightarrow{\phi_{D',C'}} & \mathcal{C}^{op}(D', F^{op}(C'))
\end{array}$$

Taking the dual we obtain the diagram (1.4).

Conversely, suppose that (iii) holds. For any morphisms $g : D \to D'$ and $f : C' \to C$ consider the corresponding diagram of (1.4) in the dual category

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathcal{D}^{op}}(G^{op}(D), C) & \xrightarrow{\phi^{op}_{D,C}} & \mathrm{Mor}\,\mathcal{C}^{op}(D, F^{op}(C)) \\
\Big\downarrow{\scriptstyle G^{op}(g^{op}) \circ - \circ f^{op}} & & \Big\downarrow{\scriptstyle g^{op} \circ - \circ F^{op}(f^{op})} \\
\mathrm{Mor}_{\mathcal{D}^{op}}(G^{op}(D'), C') & \xrightarrow{\phi^{op}_{D',C'}} & \mathrm{Mor}\,\mathcal{C}^{op}(D', F^{op}(C'))
\end{array}$$

Since $(iii) \Rightarrow (i)$, G^{op} is the left adjoint of F^{op}. By lemma 1.3.3 G is the right adjoint of F. $\square$

The following example is basic adjunction in commutative algebra.

Example 1.3.5 (Adjunction in **RMod**)**.** Let R be a commutative ring. One can show that in the category **RMod** the the functor $N \otimes - : \mathbf{RMod} \to \mathbf{RMod}$ is left adjoint to the functor $\mathrm{Mor}_R(-, N) : \mathbf{RMod} \to \mathbf{RMod}$. This means that for all R-modules M and P the bijections

$$\mathrm{Mor}_{\mathbf{RMod}}(M \otimes_R N, P) \cong \mathrm{Mor}_{\mathbf{RMod}}(M, \mathrm{Mor}_{\mathbf{RMod}}(P, N))$$

are natural in both M and P so that any diagram of the form (1.4) commutes. Actually, since the category **RMod** is R-linear, meaning that all sets of morphisms admit a natural R-module structure, these bijections between sets of morphisms are isomorphisms of R-modules. For more details, see [Bor94a, Example 3.1.6.e].

Chapter 2

Abelian categories

In this chapter we introduce the main mathematical objects to study in this thesis, additive and abelian categories. Abelian categories can be seen as categorical generalization of the categories of R-modules, because every small abelian category admits a full, faithful, and exact embedding to **RMod**, for some commutative ring R [Bor94b, Theorem 1.14.9]. Thus one can use intuition from the category of R-modules to study abelian categories. Every abelian category is additive, by theorem 2.3.3, but not every additive category is abelian.

2.1 Additive categories

Let us start with the definitions of preadditive and additive categories.

Definition 2.1.1 (Preadditive and additive categories)**.** A *preadditive* category $\mathcal{A}$ is a category such that for any $X, Y \in \operatorname{Ob}\mathcal{A}$ the set $\operatorname{Mor}_{\mathcal{A}}(X, Y)$ has a structure of an abelian group, and for any morphisms $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$ in $\mathcal{A}$ we have

$$(g_1 + g_2)(f_1 + f_2) = g_1 f_1 + g_1 f_2 + g_2 f_1 + g_2 f_2.$$

We say that a preadditive category $\mathcal{A}$ is *additive* if it has a zero object, denoted by 0, and biproducts, that is, for any $X, Y \in \operatorname{Ob}\mathcal{A}$ there exists an object $X \oplus Y$ and morphisms $i_1 : X \to X \oplus Y$, $p_1 : X \oplus Y \to X$, $i_2 : Y \to X \oplus Y$, and $p_2 : X \oplus Y \to Y$ such that the following equalities hold

$$p_1 i_1 = \mathrm{Id}_X, \qquad p_2 i_2 = \mathrm{Id}_Y, \qquad p_1 i_2 = 0, \qquad p_2 i_1 = 0, \qquad i_1 p_1 + i_2 p_2 = \mathrm{Id}_{X \oplus Y},$$

and for any object Z and any morphisms $f : X \to Z$ and $g : Y \to Z$ there exist a unique morphism $f \oplus g : X \oplus Y \to Z$ such that the following diagram is commutative

$$\begin{tikzcd}
& Z & \\
X \arrow[ur, "f"] \arrow[r, shift left, "i_1"] & X \oplus Y \arrow[u, "f\oplus g"] \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & Y \arrow[ul, "g"'] \arrow[l, shift right, "i_2"']
\end{tikzcd}$$

One can easily verify that the morphism $f \oplus g$ is given by $f p_1 + g p_2$.

Example 2.1.2 (Preadditive and additive categories)**.** Let us give a few examples concerning preadditive and additive categories.

(i) The category of groups is not preadditive. The reason is that there is no natural way to define abelian category structure on the set of morphisms. For details, see [Bor94b, Example 1.2.9.b].

(ii) Consider the full subcategory $\mathcal{C}$ of **CRing** consisting of the object $\mathbb{Z}$. Clearly $\mathrm{Mor}_{\mathcal{C}}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$ and this set has a natural structure of an abelian group given by sum of ring homomorphisms. For any ring homomorphisms $f_1, f_2, g_1, g_2 : \mathbb{Z} \to \mathbb{Z}$ one has

$$(f_1 + f_2)(g_1 + g_2) = f_1g_1 + f_1g_2 + f_2g_1 + f_2g_2.$$

Hence the category $\mathcal{C}$ is preadditive.

On the other hand, this category is not additive. Indeed, it does not have the zero object because $\mathrm{Mor}_{\mathcal{C}}(\mathbb{Z},\mathbb{Z})$ consists of more than one morphism. Also, $\mathcal{C}$ does not have biproducts. Indeed, suppose $(\mathbb{Z}, i_1, i_2, p_1, p_2)$ is the biproduct of $\mathbb{Z}$ and $\mathbb{Z}$. From the identities $p_1i_1 = \mathrm{Id}_{\mathbb{Z}}$ and $p_2i_2 = \mathrm{Id}_{\mathbb{Z}}$ it follows that i_1 and i_2 send the identity element 1 of $\mathbb{Z}$ to -1 or 1, because multiplications by 1 and -1 are the only automorphisms of $\mathbb{Z}$. Let $2 : \mathbb{Z} \to \mathbb{Z}$ and $3 : \mathbb{Z} \to \mathbb{Z}$ be the multiplications by 2 and 3. Then there cannot be a morphism h such that the following diagram would commute

$$\begin{tikzcd}
& \mathbb{Z} & \\
\mathbb{Z} \arrow[ur, "2"] \arrow[r, "i_1"'] & \mathbb{Z} \arrow[u, "h"] & \mathbb{Z} \arrow[l, "i_2"] \arrow[ul, "3"']
\end{tikzcd}$$

because h would need to map the element 1 to (2 or -2) and (3 or -3). Hence the category $\mathcal{C}$ is only preadditive.

(iii) Let us show that it is not enough that a preadditive category to have a zero object to be additive. Consider the full subcategry $\mathcal{C}$ of **CRing** consisting of the objects 0 and $\mathbb{Z}$. Clearly 0 cannot be the biproduct of $\mathbb{Z}$ and $\mathbb{Z}$, so the argument of the previous example (ii) shows that the the biproduct of $\mathbb{Z}$ and $\mathbb{Z}$ does not exist in $\mathcal{C}$.

(iv) For only this example, to keep the argument readable, we abuse introduced notation and write i_n and p_n for the inclusion and projection of the nth component of biproduct and not care in which order the biproduct is formed.

In this example we show that there exists only one finite additive category, up to equivalence of categories, the category having only the zero object. Clearly the category having only the zero object is additive. Suppose that $\mathcal{A}$ is an additive category, and let X be a nonzero object of $\mathcal{A}$. Then $\mathrm{Mor}_{\mathcal{A}}(X, X) \geqslant 2$ because this set must contain at least the zero morphism and the identity morphism. Here it cannot be the case that Id_X equals the zero morphism because otherwise X would be isomorphic to zero object and hence a zero object. This contradicts the assumption that X is not a zero object.

Let $n \geqslant 2$ and let $f : \bigoplus_{i=1}^n X \to \bigoplus_{i=1}^n X$ be a nonzero morphism. This means that $fi_j \neq 0$ for some $1 \leqslant j \leqslant n$, because $f = f(i_1p_1 + \ldots + i_np_n)$. The morphisms $f(i_1p_1 + \ldots + i_np_n), f(i_1p_1 + \ldots + \widehat{i_jp_j} + \ldots + i_{n+1}p_{n+1}) : \bigoplus_1^{n+1} X \to \bigoplus_1^n X$ are different. Indeed, we have

$$f(i_1p_1 + \ldots + i_np_n)i_j = fi_j \neq 0 = f(i_1p_1 + \ldots + \widehat{i_jp_j} + \ldots + i_{n+1}p_{n+1})i_j.$$

Hence

$$\#(\mathrm{Mor}_{\mathcal{A}}(\textstyle\bigoplus_1^{n+1} X, \bigoplus_1^n X)) \geqslant 2 \cdot \#(\mathrm{Mor}_{\mathcal{A}}(\bigoplus_1^n X, \bigoplus_1^n X)) - 1.$$

For any two distinct morphisms $\phi, \phi' : \oplus_1^{n+1} X \to \oplus_1^n X$, we have $p_j\phi \neq p_j\phi'$, for some $1 \leqslant j \leqslant n$, and the morphisms $(i_1p_1 + \ldots + i_np_n)\phi, (i_1p_1 + \ldots + i_np_n)\phi' : \oplus_1^{n+1} X \to \oplus_1^{n+1} X$ are not equal because $p_j(i_1p_1 + \ldots i_np_n)\phi = p_j\phi \neq p_j\phi' = (i_1p_1 + \ldots + i_np_n)\phi'$. Therefore

$$\#(\mathrm{Mor}_{\mathcal{A}}(\oplus_1^{n+1} X, \oplus_1^{n+1} X)) \geqslant \#(\mathrm{Mor}_{\mathcal{A}}(\oplus_1^{n+1} X, \oplus_1^n X)).$$

Combining the above inequalities we get

$$\#(\mathrm{Mor}_{\mathcal{A}}(\oplus_1^{n+1} X, \oplus_1^{n+1} X)) > \#(\mathrm{Mor}_{\mathcal{A}}(\oplus_1^n X, \oplus_1^n X)). \tag{2.1}$$

Since $\mathcal{A}$ is finite we can choose m to be the smallest positive integer such that $\oplus_{i=1}^m X = \oplus_{i=1}^n X$ for some positive integer $n < m$. By induction, the inequality (2.1) now says

$$\#(\mathrm{Mor}_{\mathcal{A}}(\oplus_{i=1}^m X, \oplus_{i=1}^m X)) > \#(\mathrm{Mor}_{\mathcal{A}}(\oplus_{i=1}^n X, \oplus_{i=1}^n X)).$$

This is a contradiction. Thus a finite additive category $\mathcal{A}$ cannot contain a nonzero object.

The following proposition shows that in a preadditive category, the biproduct of two objects is both the product and the coproduct of the objects.

Proposition 2.1.3. *Let $\mathcal{A}$ be a preadditive category and let A and B be objects of $\mathcal{A}$. Then $\mathcal{A}$ has the biproduct of A and B if and only if it has the product and the coproduct of A and B.*

Proof. Let $A \oplus B$ be the biproduct of A and B together with the morphisms i_1, p_1, i_2, and p_2 like in definition 2.1.1. First we show that $\mathcal{A}$ has the product of A and B. Let $C \in \mathrm{Ob}\,\mathcal{A}$, and let $f : C \to A$ and $g : C \to B$ be morphisms in $\mathcal{A}$. Then $i_1f + i_2g$ is a morphism to $A \oplus B$ such that $p_1(i_1f + i_2g) = f$ and $p_2(i_1f + i_2g) = g$. To show the uniqueness of this morphism, let $h : C \to A \oplus B$ be a morphism such that $p_1h = f$ and $p_2h = g$. Then $h = (i_1p_1 + i_2p_2)h = i_1f + i_2g$. This shows that $\mathcal{A}$ has $A \prod B$.

To show that $\mathcal{A}$ has the coproduct of A and B, let $\mathcal{D}$ be an object of $\mathcal{A}$ and let $f : A \to D$ and $g : B \to D$ be morphisms in $\mathcal{A}$. Now, $fp_1 + gp_2$ is a morphism such that $f = (fp_1 + gp_2)i_1$ and $g = (fp_1 + gp_2)i_2$. To show uniqueness of this morphism, let $h : A \oplus B \to D$ be a morphism such that $hi_1 = f$ and $hi_2 = g$. Now $h = h(i_1p_1 + i_2p_2) = fp_1 + gp_2$. Therefore $\mathcal{A}$ has $A \coprod B$.

Conversely, let $A \prod B$ be the product of A and B together with morphisms $p_1 : A \prod B \to A$ and $p_2 : A \prod B \to B$. The identity morphism $\mathrm{Id}_A : A \to A$ gives rise to a unique morphism $i_1 : A \to A \prod B$ such that $p_1i_1 = \mathrm{Id}_A$ and $p_2i_1 = 0$. Similarly, the identity morphism $\mathrm{Id}_B : B \to B$ gives a unique morpism $i_2 : B \to A \oplus B$ such that $p_1i_2 = 0$ and $p_2i_2 = \mathrm{Id}_B$. Now

$$\begin{cases} p_1(i_1p_1 + i_2p_2) &= p_1, \\ p_2(i_1p_1 + i_2p_2) &= p_2 \end{cases} \quad \text{and} \quad \begin{cases} p_1 \mathrm{Id}_{A\prod B} &= p_1, \\ p_2 \mathrm{Id}_{A\prod B} &= p_2 \end{cases}$$

and by the universal property of product $i_1p_1 + i_2p_2 = \mathrm{Id}_{A\prod B}$. Hence $(A \prod B, i_1, i_2, p_1, p_2)$ is the coproduct of A and B. This completes the proof. ☐

A functor which preserves the structure of an additive category is called an additive functor.

Definition 2.1.4 (Additive functor)**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. If for all objects A_1, A_2 of $\mathcal{A}$ the map

$$\mathrm{Mor}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Mor}_{\mathcal{B}}(F(A_1), F(A_2)), \qquad f \mapsto F(f)$$

is a group homomorphism, then F is *additive*.

The following proposition gives a characterization of an additive functor by biproducts.

Proposition 2.1.5. *A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is additive if and only if preserves biproducts, that is, if $(A \oplus B, i_1, i_2, p_1, p_2)$ is a biproduct in $\mathcal{A}$, then $(F(A \oplus B), F(i_1), F(i_2), F(p_1), F(p_2))$ is a biproduct in $\mathcal{B}$. In particular, an additive functor preserves the zero object.*

Proof. $\Rightarrow$: Let F be additive and let $(A \oplus B, i_1, i_2, p_1, p_2)$ be a biproduct in $\mathcal{A}$. Now, $F(p_1 i_1) = \mathrm{Id}_{F(A)}$, $F(p_2 i_1) = \mathrm{Id}_{F(b)}$, $F(p_1 i_2) = 0$, $F(p_2 i_1) = 0$, and $\mathrm{Id}_{F(A \oplus B)} = F(i_1 p_1 + i_2 p_2) = F(i_1)F(p_1) + F(i_2)F(p_2)$. Indeed, to see that $F(p_1 i_2) = F(i_2 p_1) = 0$ it suffices to show that $F(0)$ is the zero object in $\mathcal{B}$. Since

$$\mathrm{Mor}_{\mathcal{A}}(0_{\mathcal{A}}, 0_{\mathcal{A}}) \to \mathrm{Mor}_{\mathcal{B}}(F(0_{\mathcal{A}}), F(0_{\mathcal{A}}))$$

is a group homomorphism, $F(\mathrm{Id}_0)$ factors through the zero object of $\mathcal{B}$. Thus there exist morphisms $f : F(0) \to 0$ and $g : 0 \to F(0)$, which are unique, such that $gf = \mathrm{Id}_{F(0)}$. Now $gf = \mathrm{Id}_0$ by uniqueness of the morphism from the zero object to itself. Hence $F(0)$ is isomorphic to the zero object in $\mathcal{B}$ and is itself the zero object.

To see that $(F(A \oplus B), F(i_1), F(i_2), F(p_1), F(p_2))$ is the biproduct of $F(A)$ and $F(B)$ in $\mathcal{B}$ it suffices to show that it is isomorphic to the object of the biproduct $(F(A) \oplus F(B), i_1, i_2, p_1, p_2)$. By the definition of biproduct there exists a unique morphism $h : F(A \oplus B) \to F(A) \oplus F(B)$ such that $p_1 h = F(p_1)$ and $p_2 h = F(p_2)$. Now

$$\mathrm{Id}_{F(A \oplus B)} = F(i_1)F(p_1) + F(i_2)F(p_2) = F(i_1)p_1 h + F(i_2)p_2 h = (F(i_1)p_1 + F(i_2)p_2)h$$

and

$$\begin{aligned}\mathrm{Id}_{F(A) \oplus F(B)} &= i_1 p_1 + i_2 p_2 = i_1 \mathrm{Id}_{F(A)} p_1 + i_2 \mathrm{Id}_{F(B)} p_2 = i_1 F(p_1)F(i_1)p_1 + i_2 F(p_2)F(i_2)p_2 \\ &= i_1 p_1 h F(i_1)p_1 + i_2 p_2 h F(i_2)p_2 = hF(i_1)p_1 + hF(i_2)p_2 = h(F(i_1)p_1 + F(i_2)p_2).\end{aligned}$$

This shows that h is an isomorphism and that $F(A \oplus B)$ is the biproduct of $F(A)$ and $F(B)$. Hence F preserves biproducts.

$\Leftarrow$: First we show that F preserves the zero object. By assumption $(F(0 \oplus 0), F(i_1), F(i_2), F(p_1), F(p_2))$ is the biproduct of $F(0)$ and $F(0)$ in $\mathcal{B}$. Since 0 is the zero object in $\mathcal{A}$, $F(i_1) = F(i_2)$ and $F(p_1) = F(p_2)$. Let $f_1, f_2 : B \to F(0)$ and $g_1, g_2 : F(0) \to C$ be morphisms in $\mathcal{B}$, and let $h_1 : B \to F(0 \oplus 0)$ and $h_2 : F(0 \oplus 0) \to C$ be the unique morphisms such that $F(p_1)h_1 = f_1$, $F(p_2)h_1 = f_2$, $h_2 F(i_1) = g_1$, and $h_2 F(i_2) = g_2$. Now

$$f_1 - f_2 = F(p_1)h_1 - F(p_2)h_1 = (F(p_1) - F(p_2))h_1 = 0$$

and

$$g_1 - g_2 = h_2 F(i_1) - h_2 F(i_2) = h_2(F(i_1) - F(i_2)) = 0.$$

Hence $f_1 = f_2$ and $g_1 = g_2$. This shows that F preserves the zero object.

It remains to show that F preserves difference of two morphisms. Let $f, g : A_1 \to A_2$ be morphisms in $\mathcal{A}$. From

$$F(f - g) = F((p_1 - p_2)(i_1 f + i_2 g)) = F(p_1 - p_2)F(i_1 f + i_2 g) = F(p_1 - p_2)(F(i_1)F(f) + F(i_2)F(g)),$$

we see that it is enough to show that F preserves the difference $p_1 - p_2$. Now

$$F(p_1 - p_2)F(i_1) = F((p_1 - p_2)i_1) = F(\mathrm{Id}_{A_1}) = \mathrm{Id}_{F(A_1)} = (F(p_1) - F(p_2))F(i_1) = \mathrm{Id}_{A_2}$$

and

$$F(p_1 - p_2)F(i_2) = F((p_1 - p_2)i_2) = F(\mathrm{Id}_{A_2}) = \mathrm{Id}_{F(A_2)} = (F(p_1) - F(p_2))F(i_2) = -\mathrm{Id}_{A_2},$$

so by the universal property of the biproduct $F(p_1 - p_2) = F(p_1) - F(p_2)$. This completes the proof. $\square$

Example 2.1.6 (Additive representable functor)**.** Let us give an example how to construct additive functors on an additive category. Suppose we have an additive category $\mathcal{A}$. Then for any object $A \in \mathrm{Ob}\,\mathcal{A}$ we can consider the representable functor $\mathrm{Mor}_{\mathcal{A}}(A,-)$, item (ii). Let us show that this is an additive functor. Fix two morphisms $f, g : X \to Y$. Then for any morphism $h : A \to X$ we have

$$\mathrm{Mor}_{\mathcal{A}}(A,-)(f-g)(h) = (f-g)(h) = \mathrm{Mor}_{\mathcal{A}}(A,-)(f)(h) - \mathrm{Mor}_{\mathcal{A}}(A,-)(g)(h).$$

This shows that the functor $\mathrm{Mor}_{\mathcal{A}}(A,-)$ is additive.

We conclude this section by an additive version of Yoneda's lemma 1.1.7. This is a special case of a more general version, the enriched Yoneda's lemma [Bor94b, Theorem 6.3.5].

Proposition 2.1.7 (Additive Yoneda's lemma)**.** *Let $\mathcal{A}$ be an additive category and $F : \mathcal{A} \to \mathbf{Ab}$ an additive functor. For any object A of $\mathcal{A}$ we have an isomorphism of abelian groups*

$$\theta_{F,A} : Nat(\mathrm{Mor}_{\mathcal{A}}(A,-), F) \to F(A),$$

which is naural in both A and F. This means that for any morphism $\phi : A \to A'$ in $\mathcal{A}$ and for any natural transformation $\eta : F \to G$ the following diagrams are commutative

$$\begin{array}{ccc} Nat(\mathrm{Mor}_{\mathcal{A}}(A,-), F) & \xrightarrow{\theta_{F,A}} & F(A) \\ \Big\downarrow{\scriptstyle -\circ(-\circ\phi)} & & \Big\downarrow{\scriptstyle F(\phi)} \\ Nat(\mathrm{Mor}_{\mathcal{A}}(A',-), F) & \xrightarrow{\theta_{F,A'}} & F(A') \end{array} \qquad \begin{array}{ccc} Nat(\mathrm{Mor}_{\mathcal{A}}(A,-), F) & \xrightarrow{\theta_{F,A}} & F(A) \\ \Big\downarrow{\scriptstyle \eta\circ -} & & \Big\downarrow{\scriptstyle \eta(A)} \\ Nat(\mathrm{Mor}_{\mathcal{A}}(A,-), G) & \xrightarrow{\theta_{G,A}} & G(A) \end{array}.$$

Proof. Let $\alpha : \mathcal{A}(A,-) \to F$ be a natural transformation and define $\theta_{F,A}(\alpha) = \alpha(A)(\mathrm{Id}_A)$. Conversely, for any element $a \in F(A)$ we assign a natural transformation

$$\tau(a)(B) : \mathrm{Mor}_{\mathcal{A}}(A,B) \to F(B), \qquad \tau(a)(B)(f) = F(f)(a).$$

Here the morphism $\tau(a)(B)$ is a group homomorphism because F is an additive functor. By the proof of theorem 1.1.7 this is map is bijective, hence isomorphism of abelian groups.

To see that this map is natural in A and F, let $\phi : A \to A'$ be a morphism in $\mathcal{A}$, $\eta : F \to G$ a natural transformation, and let $\Psi \in Nat(\mathrm{Mor}(A,-), F)$. Then

$$\begin{aligned}(F(\phi) \circ \theta_{F,A})(\Psi) &= F(\phi)(\Psi(A)(\mathrm{Id}_A)) = \Psi(A')(\phi) \\ &= \theta_{F,A'}(\Psi \circ (- \circ \phi)) = (\theta_{F,A'} \circ (- \circ \phi))(\Psi),\end{aligned}$$

where the second equality follows from the fact that Ψ is a natural transformation, so the first diagram is commutative. Also

$$\begin{aligned}(\eta(A) \circ \theta_{F,A})(\Psi) &= \eta(A)(\Psi(A)(\mathrm{Id}_A)) = \theta_{G,A}(\eta \circ \Psi) \\ &= (\theta_{G,A} \circ (\eta \circ -))(\Psi),\end{aligned}$$

so the second diagram is also commutative. $\square$

2.2 Abelian categories

In this section we introduce abelian categories and develop some of the most elementary properties about them. These are the image factorization theorem 2.2.10 and the fact that abelian categories is additive by theorem 2.3.3.

Definition 2.2.1. A category $\mathcal{A}$ is *abelian* if the following conditions hold.

AB 1 $\mathcal{A}$ has the zero object.

AB 2 Every pair of objects of $\mathcal{A}$ has the product and coproduct.

AB 3 Every morphism of $\mathcal{A}$ has the kernel and the cokernel.

AB 4 Every monomorphism is the kernel of some morphism of $\mathcal{A}$ and every epimorphism is the cokernel of some morphism of $\mathcal{A}$.

Remark 2.2.2. From the definition of an abelian category it is clear that the opposite category of an abelian category is an abelian category. This allows us to prove results about abelian categories by duality. More precisely, if we give some proof about an abelian category $\mathcal{A}$ only using the axioms of the abelian category, then the same proof applies for $\mathcal{A}^{op}$. Since limit (resp. colimit) in $\mathcal{A}$ corresponds to colimit (resp. limit) in $\mathcal{A}^{op}$, constructions using limits and colimits have also their dual version in $\mathcal{A}$.

Proposition 2.1.3 shows that the condition AB2 can be replaced by the condition that $\mathcal{A}$ has all biproducts. Proposition 1.2.6 implies that an abelian category has all finite products, and by duality all finite coproducts.

First, let us note that, in an abelian category a morphism which is both a monomorphism and an epimorphism, is an isomorphism. Indeed, let $f : X \to Y$ be a morphism which is a monomorphism and an epimorphism. Then by AB4 f is the kernel of some morphism $g : Y \to Z$, so $gf = 0$. Since f is an epimorphism, we get $g = 0$. Now $g \operatorname{Id}_Y = 0$, so there exists a unique morphism $h_1 : Y \to X$ such that $fh_1 = \operatorname{Id}_Y$. By duality there exists a unique morphism $h_2 : Y \to X$ such that $h_2 f = \operatorname{Id}_X$. Putting these together yields

$$h_1 = (h_2 g)h_1 = h_2(gh_1) = h_2.$$

This shows that f is an isomorphism.

Example 2.2.3. The categories **Ab**, and **RMod** are abelian categories. Moreover, we will see that every abelian category has all finite limits and finite colimits 2.2.5, but the category **RMod** has more. It has all small limits and all small colimits, [Bor94b, Example 1.4.6a].

Let $\mathcal{A}$ be an abelian category and consider an object A of $\mathcal{A}$. The monomorphisms in $\mathcal{A}$ with codomain the object A are called the *subobjects* under the following equivalence relation: let $f : X \to A$ and $g : Y \to A$ be monomorphisms. They are equivalent as subobjects if there exists an isomorphism $h : X \to Y$ such that $f = gh$. Since g is a monomorphism, one sees that the morphism h is unique.

Lemma 2.2.4. *An abelian category $\mathcal{A}$ has the pullback of two subobjects of any object.*

Proof. Let a, b be two monomorphisms with equal codomains, and fix morphisms f and g so that $a = \ker f$ and $b = \ker g$ by AB4. Denote by k the morphism of $\ker(f, g)$. Now, $fk = p_1(f, g)k = 0$, so there exists a unique morphism a' such that $k = aa'$. Similarly, from $gk = p_2(f, g)k = 0$, we get a unique morphism b' with $k = bb'$. We have the following commutative diagram

$$\begin{tikzcd}
B' \arrow[dr, "b"] & C \arrow[l, "b'"'] \arrow[r, "a'"] \arrow[d, "k"] & A' \arrow[dl, "a"'] \\
& A \arrow[dl, "f"'] \arrow[d, "(f,g)"] \arrow[dr, "g"] & \\
X & X\prod Y \arrow[l, "p_2"] \arrow[r, "p_1"'] & Y
\end{tikzcd}$$

We prove that (C, a', b') is the pullback of a and b.

Let u, v be morphisms with the same domain such that $au = bv$. Then $fbv = fau = gbv = gau = 0$, so by uniqueness of the product we have that $(f,g)bv = (f,g)au = 0$. Hence there exists a unique morphism w such that $kw = bv = au$. Since a and b are monomorphisms, by commutativity we have $v = b'w$ and $u = a'w$. Uniqueness of w follows from commutativity and the fact that k is a monomorphism. □

The following lemma shows that all constructions like equalizers, coequalizers, pullbacks and pushouts constructed by finite limits and finite colimits exist in abelian categories.

Lemma 2.2.5. *An abelian category $\mathcal{A}$ is finitely complete and finitely cocomplete.*

Proof. By duality for abelian categories, it suffices to show that $\mathcal{A}$ is finitely complete. By proposition 1.2.6 $\mathcal{A}$ has finite products, and by proposition 1.2.7 it remains to show that $\mathcal{A}$ has equalizers for all pairs of parallel morphisms.

Let $f, g : A \to B$ be morphisms. Now (Id_A, f) and (Id_A, g) are monomorphisms, so by lemma 2.2.4, they have a pullback (P, u, v). From

$$\begin{cases} u = p_1(\mathrm{Id}_A, f)u = p_1(\mathrm{Id}_A, g)v = v \\ fu = p_2(\mathrm{Id}_A, f)u = p_2(\mathrm{Id}_A, g)v = gv, \end{cases}$$

one gets that $fu = gu$. Suppose $fx = gx$ for some morphism x. Then $(\mathrm{Id}_A, f)x = (\mathrm{Id}_A, g)x$, by the uniqueness property of the product. Hence, by the uniqueness property of the pullback P there exists a unique morphism y such that $uy = x = vy$. This shows that (P, u, v) is the equalizer of f and g. □

In a general category, a morphism with kernel 0 need not be a monomorphism, but in abelian categories vanishing of the kernel implies that the morphism is a monomorphism.

Lemma 2.2.6. *Let $\mathcal{A}$ be an abelian category. A morphism f is a monomorphism (resp. an epimorphism) if and only if* $\ker f = 0$ *(resp.* $\operatorname{coker} f = 0$*).*

Proof. By duality it suffices to prove that f is a monomorphism if and only if $\ker f = 0$. Suppose $f : X \to Y$ is a monomorphism. If $fg = 0$, for some morphism g, then $g = 0$. Thus 0 is the kernel of f.

Conversely, let $\ker f = 0$. Let u, v be morphisms such that $fu = fv$. Let q be the coequalizer of u and v and m the unique morphism such that $f = mq$. Since a coequalizer is an epimorphism, $q = \operatorname{coker} w$ for some morphism w. Let k be the kernel of f. From $fw = mqw = 0$ we get a unique morphism n such that $kn = w$. Thus $w = 0$ because $\ker f = 0$. The morphism q is an isomorphism because the cokernel of the zero morphism is an isomorphism. Thus $qu = qv$ implies $u = v$ which shows that f is a monomorphism. □

To prove factorization of morphisms in abelian categories, we introduce strong epimorphisms. We will see that in an abelian category every epimorphism is strong.

Definition 2.2.7 (Strong and regular epimorphisms)**.** An epimorphism $f : A \to B$ is *strong* if for every commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "g'"] \arrow[dl, dashed, "h"'] \\ C \arrow[r, "f'"] & D \end{tikzcd}$$

where f' is a monomorphism, there exists a unique morphism $h : B \to C$ making the diagram commutative.

An epimorphism is said *regular* if it is the coequalizer of some pair of morphisms.

Lemma 2.2.8. *A regular epimorphism is a strong epimorphism.*

Proof. Let f be the coequalizer of $a, b : A \to B$ such that the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "{a,b}"] & B \arrow[r, "f"] \arrow[d, "g"] & C \arrow[d, "g'"] \arrow[dl, dashed, "h"'] \\ & X \arrow[r, "f'"] & Y \end{tikzcd}$$

where f' is a monomorphism. By commutativity $g'fa = f'ga$ and $g'fb = f'gb$. Since f' is a monomorphism, we get $ga = gb$. Hence there exists a unique morphism h such that $fh = g$. Since f is an epimorphism, by commutativity $g' = f'h$. From the fact f' is a monomorphism, the morphism h is unique. This shows that f is a strong epimorphism. ☐

Since all epimorphisms in an abelian category are regular by AB4, the above lemma implies that all epimorphisms in an abelian category are strong.

Lemma 2.2.9. *Let $f = ip$ for some strong epimorphism p and some monomorphism i. This factorization is unique up to unique isomorphism, more precisely, if $ip = i'p'$ where p' is a strong epimorphism and i' is a monomorphism, then there exists a unique isomorphism h such that $p = hp'$ and $i' = ih$.*

Proof. Let $f = ip = i'p'$, with p and p' strong epimorphisms, and i and i' monomorphisms. By definition of strong epimorphism there exist unique morphisms h and h' such that $hi = i'$, $h'i' = i$, $hp = p'$, and $h'p' = p$. By uniqueness $hh' = \mathrm{Id}$ and $h'h = \mathrm{Id}$. Thus h is an isomorphism. ☐

We are ready to prove the epimorphism monomorphism factorization of morphisms in an abelian category. Define the *image* of a morphism f to be $\ker(\operatorname{coker} f)$ and the *coimage* of a morphism f to be $\operatorname{coker}(\ker f)$.

Theorem 2.2.10 (Factorization of morphisms)**.** *Let $\mathcal{A}$ be an abelian category. Every morphism $f : X \to Y$ in $\mathcal{A}$ can be factorized uniquely, up to unique isomorphism, as $f = me$, where e is an epimorphism and m is a monomorphism. In particular, we have $f = ip$, where $i = \ker(\operatorname{coker} f)$ and $e = \operatorname{coker}(\ker f)$.*

Moreover, if we have the following commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "e_1"] \arrow[d, "f"] & X_2 \arrow[r, "m_1"] \arrow[d, dashed, "h"] & X_3 \arrow[d, "g"] \\ Y_1 \arrow[r, "e_2"] & Y_2 \arrow[r, "m_2"] & Y_3 \end{tikzcd}$$

where e_1 and e_2 are epimorphisms and m_1 and m_2 are monomorphisms, then there exists a unique morphism h which makes the diagram commutative.

Proof. **Factorization:** Fix notation by the following diagram

$$\begin{tikzcd}[column sep=small]
 & X \arrow[rrr, "f"] \arrow[dr, "f_2"] & & & Y \arrow[dr, "f_4"] & \\
\ker f \arrow[ur, "f_1"] & & \operatorname{Coim} f \arrow[r, dashed, "g"] & \operatorname{Im} f \arrow[ur, "f_3"] & & \operatorname{coker} f
\end{tikzcd} \tag{2.2}$$

Since $ff_1 = 0$, there exists a unique morphism ϕ such that $f = \phi f_2$. Now $f_4\phi f_2 = f_4 f = 0$, and f_2 is an epimorphism, so $f_4\phi = 0$. Thus there exists a unique morphism g such that $f = \phi f_2 = f_4 g f_2$.

To show that g is an isomorphism it suffices to show that g is an epimorphism and a monomorphism. Let v be a morphism such that $f_3 g v = 0$ and let q be the cokernel of v. Hence there exists a unique morphism r such that $rq = f_3 g$. Now qf_2 is an epimorphism, so by AB4 there exists a morphism u such that $qf_2 = \operatorname{coker} u$. From $fu = f_3 g f_2 u = rqf_2 u = 0$ one obtains that $u = f_1 l$ for some unique morphism l. Next, $f_2 u = f_2 f_1 l = 0$, so $f_2 = sqf_2$ for some unique morphism s and $sq = \mathrm{Id}$ by the fact that f_2 is an epimorphism. From this it follows that q is a monomorphism, so $qv = 0$ implies $v = 0$. This shows that $f_3 g$ is a monomorphism. But f_3 as a kernel is a monomorphism, so g is a monomorphism. Dually one gets that g is an epimorphism and hence an isomorphism.

Lemma 2.2.9 implies that the factorization is unique up to unique isomorphism.

Morphism between factorizations: Since all epimorphisms in an abelian category are strong, by definition of strong epimorphism there exists a unique morphism h making the following diagram commutative.

$$\begin{tikzcd}
X_1 \arrow[r, "e_1"] \arrow[d, "e_2 f"] & X_2 \arrow[d, "g m_1"] \arrow[dl, "h"] \\
Y_2 \arrow[r, "m_2"] & Y_3
\end{tikzcd}$$

□

The following are useful corollaries of the above result.

Corollary 2.2.11. *In an abelian category $\mathcal{A}$, every monomorphism (resp. epimorphism) is the kernel of its cokernel (resp. cokernel of its kernel).*

Proof. Let $f : A \to B$ be a monomorphism. By theorem 2.2.10 we have $f = \ker(\operatorname{coker}(f))e$ for some epimorphism e. Since f is a monomorphism, e is a monomorphism, and hence an isomorphism. Thus f is the kernel of its cokernel. By duality one obtains that an epimorphism is the cokernel of its kernel. □

Corollary 2.2.12. *Let $\mathcal{A}$ be an abelian category and $f = me$ a morphism of $\mathcal{A}$ where m is a monomorphism and e is an epimorphism. Then $\ker f \cong \ker e$ and $\operatorname{coker} f \cong \operatorname{coker} m$.*

Proof. By duality it suffices to show that $\ker f \cong \ker e$. Let $k_1 : \ker f \to f$ and $k_2 : \ker e \to e$ be the corresponding morphisms. Then $fk_1 = 0$ implies $ek_1 = 0$ and $ek_2 = 0$ implies $fk_2 = 0$. From the universal properties if follows that $\ker f \cong \ker e$. □

The rest of this section is devoted for a quick introduction to exact sequences in abelian categories.

Definition 2.2.13 (Exact sequence)**.** Let $\mathcal{A}$ be an abelian category. A sequence of objects and morphisms of the form

$$\dots \xrightarrow{\phi^{i-2}} X^{i-1} \xrightarrow{\phi^{i-1}} X^i \xrightarrow{\phi^i} X^{i+1} \xrightarrow{\phi^{i+1}} \dots$$

is an *exact sequence* if $\operatorname{Im}\phi^{i-1} \cong \ker\phi^i$, where $\operatorname{Im}\phi^{i-1} = \ker(\operatorname{coker}\phi^{i-1})$, for all $i \in \mathbb{Z}$.

In case an exact sequence consists of at most 3 consecutive nonzero objects the sequence is called a *short exact sequence*. In this case we do not write all the 0 objects but only

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \ .$$

We have the following useful criterion for exact sequences

Proposition 2.2.14. *Let $\mathcal{A}$ be an abelian category. Then*

(i) $0 \to A \xrightarrow{f} B$ *is an exact sequence if and only if f is a monomorphism.*

(ii) $B \xrightarrow{f} A \to 0$ *is an exact sequence if and only if f is an epimorphism.*

(iii) $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ *is an exact sequence if and only if* $f = \ker g$.

(iv) $C \xrightarrow{g} B \xrightarrow{f} A \to 0$ *is an exact sequence if and only if* $f = \operatorname{coker} g$.

Proof. Let us prove (i). The morphism $0 \to A$ is a monomorphism because 0 is the terminal object in $\mathcal{A}$. Hence by corollary 2.2.11, the morphism $0 \to A$ is image of itself. Thus by exactness $\ker f = 0$. By corollary 2.2.11 it follows that f is a monomorphism. Conversely, if f is a monomorphism, then by corollary 2.2.11 $0 \to A$ is its kernel. Thus the sequence is exact. By duality (ii) is also proven.

The statement of (iii) is a direct consequence of (i) and corollary 2.2.11. By duality (iv) holds. $\square$

The following lemma will be used later in derived functors.

Lemma 2.2.15. *Consider the following diagram in an abelian category $\mathcal{A}$.*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e_1"'] & & Y \arrow[rr, "g"] \arrow[dr, "e_2"'] & & Z \\
& \ker(\operatorname{coker} f) \arrow[ur, "m_1"'] & & \ker(\operatorname{coker} g) \arrow[ur, "m_2"'] &
\end{tikzcd} .$$

Then (f, g) is exact if and only if (m_1, e_2) is exact. If this is true, we have the following short exact sequence

$$0 \longrightarrow \ker g \xrightarrow{m_1} Y \xrightarrow{e_2} \ker(\operatorname{coker} g) \longrightarrow 0 \ .$$

Proof. Follows from 2.2.14, 2.2.12, and 2.2.6. $\square$

Definition 2.2.16 (Split short exact sequence)**.** Let $\mathcal{A}$ be an abelian category. We say that a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

splits, or that it is a split short exact sequence, if there exist morphisms $i : Y \to X$ and $j : Z \to Y$ such that $if = \mathrm{Id}_X$ and $gj = \mathrm{Id}_Z$.

Example 2.2.17. Not every short exact sequence splits. Consider the following short exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} \mathbb{Z}/4\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

in the category **Ab**, where $f(1) = 2$ and $g(1) = 1$. The only nonzero morphism from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/4\mathbb{Z}$ sends 1 to 2, so there cannot be a morphism $j : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ with $gj = \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}}$.

Definition 2.2.18 (Semisimple)**.** An abelian category $\mathcal{A}$ where all the short exact sequences split is called *semisimple.*

Example 2.2.19. The category of finite dimensional vector spaces over a field is semisimple.

2.3 An abelian category is additive

In this section we follow [Bor94b, Section 1.6] to prove that an abelian category is additive. This result is interesting in the sense that in the definition of abelian category 2.2.1 we didn't touch the set of morphisms between objects directly. Nevertheless, this definition gives the sets of morphisms between objects the structure of abelian groups, that is, $\mathrm{Mor}(X, Y)$ is an abelian group for any objects X and Y, which are compatible with composition of morphisms. This means that for any morphisms $f, g : X \to Y$ and $h, k : Y \to Z$ we have $h(f + g) = hf + hg$ and $(h + k)f = hf + kf$.

Lemma 2.3.1. *Let $\mathcal{A}$ be an abelian category. For any object A of $\mathcal{A}$ the cokernel $q : A \times A \to Q$ of $\Delta : A \to A \times A$, the unique morphism induced by the morphisms Id_A and Id_A, has the property that $Q \cong A$.*

Proof. Denote by $r : A \to Q$ the composite $q(\mathrm{Id}_A, 0)$. We prove that r is an isomorphism. Fix notation by the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[d, tail, "{(\mathrm{Id}_A,0)}"] \arrow[dr, "r"] & \\
A \arrow[r, tail, "\Delta"] \arrow[d, "\mathrm{Id}_A"] \arrow[dr, "\mathrm{Id}_A"] & A \times A \arrow[r, two heads, "q"] \arrow[d, "p_2"] \arrow[dl, "p_1"] & Q \; . \\
A & A &
\end{tikzcd}$$

We have $\Delta = \ker(\mathrm{coker}(\Delta)) = \ker(q)$ by corollary 2.2.11 because Δ is a monomorphism. From $p_1(\mathrm{Id}_A, 0) = \mathrm{Id}_A = p_2(0, \mathrm{Id}_A)$ it follows that p_1 and p_2 are epimorphisms and $(\mathrm{Id}_A, 0)$ and $(0, \mathrm{Id}_A)$ are monomorphisms. Let $v : V \to A \times A$ be a morphism such that $p_2 v = 0$. Now

$$\begin{cases} p_1(\mathrm{Id}_A, 0)p_1 v &= p_1 v \\ p_2(\mathrm{Id}_A, 0)p_1 v &= 0 \end{cases} \qquad \begin{cases} p_1 v &= p_1 v \\ p_2 v &= 0 \end{cases}$$

so by the universal property of product, $(\mathrm{Id}_A, 0)p_1 v = v$. The factorization $p_1 v : V \to A$ is unique, because $(\mathrm{Id}_A, 0)$ is a monomorphism. This shows that $(\mathrm{Id}_A, 0) : A \to A \times A$ is the kernel of p_2.

Similarly, if $p_1 v = 0$, then from

$$\begin{cases} p_1(0, \mathrm{Id}_A)p_2 v &= 0 \\ p_2(0, \mathrm{Id}_A)p_2 v &= p_2 v \end{cases} \qquad \begin{cases} p_1 v &= 0 \\ p_2 v &= p_2 v \end{cases}$$

the universal property gives $(0, \mathrm{Id}_A)p_2 v = v$. The factorization $p_2 v : V \to A$ is unique because $(0, \mathrm{Id}_A)$ is a monomorphism, hence $(0, \mathrm{Id}_A) = \ker p_1$. Since p_1 and p_2 are epimorphisms, by corollary 2.2.11 $p_2 = \mathrm{coker}(\mathrm{Id}_A, 0)$ and $p_1 = \mathrm{coker}(0, \mathrm{Id}_A)$.

To show that r is a monomorphism, let $x : X \to A$ be a morphism such that $rx = 0$. Since $\Delta = \ker q$, we obtain a morphism $y : X \to A$ such that $(\mathrm{Id}_A, 0)x = \Delta y$. From $y = p_2\Delta y = p_2(\mathrm{Id}_A, 0)x = 0$, we get $0 = y = p_1\Delta y = p_1(\mathrm{Id}_A, 0)x = x$. By 2.2.6, this shows that r is a monomorphism.

To show that r is an epimorphism, let $x : Q \to X$ be a morphism such that $xr = 0$. Since $p_2 = \mathrm{coker}(\mathrm{Id}_A, 0)$, we obtain a morphism $y : A \to X$ such that $yp_2 = xq$. From $y = yp_2\Delta = xq\Delta = 0$, one gets $xq = 0$, and thus $x = 0$, because q is an epimorphism. Therefore r is an epimorphism, by 2.2.6, and we conclude that r is an isomorphism. □

Let us introduce the sum of morphisms for an abelian category $\mathcal{A}$. Denote by q the cokernel of the diagonal morphism $\Delta : A \to A \times A$ and write σ_A for the composite $r^{-1}q$, where r is the composite $q(\mathrm{Id}_A, 0)$ as in lemma 2.3.1. The composite

$$B \xrightarrow{(f,g)} A \times A \xrightarrow{\sigma_A} A$$

is denoted by $f - g$. We define $f + g = f - (0 - g)$ and show that this definition makes an abelian category additive. To do that, we need the following lemma.

Lemma 2.3.2. *Let $\mathcal{A}$ be an abelian category and keep the notation above. For a morphism $f : B \to A$ in $\mathcal{A}$, we have $f \circ \sigma_B = \sigma_A \circ (f \times f)$.*

Proof. Fix notation by the following commutative diagram

$$\begin{tikzcd}
& B \arrow[r, "f"] \arrow[d, tail, "\Delta_B"] & A \arrow[d, tail, "\Delta_A"] & \\
& B \times B \arrow[r, "f \times f"] \arrow[d, "\sigma_B"] & A \times A \arrow[d, "\sigma_A"] & \\
B \arrow[ur, "(\mathrm{Id}_B,0)"] \arrow[r, "\mathrm{Id}_B"] & B \arrow[r, dashed, "g"] & A & A \arrow[l, "\mathrm{Id}_A"] \arrow[ul, "(\mathrm{Id}_A,0)"']
\end{tikzcd}$$

where the morphism $g : B \to A$ is the unique morphism such that $g\sigma_b = \sigma_A(f \times f)$ given by the cokernel property of σ_B. It suffices to show that $f = g$.

Now $q(\mathrm{Id}_A\, 0) = r$, so $\sigma_A(\mathrm{Id}_A, 0) = r^{-1}q(\mathrm{Id}_A, 0) = \mathrm{Id}_A$. Similarly $\sigma_B(\mathrm{Id}_B, 0) = \mathrm{Id}_B$. From

$$\begin{cases} p_1(f \times f)(\mathrm{Id}_B, 0) & = f \\ p_2(f \times f)(\mathrm{Id}_B, 0) & = 0 \end{cases} \qquad \begin{cases} p_1(\mathrm{Id}_A, 0)f & = f \\ p_2(\mathrm{Id}_A, 0)f & = 0 \end{cases}$$

the universal property for product implies $(f \times f)(\mathrm{Id}_B, 0) = (\mathrm{Id}_A, 0)f$. Putting these together we get

$$g = g\sigma_B(\mathrm{Id}_B, 0) = \sigma_A(f \times f)(\mathrm{Id}_B\, 0) = \sigma_A(\mathrm{Id}_A, 0)f = f.$$

□

We are ready to prove that an abelian category is additive.

Theorem 2.3.3. *An abelian category $\mathcal{A}$ is additive.*

Proof. Let A, B, C be any objects in $\mathcal{A}$, and $f : C \to B$ a morphism in $\mathcal{A}$.

$\mathrm{Mor}_{\mathcal{A}}(A, B)$ **abelian:** Let $a, b, c, d : C \to A$ be any morphisms in $\mathcal{A}$. By lemma 2.3.2 for $p_1 : A \times A \to A$ and $p_2 : A \times A \to A$ we have the following commutative diagrams

$$\begin{array}{ccc} (A\times A)\times(A\times A) & \xrightarrow{p_1\times p_1} & A\times A \\ \Big\downarrow{\scriptstyle \sigma_{A\times A}} & & \Big\downarrow{\scriptstyle \sigma_A} \\ A\times A & \xrightarrow{\quad p_1 \quad} & A \end{array} \qquad \begin{array}{ccc} (A\times A)\times(A\times A) & \xrightarrow{p_2\times p_2} & A\times A \\ \Big\downarrow{\scriptstyle \sigma_{A\times A}} & & \Big\downarrow{\scriptstyle \sigma_A} \\ A\times A & \xrightarrow{\quad p_2 \quad} & A \end{array}$$

Now, by commutativity

$$\begin{cases} p_1((a,b)-(c,d)) = p_1\sigma_{A\times A}((a,b),(c,d)) = \sigma_A(p_1\times p_1)((a,b),(c,d)) = \sigma_A(a,c) \\ p_2((a,b)-(c,d)) = p_2((a,b),(c,d))\sigma_{A\times A} = \sigma_A(p_1\times p_1)((a,b),(c,d)) = \sigma_A(b,d) \end{cases}$$

Since

$$\begin{cases} p_1((a-c),(b-d)) = a-c = \sigma_A(a,c) \\ p_2((a-c),(b-d)) = b-d = \sigma_A(b,d) \end{cases}$$

by uniqueness we have

$$(a,b)-(c,d) = (a-c, b-d).$$

Lemma 2.3.2 applied to $\sigma_A : A \times A \to A$, we get the following commutative diagram

$$\begin{array}{ccc} (A\times A)\times(A\times A) & \xrightarrow{\sigma_A\times\sigma_A} & A\times A \\ \Big\downarrow{\scriptstyle \sigma_{A\times A}} & & \Big\downarrow{\scriptstyle \sigma_A} \\ A\times A & \xrightarrow{\quad \sigma_A \quad} & A \end{array}$$

Now

$$\begin{aligned} (a-c)-(b-d) &= \sigma_A((a-c),(b-d)) \\ &= \sigma_A((a,b)-(c,d)) \\ &= \sigma_A\sigma_{A\times A}((a,b),(c,d)) \\ &= \sigma_A(\sigma_A\times\sigma_A)((a,b),(c,d)) \\ &= \sigma_A(\sigma_A(a,b),\sigma_A(c,d)) \\ &= \sigma_A((a-b),(c-d)) \\ &= (a-b)-(c-d), \end{aligned}$$

where the equality $(\sigma_A \times \sigma_A)((a,b),(c,d)) = (\sigma_A(a,b), \sigma_A(c,d))$ follows from the universal property of the product. Indeed, suppose we have a morphism $\phi : Y \to Z$ and morphisms $x, y : X \to Y$. Then

$$\begin{cases} p_1(\phi x, \phi y) &= \phi x \\ p_2(\phi x, \phi y) &= \phi y \end{cases}$$

and from commutativity of the following diagram

$$\begin{array}{ccccc}
X & \xrightarrow{x} & Y & \xrightarrow{\phi} & Z \\
{\scriptstyle p_1}\uparrow & & {\scriptstyle p_1}\uparrow & & {\scriptstyle p_1}\uparrow \\
X \times X & \xrightarrow{(x,y)} & Y \times Y & \xrightarrow{\phi\times\phi} & Z \times Z \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} \\
X & \xrightarrow{y} & Y & \xrightarrow{\phi} & Z
\end{array}$$

we get that

$$\begin{cases} p_1(\phi \times \phi)(x,y) & = \phi x \\ p_2(\phi \times \phi)(x,y) & = \phi y. \end{cases}$$

Hence $(\phi \times \phi)(x,y) = (\phi x, \phi y)$. Denoting B by x, $A \times A$ by Y and Z, σ_A by ϕ, and a and b by x and y one gets the claimed equality.

We are ready to verify the axioms of an abelian group for $\mathrm{Mor}_{\mathcal{A}}(B, A)$.

Zero element The zero element of $\mathrm{Mor}_{\mathcal{A}}(B, A)$ is naturally the zero morphism $0 : B \to A$. From the identity $\sigma_A(\mathrm{Id}_A, 0) = \mathrm{Id}_A$ proved in lemma 2.3.2 and $(a, 0) = (\mathrm{Id}_A, 0)a$ we get $a - 0 = a$.

Inverse From the identity $\sigma_A \Delta_A = 0$, proved in lemma 2.3.2, and $(a, a) = \Delta_A a$ we have

$$a - a = \sigma_A(a, a) = 0.$$

Therefore

$$(0 - a) + a = (0 - a) - (0 - a) = (0 - 0) - (a - a) = 0.$$

Commutativity From the identities

$$(0 - b) - c = (0 - b) - (c - 0) = (0 - c) - (b - 0) = (0 - c) - b$$

and

$$0 - (0 - d) = (d - d) - (0 - d) = (d - 0) - (d - d) = (d - 0) - 0 = d$$

we get

$$\begin{aligned} b + c = b - (0 - c) &= (0 - (0 - b)) - (0 - c) = (0 - 0) - ((0 - b) - c) \\ &= (0 - 0) - ((0 - c) - b) = (0 - (0 - c)) - (0 - b) = c - (0 - b) \\ &= c + b. \end{aligned}$$

Associativity Finally, by using the following 5 identities

$$\begin{aligned} &b + (0 - c) = b - (0 - (0 - c)) = b - c, \\ &b + (0 - b) = b - b = 0, \\ &0 - (c - d) = (0 - 0) - (c - d) = (0 - c) - (0 - d) = (0 - c) + d \\ &0 - (c + d) = 0 - (c - (0 - d)) = (0 - c) + (0 - d) = (0 - c) - d, \\ &(a - b) + d = (a - b) - (0 - d) = (a - 0) - (b - d) = a - (b - d), \end{aligned}$$

we get

$$\begin{aligned}(a+b)+d &= (a-(0-b))+d = a-((0-b)-d) = a-((0-b)-(0-(0-d)) \\ &= a-((0-0)-(b-(0-d))) = a-(0-(b+d)) = a+(b+d).\end{aligned}$$

Composition: To show that addition is bilinear with respect to composition of morphisms, let $x : X \to C$ and $y : A \to Y$ be morphisms in $\mathcal{A}$. Now

$$(a-b)x = \sigma_A(a,b)x = \sigma_A(ax,bx) = ax - bx$$

and by lemma 2.3.2 we have

$$y(a-b) = y\sigma_A(a,b) = \sigma_Y(y \times y)(a,b) = \sigma_Y(ya,yb) = ya - yb.$$

This shows that addition is bilinear under composition of morphisms.

By proposition 2.1.3, $\mathcal{A}$ has the biproduct of A and B. Since A, B, C, and f were arbitrary, we have proven that $\mathcal{A}$ is an additive category. □

Corollary 2.3.4. *Let $\mathcal{A}$ be an abelian category and let $f : A \to B$ and $g : A \to C$ be morphisms. The following sequence*

$$A \xrightarrow{i_1f-i_2g} B \oplus C \xrightarrow{c} W \longrightarrow 0$$

is exact, where $c : B \oplus C \to W$ is the coequalizer of i_1f and i_2g. In particular, the coequalizer W here is the pushout of f and g by the dual version of the proof of proposition 1.2.7.

Proof. The morphism c is an epimorphism because it is a coequalizer. By proposition 2.2.14 it suffices to show that c is the cokernel of $i_1f - i_2g$. But the coequalizer W of i_1f and i_2g is also the cokernel of $i_1f - i_2g$. Indeed, if $d : B \oplus C \to W'$ is a morphism such that $di_1f = di_2g$, equivalently $d(i_1f - i_2g) = 0$, then there is a unique morphism e such that $ec = d$. Thus the sequence is exact. □

Corollary 2.3.5. *Let $\mathcal{A}$ be an abelian category. Then every pushout square is a pullback square. Moreover, pushout of a monomorphism is a monomorphism and pullback of an epimorphism is an epimorphism.*

Proof. By duality, it suffices to show that pushout of a monomorphism is a monomorphism, because pushout squares correspond to pullback squares and monomorphisms to epimorphisms in the opposite category.

Let $f : X \to Y$ be a monomorphism and $g : X \to Z$ a morphism. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i_2g-i_1f"] \arrow[dd, "g"'] & & Y \arrow[dl, "i_1"] \\
& Y \oplus Z \arrow[dr, "c"] & \\
Z \arrow[ur, "i_2"] & & W
\end{tikzcd}$$

where $c : Y \oplus Z \to W$ is the cokernel of $i_2g - i_1f$. Then $(W, ci_1 : Y \to W, ci_2 : Z \to W)$ is the pushout of f and g by the proof of 1.2.7.

Let us show that $i_2g - i_1f$ is a monomorphism. Let $x : D \to X$ be a morphism such that $(i_2g - i_1f)x = 0$. Now $0 = p_1(i_2g - i_1f)x = -fx$, so $x = 0$ by the fact that f is a monomorphism. This shows that $i_2g - i_1f$ is a monomorphism. By 2.2.11 we have $i_2g - i_1f = \ker c$.

To show that ci_2 is a monomorphism, let $y : D \to Y$ be a morphism such that $ci_2 y = 0$. Then there exists a unique morphism $\phi : D \to X$ such that $(i_2 g - i_1 f)\phi = i_2 y$. Now $-f\phi = p_1(i_2 g - i_1 f)\phi = p_1 i_2 y = 0$, so $\phi = 0$ by the fact that f is a monomorphism. From $i_1 y = (i_2 g - i_1 f)\phi = 0$, we get $y = 0$, because i_2 is a monomorphism. This shows that ci_2 is a monomorphism.

It remains to show that a pushout square is a pullback square. Keep the notation of the diagram and let $y : D \to Y$ and $z : D \to Z$ be morphisms such that $ci_1 y = ci_2 z$. Let $q : Y \to V$ be the cokernel of f. Now $qf = 0g = 0$, so there is a unique morphism r such that $rci_1 = q$ and $rci_2 = 0$. From $qy = rci_1 y = rci_2 z = 0$, we get that there is a unique morphism x such that $fx = y$, because $f = \ker q$. Now $ci_2 z = ci_1 y = ci_1 fx = ci_2 gx$, so $z = gx$ by the fact that ci_2 is a monomorphism. Uniqueness of the morphism x follows from the fact that f is a monomorphism. This shows that (X, f, g) is the pullback of (W, ci_1, ci_2). □

2.4 Formalism of pseudo-elements

We introduce the formalism of pseudo-elements for abelian categories following [Bor94b, Chapter 1, Section 9] and [ML78, Chapter VIII]. This formalism allows us to reason about morphisms of abelian categories similarly as one reasons of morphisms of abelian groups by using elements. We will use this formalism to prove exactness of some sequences.

Definition 2.4.1 (Pseudo-element)**.** Let $\mathcal{A}$ be an abelian category. A *pseudo-element* a of $A \in \operatorname{Ob}\mathcal{A}$, written $a \in^* A$, is a morphism with codomain A. Two pseudo-elements $a : X \to A$, $a' : X' \to A$ of A are *pseudo-equal*, written $a =^* a'$, if there exist epimorphisms $p : Y \to X$ and $p' : Y \to X'$ such that $ap = a'p'$.

A pseudo-image under a morphism $f : A \to B$ of a pseudo-element $a \in^* A$ is the composite fa, also denoted $f(a)$. The following proposition proves some basic facts about pseudo-elements.

Proposition 2.4.2. *Let $f : A \to B$, $g : B \to C$ be morphisms in an abelian cateogry $\mathcal{A}$.*

(i) pseudo-equality is an equivalence relation.

(ii) $a =^* b$ implies $f(a) =^* f(b)$.

(iii) $f(g(a)) =^* (fg)(a)$ for $a \in^* A$.

(iv) There is an equivalence class of pseudo-elements of A consisting of zero morphisms with codomain A. In particular, $a =^* 0$ if and only if $a = 0$.

Proof. (i) Obviously pseudo-equality is reflexive and symmetric. Let $a, b, c \in^* A$, so that $au_1 = bu_2$ and $bv_1 = cv_2$ for some epimorphisms u_1, u_2, v_1 and v_2. Consider the following pullback diagram

$$\begin{tikzcd} Z_1 \arrow[r, "u_2'"] \arrow[d, "v_1'"] & Y_1 \arrow[d, "u_2"] \\ Y_2 \arrow[r, "v_1"] & X_2 \end{tikzcd}$$

By corollary 2.3.5 u_2' and v_1' are epimorphisms and $au_1u_2' = bu_2u_2' = bv_1v_1' = cv_2v_1'$. Hence $a =^* b$.

(ii) Obvious from definition.

(iii) Obvious from definition.

(iv) Suppose $a \in^* A$ and $a =^* 0$. Then $av = 0$ for some epimorphism v, so $a = 0$. Conversely, if $a : X \to A$ $b : Y \to A$ are zero morphisms, then $p_1 : X \oplus Y \to X$ and $p_2 : X \oplus Y \to Y$ are epimorphisms and $ap_1 = 0 = bp_2$. Hence, any two zero morphisms with codomain A are pseudo-equal.

□

Next we prove some diagram chasing results for pseudo-elements.

Proposition 2.4.3. *Let $\mathcal{A}$ be an abelian category.*

(i) $f : A \to B$ is the zero morphism if and only if $f(a) = 0$ for all $a \in^* A$.

(ii) $f : A \to B$ is a monomorphism if and only if $f(a) =^* f(a')$ implies $a =^* a'$ for all $a, a' \in^* A$.

(iii) $f : A \to B$ is an epimorphism if and only if for all $b \in^* B$ there exist $a \in^* A$ such that $f(a) =^* b$.

(iv) A short exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if for every $a \in^* A$ we have $g(f(a)) = 0$ and for every $b \in^* B$ with $g(b) = 0$ there exists a pseudo-element $a' \in^* A$ such that $f(a) =^* b$.

(v) For $f : A \to B$ and $a, a' \in^* A$ with $f(a) =^* f(a')$ there exists $a'' \in^* A$ such that $f(a'') = 0$ and for all $g : A \to C$ condition $g(a') = 0$ implies $g(a'') =^* g(a)$.

(vi) Let

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

be a pullback diagram in $\mathcal{A}$ and let $a : A' \to A$ and $b : B' \to B$ be pseudo-elements such that $f(a) =^* g(a)$. Then there is a pseudo-element $d \in^* D$, unique up to pseudo-equality, such that $g'(d) =^* a$ and $f'(d) =^* b$.

Proof. (i) Clear from the definition of pseudo-element and proposition 2.4.2

(ii) Clear from the definition of monomorphism.

(iii) Suppose f is an epimorphism and let $b \in^* B$. Take the pullback of f and b so that $fa = bp$, where p is an epimorphism by corollary 2.3.5. This shows that $f(a) =^* b$. Conversely, suppose that for every pseudo-element $b \in^* B$ there exists a pseudo-element $a \in^* A$ such that for some epimorphisms p and q we have $fap = bq$. Let x be a morphism such that $xf = 0$. Since Id_B is a pseudo-element of B we have $fa'p = q$ for some pseudo-element a' of A and some epimorphisms p and q. This implies that $fa'p$ is an epimorphism and hence fa' is an epimorphism. Now $xfa' = 0$, so $x = 0$. Thus f is an epimorphism.

(iv) Suppose that

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.3}$$

is an exact sequence and let $f = em$ be the factorization given by theorem 2.2.10. Since $gf = 0$, we have $gf(a) = 0$ for all pseudo-elements a of A. Let $b \in^* B$ such that $gb = 0$. Then there exists a unique morphism

c such that $mc = b$ because by exactness $m = \ker g$. By taking the pullback of e and c, we obtain a the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "a"] & X \arrow[d, "c"] \arrow[dr, "b"] & & \\ A \arrow[r, "e"] & I \arrow[r, "m"] & B \arrow[r, "g"] & C \end{tikzcd}$$

where a and q are epimorphisms by corollary 2.3.5. By commutativity $fa = qb$, so $f(a) =^* b$.

Conversely, to show that (2.3) is exact, let $f = me$ be the epimorphism monomorphism factorization of f. It suffices to show that $m = \ker g$. Now $gf = 0$, so $gm = 0$ because e is an epimorphism. If b is a morphism with $gb = 0$, then there exists a pseudo-element $a \in^* A$ such that $fap = bq$ for some epimorphisms p and q. Consider the following pullback square

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "c"] & X \arrow[d, "b"] \\ P \arrow[r, "m"] & B \end{tikzcd}$$

The morphism j is a monomorphism by proposition 1.2.8. Since $bq = meap$, there exists a unique morphism z such that $jz = q$ and $cz = eap$. The morphism q is an epimorphism, and thus j is an epimorphism. Hence j is an isomorphism, and $b = mcj^{-1}$. This shows that m is the kernel of g, because cj^{-1} is unique by the fact that m is a monomorphism.

(v) Let $a, a' \in^* A$ such that $fap = fa'q$ for some epimorphisms p and q. Let $a'' = ap - a'q$. Clearly $f(a'') = 0$, and for all $g : A \to C$ with $ga'q_2 = 0$, for some epimorphism q_2, we have $ga''q_2 = gaq_2$, so $g(a'') =^* g(a)$.

(vi) Let $a \in^* A$, $a : A' \to A$, and $b \in^* B$, $b : B' \to B$ such that $fap = gbq$ for some epimorphisms p and q. Then there exists a unique morphism d such that $g'd = ap$ and $f'd = bq$. Now $d \in^* D$, $D' \to D$ is a pseudo-element such that $f'(d) =^* b$ and $g'(d) =^* a$.

To show uniqueness of the pseudo-element d up to pseudo-equality, let $d' : D'' \to D$ be another such pseudo-element. We can assume that $g'd' = ap'$ and $f'd' = bq'$ for some epimorphisms p' and q'. By taking the following pullbacks and using corollary 2.3.5

$$\begin{tikzcd} W_1 \arrow[r] \arrow[d] & D' \arrow[d, "p"] \\ D'' \arrow[r, "p'"] & A' \end{tikzcd} \qquad \begin{tikzcd} W_2 \arrow[r] \arrow[d] & D' \arrow[d, "q"] \\ D'' \arrow[r, "q'"] & B' \end{tikzcd} \qquad \begin{tikzcd} W_3 \arrow[r] \arrow[d] & W_1 \arrow[d] \\ W_2 \arrow[r] & D' \end{tikzcd}$$

we get that all the morphisms in the above diagrams are epimorphisms and

$$W_3 \to W_1 \to D' \to B' \to B = W_3 \to W_2 \to D'' \to B' \to B$$
$$W_3 \to W_2 \to D' \to A' \to A = W_3 \to W_2 \to D'' \to A' \to A,$$

so by uniqueness of pullback $W_3 \to W_1 \to D' \to D = W_3 \to W_1 \to D'' \to D$. Hence $d =^* d'$.

$\square$

Remark 2.4.4. Note that pseudo-elements cannot be used to prove equality of morphisms in general. Let $a \in^* A$ be a pseudo-element not pseudo-equal to 0. Then $a =^* -a$, but as a morphism $a + a$ need not be equal to 0. Indeed, the morphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ in the category of $\mathbb{Z}$-modules is not equal to $\mathbb{Z} \xrightarrow{-2} \mathbb{Z}$.

2.5 Category of complexes

In this section we construct a new category from an additive category, the category of complexes. The objects of this category are sequences of objects of the underlying category connected with differentials. It turns out that the category of complexes over an additive category is additive and the category of complexes over an abelian category is abelian, see lemma 2.5.6 and theorem 2.5.7. The language of complexes allows us to define cohomology, which is central in homological algebra.

Definition 2.5.1 (Category of complexes)**.** For any additive category $\mathcal{A}$ we can construct the category of *(cochain) complexes* $C(\mathcal{A})$. The objects of $C(\mathcal{A})$ are sequences $A^\bullet = (A^i, d^i_{A^\bullet} : A^i \to A^{i+1})_{i\in\mathbb{Z}}$, where A^i are objects of $\mathcal{A}$ and $d^i_{A^\bullet} : A^i \to A^{i+1}$ are morphisms of $\mathcal{A}$, called *differentials*, such that $d^{i+1}_{A^\bullet} d^i_{A^\bullet} = 0$ for all i. A morphism of complexes $f : A^\bullet \to B^\bullet$ is a sequence of morphisms $f^i : A^i \to B^i$ in $\mathcal{A}$, $i \in \mathbb{Z}$, such that the following diagram is commutative

$$\begin{tikzcd}
\dots \arrow[r, "d^{i-2}_{A^\bullet}"] & A^{i-1} \arrow[r, "d^{i-1}_{A^\bullet}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d^{i}_{A^\bullet}"] \arrow[d, "f^{i}"] & A^{i+1} \arrow[r, "d^{i+1}_{A^\bullet}"] \arrow[d, "f^{i+1}"] & \dots \\
\dots \arrow[r, "d^{i-2}_{B^\bullet}"] & B^{i-1} \arrow[r, "d^{i-1}_{B^\bullet}"] & B^i \arrow[r, "d^{i}_{B^\bullet}"] & B^{i+1} \arrow[r, "d^{i+1}_{B^\bullet}"] & \dots
\end{tikzcd}$$

We denote by $C^+(\mathcal{A})$ (resp. $C^-(\mathcal{A})$, resp. $C^b(\mathcal{A})$) the full subcategory of $C(\mathcal{A})$ consisting of objects $X \in C(\mathcal{A})$ such that for some $n \in \mathbb{N}$, $X^i = 0$ for all $i < -n$ (resp. $i > n$, resp. $i < -n$ and $i > n$).

The construction of cochain complexes can be expressed also in terms of functors. Let I be a category where objects are the elements of $\mathbb{Z} \cup \{*\}$. Suppose $*$ is the zero object in I, and for any $i \in \mathbb{Z}$, there exists a morphism $i \to i+1$ such that $i \to i+1 \to i+2 = i \to * \to i+2$. Then a complex over an additive category $\mathcal{A}$ is a functor $F : I \to \mathcal{A}$ which preserves the zero object. A morphism of complexes $F : I \to \mathcal{A}$ and $G : I \to \mathcal{A}$ is a natural transformation $F \to G$.

Definition 2.5.2 (Cohomology complex)**.** Let $C(\mathcal{A})$ be the category of chain complexes over an abelian category. Let $X^\bullet \in \operatorname{Ob} C(\mathcal{A})$ and consider the commutative diagram

$$\begin{tikzcd}
 & \operatorname{coker} d^{i-1}_{X^\bullet} \arrow[dr, dashed, "b^i"] & \\
X^{i-1} \arrow[r, "d^{i-1}_{X^\bullet}"] \arrow[dr, dashed, "a^i"'] & X^i \arrow[u, "\phi^i_2"] \arrow[r, "d^i_{X^\bullet}"] & X^{i+1} \\
 & \ker d^i_{X^\bullet} \arrow[u, "\phi^i_1"] &
\end{tikzcd} \tag{2.4}$$

where existence of morphisms a^i and b^i follows from the universal property of kernel and cokernel and the equation $d^i_{X^\bullet} d^{i-1}_{X^\bullet} = 0$. The *i-th cohomology* of $X^\bullet$ is the object

$$H^i(X^\bullet) = \operatorname{coker} a^i \cong \ker b^i$$

and the *cohomology complex* of $X^\bullet$ is the complex

$$H^\bullet(X^\bullet) : \ \dots \xrightarrow{0} H^{i-1}(X^\bullet) \xrightarrow{0} H^i(X^\bullet) \xrightarrow{0} H^{i+1}(X^\bullet) \xrightarrow{0} \dots .$$

We need to verify that the above definition is well-defined, that is $\operatorname{coker} a^i \cong \ker b^i$. Let $d_{X^\bullet}^{i-1} = m_1 e_1$ and $d_{X^\bullet}^i = m_2 e_2$ be the factorizations given by theorem 2.2.10. We get the following commutative diagram where the morphisms a' and b' are obtained from equations $m_2 e_2 m_1 = 0$ and $e_2 m_1 e_1 = 0$.

$$\begin{tikzcd}
 & & \operatorname{coker} d_{X^\bullet}^{i-1} \arrow[dr, "b'"] \arrow[drr, "b^i"] & & \\
X^{i-1} \arrow[r, "e_1"] \arrow[drr, "a^i"'] & Y^{i-1} \arrow[r, "m_1"] \arrow[dr, "a'"] & X^i \arrow[u, "\phi_2^i"] \arrow[r, "e_2"] & Y^i \arrow[r, "m_2"] & X^{i+1} \\
 & & \ker d_{X^\bullet}^i \arrow[u, "\phi_1^i"] & &
\end{tikzcd}$$

From $m_1 = \phi_1^i a'$ and $e_2 = b' \phi_2^i$ we get that a' is a monomorphism and b' is an epimorphism. By corollary 2.2.12 $\operatorname{coker} a' = \operatorname{coker} a^i$ and $\ker b' = \ker b^i$. Thus it suffices to show that $\operatorname{coker} a' \cong \ker b'$.

We show that a' is the kernel of $\phi_2^i \phi_1^i$. Let ψ be a morphism such that $\phi_2^i \phi_1^i \psi = 0$. By corollaries 2.2.11 and 2.2.12 $m_1 = \ker(\operatorname{coker} m_1) = \ker \phi_2^i$, so there exists a unique morphism h such that $m_1 h = \phi_1^i a' h = \phi_1^i \psi$. Now ϕ_1^i is a monomorphism, so $a' h = \psi$. The morphism h is unique because a' is a monomorphism. This shows that a' is the kernel of $\phi_2^i \phi_1^i$. Similarly one shows that b' is the cokernel of $\phi_2^i \phi_1^i$. Putting these together and using theorem 2.2.10 we get

$$\operatorname{coker} a^i = \operatorname{coker} a' = \operatorname{coker}(\ker \phi_2^i \phi_1^i) \cong \ker(\operatorname{coker} \phi_2^i \phi_1^i) = \ker b' = \ker b^i.$$

This shows that cohomology is well-defined.

We prove an alternative characterization for cohomology which will be used to obtain a long exact sequence from a short exact sequence of complexes.

Lemma 2.5.3. *For any complex $X^\bullet$ there exists a unique morphism h such that the following diagram*

$$\begin{tikzcd}
X^i \arrow[r, "\phi_1^i"] \arrow[dr, "\phi_2^i"'] & \operatorname{coker} d_{X^\bullet}^{i-1} \arrow[d, dashed, "h"] \arrow[dr, "\psi_1^i"] & \\
 & \ker d_{X^\bullet}^{i+1} \arrow[r, "\psi_2^i"] & X^{i+1}
\end{tikzcd} \tag{2.5}$$

is commutative. Here $\psi_1^i \phi_1^i = \psi_2^i \phi_2^i = d_{X^\bullet}^i$, ϕ_1^i is an epimorphism, and ψ_2^i is a monomorphism. Moreover, $H^i(X^\bullet) \cong \ker h$ and $H^{i+1}(X^\bullet) \cong \operatorname{coker} h$.

Proof. **Construction of h:** From $d_{X^\bullet}^{i+1} \psi_1^i \phi_1^i = d_{X^\bullet}^{i+1} d_{X^\bullet}^i = 0$ we get $d_{X^\bullet}^{i+1} \psi_1^i = 0$, because ϕ_1^i is an epimorphism. Hence, there exists a unique morphism $h : \operatorname{coker} d_{X^\bullet}^{i-1} \to \ker d_{X^\bullet}^{i+1}$ such that $\psi_1^i = \psi_2^i h$. Now $\psi_2^i \phi_2^i = \psi_1^i \phi_1^i = \psi_2^i h \phi_1^i$ and $\phi_2^i = h \phi_1^i$ by the fact that ψ_2^i is a monomorphism. This shows that the resulting diagram is commutative.

$H^i(X^\bullet) \cong \ker h$: Let $\alpha_1 : \ker h \to \operatorname{coker} d_{X^\bullet}^{i-1}$ be the kernel of h and $\alpha_2 : H^i(X^\bullet) \to \operatorname{coker} d_{X^\bullet}^{i-1}$ the kernel of $\phi_1^i : \operatorname{coker} d_{X^\bullet}^{i-1} \to X^{i+1}$. Now $h\alpha_2 = 0$, because ψ_2^i is a monomorphism, and $\psi_1^i \alpha_1 = \psi_2^i h \alpha_1 = 0$. Thus there exist unique morphisms $\gamma_1 : \ker h \to H^i(X^\bullet)$ and $\gamma_2 : H^i(X^\bullet) \to \ker h$ such that $\alpha_2 \gamma_1 = \alpha_1$ and $\alpha_1 \gamma_2 = \alpha_2$. Hence $\alpha_1 = \gamma_2 \gamma_1 \alpha_1$, so $\gamma_2 \gamma_1 = \mathrm{Id}_{\ker h}$, and $\alpha_2 = \gamma_1 \gamma_2 \alpha_2$, so $\gamma_1 \gamma_2 = \mathrm{Id}_{H^i(X^\bullet)}$. This shows that $H^i(X^\bullet) \cong \ker h$.

$H^{i+1}(X^\bullet) \cong \operatorname{coker} h$: Let $\beta_1 : \ker d_{X^\bullet}^{i+1} \to \operatorname{coker} h$ be the cokernel of h and $\beta_2 : \ker d_X^{i+1\bullet} \to H^{i+1}(X^\bullet)$ be the cokernel of $X^i \to \ker d_{X^\bullet}^{i+1}$. Now $\beta_2 h \phi_1^i = \beta_2 \phi_2^i = 0$, so $\beta_2 h = 0$, by the fact that ϕ_1^i is an epimorphism, and

$\beta_1\phi_2^i = \beta_1 h\phi_1^i = 0$. Therefore there exist unique morphisms $\gamma_1 : \operatorname{coker} h \to H^{i+1}(X^\bullet)$ and $\gamma_2 : H^{i+1}(X^\bullet) \to \operatorname{coker} h$ such that $\gamma_1\gamma_2\beta_2 = \beta_2$ and $\gamma_2\gamma_1\beta_1 = \beta_1$, so $\gamma_1\gamma_2 = \mathrm{Id}_{H^{i+1}(X^\bullet)}$ and $\gamma_2\gamma_1 = \mathrm{Id}_{\operatorname{coker} h}$.

□

For any morphism $f : X^\bullet \to Y^\bullet$ of complexes over an abelian category, we have an induced morphism of complexes $H^i(f) : H^i(X) \to H^i(Y)$ for all i. Indeed,

$$\ker d_X^i \to X^i \to Y^i \to Y^{i+1} = \ker d_X^i \to X^i \to X^{i+1} \to Y^{i+1} = 0,$$

so there exists a unique morphism $\ker f^i : \ker d_X^i \to \ker d_Y^i$ such that the following diagram is commutative

$$\begin{tikzcd} \ker d_X^i \arrow[r, "\ker f^i"] \arrow[d] & \ker d_Y^i \arrow[d] \\ X^i \arrow[r, "f^i"] & Y^i \end{tikzcd} \tag{2.6}$$

By commutativity

$$\begin{aligned} X^{i-1} \to \ker d_X^i \to \ker d_Y^i \to Y^i &= X^{i-1} \to \ker d_X^i \to X^i \to Y^i \\ &= X^{i-1} \to X^i \to Y^i \\ &= X^{i-1} \to Y^{i-1} \to Y^i \\ &= X^{i-1} \to Y^{i-1} \to \ker d_Y^i \to Y^i. \end{aligned}$$

The morphism $\ker d_Y^i \to Y^i$ is a monomorphism, so we have

$$X^{i-1} \to \ker d_X^i \to \ker d_Y^i = X^{i-1} \to Y^{i-1} \to \ker d_Y^i. \tag{2.7}$$

Therefore

$$X_1^{i-1} \to \ker d_X^i \to \ker d_Y^i \to H^i(Y) = X^{i-1} \to Y^{i-1} \to \ker d^i \to H^i(Y) = 0,$$

and by the cokernel property for $H^i(X)$ there exists a unique morphism $H^i(f) : H^i(X) \to H^i(Y)$ such that the following diagram is commutative

$$\begin{tikzcd} \ker d_X^i \arrow[r, "\ker f^i"] \arrow[d] & \ker d_Y^i \arrow[d] \\ H^i(X^\bullet) \arrow[r, "H^i(f)"] & H^i(Y^\bullet) \end{tikzcd} \tag{2.8}$$

By fixing kernels and cokernels of each morphism, this allows us to define a functor $C(\mathcal{A}) \to C(\mathcal{A})$ which maps a complex to its cohomology complex and morphisms to induced morphisms. This functor is called the *cohomology functor*. Indeed, by using the property that the morphisms $\ker f^i$ and $H^i(f)$ in the commutative diagrams (2.6) and (2.8) are unique it is easy to see that this map preserves identity morphisms and composition of morphisms. We have proved the following proposition.

Proposition 2.5.4. *Let $\mathcal{A}$ be an abelian category. There exists a functor $H^\bullet : C(\mathcal{A}) \to C(\mathcal{A})$ which sends a complex $X^\bullet$ to the cohomology complex $H^\bullet(X^\bullet)$ and a morphism $f : X^\bullet \to Y^\bullet$ to the induced morphism $H^\bullet(f) : H^\bullet(X^\bullet) \to H^\bullet(Y^\bullet)$.*

The following proposition gives an alternative characterization of the pseudo-elements of i-th cohomology of a complex.

Proposition 2.5.5. *Let $X^\bullet \in \operatorname{Ob} C(\mathcal{A})$ where $\mathcal{A}$ is an abelian category. The pseudo-elements of $H^i(X^\bullet)$ are in natural one-to-one correspondence with pseudo-elements $x \in^* X^i$ such that $d^i_{X^\bullet}(x) = 0$ under the following equivalence relation $\sim$: for $x : Y_1 \to X^i, y : Y_2 \to X^i \in^* X^i$ we have $x \sim y$ if and only if there exist epimorphisms $e_1 : V \to Y_1$ and $e_2 : V \to Y_2$ such that $xe_1 - ye_2 = d^{i-1}_{X^\bullet} z$ for some pseudo-element $z \in^* X^{i-1}$.*

This one-to-one correspondence ψ is given for a pseudo-element $x : U \to X^i$ such that $d^i_{X^\bullet} x = 0$ by first taking the unique morphism $x' : U \to \ker d^i_{X^\bullet}$ such that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "x"] \arrow[dr, "x'"'] & X^i \\ & \ker d^i_{X^\bullet} \arrow[u, "\phi^i_1"'] \end{tikzcd}$$

and then we let $\psi(x)$ to be the composite of x' followed by the morphism $\gamma : \ker d^i_{X^\bullet} \to H^i(X^\bullet)$, the cokernel of the morphism $a^i : X^{i-1} \to \ker d^i_{X^\bullet}$ constructed in the diagram (2.4).

Proof. Fix notation by the following commutative diagram

$$\begin{tikzcd} X^{i-1} \arrow[r, "d^{i-1}_{X^\bullet}"] \arrow[dr, "a^i"'] & X^i & \\ & \ker d^i_{X^\bullet} \arrow[u, "\phi^i_1"'] \arrow[dr, "\gamma"] & \\ & & H^i(X^\bullet) \end{tikzcd}$$

First we need to verify that the map ψ is well-defined. Let $x, y \in^* X^i$ be pseudo-elements such that $d^i_{X^\bullet} x = d^i_{X^\bullet} y = 0$ and $xe_1 - ye_2 = d^{i-1}_{X^\bullet} z$ for some pseudo-element z of X^{i-1} and epimorphisms e_1 and e_2. By definition of kernel there exist unique morphisms x' and y' such that $\phi^i_1 x' = x$ and $\phi^i_1 y' = y$. Since $\phi^i_1(x'e_1 - y'e_2) = d^{i-1}_{X^\bullet} z = \phi^i_1 a^i z$, and $x'e_1 - y'e_2$ is the unique morphism with this property, we have $x'e_1 - y'e_2 = a^i z$, because ϕ^i_1 is a monomorphism. Now $\gamma a^i z = 0$, by the fact that γ is the cokernel of a^i, and we find that $\gamma x' e_1 = \gamma y' e_2$. This shows that $\psi(x) =^* \psi(y)$ and ψ is well-defined.

To show that every pseudo-element of $H^i(X^\bullet)$ is an image of a pseudo-element of X^i, let $w \in^* H^i(X^\bullet)$. By 2.4.3 (iii) there exists $y \in^* \ker d^i_{X^\bullet}$ such that $\phi^i_1(y) =^* w$. Then $\phi^i_1(y)$ is a pseudo-element of X^i such that $d^i_{X^\bullet}(\phi^i_1(y)) = 0$. By construction of ψ, $\phi^i_1(y)$ is sent to a pseudo-element of $\ker d^{i-1}_{X^\bullet}$ pseudo-equal to y. Thus ψ is surjective.

It remains to show that ψ is injective. Let $x, y \in^* X^i$ be pseudo-elements which are mapped to pseudo-equal pseudo-elements of $H^i(X^\bullet)$. We have to show that $xe_1 - ye_2 = d^{i-1}_{X^\bullet} z$ for some pseudo-element z of X^{i-1} and some epimorphisms e_1 and e_2. Let x' and y' be the unique morphisms such that $\phi^i_1 x' = x$ and $\phi^i_1 y' = y$. By assumption $\gamma x' p_1 = \gamma y' p_2$ for some epimorphisms p_1 and p_2. Now $\gamma(x'p_1 - y'p_2) = 0$ and by 2.4.3 (iv) there exists a pseudo-element z' of X^{i-1} such that $a^i z q_1 = (x'p_1 - y'p_2)q_2$ for some epimorphisms q_1 and q_2. Now $d^{i-1}_{X^\bullet}(z'q_1) = \phi^i_1(x'p_1 - y'p_2)q_2 = xp_1q_2 - yp_2q_2$. Hence we can take $z = z'q_1$. This shows that ψ is injective. $\square$

Lemma 2.5.6. *If $\mathcal{A}$ is an additive category, then $C(\mathcal{A})$ is additive.*

Proof. **Abelian group:** Let $f, g : K^\bullet \to L^\bullet$ be two morphisms in $C(\mathcal{A})$. The following diagram commutes

$$\begin{tikzcd}
\dots \arrow[r, "d_{A^\bullet}^{i-2}"] & A^{i-1} \arrow[r, "d_{A^\bullet}^{i-1}"] \arrow[d, "f^{i-1}+g^{i-1}"] & A^i \arrow[r, "d_{A^\bullet}^{i}"] \arrow[d, "f^{i}+g^{i}"] & a^{i+2} \arrow[r, "d_{A^\bullet}^{i+1}"] \arrow[d, "f^{i+1}+g^{i+1}"] & \dots \\
\dots \arrow[r, "d_{B^\bullet}^{i-2}"] & B^{i-1} \arrow[r, "d_{B^\bullet}^{i-i}"] & B^i \arrow[r, "d_{B^\bullet}^{i}"] & B^{i+1} \arrow[r, "d_{B^\bullet}^{i+1}"] & \dots
\end{tikzcd}$$

so $f + g := (f^i + g^i)_{i\in\mathbb{Z}}$ is a morphism of complexes. If $\alpha, \beta : L^\bullet \to M^\bullet$ are two morphisms in $C(\mathcal{A})$ we have

$$\begin{aligned}
((f+g)\circ(\alpha+\beta))^i &= (f^i + g^i)\circ(\alpha^i + \beta^i) \\
&= (f^i \circ \alpha^i) + (f^i \circ \beta^i) + (g^i \circ \alpha^i) + (g^i \circ \beta^i) \\
&= (f\circ\alpha)^i + (f\circ\beta)^i + (g\circ\alpha)^i + (g\circ\beta)^i.
\end{aligned}$$

This shows that addition is bilinear with respect to composition.

Zero object: Clearly the complex $(0_{\mathcal{A}}, 0^i_{\mathcal{A}})_{i\in\mathbb{Z}}$ is the zero complex in $C(\mathcal{A})$.

Biproduct: To construct the biproduct in $C(\mathcal{A})$, let $A^\bullet, B^\bullet \in \operatorname{Ob} C(\mathcal{A})$ be objects in $C(\mathcal{A})$. For any $i \in \mathbb{Z}$, let $(A \oplus B)^i = A^i \oplus B^i$ and let the differential $d^i_{A^\bullet \oplus B^\bullet}$ to be the unique morphism $i_1 d^i_{A^\bullet} p_1 + i_2 d^i_{B^\bullet} p_2$ which makes the following diagram commutative

$$\begin{tikzcd}
 & A^{i+1} \oplus B^{i+1} & \\
A^i \arrow[ur, "i_1 d^i_{A^\bullet}"] \arrow[r, "i_1"'] & A^i \oplus B^i \arrow[u, dashed, "d^i_{A^\bullet\oplus B^\bullet}"] & B^i \arrow[ul, "i_2 d^i_{B^\bullet}"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Since $d^{i+1}_{A^\bullet} d^i_{A^\bullet} = d^{i+1}_{B^\bullet} d^i_{B^\bullet} = 0$, we have $d^{i+1}_{A^\bullet \oplus B^\bullet} d^i_{A^\bullet \oplus B^\bullet} = 0$. Hence $(A^\bullet \oplus B^\bullet, d_{A^\bullet \oplus B^\bullet})$ is a complex.
We define $i_1 : A^\bullet \to A^\bullet \oplus B^\bullet$, $(i_1)^i := i_1 : A^i \to A^i \oplus B^i$, $i_2 : B^\bullet \to A^\bullet \oplus B^\bullet$, $(i_2)^i := i_2 : B^i \to A^i \oplus B^i$, $p_1 : A^\bullet \oplus B^\bullet \to A^\bullet$, $(p_1)^i := p_1 : A^i \oplus B^i \to A^i$, and $p_2 : A^\bullet \oplus B^\bullet \to B^\bullet$, $(p_2)^i := p_2 : A^i \oplus B^i \to B^i$. One can easily check that these are morphisms in $C(\mathcal{A})$ satisfying the following formulas

$$p_1 i_1 = \mathrm{Id}_{A^\bullet}, \qquad p_2 i_2 = \mathrm{Id}_{B^\bullet}, \qquad p_1 i_2 = 0, p_2 i_1 = 0, \qquad p_1 i_1 + p_2 i_2 = \mathrm{Id}_{A^\bullet \oplus B^\bullet}.$$

Let $C^\bullet \in C(\mathcal{A})$ be a complex and $f : A^\bullet \to C^\bullet$ and $g : B^\bullet \to C^\bullet$ be morphisms of complexes. Then $h^\bullet = \{f^i p_1 + g^i p_2\}_{i\in\mathbb{Z}}$ is the unique morphism from $A^\bullet \oplus B^\bullet$ to $C^\bullet$ making the following diagram commutative

$$\begin{tikzcd}
 & C^\bullet & \\
A^\bullet \arrow[ur, "f"] \arrow[r, "i_1"'] & A^\bullet \oplus B^\bullet \arrow[u, "h"] & B^\bullet \arrow[ul, "g"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Indeed, the fact that $h^\bullet$ is a morphism follows from the equations

$$\begin{aligned}
h^{i+1} d^i_{A^\bullet \oplus B^\bullet} &= (f^{i+1} p_1 + g^{i+1} p_2)(i_1 d^i_{A^\bullet} p_1 + i_2 d^i_{B^\bullet} p_2) \\
&= (f^{i+1} d^i_{A^\bullet} p_1 + g^{i+1} d^i_{B^\bullet} p_2) \\
&= d^i_{C^\bullet}(f^{i+1} p_1 + g^{i+1} p_2) = d^i_{C^\bullet} h^i.
\end{aligned}$$

Uniqueness of the morphism $h^\bullet$ follows from uniqueness of each of the h^i. This shows that $C(\mathcal{A})$ has biproducts. □

Theorem 2.5.7. *Let $\mathcal{A}$ be an abelian category. Then $\mathcal{C}(\mathcal{A})$ is abelian.*

Proof. **Zero object:** Clearly the complex

$$\dots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \dots$$

is the zero object in $C(\mathcal{A})$.

Product: Since every abelian category is additive, by theorem 2.3.3, by lemma 2.5.6 the category $C(\mathcal{A})$ is additive. By proposition 2.1.3, $C(\mathcal{A})$ has the product and coproduct of each pair of objects because it has the biproduct of each pair of objects.

Kernel: By duality it suffices to show that every morphism has the kernel. Consider a morphism $f : K^\bullet \to L^\bullet$ of complexes. Let $\ker f$ denote the complex

$$\dots \xrightarrow{d^{i-2}_{\ker f}} \ker f^{i-1} \xrightarrow{d^{i-1}_{\ker f}} \ker f^i \xrightarrow{d^{i}_{\ker f}} \ker f^{i+1} \xrightarrow{d^{i+1}_{\ker f}} \dots$$

where $d^i_{\ker f}$ is the unique morphism making the following diagram commutative

$$\begin{tikzcd}
\ker f^i \arrow[r, dashed, "d^i_{\ker f}"] \arrow[d, "\gamma^i"] & \ker f^{i+1} \arrow[d, "\gamma^{i+1}"] \\
K^i \arrow[r, "d^i_{K^\bullet}"] \arrow[d, "f^i"] & K^{i+1} \arrow[d, "f^{i+1}"] \\
L^i \arrow[r, "d^i_{L^\bullet}"] & L^{i+1}
\end{tikzcd}$$

The equation $d^{i+1}_{\ker f} d^i_{\ker f} = 0$ follows from commutativity, $d^{i+1}_{K^\bullet} d^i_{K^\bullet} = 0$, and the fact that $\ker f^i \to K^i$ is a monomorphism.

Now, given any morphism $g : M^\bullet \to K^\bullet$ such that $fg = 0$, there exist unique morphisms $\phi^i : M^i \to \ker f^i$ such that $\gamma^i \phi^i = g^i$. It remains to verify that these ϕ^i define a morphisms of complexes. Now

$$\gamma^{i+1}\phi^{i+1} d^i_{M^\bullet} = g^{i+1} d^i_{M^\bullet} = d^i_{K^\bullet} g^i = d^i_{K^\bullet}\gamma^i\phi^i = \gamma^{i+1} d^i_{\ker f}\phi^i,$$

so by the fact that γ^{i+1} is a monomorphism ϕ commutes with differentials. This shows that the complex $\ker f$ satisfies the universal property for kernel of f in $C(\mathcal{A})$.

Monomorphism is kernel: By duality it suffices to show that every monomorphism is the kernel of some morphism. Let $f : K^\bullet \to L^\bullet$ be a monomorphism. Then each f^i is a monomorphism. Let $\gamma : L^\bullet \to \operatorname{coker} f$ be the cokernel of f. By the construction of $\operatorname{coker} f$ in $C(\mathcal{A})$, γ^i is the cokernel of f^i. By corollary 2.2.11, $f^i = \ker \gamma^i$. By construction of kernels in $C(\mathcal{A})$ this shows that f is the kernel of its cokernel.

□

2.6 Diagram lemmas

In this section we prove two well-known and important results of homological algebra for abelian catetories. The 5-lemma and the snake lemma. These results allow us to associate long exact sequences to short exact sequences of complexes over an abelian category, in a functorial way. Later we prove analogous results for triangulated categories and the homotopy categories, respectively. We mainly follow [Bor94b, 1.10].

We begin by proving the 5-lemma.

Lemma 2.6.1 (5-lemma)**.** *Given a commutative diagram*

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{a_1} & B_1 & \xrightarrow{b_1} & C_1 & \xrightarrow{c_1} & D_1 & \xrightarrow{d_1} & E_1 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
A_2 & \xrightarrow{a_2} & B_2 & \xrightarrow{b_2} & C_2 & \xrightarrow{c_2} & D_2 & \xrightarrow{d_2} & E_2
\end{array}$$

such that the rows are exact and f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is an isomorphism.

Proof. By duality if suffices to prove that f_3 is a monomorphism. We use 2.4.3 (ii). Let $x \in^* C_1$ be a pseudo-element such that $f_3(x) = 0$. We have

$$0 = (c_2 f_3)(x) = (f_4 c_1)(x).$$

Thus $c_1(x) = 0$ because f_4 is monomorphism. By 2.4.3 (iv) there exists $y \in^* B_1$ such that $b_1(y) =^* x$. From

$$0 = (f_3 b_1)(y) = (b_2 f_2)(y)$$

and 2.4.3 (iv) we get $z \in^* A_2$ such that $a_2(z) =^* f_2(y)$. Now f_1 is an epimorphism, so by 2.4.3 (iv) there exists $z' \in^* A_1$ such that $f_1(z') =^* z$ and by commutativity and the fact that f_2 is a monomorphism $a_1(z') =^* y$. Now by 2.4.3 (iv)

$$x =^* (b_1)(y) =^* (b_1 a_1)(z') = 0.$$

This shows that f_3 is a monomorphism. $\square$

To give a proof of the snake lemma, we need the following lemma.

Lemma 2.6.2. *Let $\mathcal{A}$ be an abelian category. Consider a pullback (resp. pushout) diagram*

$$\begin{tikzcd}
 & A \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\
K \arrow[ur, dashed, "k'"] \arrow[r, "k"] & C \arrow[r, "f"] & D
\end{tikzcd}
\qquad
\left(resp. \begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[r, "l"] \arrow[d, "g"] & L \\
C \arrow[r, "f"] & D \arrow[ur, dashed, "l'"] &
\end{tikzcd} \right)$$

of f and g (resp. f' and g'), where $k = \ker f$ (resp. $l = \operatorname{coker} f'$). Then $k = g'k'$ (resp. $l = l'g$), where $k' = \ker f'$ (resp. $l' = \operatorname{coker} f$).

Proof. By duality it is enough to prove the claim about the pullback diagram. Consider the morphisms $k : K \to C$ and $0 : K \to B$. By the pullback property there exists a unique morphism k' such that $g'k' = k$ and $f'k' = 0$. To show that k' is the kernel of f', let h be a morphism such that $f'h = 0$. Commutativity of the diagram gives that $fg'h = 0$, so there exists a unique morphism j such that $g'k'j = kj = g'h$. Uniqueness of the pullback gives $k'j = h$. This shows that k' is the kernel of f'. $\square$

We are ready to prove a restricted version of the snake lemma.

Theorem 2.6.3 (Restricted Snake lemma)**.** *Let $\mathcal{A}$ be an abelian category and*

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "m"] \arrow[d, "f"] & B_1 \arrow[r, "e"] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "p"] & B_2 \arrow[r, "n"] & C_2 \arrow[r] & 0
\end{tikzcd}$$

a commutative diagram in $\mathcal{A}$ with exact rows. Then we have an exact sequence

$$0 \longrightarrow \ker f \xrightarrow{\tilde{f}_1} \ker g \xrightarrow{\tilde{g}_1} \ker h \xrightarrow{\tilde{\delta}} \operatorname{coker} f \xrightarrow{\tilde{f}_2} \operatorname{coker} g \xrightarrow{\tilde{g}_2} \operatorname{coker} h \longrightarrow 0\ . \tag{2.9}$$

Proof. **Construction of $\tilde{f}_1$, $\tilde{g}_1$, $\tilde{f}_2$, and $\tilde{g}_2$:** Fix notation by the following commutative diagram

$$\begin{tikzcd}
& \ker f \arrow[r, dashed, "\tilde{f}_1"] \arrow[d, "k_1"] & \ker g \arrow[r, dashed, "\tilde{g}_1"] \arrow[d, "k_2"] & \ker h \arrow[d, "k_3"] & \\
0 \arrow[r] & A_1 \arrow[r, "m"] \arrow[d, "f"] & B_1 \arrow[r, "e"] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "p"] \arrow[d, "c_1"] & B_2 \arrow[r, "n"] \arrow[d, "c_2"] & C_2 \arrow[r] \arrow[d, "c_3"] & 0 \\
& \operatorname{coker} f \arrow[r, dashed, "\tilde{f}_2"] & \operatorname{coker} g \arrow[r, dashed, "\tilde{g}_2"] & \operatorname{coker} h &
\end{tikzcd} \tag{2.10}$$

The composite $\ker f \to A_1 \to B_1 \to B_2$ is the zero morphism by commutativity, so we obtain a unique morphism $\tilde{f}_1 : \ker f \to \ker g$ such that $k_2\tilde{f}_1 = mk_1$. The composite $A_1 \to A_2 \to B_2 \to \operatorname{coker} g$ is the zero morphism by commutativity, so there exists a unique morphism $\tilde{f}_2 : \operatorname{coker} f \to \operatorname{coker} g$ such that $\tilde{f}_2 c_1 = c_2 p$. Similarly we obtain the unique morphisms $\tilde{g}_1$ and $\tilde{g}_2$ making the diagram commutative.

Existence of $\tilde{\delta}$: Consider the diagram

$$\begin{tikzcd}
& & X_1 \arrow[r, "e'"] \arrow[d, "k'"] & \ker h \arrow[d, "k"] & \\
0 \arrow[r] & A_1 \arrow[ur, dashed, "s"] \arrow[r, "m"] \arrow[d, "f"] & B_1 \arrow[r, "e"] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "p"] \arrow[d, "u"] & B_2 \arrow[r, "n"] \arrow[d, "u'"] & C_2 \arrow[r] & 0 \\
& \operatorname{coker} f \arrow[r, "p'"] & X_2 \arrow[ur, dashed, "s'"] & &
\end{tikzcd} \tag{2.11}$$

where X_1 is the pullback of e and k and X_2 is the pushout of p and u. From lemma 2.6.2 it follows that e' is the cokernel of s and p' is the kernel of s'. Now

$$A_1 \to X_1 \to B_1 \to B_2 \to X_2 = 0$$

by commutativity, so there exists a unique morphism $\phi : \ker h \to X_2$ such that

$$X_1 \to \ker h \to X_2 = X_1 \to B_1 \to B_2 \to X_2. \tag{2.12}$$

By commutativity and the fact that e' is an epimorphism

$$\ker h \to X_2 \to C_2 = 0,$$

so there exists a unique morphism $\tilde{\delta} : \ker h \to \operatorname{coker} f$ such that

$$\ker h \to \operatorname{coker} f \to X_2 = \ker h \to X_2.$$

$\tilde{\delta}$ on pseudo-elements: Consider the following diagram of pseudo-elements

$$\begin{tikzcd} & & x \in^* \ker h \arrow[d, mapsto] \\ & y \arrow[r, mapsto] \arrow[d, mapsto] & e(y) =^* k(x) \arrow[d, mapsto] \\ z \arrow[r, mapsto] \arrow[d, mapsto] & g(y) \arrow[r, mapsto] & 0 \\ u(z) & & \end{tikzcd} \tag{2.13}$$

where existence of y follows from 2.4.3 (iii), because e is an epimorphism, and existence of z follows from 2.4.3 (iv), because $g(y)$ is mapped to zero.

We show that $\tilde{\delta}(x) =^* u(z)$. Since X_1 is the pullback of k and e, by 2.4.3 (vi) there exists a pseudo-element $x_0 \in^* X_1$ such that $k'(x_0) =^* y$ and $e'(x_0) =^* x$. Now

$$p'\tilde{\delta}(x) =^* \phi(x) =^* u'gk'(x_0) =^* u'p(z) =^* p'u(z).$$

The morphism p' is a monomorphism by lemma 2.6.2, so by 2.4.3 (ii) we have that $\tilde{\delta}(x) =^* u(z)$.

Exactness at $\ker f$**:** We need to show that $\tilde{f}_1$ is a monomorphism. But this follows from the equality

$$k_2\tilde{f}_1 = mk_1,$$

because k_2, m, and k_1 are monomorphisms.

Exactness at $\ker g$**:** We use 2.4.3 (iv). By commutativity and the fact that k_3 is a monomorphism, $\tilde{g}_1\tilde{f}_1$ is the zero morphism. Let $x \in^* \ker g$ such that $\tilde{g}_1(x) = 0$. Now, $ek_2(x) = 0$, so by exactness of the second row, there exists a pseudo-element $a_1 \in^* A_1$ such that $m(a_1) =^* k_2(x)$. We have $f(a_1) = 0$ because $p(f(a_1)) =^* g(k_2(x)) = 0$ and p is a monomorphism. Thus, there exists a pseudo-element $y \in^* \ker f$ such that $k_1(y) =^* a_1$. Since k_2 is a monomorphism, $\tilde{f}_1(y) =^* x$.

Exactness at $\ker h$**:** We use 2.4.3 (iv). Let $w \in^* \ker g$. Now $gk_2(w) =^* 0$, and by 2.4.3 (ii) $0 \in^* A_2$ is the unique pseudo-element, up to pseudo-equality, with pseudo-image $gk_2(w)$. From the diagram (2.13) we get $\tilde{\delta}\tilde{g}_1(w) =^* c_1(0) = 0$.

Next, let $x \in^* \ker h$ such that $\tilde{\delta}(x) = 0$. By the diagram (2.13) and exactness of the sequence

$$A_1 \xrightarrow{f} A_2 \xrightarrow{c_1} \operatorname{coker} f \longrightarrow 0$$

there exists $a_1 \in^* A_1$ such that $f(a_1) =^* z$. By commutativity $gm(a_1) =^* g(y)$ and by exactness of the second row $em(a_1) =^* 0$. Thus, from 2.4.3 (v) there exists $b_1 \in^* B_1$ such that $g(b_1) = 0$ and $e(b_1) =^* e(y) =^* k_3(x)$. Since in the diagram (2.13) the choice of the pseudo-element y mapping to $k_3(x)$ was arbitrary, we can choose y to be b_1. Finally, from exactness of

$$0 \longrightarrow \ker g \xrightarrow{k_2} B_1 \xrightarrow{g} B_2$$

by 2.4.3 (iv) we get a pseudo-element $w \in^* \ker g$ such that $k_2(w) =^* b_1$. By commutativity $k_3\tilde{g}_2(w) =^* k_3(x)$ and 2.4.3 (ii) applied to k_3 we get $\tilde{g}_2(w) =^* x$. This shows that the sequence is exact at $\ker h$.

Exactness at $\operatorname{coker} f$**:** We use 2.4.3 (iv). Let $x \in^* \ker h$. From the diagram (2.13) and commutativity, we see that $\tilde{f}_2(\tilde{\delta}(x)) =^* c_2(g(y)) =^* 0$, because $c_2 g = 0$.

Let $x' \in^* \operatorname{coker} f$ such that $\tilde{f}_2(x') =^* 0$. By 2.4.3 (iii) there exists $z \in^* A_2$ with $c_1(z) =^* x'$. From commutativity of the diagram (2.11) and exactness of

$$B_1 \xrightarrow{g} B_2 \xrightarrow{c_2} \operatorname{coker} g \longrightarrow 0$$

we get a pseudo-element $y \in^* B_1$ such that $g(b_1) =^* p(z)$. Now $ng(b_1) =^* 0$ by exactness of the third row, so from the exact sequence

$$0 \longrightarrow \ker h \xrightarrow{k_3} C_1 \xrightarrow{h} C_2$$

we get a pseudo-element $x \in^* \ker h$ with $k_3(x) =^* e(y)$. The choices of z, y, and x fit into the diagram (2.13), showing that $\tilde{\delta}(x) =^* x'$. This shows that the sequence is exact at $\operatorname{coker} f$.

Exactness at $\operatorname{coker} g$**:** We use 2.4.3 (iv). By commutativity $\tilde{g}_2\tilde{f}_2 = 0$. Let $x \in^* \operatorname{coker} g$ such that $\tilde{g}_2(x) = 0$. By 2.4.3 (iii) there exists $b_2 \in^* B_2$ such that $c_2(b_2) =^* x$. By exactness of the following sequence

$$C_1 \xrightarrow{h} C_2 \xrightarrow{c_3} \operatorname{coker} h \longrightarrow 0 \ ,$$

there exists $c_1 \in^* C_1$ with $h(c_1) = n(b_2)$. By 2.4.3 (iii) there exists $b_1 \in^* B_1$ such that $e(b_1) =^* c_1$. By commutativity, $ng(b_1) =^* n(b_2) =^* c_2$. By 2.4.3 (v), there exists $b_2' \in^* B_2$ such that $n(b_2') = 0$ and $c_2(b_2') =^* x$. By exactness of the third row, there is $a_2 \in^* A_2$ such that $p(a_1) =^* b_2'$. Therefore $c_1(a_2) \in^* \operatorname{coker} f$ is the pseudo-element which is mapped to x. This shows exactness at $\operatorname{coker} g$.

Exactness at $\operatorname{coker} h$**:** It suffices to show that $\tilde{g}_2$ is an epimorphism. By commutativity

$$\tilde{g}_2 c_2 = c_3 n$$

where c_2, c_3, and n are epimorphisms. Hence $\tilde{g}_2$ is an epimorphism.

□

To see that the morphism $\tilde{\delta}$ in (2.9) is not in general the zero morphism, consider the following morphism of short exact sequences over **Ab**.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We see that in (2.9) the last morphisms between kernels $0 \cong \ker(\mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z}) \to \ker(\mathbb{Z} \to \mathbb{Z}/2) \cong 2\mathbb{Z} \cong \mathbb{Z}$ is not surjective and $\mathbb{Z} \cong \operatorname{coker}(0 \to \mathbb{Z}) \to \operatorname{coker}(\mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z}) \cong 0$ is not injective. In this case the sequence (2.9) is exact when $\tilde{\delta} = \mathrm{Id}_{\mathbb{Z}}$, which is not the zero morphism. If one embeddes this diagram to complexes over **Ab** in degree 0, one gets that also in this case the morphism $\tilde{\delta}$, now between complexes, is not the zero morphism.

We wish to apply the previous result to certain kernel and cokernel sequences. But these are not short exact sequences as the above argument shows. Therefore we need to generalize theorem 2.6.3 to the following

Corollary 2.6.4 (Snake lemma)**.** *Let $\mathcal{A}$ be an abelian category. Given a diagram*

$$\begin{tikzcd}
 & A_1 \arrow[r, "m"] \arrow[d, "f"] & B_1 \arrow[r, "e"] \arrow[d, "g"] & C_1 \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "p"] & B_2 \arrow[r, "n"] & C_2 &
\end{tikzcd}$$

with exact rows, there exists a morphism $\delta : \ker h \to \operatorname{coker} f$ such that the following sequence is an exact sequence

$$\ker f \xrightarrow{\tilde{f}_1} \ker g \xrightarrow{\tilde{g}_1} \ker h \xrightarrow{\delta} \operatorname{coker} f \xrightarrow{\tilde{f}_2} \operatorname{coker} g \xrightarrow{\tilde{g}_2} \operatorname{coker} h \ .$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
 & \ker f \arrow[rr, "\alpha"] \arrow[d] & & \ker g \arrow[rr, "\beta"] \arrow[d] & & \ker h \arrow[d] & \\
 & A_1 \arrow[r] \arrow[d, "f"] & A_1' \arrow[r] \arrow[dl] & B_1 \arrow[rr] \arrow[d, "g"] & & C_1 \arrow[r] \arrow[dl] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[rr] \arrow[d] & & B_2 \arrow[r] \arrow[d] & B_2' \arrow[r] & C_2 \arrow[d] & \\
 & \operatorname{coker} f \arrow[rr, "\tau"] & & \operatorname{coker} g \arrow[rr, "\phi"] & & \operatorname{coker} h &
\end{tikzcd}$$

where the objects A_1' and B_2' are given by theorem 2.2.10 applied to the morphisms $A_1 \to B_1$ and $B_2 \to C_2$. By theorem 2.2.10 we get unique morphisms $A_1' \to A_2$ and $C_1 \to B_2'$ which keep the diagram commutative. Let X_1' be the kernel of $C_1 \to B_2'$ and X_2' the cokernel of $A_1' \to B_2$. It is not hard to see that we obtain factorizations $\beta = \beta_2\beta_1$ and $\tau = \tau_2\tau_1$ through the kernel X_1' and the cokernel X_2'. Let us denote by X_1 the pullback of the morphisms η and e and by X_2 the pushout of the morphisms p and $A_2 \to X_2'$. We have obtained the following commutative diagram

$$\begin{tikzcd}
 & & & X_1 \arrow[r] \arrow[d] & X_1' \arrow[r, "\beta_2"] \arrow[dr, "\eta"] & \ker h \arrow[d, "\epsilon"] & \\
 & A_1 \arrow[r] \arrow[d] & A_1' \arrow[r] \arrow[dl] \arrow[ur, dashed] & B_1 \arrow[rr, "e"] \arrow[d] & & C_1 \arrow[r] \arrow[dl, "\gamma"'] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A_2 \arrow[rr, "p"] \arrow[d] \arrow[dr] & & B_2 \arrow[r] \arrow[d] & B_2' \arrow[r, "\eta_2"] & C_2 & \\
 & \ker f \arrow[r] & X_2' \arrow[r] & X_2 \arrow[ur, dashed] & & &
\end{tikzcd} \tag{2.14}$$

The dashed morphisms exist by lemma 2.6.2 and are the kernel and the cokernel of $X_1 \to X_1'$ and $X_2' \to X_2$, respectively. From this diagram we obtain the following commutative diagram

$$
\begin{tikzcd}
 & & X_1 \arrow[r] \arrow[d] & X_1' \arrow[d] & \\
0 \arrow[r] & A_1' \arrow[r] \arrow[d] \arrow[ur, dashed] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_2' \arrow[r] & 0 \\
 & X_2' \arrow[r] & X_2 \arrow[ur, dashed] & &
\end{tikzcd}
$$

By applying theorem 2.6.3 we obtain the morphism $\tilde{\delta} : X_1' \to X_2'$. We show that the morphism $\beta_2 : X_1' \to \ker h$ is an epimorphism, and by duality that $\tau_1 : \operatorname{coker} f \to X_2'$ is a monomorphism, so that we can define the map δ to be the composite $\tau_1^{-1}\tilde{\delta}\beta_2^{-1}$.

Let $c \in^* \ker h$, $(\eta_2\gamma\epsilon)(c) =^* (h\epsilon)(c) =^* 0$ so that $(\gamma\epsilon)(c) = 0$ by 2.4.3 (ii) since η_2 is a monomorphism. Since η is the kernel of γ, by 2.4.3 (iv), there exists $u \in^* X_1'$ such that $\eta(u) =^* \epsilon(c)$. Finally, by commutativity $(\epsilon\beta_2)(u) =^* \eta(u) =^* \epsilon(c)$, thus $\beta_2(u) =^* c$ by 2.4.3 (ii), because ϵ is a monomorphism. This shows that β_2 is an epimorpism and thus an isomorphism. By duality, the morphism τ_1 is also an isomorphism. $\square$

Lemma 2.6.5. *Consider a morphism $f : X^\bullet \to Y^\bullet$ of two complexes over an abelian category. Then the diagram*

$$
\begin{tikzcd}
\operatorname{coker} d_{X^\bullet}^{i-1} \arrow[r, "\tilde{f}_2"] \arrow[d, "\phi_1"] & \operatorname{coker} d_{Y^\bullet}^{i-1} \arrow[d, "\phi_2"] \\
\ker d_{X^\bullet}^{i+1} \arrow[r, "\tilde{f}_1"] & \ker d_{Y^\bullet}^{i+1}
\end{tikzcd}
\tag{2.15}
$$

is commutative for all $i \in \mathbb{Z}$, where the morphisms ϕ_1 and ϕ_2 are from the diagram (2.5) *and the morphisms $\tilde{f}_1$ and $\tilde{f}_2$ are from the diagram* (2.10).

Proof. Recall that the morphisms ϕ_1, ϕ_2, $\tilde{f}_1$ and $\tilde{f}_2$ are the unique morphisms which make the following diagrams commutative

$$
\begin{tikzcd}
X^i \arrow[r] \arrow[d] & Y^i \arrow[d] \\
\operatorname{coker} d_{X^\bullet}^{i-1} \arrow[r, "\tilde{f}_1"] & \operatorname{coker} d_{Y^\bullet}^{i-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\ker d_{X^\bullet}^{i+1} \arrow[r, "\tilde{f}_2"] \arrow[d] & \ker d_{Y^\bullet}^{i+1} \arrow[d] \\
X^{i+1} \arrow[r] & Y^{i+1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
X^i \arrow[r] \arrow[dr] & \operatorname{coker} d_{X^\bullet}^{i-1} \arrow[d, "\phi_1"] \arrow[dr] & \\
 & \ker d_{X^\bullet}^{i+1} \arrow[r] & X^{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
Y^i \arrow[r] \arrow[dr] & \operatorname{coker} d_{Y^\bullet}^{i-1} \arrow[d, "\phi_2"] \arrow[dr] & \\
 & \ker d_{Y^\bullet}^{i+1} \arrow[r] & Y^{i+1}
\end{tikzcd}
,
$$

where the lower two diagrams are of the from (2.5). By the fact that f is a morphism of complexes and commutativity of the above diagrams, we have

$$\begin{aligned} X^i &\to \operatorname{coker} d_{X^\bullet}^{i-1} \to \ker d_{X^\bullet}^{i+1} \to \ker d_{Y^\bullet}^{i+1} \to Y^{i+1} \\ &= X^i \to \operatorname{coker} d_{X^\bullet}^{i-1} \to \ker d_{X^\bullet}^{i+1} \to X^{i+1} \to Y^{i+1} \\ &= X^i \to X^{i+1} \to Y^{i+1} \\ &= X^i \to Y^i \to Y^{i+1} \\ &= X^i \to Y^i \to \operatorname{coker} d_{Y^\bullet}^{i-1} \to \ker d_{Y^\bullet}^{i+1} \to Y^{i+1} \\ &= X^i \to \operatorname{coker} d_{X^\bullet}^{i-1} \to \operatorname{coker} d_{Y^\bullet}^{i-1} \to \ker d_{Y^\bullet}^{i+1} \to Y^{i+1} \end{aligned}$$

Since $X^i \to \operatorname{coker} d_{X^\bullet}^{i-1}$ is an epimorphism, and $\ker d_{Y^\bullet}^{i+1} \to Y^{i+1}$ is a monomorphism, these can be cancelled out in the equation. Thus we obtain that the diagram (2.15) is commutative. □

Next we prove the main theorem of this section which assings to every short exact sequence of complexes over an abelian category $\mathcal{A}$ a long exact sequence over $\mathcal{A}$ in a functorial way.

Theorem 2.6.6 (Functorial long exact sequence)**.** *Let $\mathcal{A}$ be an abelian category. For any short exact sequence of objects in $C(\mathcal{A})$*

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

we have a long exact sequence

$$\ldots \xrightarrow{H^i(f)} H^i(Y^\bullet) \xrightarrow{H^i(g)} H^i(Z^\bullet) \xrightarrow{\delta^i} H^{i+1}(X^\bullet) \xrightarrow{H^{i+1}(f)} H^{i+1}(Y^\bullet) \xrightarrow{H^{i+1}(g)} \ldots . \tag{2.16}$$

This long exact sequence is functorial in the sense that given a morphism of two short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & X_1^\bullet \arrow[r, "f_1"] \arrow[d, "h_1"] & Y_1^\bullet \arrow[r, "g_1"] \arrow[d, "h_2"] & Z_1^\bullet \arrow[r] \arrow[d, "h_3"] & 0 \\ 0 \arrow[r] & X_2^\bullet \arrow[r, "f_2"] & Y_2^\bullet \arrow[r, "g_2"] & Z_2^\bullet \arrow[r] & 0 \end{tikzcd} \tag{2.17}$$

we have a morphism of complexes

$$\begin{tikzcd} \ldots \arrow[r] & H^i(Y_1^\bullet) \arrow[r, "H^i(g_1)"] \arrow[d, "H^i(h_2)"] & H^i(Z_1^\bullet) \arrow[r, "\delta_1"] \arrow[d, "H^i(h_3)"] & H^{i+1}(X_1^\bullet) \arrow[r, "H^{i+1}(f_1)"] \arrow[d, "H^i(h_1)"] & H^{i+1}(Y_1^\bullet) \arrow[r] \arrow[d, "H^i(h_2)"] & \ldots \\ \ldots \arrow[r] & H^i(Y_2^\bullet) \arrow[r, "H^i(g_2)"] & H^i(Z_2^\bullet) \arrow[r, "\delta_2"] & H^{i+1}(X_2^\bullet) \arrow[r, "H^{i+1}(f_2)"] & H^{i+1}(Y_2) \arrow[r] & \ldots \end{tikzcd}$$

Proof. By theorem 2.6.3 and lemma 2.6.5 we have following commutative diagram with exact rows

$$\begin{tikzcd} & \operatorname{coker} d_{X^\bullet}^{i-1} \arrow[r, "\tilde{f}_2"] \arrow[d, "\phi_1"] & \operatorname{coker} d_{Y^\bullet}^{i-1} \arrow[r, "\tilde{g}_2"] \arrow[d, "\phi_2"] & \operatorname{coker} d_{Z^\bullet}^{i-1} \arrow[r] \arrow[d, "\phi_3"] & 0 \\ 0 \arrow[r] & \ker d_{X^\bullet}^{i+1} \arrow[r, "\tilde{f}_1"] & \ker d_{Y^\bullet}^{i+1} \arrow[r, "\tilde{g}_1"] & \ker d_{Z^\bullet}^{i+1} & \end{tikzcd}$$

for all $i \in \mathbb{Z}$. By lemma 2.5.3 and corollary 2.6.4 we get long exact sequence (2.16).

Let us show that the long exact sequence is functorial. By definition of a functor and proposition 2.5.4, the following diagrams are commutative for all $i \in \mathbb{Z}$

$$\begin{tikzcd} H^i(X_1^\bullet) \arrow[r, "H^i(f_1)"] \arrow[d, "H^i(h_1)"] & H^i(Y_1^\bullet) \arrow[d, "H^i(h_2)"] \\ H^i(X_2^\bullet) \arrow[r, "H^i(f_2)"] & H^i(Y_2^\bullet) \end{tikzcd} \qquad \begin{tikzcd} H^i(Y_1^\bullet) \arrow[r, "H^i(g_1)"] \arrow[d, "H^i(h_2)"] & H^i(Z_1^\bullet) \arrow[d, "H^i(h_3)"] \\ H^i(Y_2^\bullet) \arrow[r, "H^i(g_2)"] & H^i(Z_2^\bullet) \end{tikzcd} \tag{2.18}$$

It remais to show that the following diagram is commutative for all i

$$\begin{tikzcd} H^i(Z_1^\bullet) \arrow[r, "\delta_1^i"] \arrow[d, "H^i(h_3)"] & H^{i+1}(X_1^\bullet) \arrow[d, "H^{i+1}(h_1)"] \\ H^i(Z_2^\bullet) \arrow[r, "\delta_2^i"] & H^{i+1}(X_1^\bullet) \end{tikzcd} \tag{2.19}$$

By construction of δ_1 and δ_2 in corollary 2.6.4, we need to show that the following diagram is commutative

$$\begin{tikzcd} H^i(Z_1^\bullet) \arrow[r, "\beta_2^{-1}"] \arrow[d, "H^i(h_3)"] & W_1 \arrow[r, "\tilde{\delta}_1"] & W_2 \arrow[r, "\tau_1^{-1}"] & H^{i+1}(X_1^\bullet) \arrow[d, "H^{i+1}(h_1)"] \\ H^i(Z_2^\bullet) \arrow[r, "\delta_2^i"] & \tilde{W}_1 \arrow[r, "\tilde{\delta}_2"] & \tilde{W}_2 \arrow[r, "\tau_1^{-1}"] & H^{i+1}(X_1^\bullet) \end{tikzcd} \tag{2.20}$$

Recall from the proof of corollary 2.6.4, the morphisms $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are constructed from the following commutative diagrams

$$\begin{tikzcd} & & X_1 \arrow[r] \arrow[d] & W_1 \arrow[d] & \\ 0 \arrow[r] & A_1 \arrow[r] \arrow[ur] \arrow[d] & \operatorname{coker} d_{Y_1}^{i-1} \arrow[r] \arrow[d] & \operatorname{coker} d_{Z_1}^{i-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker d_{X_1}^{i+1} \arrow[r] \arrow[d] & \ker d_{Y_1}^{i+1} \arrow[r] \arrow[d] & C_2 \arrow[r] & 0 \\ & W_2 \arrow[r] & X_2 \arrow[ur] & & \end{tikzcd} \tag{2.21}$$

$$\begin{tikzcd} & & \tilde{X}_1 \arrow[r] \arrow[d] & \tilde{W}_1 \arrow[d] & \\ 0 \arrow[r] & \tilde{A}_1 \arrow[r] \arrow[ur] \arrow[d] & \operatorname{coker} d_{Y_2}^{i-1} \arrow[r] \arrow[d] & \operatorname{coker} d_{Z_2}^{i-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker d_{X_2}^{i+1} \arrow[r] \arrow[d] & \ker d_{Y_2}^{i+1} \arrow[r] \arrow[d] & \tilde{C}_2 \arrow[r] & 0 \\ & \tilde{W}_2 \arrow[r] & \tilde{X}_2 \arrow[ur] & & \end{tikzcd} \tag{2.22}$$

Since h_1 and h_3 are morphisms of complexes and by theorem 2.2.10, the following diagrams are commutative

$$\begin{tikzcd}
\ker d_{Y_1}^{i+1} \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & \ker d_{Z_1}^{i+1} \arrow[d] \\
\ker d_{Y_2}^{i+1} \arrow[r] & \tilde{C}_2 \arrow[r] & \ker d_{Z_2}^{i+1}
\end{tikzcd} \qquad
\begin{tikzcd}
\operatorname{coker} d_{X_1}^{i-1} \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & \operatorname{coker} d_{Y_1}^{i-1} \arrow[d] \\
\operatorname{coker} d_{X_2}^{i-1} \arrow[r] & \tilde{A}_1 \arrow[r] & \operatorname{coker} d_{Y_2}^{i-1}
\end{tikzcd}$$

By the proof of corollary 2.6.4 $W_1 \to \operatorname{coker} d_{Z_1}^{i-1}$ and $\tilde{W}_1 \to \operatorname{coker} d_{Z_2}^{i-1}$ are the kernels of $\operatorname{coker} d_{Z_1}^{i-1} \to C_2$ and $\operatorname{coker} d_{Z_2}^{i-1} \to \tilde{C}_2$ and the morphisms $\ker d_{X_1}^{i+1} \to \tilde{W}_1$ and $\ker d_{X_2}^{i+1} \to \tilde{W}_2$ are cokernels of $A_1 \to \ker d_{X_1}^{i+1}$ and $\tilde{A}_1 \to \ker d_{X_2}^{i+1}$. By commutativity and the kernel property, there exists unique morphisms $W_1 \to \tilde{W}_1$ and $W_2 \to \tilde{W}_2$ such that the following diagrams are commutative

$$\begin{tikzcd}
W_1 \arrow[r] \arrow[d] & \tilde{W}_1 \arrow[d] \\
\operatorname{coker} d_{Z_1}^{i-1} \arrow[r] & \operatorname{coker} d_{Z_2}^{i-1}
\end{tikzcd} \qquad
\begin{tikzcd}
\ker d_{X_1}^{i+1} \arrow[r] \arrow[d] & \ker d_{X_2}^{i+1} \arrow[d] \\
W_2 \arrow[r] & \tilde{W}_2
\end{tikzcd} \tag{2.23}$$

We need to show that the following cubes commute

$$\begin{tikzcd}
 & X_1 \arrow[rr] \arrow[dl, dashed] \arrow[dd] & & W_1 \arrow[dl] \arrow[dd] \\
\tilde{X}_1 \arrow[rr, crossing over] \arrow[dd] & & \tilde{W}_1 & \\
 & \operatorname{coker} d_{Y_1}^{i-1} \arrow[rr] \arrow[dl] & & \operatorname{coker} d_{Z_1}^{i-1} \arrow[dl] \\
\operatorname{coker} d_{Y_2}^{i-1} \arrow[rr] & & \operatorname{coker} d_{Z_2}^{i-1} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \ker d_{X_1}^{i+1} \arrow[rr] \arrow[dl] \arrow[dd] & & \ker d_{Y_1}^{i+1} \arrow[dl] \arrow[dd] \\
\ker d_{X_2}^{i+1} \arrow[rr, crossing over] \arrow[dd] & & \ker d_{Y_2}^{i+1} & \\
 & W_2 \arrow[rr] \arrow[dl] & & X_2 \arrow[dl, dashed] \\
\tilde{W}_2 \arrow[rr] & & \tilde{X}_2 \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{2.24}$$

We see that both of the commutative diagrams (2.23) are embedded in both of the above cubes. Commutativity of the following squares

$$\begin{array}{ccccccc} \ker d_{X_1}^{i+1} & \longrightarrow & \ker d_{Y_1}^{i+1} & \quad & \operatorname{coker} d_{Y_1}^{i-1} & \longrightarrow & \operatorname{coker} d_{Z_1}^{i-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ker d_{X_2}^{i+1} & \longrightarrow & \ker d_{Y_2}^{i+1} & & \operatorname{coker} d_{Y_2}^{i-1} & \longrightarrow & \operatorname{coker} d_{Z_2}^{i-1} \end{array}$$

follow from the equations

$$\begin{aligned} \ker d_{X_1}^{i+1} \to \ker d_{Y_1}^{i+1} \to \ker d_{Y_2}^{i+1} \to Y_2^{i+1} &= \ker d_{X_1}^{i+1} \to X_1^{i+1} \to Y_1^{i+1} \to Y_2^{i+1} \\ &= \ker d_{X_1}^{i+1} \to X_1^{i+1} \to X_2^{i+1} \to Y_2^{i+1} \\ &= \ker d_{X_1}^{i+1} \to \ker d_{X_2}^{i+1} \to \ker d_{Y_2}^{i+1} \to Y_2^{i+1} \end{aligned}$$

and

$$\begin{aligned} Y_1^i \to \operatorname{coker} d_{Y_1}^{i-1} \to \operatorname{coker} d_{Z_1}^{i-1} \to \operatorname{coker} d_{Z_2}^{i-1} &= Y_1^i \to Z_1^i \to Z_2^i \to \operatorname{coker} d_{Z_2}^{i-1} \\ &= Y_1^i \to Y_2^i \to Z_2^i \to \operatorname{coker} d_{Z_2}^{i-1} \\ &= Y_1^i \to \operatorname{coker} d_{Y_1}^{i-1} \to \operatorname{coker} d_{Y_2}^{i-1} \to \operatorname{coker} d_{Z_2}^{i-1} \end{aligned}$$

because $\ker d_{Y_2}^{i+1} \to Y_2^{i+1}$ is a monomorphism and $Y_1^i \to \operatorname{coker} d_{Y_1}^{i-1}$ is an epimorphism. The squares

$$\begin{array}{ccccccc} X_1 & \longrightarrow & W_1 & \quad & \tilde{X}_1 & \longrightarrow & \tilde{W}_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{coker} d_{Y_1}^{i-1} & \longrightarrow & \operatorname{coker} d_{Z_1}^{i-1} & & \operatorname{coker} d_{Y_2}^{i-1} & \longrightarrow & \operatorname{coker} d_{Z_2}^{i-1} \end{array}$$

are commutative because they are pullback squares. Similarly, the squares

$$\begin{array}{ccccccc} \ker d_{X_1}^{i+1} & \longrightarrow & \ker d_{Y_1}^{i+1} & \quad & \ker d_{X_2}^{i+1} & \longrightarrow & \ker d_{Y_2}^{i+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W_2 & \longrightarrow & X_2 & & \tilde{W}_2 & \longrightarrow & \tilde{X}_2 \end{array}$$

are commutative since they are pushout squares. From the following identities

$$\begin{aligned} & X_1 \to \operatorname{coker} d_{Y_1}^{i-1} \to \operatorname{coker} d_{Y_2}^{i-1} \to \operatorname{coker} d_{Z_2}^{i-1} \\ &\quad = X_1 \to \operatorname{coker} d_{Y_1}^{i-1} \to \operatorname{coker} d_{Z_1}^{i-1} \to \operatorname{coker} d_{Z_2}^{i-1} \\ &\quad = X_1 \to W_1 \to \operatorname{coker} d_{Z_1}^{i-1} \to \operatorname{coker} d_{Z_2}^{i-1} \\ &\quad = X_1 \to W_1 \to \tilde{W}_1 \to \operatorname{coker} d_{Z_2}^{i-1} \end{aligned}$$

and

$$\begin{aligned} & \ker d_{X_1}^{i+1} \to W_2 \to \tilde{W}_2 \to \tilde{X}_2 \\ &\quad = \ker d_{X_1}^{i+1} \to \ker d_{X_2}^{i+1} \to \tilde{W}_2 \to \tilde{X}_2 \\ &\quad = \ker d_{X_1}^{i+1} \to \ker d_{X_2}^{i+1} \to \ker d_{Y_2}^{i+1} \to \tilde{X}_2 \\ &\quad = \ker d_{X_1}^{i+1} \to \ker d_{Y_1}^{i+1} \to \ker d_{Y_2}^{i+1} \to \tilde{X}_2 \end{aligned}$$

and the pullback and pushout universality of $\tilde{X}_1$ and $\tilde{X}_2$, we obtain unique morphisms $X_1 \to \tilde{X}_1$ and $X_2 \to \tilde{X}_2$ which make the remaining squares commutative.

Putting these together the left and right squares of the following diagram are commutative.

$$\begin{array}{ccccccc} X_1 & \longrightarrow & \operatorname{coker} d_{Y_1}^{i-1} & \longrightarrow & \ker d_{Y_1}^{i+1} & \longrightarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{X}_1 & \longrightarrow & \operatorname{coker} d_{Y_2}^{i-1} & \longrightarrow & \ker d_{Y_2}^{i+1} & \longrightarrow & \tilde{X}_2 \end{array} \tag{2.25}$$

The middle square commutes by lemma 2.6.5, and so the diagram is commutative. Finally, from bottom square of the cube (2.24), commutativity of the right square of (2.23), definition of $H^{i+1}(h_3)$, and the fact that $W_2 \cong H^{i+1}(X_1^\bullet)$ and $W_1 \cong H^{i+1}(X_2^\bullet)$ (as shown in corollary 2.6.4), we have the following commutative diagrams

$$\begin{array}{ccccc} X_1 & \longrightarrow & W_1 & \xrightarrow{\cong} & H^i(Z_1^\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{X}_1 & \longrightarrow & \tilde{W}_1 & \xrightarrow{\cong} & H^i(Z_2^\bullet) \end{array} \qquad \begin{array}{ccccc} H^{i+1}(X_1^\bullet) & \xrightarrow{\cong} & W_2 & \longrightarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow \\ H^{i+1}(X_2^\bullet) & \xrightarrow{\cong} & \tilde{W}_2 & \longrightarrow & \tilde{X}_2 \end{array} \tag{2.26}$$

We have the following equalities

$$\begin{aligned} & X_1 \to H^i(Z_1^\bullet) \to W_1 \to W_2 \to H^{i+1}(X_1^\bullet) \to H^{i+1}(X_2^\bullet) \to \tilde{X}_2 \\ &= X_1 \to W_1 \to W_2 \to X_2 \to \tilde{X}_2 && (2.26) \\ &= X_1 \to \operatorname{coker} d_{Y_1}^{i-1} \to \ker d_{Y_1}^{i+1} \to X_2 \to \tilde{X}_2 && (2.21) \\ &= X_1 \to \tilde{X}_1 \to \operatorname{coker} d_{Y_2}^{i-1} \to \ker d_{Y_2}^{i+1} \to \tilde{X}_2 && (2.25) \\ &= X_1 \to \tilde{X}_1 \to \tilde{W}_1 \to \tilde{W}_2 \to \tilde{X}_2 && (2.22) \\ &= X_1 \to H^i(Z_1^\bullet) \to H^i(Z_2^\bullet) \to \tilde{W}_1 \to \tilde{W}_2 \to H^{i+1}(X_2^\bullet) \to \tilde{X}_2. && (2.26) \end{aligned}$$

The morphism $X_1 \to H^i(Z_1^\bullet)$ is an epimorphism and $H^{i+1}(X_2^\bullet) \to X_2$ is a monomorphism, so these can be cancelled out. Therefore we obtain commutativity of the diagram (2.20). This finishes the proof. □

2.7 Homotopy category

In this section we introduce the homotopy category over an additive category, which is the quotient of the category of complexes over an additive category by homotopies between morphisms. In particular, it is an additive category by proposition 2.7.12. This category is used to prove that the derived category of an abelian category is a triangulated category. The reason we need it is that the class of quasi-isomorphisms in the homotopy category of an abelian category forms a localizing class, see definition 3.2.1, which is not true in general for the class of quasi-isomorphisms in the category of complexes over an abelian category.

Definition 2.7.1 (Quasi-isomorphism)**.** Let $\mathcal{A}$ be an abelian category and $X^\bullet, Y^\bullet \in \operatorname{Ob} C(\mathcal{A})$. A morphism $f : X^\bullet \to Y^\bullet$ is a *quasi-isomorphsm* if for all $i \in \mathbb{Z}$ the induced morphism $H^i(f) : H^i(X^\bullet) \to H^i(Y^\bullet)$ between the i-th cohomology of $X^\bullet$ and $Y^\bullet$ is an isomorphism.

Definition 2.7.2 (Homotopy of morphisms)**.** Let $\mathcal{A}$ be an additive category and $f : X^\bullet \to Y^\bullet$ a morphism in $C(\mathcal{A})$. We define f to be *null-homotopic* if there exists a family of morphisms $h^i : X^i \to Y^{i-1}$ in $\mathcal{A}$ such that

$f^i = d_{Y^\bullet}^{i-1} h^i + h^{i+1} d_{X^\bullet}^i$ for all $i \in \mathbb{Z}$. Two morphisms $f, g : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ are defined to be *homotopic*, denoted $f \sim g$, if $f - g$ is null-homotopic.

Lemma 2.7.3. *Let $\mathcal{A}$ be an additive category.*

(a) *For any objects $X^\bullet, Y^\bullet$ of $C(\mathcal{A})$, and any collection $h^i : K^i \to L^{i-1}$ of morphisms in $\mathcal{A}$, $h^{i+1} d_{X^\bullet}^i + d_{Y^\bullet}^{i-1} h^i : K^i \to L^i$ is a morphism of complexes.*

(b) *For any complexes $X^\bullet$ and $Y^\bullet$ over $\mathcal{A}$, the subset $I_{X^\bullet, Y^\bullet} \subset \operatorname{Mor}_{\mathcal{A}}(X^\bullet, Y^\bullet)$ consisting of morphisms of the form $h^{i+1} d_{X^\bullet}^i + d_{Y^\bullet}^{i-1} h^i$, where $h^i : X^i \to Y^{i-1}$ is any family of morphisms in $\mathcal{A}$, is an abelian group and stable under composition of morphisms. This means that for any morphisms $h_1 : Z^\bullet \to X^\bullet$, $h_2 : Y^\bullet \to W^\bullet$, and $i \in I_{X^\bullet, Y^\bullet}$, we have $i h_1 \in I_{Z^\bullet, Y^\bullet}$ and $h_2 i \in I_{X^\bullet, W^\bullet}$.*

(c) *Suppose $\mathcal{A}$ is an abelian category and let $f, g : X^\bullet \to Y^\bullet$ be morphisms in $C(\mathcal{A})$. If $f^i - g^i = h^{i+1} d_{X^\bullet}^i + d_{Y^\bullet}^{i-1} h^i$, for some family of morphisms $h^i : X^i \to Y^{i-1}$, then $H^i(f) \cong H^i(g)$ for all $i \in \mathbb{Z}$.*

Proof. (a) For any $i \in \mathbb{Z}$ we have

$$d_{Y^\bullet}^i (h^{i+1} d_{X^\bullet}^i + d_{Y^\bullet}^{i-1} h^i) = d_{Y^\bullet}^i h^{i+1} d_{X^\bullet}^i = (d_{Y^\bullet}^i h^{i+1} + h^{i+2} d_{X^\bullet}^{i+1}) d_{X^\bullet}^i,$$

so $h^{i+1} d_{X^\bullet}^i + d_{Y^\bullet}^{i-1} h^i$ is a morphism of complexes.

(b) For two families $h_1^i, h_2^i : X^i \to Y^{i-1}$ of morphisms, we have

$$(d_{Y^\bullet}^{i-1} h_1^i + h_1^{i+1} d_{X^\bullet}^{i-1}) + (d_{Y^\bullet}^{i-1} h_2^i + h_2^{i+1} d_{X^\bullet}^{i-1}) = d_{Y^\bullet}^{i-1} (h_1^i + h_2^i) + (h_1^{i+1} + h_2^{i+1}) d_{X^\bullet}^{i-1}.$$

From this it is easy to see that the class of morphisms $I_{X^\bullet, Y^\bullet}$ has the structure of an abelian group.

Let $h_1 : Z^\bullet \to X^\bullet$ and $h_2 : Y^\bullet \to W^\bullet$ be morphisms of complexes. Now

$$(h^{i+1} d_{X^\bullet}^i + d_{Y^\bullet}^{i-1} h^i) h_1^i = (h^{i+1} h_1^{i+1}) d_{Z^\bullet}^i + d_{Y^\bullet}^{i-1} (h^i h_1^i)$$

and

$$h_2^i (d_{Y^\bullet}^{i-1} h^i + h^{i+1} d_{X^\bullet}^i) = d_{W^\bullet}^{i-1} (h_2^{i-1} h^i) + (h_2^i h^{i+1}) d_{X^\bullet}^i.$$

so the classes $I_{X^\bullet, Y^\bullet}$ are closed under composition of morphisms.

(c) By the universal property used to define the induces morphisms between cohomology groups, one can verify that $H^i(f - g) \cong H^i(f) - H^i(g)$. Hence, to show that $H^i(f) \cong H^i(g)$, it is enough to show that $H^i(d_{Y^\bullet}^{i-1} h^i + h^{i+1} d_{X^\bullet}^i) \cong 0$.

By definition of the cohomology complex definition 2.5.2, 2.4.3 (i), and proposition 2.5.5 it suffices to show that for any pseudo-element $a \in^* X^i$ such that $d_{X^\bullet}^i(a) = 0$ we have $(d_{Y^\bullet}^{i-1} h^i + h^{i+1} d_{X^\bullet}^i)(a) =^* d_{Y^\bullet}^{i-1}(b)$ for some pseudo-element $b \in^* Y^{i-1}$. Now

$$(d_{Y^\bullet}^{i-1} h^i + h^{i+1} d_{X^\bullet}^i)(a) = (d_{Y^\bullet}^{i-1} h^i)(a) + (h^{i+1} d_{X^\bullet}^i)(a) =^* (d_{L^\bullet}^{i-1} h^i)(a),$$

because $(h^{i+1} d_{X^\bullet}^i)(a) = 0$ by assumption on the pseudo-element a. Therefore we can choose $b = h^i a$. □

Definition 2.7.4 (Translation)**.** Let $\mathcal{A}$ be an additive category. Define a functor $T : C(\mathcal{A}) \to C(\mathcal{A})$ as follows. For an object $X^\bullet \in C(\mathcal{A})$, let $T(X)$, denoted also by $X[1]$, to be the complex

$$T(X)^i = X^{i+1}, \qquad T(d^i_{X^\bullet})^i = -d^{i+1}_{X^\bullet}.$$

For a morphism $f : X^\bullet \to Y^\bullet$, let $T(f)^i = f^{i+1}$. This is also denoted by $f[1]$. Clearly this defines a functor, which is an isomorphism on $C(\mathcal{A})$, where the inverse T^{-1} is given by translation to other direction.

For any positive integer $n \in \mathbb{Z}$ we denote by $X^\bullet[n]$ and by $f[n]$ the functor T applied n times to object $X^\bullet$ and to morphism f. For a negative n, $X^\bullet[n]$ and $f[n]$ denotes T^{-1} applied $-n$ times.

The reader can verify that the translation functor defined above for the category $C(\mathcal{A})$, also works for the category $K(\mathcal{A})$, because homotopies are translated to homotopies.

Definition 2.7.5 (Mapping cone)**.** Let $\mathcal{A}$ be an additive category, and $f : X^\bullet \to Y^\bullet$ a morphism of complexes in $C(\mathcal{A})$. We define the mapping cone $C(f)$ of f to be the complex $C(f) = X^\bullet[1] \oplus Y^\bullet$ with differential

$$d^i_{C(f)} = -i_1 d^{i+1}_{X^\bullet} p_1 + i_2 f^{i+1} p_1 + i_2 d^i_{Y^\bullet} p_2.$$

The following shows that this is indeed a differential

$$\begin{aligned} d^{i+1}_{C(f)} d^i_{C(f)} &= (-i_1 d^{i+2}_{X^\bullet} p_1 + i_2 f^{i+2} p_1 + i_2 d^{i+1}_{Y^\bullet} p_2)(-i_1 d^{i+1}_{X^\bullet} p_1 + i_2 f^{i+1} p_1 + i_2 d^i_{Y^\bullet} p_2) \\ &= -i_2 f^{i+2} d^i_{X^\bullet} p_2 + i_2 d^{i+1}_{Y^\bullet} f^{i+1} p_2 \\ &= 0. \end{aligned}$$

Similarly one defines mapping cones in $K(\mathcal{A})$. We have the following important short exact sequence of complexes for mapping cones.

Lemma 2.7.6. *Let $\mathcal{A}$ be an abelian category. For a morphism $f : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$, the sequence*

$$0 \longrightarrow Y^\bullet \xrightarrow{i_2} C(f) \xrightarrow{p_1} X^\bullet[1] \longrightarrow 0$$

is a short exact sequence.

Proof. Clearly i_2 is a monomorphism and p_2 is an epimorphism. Since $p_1 i_2 = 0$, by 2.4.3 (iv) it suffices to show that for any $a \in^* C(f)$ with $p_1 a = 0$, there exists a pseudo element of $Y^\bullet$ mapping to a. Now $\mathrm{Id}_{C(f)} = i_1 p_1 + i_2 p_2$, so $a = i_2 p_2 a$. Hence the pseudo element $p_2 a \in^* Y^\bullet$ maps to a. This shows that the sequence is exact.

To show that i_2 and p_1 are morphisms of complexes, we have to show that they commute with differentials. Now

$$i_2 d^i_{Y^\bullet} = (-i_1 d^{i+1}_{X^\bullet} p_1 + i_2 f^{i+1} p_1 + i_2 d^i_{Y^\bullet} p_2) i_2 = d^i_{C(f)} i_2$$

which shows that i_2 commutes with differentials. From

$$p_1 d^i_{C(f)} = -d^{i+1}_{X^\bullet} p_1 = d^i_{X^\bullet[1]} p_1$$

we see that p_1 commutes with differentials. □

Definition 2.7.7 (Mapping cylinder)**.** Let $\mathcal{A}$ be an additive category and let $f : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C(\mathcal{A})$. We define the mapping cylinder $Cyl(f)$ of f to be the complex $Cyl(f) = X^\bullet \oplus C(f)$ with differential

$$d^i_{Cyl(f)} = i_1 d^i_{X^\bullet} - i_1 p_1 p_2 + i_2 d^i_{C(f)} p_2 = i_1 d^i_{X^\bullet} p_1 - i_1 p_1 p_2 - i_2 i_1 d^{i+1}_{X^\bullet} p_1 p_2 + i_2 i_2 f^{i+1} p_1 p_2 + i_2 i_2 d^i_{Y^\bullet} p_2 p_2.$$

The following shows that this is a differential

$$\begin{aligned} d^{i+1}_{Cyl(f)} d^i_{Cyl(f)} &= (i_1 d^{i+1}_{X^\bullet} p_1 - i_1 p_1 p_2 - i_2 i_1 d^{i+2}_{X^\bullet} p_1 p_2 + i_2 i_2 f^{i+2} p_1 p_2 + i_2 i_2 d^{i+1}_{Y^\bullet} p_2 p_2) \\ &\quad (i_1 d^i_{X^\bullet} p_1 - i_1 p_1 p_2 - i_2 i_1 d^{i+1}_{X^\bullet} p_1 p_2 + i_2 i_2 f^{i+1} p_1 p_2 + i_2 i_2 d^i_{Y^\bullet} p_2 p_2) \\ &= i_1 d^{i+1}_{X^\bullet} p_1 p_2 - i_1 d^{i+1}_{X^\bullet} p_1 p_2 - i_2 i_2 f^{i+2} d^{i+1}_{X^\bullet} p_2 p_2 + i_2 i_2 d^{i+1}_{Y^\bullet} f^{i+1} p_2 p_2 \\ &= 0. \end{aligned}$$

Similarly one defines mapping cylinders in $K(\mathcal{A})$. For mapping cylinders we have the following important short exact sequence

Lemma 2.7.8. *Let $\mathcal{A}$ be an abelian category. For a morphism $f : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$, the sequence*

$$0 \longrightarrow X^\bullet \xrightarrow{i_1} Cyl(f) \xrightarrow{p_2} C(f) \longrightarrow 0$$

is a short exact sequence.

Proof. First, let us verify that the morphisms are morphisms of complexes. From

$$i_1 d^i_{X^\bullet} = (i_1 d^i_{X^\bullet} p_1 + i_1 p_2 - i_2 i_1 d^{i+1}_{X^\bullet} p_1 p_2 + i_2 i_2 f^{i+1} p_1 p_2 + i_2 i_2 d^i_{Y^\bullet} p_2 p_2) i_1 = d^i_{Cyl(f)} i_1$$

and

$$p_2 d^i_{Cyl(f)} = -i_1 d^{i+1}_{X^\bullet} p_1 p_2 + i_2 f^{i+1} p_1 p_2 + i_2 d^i_{Y^\bullet} p_2 p_2 = d^i_{C(f)} p_2,$$

we see that i_1 and p_2 commute with differentials.

Next, we verify that the sequences is exact. Clearly i_1 is a monomorphism and p_2 is an epimorphism. Since $p_2 i_1 = 0$, by 2.4.3 (iv) it suffices to show that for $a \in^* Cyl(f)$ such that $p_2 a = 0$ there exists a pseudo-element of $X^\bullet$ mapping to a. Since $a = \mathrm{Id}_{Cyl(f)}\, a = (i_1 p_1 + i_2 p_2) a = i_1 p_1 a$, the pseudo-element $p_1 a$ maps to a. Hence the short sequence is exact. $\square$

The following lemma captures the main properties of mapping cone and mapping cylinder of a morphism.

Lemma 2.7.9. *Let $\mathcal{A}$ be an abelian category. For any morphism $f : X^\bullet \to Y^\bullet$ there exists the following commutative diagram in $C(\mathcal{A})$ with exact rows*

$$\begin{tikzcd} & 0 \arrow[r] & Y^\bullet \arrow[r, "i_2"] \arrow[d, "\alpha"] & C(f) \arrow[r, "p_1"] \arrow[d, equal] & X^\bullet[1] \arrow[r] & 0 \\ 0 \arrow[r] & X^\bullet \arrow[r, "i_1"] \arrow[d, equal] & Cyl(f) \arrow[r, "p_2"] \arrow[d, "\beta"] & C(f) \arrow[r] & 0 & \\ & X^\bullet \arrow[r, "f"] & Y^\bullet & & & \end{tikzcd},$$

where $\alpha^i = i_2 i_2$ and $\beta^i = f p_1 + p_2 p_2$ are quasi-isomorphisms, $\beta\alpha = Id_L$ and $\alpha\beta$ is homotopic to $Id_{Cyl(f)}$.

Proof. From lemmas 2.7.6 and 2.7.8 it follows that the rows are exact. Now $\beta i_1 = (f^i p_1 + p_2 p_2) i_1 = f^i$ and $i_2 = p_2 i_2 i_2 = p_2 \alpha$, so both of the squares in the diagram are commutative. From

$$\alpha^{i+1} d^i_{Y^\bullet} = i_2 i_2 d^i_{Y^\bullet} = d^i_{Cyl(f)} \alpha^i$$

and

$$\begin{aligned}
\beta^{i+1} d^i_{Cyl(f)} &= f^{i+1} d^i_{X^\bullet} p_1 - f^{i+1} p_1 p_2 + f^{i+1} p_1 p_2 + d^i_{Y^\bullet} p_2 p_2 \\
&= d^i_{Y^\bullet} f^i p_1 + d_{Y^\bullet} p_2 p_2 \\
&= d^i_{Y^\bullet} \beta^i
\end{aligned}$$

it follows that α and β are morphisms of complexes.

Let $\chi^i : Cyl(f)^i \to Cyl(f)^{i-1}, \chi^i = -i_2 i_1 p_1$. Now $\beta^i \alpha^i = (f^i p_1 + p_2 p_2) i_2 i_2 = \mathrm{Id}_{Y^\bullet}$ and

$$\begin{aligned}
\alpha^i \beta^i &= i_2 i_2 (f^i p_1 + p_2 p_2), \\
d^{i-1}_{Cyl(f)} \chi^i &= i_1 p_1 + i_2 i_1 d^i_{X^\bullet} p_1 - i_2 i_2 f^i p_1, \\
\chi^{i+1} d^i_{Cyl(f)} &= -i_2 i_1 d^i_{X^\bullet} p_1 + i_2 i_1 p_1 p_2,
\end{aligned}$$

so

$$\begin{aligned}
\mathrm{Id}_{Cyl(f)^i} - (\alpha\beta)^i &= i_1 p_1 + i_2 i_1 p_1 p_2 - i_2 i_2 f^i p_1 \\
&= d^{i-1}_{Cyl(f)} \chi^i + \chi^{i+1} d^i_{Cyl(f)}.
\end{aligned}$$

This shows that $\alpha\beta$ is homotopic to $\mathrm{Id}_{Cyl(f)}$. ☐

Lemma 2.7.10. *The relation of homotopy is an equivalence relation and respects composition of morphisms.*

Proof. Let $f, g, k : X^\bullet \to Y^\bullet$ be morphisms of complexes. Clearly $f \sim f$, because $\chi^i = 0$ satisfies $f - f = d^{i-1}_{Y^\bullet} \chi^i + \chi^{i+1} d^i_{X^\bullet}$. If $f \sim g$ and $\chi^i : X^i \to Y^{i-1}$ are the morphisms which define the homotopy, then $-\chi^i$ are morphisms which show that $g \sim f$. Suppose $f \sim g$ and $g \sim k$ and let $\chi^i_1 : X^i \to Y^{i-1}$ and $\chi^i_2 : X^i \to Y^{i-1}$ be the families of morphisms which show that $f \sim g$ and $g \sim k$, respectively. The family $\chi^i_1 + \chi^i_2$ shows that $f \sim k$. Hence homotopy is an equivalence relation.

To show that homotopy respects composition, let $f_1, f_2 : X^\bullet \to Y^\bullet$ and $g_1, g_2 : Y^\bullet \to Z^\bullet$ be morphisms in $C(\mathcal{A})$ such that $f_1 \sim f_2$ and $g_1 \sim g_2$. We show that $g_1 f_1 \sim g_2 f_2$. Let $(\chi^i) : X^i \to Y^{i-1}$ be a set of morphisms such that $(f_1 - f_2)^i = d^{i-1}_{L^\bullet} \chi^i + \chi^{i+1} d^i_{K^\bullet}$ for all i. Then

$$g^i_1 (f_1 - f_2)^i = g^i_1 d^{i-1}_{Y^\bullet} \chi^i + g^i_1 \chi^{i+1} d^i_{X^\bullet} = d^{i-1}_{Z^\bullet} (g^{i-1}_1 \chi^i) + (g^i_1 \chi^{i+1}) d^i_{X^\bullet}$$

so $g_1 f_1 \sim g_1 f_2$. If $(\chi^i) : Y^i \to Z^{i-1}$ is a set of morphisms such that $(g_1 - g_2)^i = d^{i-1}_{Z^\bullet} \chi^i + \chi^{i+1} d^i_{Y^\bullet}$, then

$$(g_1 - g_2)^i f^i_2 = d^{i-1}_{Z^\bullet} \chi^i f^i_2 + \chi^{i+1} d^i_{Y^\bullet} f^i_2 = d^{i-1}_{Z^\bullet} (\chi^i f^i_2) + (\chi^{i+1} f^{i+1}_2) d^i_{X^\bullet}$$

so $g_1 f_2 \sim g_2 f_2$. By transitivity, we get $g_1 f_1 \sim g_2 f_2$. ☐

Definition 2.7.11 (Homotopy category)**.** We define the homotopy category $K(\mathcal{A})$ of an additive category $\mathcal{A}$ to be a quotient category of $C(\mathcal{A})$ obtained by identifying morphisms which are homotopic. Precisely, $\mathrm{Ob}\, K(\mathcal{A}) = \mathrm{Ob}\, C(\mathcal{A})$, and for any objects $X, Y \in K(\mathcal{A})$ we define

$$\mathrm{Mor}_{K(\mathcal{A})}(X^\bullet, Y^\bullet) = \mathrm{Mor}_{C(\mathcal{A})}(X^\bullet, Y^\bullet)/\sim,$$

where $\sim$ is the homotopy relation. By lemma 2.7.10 this definition is well-defined.

Similarly one defines the categories $K^*(\mathcal{A})$, $* = +, -, b$, by using the categories $C^*(\mathcal{A})$, see definition 2.5.1, and the homotopy relation. Note that the categories $K^*(\mathcal{A})$, $* = +, -, b$, are full subcategories of $K(\mathcal{A})$, because $C^*(\mathcal{A})$ are full subcategories of $C(\mathcal{A})$ by definition.

In case the category $\mathcal{A}$ is an abelian category, we still have cohomology complexes for $K(\mathcal{A})$, because by lemma 2.7.3 (c), homotopic morphisms induce the same maps on cohomology. Precisely this means that the cohomology functor $H^\bullet : C(\mathcal{A}) \to C(\mathcal{A})$ factors through the homotopy category $K(\mathcal{A})$. We can compute the cohomology complex of an object of $K(\mathcal{A})$ by choosing a representative and then compute its cohomology in $C(\mathcal{A})$. By abuse of notation, we denote this cohomology functor also by $H^\bullet : K(\mathcal{A}) \to C(\mathcal{A})$.

The following proposition shows that the homotopy category of an additive category is additive.

Proposition 2.7.12. *For any additive category $\mathcal{A}$ the category $K(\mathcal{A})$ is additive. Since $K^*(\mathcal{A})$, $* = +, -, b$, are full subcategories of $K(\mathcal{A})$ it follows that these are also additive.*

Proof. From lemma 2.7.3 we see that the sets of morphisms in $K(\mathcal{A})$ are abelian groups and composition of morphisms respects addition. Hence it follows that the zero object and the biproduct of any two objects are the same as in $C(\mathcal{A})$.

To see that the morphism from the biproduct is unique, consider two morphisms $f, g : X \oplus Y \to Z$ such that $fi_1 \sim gi_1$ and $fi_2 \sim gi_2$. Then $f - g \sim (f - g)(i_1p_1 + i_2p_2) \sim (fi_1 - gi_1)p_1 + (fi_2 - gi_2)p_2 \sim 0$. Hence f and g are homotopic. Therefore $K(\mathcal{A})$ is an additive category. $\square$

2.8 Notes

Instead of developing homological algebra using only the language of category theory, one can assume the Freyd embedding, [Bor94b, Theorem 1.14.9], and use the category **RMod** to develop the theory. In particular, this allows one to use elements in the proofs. Since this approach hides how one uses universal properties to construct and identify morphisms, we decided not to take this approach. Also, the categories $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$, see chapter 5, and arbitrary triangulated category, see chapter 4, are not abelian categories in general, see 4.2.7, so later we are forced to use universal properties anyway.

Chapter 3

Localization of a category

In this chapter we introduce Gabriel-Zismann localization of categories. Localization produces a new category with the same objects and inverses are added to a given class of morphisms. Set theoretically this has the problem that morphisms between two objects in localization may not form a set but a class. We discuss this problem in remark 3.1.2.

Unfortunately the morphisms in the localization are described in a way which is hard to work with. To overcome this, we specialize to a special kind of classes, localizing classes. In this case the morphisms in the localization can be more easily understood by using the formalism of roofs. In particular, the class of quasi-isomorphisms form a localizing class in the homotopy category of an abelian category, but not necessarily in the category of complexes over it. We will see that this is important in proving that the derived category of an abelian category is a triangulated category.

3.1 Gabriel-Zisman localization

The following construction of a category ignores set theoretical problems, that is, the class of morphisms is not necessarily a set, by enlarging the definition of a category to include this case. Later we show that in the cases we are interested in the classes of morphisms are actually sets.

Construction 3.1.1. Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$. We construct the category $\mathcal{C}[S^{-1}]$ which is the *localization* of $\mathcal{C}$ by the class of morphisms S. Let $\operatorname{Ob}\mathcal{C}[S^{-1}] = \operatorname{Ob}\mathcal{C}$. For any objects $X, Y \in \operatorname{Ob}\mathcal{C}[S^{-1}]$, let

$$M(X,Y) = \operatorname{Mor}_{\mathcal{C}}(X,Y) \coprod (S \cap \operatorname{Mor}_{\mathcal{C}}(Y,X))$$

where $\coprod$ is the disjoint union of sets. The first component consists of the morphisms from X to Y in $\mathcal{C}$ and the second component denotes formal inverses to morphisms in S. This means that we denote an element $s \in (S \cap \operatorname{Mor}_{\mathcal{C}}(Y,X))$ by $s^{-1} \in M(X,Y)$, and set $\operatorname{Dom}(s^{-1}) = X$ and $\operatorname{Cod}(s^{-1}) = Y$. Extend the composition of $\mathcal{C}$ by defining

$$s^{-1}s = \mathrm{Id}_Y \qquad \text{and} \qquad ss^{-1} = \mathrm{Id}_X \,.$$

Finally, for any $X, Y \in \mathrm{Ob}\, \mathcal{C}[S^{-1}]$, let

$$\mathrm{Mor}_{\mathcal{C}[S^{-1}]}(X,Y) = \left(\bigcup_{\substack{n \in \mathbb{N}-\{0\} \\ X=X_1,\dots,X_{n+1}=Y \in \mathrm{Ob}\, \mathcal{C}[S^{-1}] \\ f_i \in M(X_i, X_{i+1}) 1 \leqslant i < n}} (f_1, \dots, f_n) \right) / \sim$$

where $\sim$ is the equivalence relation determined by the following conditions:

For $n \in \mathbb{N} - \{0\}$, $X_1, \dots, X_{n+1} \in \mathrm{Ob}\, \mathcal{C}[S^{-1}]$, and $f_i, g_i \in M(X_i, X_{i+1})$ for $1 \leqslant i < n$

(i) $(f_1, \dots, f_n) \sim (g_1, \dots, g_n)$ if $f_i = g_i$ in $\mathcal{C}$ for all $1 \leqslant i < n$.

(ii) $(f_1, \dots, f_j, f_{j+1}, \dots, f_n) \sim (f_1, \dots, f_{j+1} f_j, \dots, f_n)$ for $1 \leqslant j < n$ if one of the following conditions holds

   (a) $f_j, f_{j+1} \notin S$

   (b) $f_j \in S, f_{j+1} = f_j^{-1}$

   (c) $f_{j+1} \in S$ and $f_j = f_{j+1}^{-1}$.

For any element $(f_1, \dots, f_n) \in \mathrm{Mor}_{\mathcal{C}[S^{-1}]}$ we set $\mathrm{Dom}((f_1, \dots, f_n)) = \mathrm{Dom}(f_1)$ and $\mathrm{Cod}((f_1, \dots, f_n)) = \mathrm{Cod}(f_n)$. For any two elements $(f_1, \dots, f_n) \in \mathrm{Mor}_{\mathcal{C}[S^{-1}]}(X,Y)$ and $(g_1, \dots, g_m) \in \mathrm{Mor}_{\mathcal{C}[S^{-1}]}(Y,Z)$, composition is given by $(f_1, \dots, f_n) \circ (g_1, \dots, g_m) = (f_1, \dots, f_n, g_1, \dots, g_m)$.

By assuming that $\mathrm{Mor}_{\mathcal{C}[S^{-1}]}(X,Y)$ is a set for all objects $X, Y \in \mathcal{C}[S^{-1}]$, one can easily verify that $\mathcal{C}[S^{-1}]$ is a category. The construction shows that there exists a natural *localization functor* $Q_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$ which is identity on objects and sends a morphism f to (f).

From the construction we see that isomorphisms in $\mathcal{C}[S^{-1}]$ are given by sequences consisting of isomorphisms in $\mathcal{C}$ together with morphisms in S and their formal inverses. This can be shown by induction on length of a sequence.

Remark 3.1.2 (Set theoretical problems). As mentioned in the introduction of this chapter, it is not clear that the morphisms in the localized category form a set. Lemma 3.2.6 gives one criterion for the existence of localization. From now on, if not otherwise stated, we assume that the localization exists by extending in this case the definition of the category to include the case when the collection of morphisms between two objects is a class. This technique is also used to show that the derived category of **RMod** exists.

If one uses universes, then one can choose a universe large enough to contain the class of objects as a set. In the case of a small category, localization always exists, see lemma 3.2.6.

Theorem 3.1.3. *Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$. If $F : \mathcal{C} \to \mathcal{D}$ is a functor which send all morphisms in S to isomorphisms, then there exists a unique functor $G : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q_S"] \arrow[dr, "F"'] & \mathcal{C}[S^{-1}] \arrow[d, dashed, "G"] \\ & \mathcal{D} \end{tikzcd}$$

Proof. Set $G(X) = F(X)$ for any object $X \in \mathrm{Ob}\, \mathcal{C} = \mathrm{Ob}\, \mathcal{C}[S^{-1}]$, $G((f_1, \dots f_n)) = F(f_n) \dots F(f_1)$ for any $(f_1, \dots, f_n) \in \mathrm{Mor}\, \mathcal{C}[S^{-1}]$. These choices are forced to make the above diagram commutative. Thus we see that G is the unique functor which makes the diagram commutative. $\square$

3.2 Localizing class

In this section we introduce a formalism of roofs. This formalism can be applied when the class of morphisms to invert is a localizing class.

Definition 3.2.1 (Localizing class)**.** A class S of morphisms of $\mathcal{C}$ is a *localizing class* if it satisfies the following conditions

LS 1 For any object $X \in \mathcal{C}$, $\mathrm{Id}_X \in S$, and if $f, g \in S$ so that gf is defined, then $gf \in S$.

LS 2 Consider the following diagrams

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "t_1"] & Y \\ Z & \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[d, "g_1"] \\ Z \arrow[r, "s_1"] & W \end{tikzcd}$$

where $s_1, t_1 \in S$ and f_1, g_1 are morphisms in $\mathcal{C}$. We can complete thess diagrams to commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "t_1"] & Y \arrow[d, "t_2"] \\ Z \arrow[r, "f_2"] & X' \end{tikzcd} \qquad \begin{tikzcd} W' \arrow[r, "s_2"] \arrow[d, "g_2"] & Y \arrow[d, "g_1"] \\ Z \arrow[r, "s_1"] & W \end{tikzcd}$$

where $t_2, s_2 \in S$.

LS 3 Let $f, g : X \to Y$ be morphisms. Then there is a morphism $s \in S$ so that $sf = sg$ if and only if $ft = gt$ for some $t \in S$.

Suppose $\mathcal{S}$ is a localizing class of morphisms in $\mathcal{C}$. Then the morphisms in $\mathcal{C}[S^{-1}]$ can be described by the formalism of roofs. An S-roof representing a morphism from X to Y in $\mathcal{C}$, denoted by (s, f), is a diagram

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "f"] & \\ Y & & Z \end{tikzcd}$$

where $s \in S$ and $f \in \mathrm{Mor}\,\mathcal{C}$. When the localizing class S is understood we simply write roof. We say that two roofs (s, f) and (t, g) are equivalent, denoted $(s, f) \sim (t, g)$, if there exists a commutative diagram of the form

$$\begin{tikzcd} & & X''' \arrow[dl, "t'"'] \arrow[dr, "s'"] & & \\ & X' \arrow[dl, "s"'] \arrow[drrr, "f"] & & X'' \arrow[dlll, "t"'] \arrow[dr, "g"] & \\ X & & & & Y \end{tikzcd}$$

where $st' \in S$. A composition of two roofs (s, f) and (t, g) is a commutative diagram of the form

$$\begin{tikzcd}
 & & X''' \arrow[dl, "s'"'] \arrow[dr, "t'"] & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & X'' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

where (ss', gt') is a roof.

Lemma 3.2.2. *The relation $\sim$ for roofs is an equivalence relation.*

Proof. The relation $\sim$ is obviously reflexive and symmetric by the definition of $\sim$. Let $(u : Z_1 \to X, f : Z_1 \to Y), (v : Z_2 \to X, g : Z_2 \to Y)$ and $(w : Z_3 \to X, h : Z_3 \to Y)$ be roofs such that $(u, f) \sim (v, g)$ and $(v, g) \sim (w, h)$. Let these relations be given by the following diagram

$$\begin{tikzcd}
 & W_1 \arrow[dl, "s"'] \arrow[dr, "k"] & & W_2 \arrow[dl, "t"'] \arrow[dr, "l"] & \\
Z_1 \arrow[d, "u"] \arrow[drrrr, "f"] & & Z_2 \arrow[dll, "v"] \arrow[drr, "g"] & & Z_3 \arrow[d, "h"] \arrow[dllll, "w"] \\
X & & & & Y
\end{tikzcd}$$

By LS 2 we obtain a commutative diagram

$$\begin{tikzcd}
W_3 \arrow[r, "k'"] \arrow[d, "t'"] & W_2 \arrow[d, "vt"] \\
W_1 \arrow[r, "vk"] & X
\end{tikzcd}$$

By LS 3 there exists a morphism $W \to W_3$ in S such that

$$W \to W_3 \to W_1 \to Z_2 = W \to W_3 \to W_2 \to Z_2.$$

We have

$$\begin{aligned}
W \to W_3 \to W_1 \to Z_1 \to X &= W \to W_3 \to W_1 \to Z_2 \to X \\
&= W \to W_3 \to W_2 \to Z_2 \to X \\
&= W \to W_3 \to W_2 \to Z_3 \to X
\end{aligned}$$

and

$$\begin{aligned}
W \to W_3 \to W_1 \to Z_1 \to Y &= W \to W_3 \to W_1 \to Z_2 \to Y \\
&= W \to W_3 \to W_2 \to Z_2 \to Y \\
&= W \to W_3 \to W_2 \to Z_3 \to Y
\end{aligned}$$

so $(u, f) \sim (w, h)$. □

Lemma 3.2.3. *Composition of roofs is well-defined, i.e., composition of roofs does not depend on the choice of representatives.*

Proof. Let

$$\begin{tikzcd}
& W_1 \arrow[dl] \arrow[dr] & & & W_2 \arrow[dl] \arrow[dr] & \\
X_1 \arrow[d, "u_1"] \arrow[drr, "f_1"] & & X_2 \arrow[d, "f_2"] \arrow[dll, "u_2"] & Y_1 \arrow[d, "v_1"] \arrow[drr, "g_1"] & & Y_2 \arrow[d, "g_2"] \arrow[dll, "v_2"] \\
X & & Y & Y & & Z
\end{tikzcd}$$

be two equivalences of roofs. By LS 2 the composite of the roofs (u_2, f_2) and (v_1, g_1) can be represented by the following diagram

$$\begin{tikzcd}
& & W_3 \arrow[dl] \arrow[dr] & & \\
& X_2 \arrow[dl, "u_2"] \arrow[dr, "f_2"] & & Y_1 \arrow[dl, "v_1"] \arrow[dr, "g_1"] & \\
X & & Y & & Z
\end{tikzcd}$$

By LS 2 we obtain the following commutative squares

$$\begin{tikzcd}
V_1 \arrow[r] \arrow[d] & W_3 \arrow[d] & V_2 \arrow[r] \arrow[d] & W_2 \arrow[d] \\
W_1 \arrow[r] & X & W_3 \arrow[r] & Y
\end{tikzcd}$$

By LS 3 we get morpisms $U_1 \to V_1$ and $U_2 \to V_2$ in S such that

$$U_1 \to V_1 \to W_1 \to X_2 = U_1 \to V_1 \to W_3 \to X_2$$

and

$$U_2 \to V_2 \to W_2 \to Y_1 = U_2 \to V_2 \to W_3 \to Y_1$$

Therefore we have the following diagram

$$\begin{tikzcd}
& & U_1 \arrow[d] & & U_2 \arrow[d] & & \\
& & V_1 \arrow[dl] \arrow[dr] & & V_2 \arrow[dl] \arrow[dr] & & \\
& W_1 \arrow[dl] \arrow[dr] & & W_3 \arrow[dl] \arrow[dr] & & W_2 \arrow[dl] \arrow[dr] & \\
X_1 \arrow[dr, "u_1"] \arrow[drrr, "f_1"] & & X_2 \arrow[dl, "u_2"] \arrow[dr, "f_2"] & & Y_1 \arrow[dl, "v_1"] \arrow[dr, "g_1"] & & Y_2 \arrow[dlll, "v_2"] \arrow[dl, "g_2"] \\
& X & & Y & & Z &
\end{tikzcd}$$

Now

$$
\begin{aligned}
U_1 \to V_1 \to W_1 \to X_1 \to Y &= U_1 \to V_1 \to W_1 \to X_2 \to Y \\
&= U_1 \to V_1 \to W_3 \to X_2 \to Y \\
&= U_1 \to V_1 \to W_3 \to Y_1 \to Y,
\end{aligned}
$$

and

$$
\begin{aligned}
U_2 \to V_2 \to W_2 \to Y_2 \to Y &= U_2 \to V_2 \to W_2 \to Y_1 \to Y \\
&= U_2 \to V_2 \to W_3 \to Y_1 \to Y \\
&= U_2 \to V_2 \to W_3 \to X_2 \to Y
\end{aligned}
$$

so $(v_1, g_1) \circ (u_1, f_1) \sim (v_1, g_1) \circ (u_2, f_2)$ and $(v_1, g_1) \circ (u_2, f_2) \sim (v_2, g_2) \circ (u_2, f_2)$. Since $\sim$ is an equivalence relation by lemma 3.2.2, we conclude that $(v_1, g_1) \circ (u_1, f_1) \sim (v_2, g_2) \circ (u_2, f_2)$ and hence composite does not depend on the chosen representatives. $\square$

The above theorems show that the category consisting of objects of $\mathcal{C}$ and morphisms given by S-roofs is a well-defined category. Call this the category of S-roofs in $\mathcal{C}$ and denote it by $\tilde{\mathcal{C}}_S$. The following theorem shows that this category is isomorphic to localization of $\mathcal{C}$ by S.

Theorem 3.2.4. *Let $\mathcal{C}$ be a category, S a localizing class in $\mathcal{C}$, $\tilde{\mathcal{C}}_S$ the category of S-roofs in $\mathcal{C}$, and $F : C \to \tilde{\mathcal{C}}_S$ the functor which is identity on objects and sends a morphism $f : X \to Y$ to the morphism represented by the roof (Id_X, f). If $\mathcal{D}$ is a category and $G : \mathcal{C} \to \mathcal{D}$ is a functor such that $G(f)$ is invertible in $\mathcal{D}$ for any $f \in S$, then there exists a unique functor $Q : \tilde{\mathcal{C}}_S \to \mathcal{D}$ such that $Q \circ F = G$.*

Proof. Let $\mathcal{D}$ be a category and $G : \mathcal{C} \to \mathcal{D}$ a functor such that $G(f)$ is invertible for any $f \in S$. Define the functor $Q : \tilde{\mathcal{C}}_S \to \mathcal{D}$ as follows. Let $Q(X) = G(X)$ for any object X. For a morphism represented by a roof (s, f) we define

$$Q((s, f)) = G(f)(G(s))^{-1}. \tag{3.1}$$

To show uniqueness of this definition, let Q' be a functor such that $G = Q' \circ F$ and let $s : Y \to X$ be any morphism in S. The reader can easily verify that we have the following relations of roofs $(\mathrm{Id}_X, s) \sim (s, \mathrm{Id}_Y)$, $(\mathrm{Id}_Y, s) \circ (s, \mathrm{Id}_Y) \sim (\mathrm{Id}_X, \mathrm{Id}_X)$, and $(s, \mathrm{Id}_Y) \circ (\mathrm{Id}_Y, s) \sim (\mathrm{Id}_Y, \mathrm{Id}_Y)$. Using these relations, we have $\mathrm{Id}_{G(X)} = Q'((\mathrm{Id}_X, \mathrm{Id}_X)) = Q'((\mathrm{Id}_Y, s) \circ (s, \mathrm{Id}_Y)) = G(s) \circ Q'((\mathrm{Id}_X, s))$ and $\mathrm{Id}_{G(Y)} = Q'(\mathrm{Id}_Y, \mathrm{Id}_Y) = Q'((s, \mathrm{Id}_Y) \circ (\mathrm{Id}_Y, s)) = Q'((\mathrm{Id}_X, s)) \circ G(s)$. Hence $Q'(s, \mathrm{Id}_X) = G(s)^{-1}$. Now for a roof $(s : Y \to Z, f : Y \to Z)$ we have $(s, f)\tilde{(}\mathrm{Id}_Y, f) \circ (s, \mathrm{Id}_Y)$, so $Q'((s, f)) = Q'((\mathrm{Id}_Y, f)) \circ Q'((s, \mathrm{Id}_Y)) = G(f)G(s)^{-1}$. This justifies the definition (3.1) and shows that if Q is a functor, then it is unique.

First, let us verify that the image of a morphism does not depend on the choice of a representative. Let (u_1, f_1) and (u_2, f_2) be two equivalent roofs, equivalence is given by

$$
\begin{tikzcd}
 & W \arrow[dl, "v"'] \arrow[dr, "g"] & \\
Z_1 \arrow[d, "u_1"] \arrow[drr, "f_1"] & & Z_2 \arrow[d, "f_2"] \arrow[dll, "u_2"] \\
X & & Y
\end{tikzcd}
$$

Now

$$\begin{aligned} Q((u_1, f_1)) &= G(f_1)G(u_1)^{-1} \\ &= G(f_1 v)G(u_1 v)^{-1} \\ &= G(f_2 g)G(u_2 g)^{-1} \\ &= G(f_2)G(u_2)^{-1} \\ &= Q((u_2, f_2)), \end{aligned}$$

so the map on morphisms is well-defined.

Clearly $Q((id_X, id_X)) = \mathrm{Id}_{F(x)}$, so to show that Q is a functor it remains to show that it respects composition of morphisms. Let (u, f) and (v, g) be two roofs with the composite given by a roof

$$\begin{tikzcd} & & W \arrow[dl, "w"'] \arrow[dr, "h"] & & \\ & Z_1 \arrow[dl, "u"'] \arrow[dr, "f"] & & Z_2 \arrow[dl, "v"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

Now

$$G(u)^{-1} = G(w)G(uw)^{-1}$$

and from $fw = vh$ we get

$$G(h) = G(v)^{-1}G(f)G(w).$$

Therefore

$$\begin{aligned} Q((v, g) \circ (u, f)) &= Q((uw, gh)) \\ &= G(g)G(h)G(uw)^{-1} \\ &= G(g)G(v)^{-1}G(f)G(w)G(uw)^{-1} \\ &= G(g)G(v)^{-1}G(f)G(u)^{-1} \\ &= Q((v, g))Q((u, f)). \end{aligned}$$

This completes the proof. □

In particular, the above theorem together with theorem 3.1.3 implies that $Q : \tilde{\mathcal{C}}_S \to \mathcal{C}[S^{-1}]$ is the unique isomorphism of categories which makes the following diagram commutative

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "Q_S"'] & \tilde{\mathcal{C}}_S \arrow[d, "Q"] \\ & \mathcal{C}[S^{-1}] \end{tikzcd}$$

The proof of theorem 3.1.3 shows that the inverse of Q is given by sending (g) to the roof $(\mathrm{Id}_{\mathrm{Dom}(g)}, g)$, if g is not a formal inverse of a morphism in S, and if $g = s^{-1}$, then $Q^{-1}((g)) = (s, \mathrm{Id}_{\mathrm{Dom}(s)})$. By abuse of notation, from now on we assume that $\mathcal{C}[S^{-1}]$ equals $\tilde{\mathcal{C}}_S$ when S is a localizing class, if not otherwise stated.

Next we give a criterion for existence of localization in our universe of set theory. We follow [Wei95]. Alternatively one can choose a universe large enough so that the category one considers becomes small, see [Bor94a, Section 1.1], and then apply the lemma below to show existence of localization.

Definition 3.2.5 (Locally small localizing class)**.** Let $\mathcal{C}$ be a category and S a localizing class of $\mathcal{C}$. Then S is *locally small* if for any object X of $\mathcal{C}$ there exists a set of morphisms S_X contained in S, all having codomain X, such that for any morphism $X_1 \to X$ there exists a morphism $X_2 \to X_1$ in $\mathcal{C}$ such that the composite $X_2 \to X_1 \to X$ is in S_X.

It is not hard to see that if $\mathcal{C}$ is a small category, then all localizing classes are small. Indeed, one can take S_X to be the set of all morphisms of $\mathcal{C}$ with codomain X. Then the lemma below shows that the localization exists. We will later use the lemma below to prove the existence of the derived category of **RMod**.

Lemma 3.2.6. *Let $\mathcal{C}$ be a category and S a locally small localizing class. Then the category $\mathcal{C}[S^{-1}]$ exists.*

Proof. Fix any two objects X and Y of $\mathcal{C}[S^{-1}]$. It suffices to show that $\mathrm{Mor}_{\mathcal{C}[S^{-1}]}(X,Y)$ is a set, that is, the class of S-roofs from X to Y is a set. First we note that since S is locally small, for any morphism from X to Y there exists a roof $(s : Z \to X, f : Z \to Y)$, which represents the morphism, such that the morphism $s : Z \to X$ is in S_X. Indeed, if $(s' : X' \to X, f' : X' \to Y)$ is any roof representing some morphism from X to Y, then by definition of locally small localizing class there exists a morphism $s : Z \to X'$ such that $s's : Z \to X$ is a morphism in S_X. Then the following diagram is commutative

$$\begin{tikzcd}
& & Z \arrow[dl, "s"'] \arrow[dr, "\mathrm{Id}_Z"] & & \\
& X' \arrow[dl, "s'"'] \arrow[drrr, "f'" near end] & & Z \arrow[dlll, "s's" near end] \arrow[dr, "f's"] & \\
X & & & & Y
\end{tikzcd}$$

This shows that the roofs $(s's, f's)$ and (s', f') represent the same morphism. Hence we have a surjection from the class of all tuples $(s : Z \to X, f : Z \to Y)$, with s a morphism in S_X, to $\mathrm{Mor}_{\mathcal{C}[S^{-1}]}(X,Y)$. It suffices to show that this class is a set, which follows from the following equation

$$\{(s,f) \mid s : Z \to X \in S_X, f \in \mathrm{Mor}_{\mathcal{C}}(Z,Y)\} = \bigcup_{s:Z\to X\in S_X} \bigcup_{f\in \mathrm{Mor}_{\mathcal{C}}(Z,Y)} (s,f),$$

where the right hand side is a set because S_X is a set and for any morphism $Z \to X$ of S_X $\mathrm{Mor}_{\mathcal{C}}(Z,Y)$ is a set. This shows that $\mathrm{Mor}_{\mathcal{C}[S^{-1}]}(X,Y)$ is a set and so localization exists. ☐

The following proposition shows that localization preserves full subcategories under some conditions.

Proposition 3.2.7. *Let $\mathcal{C}$ be a category, S a localizing class in $\mathcal{C}$, and $\mathcal{B}$ a full subcategory of $\mathcal{C}$. Suppose that $S_{\mathcal{B}} = S \cap \mathcal{B}$ is a localizing class in $\mathcal{B}$. If either of the following two conditions hold*

(i) For any morphism $s \in S$ with codomain in $\mathrm{Ob}\,\mathcal{B}$*, there exists a morphisms f of $\mathcal{C}$ with domain in* $\mathrm{Ob}\,\mathcal{B}$ *such that $sf \in S$.*

(ii) For any morphism $s \in S$ with domain in $\mathrm{Ob}\,\mathcal{B}$, there exists a morphism f of $\mathcal{C}$ with codomain in $\mathrm{Ob}\,\mathcal{B}$ such that $fs \in S$.

then $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ is a full subcategory of $\mathcal{C}[S^{-1}]$.

Proof. Let $I : \mathcal{B} \to \mathcal{C}$ be the inclusion functor. Suppose that the condition (i) holds. We have to show that for any $X, Y \in \mathrm{Ob}\,\mathcal{B}$, the map

$$I_S : \mathrm{Mor}_{\mathcal{B}[S_{\mathcal{B}}^{-1}]}(X, Y) \to \mathrm{Mor}_{\mathcal{C}[S^{-1}]}(I(X), I(Y)) \tag{3.2}$$

is bijective, where I_S is the unique functor given by theorem 3.1.3. This functor makes the following diagram commutative

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "I"] \arrow[d, "Q_{\mathcal{B}}"] & \mathcal{C} \arrow[d, "Q_{\mathcal{C}}"] \\
\mathcal{B}[S_{\mathcal{B}}^{-1}] \arrow[r, "I_S"] & \mathcal{C}[S^{-1}]
\end{tikzcd}$$

To show that the map (3.2) is injective, let (s_f, f) and (s_g, g) be two roofs in $\mathcal{C}$ with domain and codomain in $\mathcal{B}$ representing the same morphism. Suppose the equivalence is given by the following diagram

$$\begin{tikzcd}
 & & W \arrow[dl, "t_1"'] \arrow[dr, "t_1"] & & \\
 & Z_1 \arrow[dl, "s_f"'] \arrow[drrr, "f"] & & Z_2 \arrow[dlll, "s_g"] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd}$$

By the condition (i), there exists a morphism $\phi : V \to W$ with $V \in \mathrm{Ob}\,\mathcal{B}$ such that $s_f t_1 \phi \in S$. Now $s_f t_1 \phi \in S_{\mathcal{B}}$ and $g t_2 \phi \in \mathrm{Mor}_B(V, Y)$, because $\mathcal{B}$ is a full subcategory of $\mathcal{C}$. These morphisms give the equivalence of the roofs in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$. Thus the map (3.2) is injective.

To show that the map (3.2) is surjective, let

$$\begin{tikzcd}
 & Z \arrow[dl, "s_f"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

be a roof which represents a morphism of $\mathcal{C}[S^{-1}]$ with domain and codomain in $\mathcal{B}$. Again by the condition (i) we obtain a morphism $\phi : V \to Z$ such that $V \in \mathcal{B}$, and $s_f \phi \in S_{\mathcal{B}}$. By the fact that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$ we get $f\phi \in \mathrm{Mor}_{\mathcal{B}}(V, Y)$. This completes the proof. If condition (ii) holds, the proposition can be proved similarly by using the formalism of coroofs and 3.2.9; see definition below. $\square$

Later, in chapter 6, we will need the formalism of coroofs. Let $\mathcal{C}$ be a category and S a localizing class for $\mathcal{C}$. A *coroof* is a diagram of the form

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "s"] \\
 & Z &
\end{tikzcd}$$

where $s \in S$, is denoted by $(s,f)^\vee$, and in this case we say that it is a coroof from X to Y. Two coroofs $(s,f)^\vee$ and $(t,g)^\vee$ are equivalent if there exists a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[drrr, "s"] & & & & Y \arrow[dlll, "g"'] \arrow[dl, "t"] \\
& Z_1 \arrow[dr, "s'"] & & Z_2 \arrow[dl, "t'"] & \\
& & W & &
\end{tikzcd}$$

where $t't \in S$. Let $\sim^\vee$ denote this relation. Composition of two coroofs $(s,f)^\vee$ and $(t,g)^\vee$ is a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[dr, "f"] & & Z \arrow[dl, "t"] \arrow[dr, "g"] & & Y \arrow[dl, "s"] \\
& Z_1 \arrow[dr, "t'"] & & Z_2 \arrow[dl, "s'"] & \\
& & W & &
\end{tikzcd}$$

where $s's$ is in S.

Lemma 3.2.8 (Equivalence of coroofs)**.** *The relation $\sim^\vee$ of coroofs is an equivalence relation and it respects composition of morphisms.*

Proof. Similar to proofs of lemmas 3.2.2 and 3.2.3. $\square$

Let us denote by $\tilde{\mathcal{C}}_S^\vee$ the category consisting of the objects of $\mathcal{C}$, morphisms are coroofs, and composition of morphisms is composition of coroofs. The following proposition shows that localization of a category along a localizing class is isomorphic to the category of coroofs.

Proposition 3.2.9. *The categories $\tilde{\mathcal{C}}_S$ and $\tilde{\mathcal{C}}_S^\vee$ are isomorphic.*

Proof. Let us define a functor $F : \tilde{\mathcal{C}}_S \to \tilde{\mathcal{C}}_S^\vee$ as follows: The map on objects is the identity. Let a morphism ϕ from X to Y be represented by a roof

$$\begin{tikzcd}
& Z_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

By LS 2, we obtain the following diagram with $s' \in S$

$$\begin{tikzcd}
X \arrow[dr, "f'"] & & Y \arrow[dl, "s'"] \\
& Z &
\end{tikzcd}$$

This is a coroof from X to Y, which represents some morphism $\psi : X \to Y$ in $\tilde{\mathcal{C}}_S^\vee$. Define $F(\phi) = \psi$.

First we verify that F is well-defined. Suppose

$$\begin{tikzcd}
& & Z_3 \arrow[dl, "s'"'] \arrow[dr, "t'"] & & \\
& Z_1 \arrow[dl, "s"'] \arrow[drrr, "f"] & & Z_2 \arrow[dlll, "t"] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd}$$

is an equivalence of roofs (s, f) and (t, g). By LS 2 we get the following commutative diagrams

$$\begin{tikzcd}
Z_3 \arrow[r, "gt'"] \arrow[d, "ss'"] & Y \arrow[d] \\
X \arrow[r] & V_1
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_1 \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & V_2
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r] \arrow[d] & V_1 \arrow[d] \\
V_2 \arrow[r] & V_3
\end{tikzcd}$$

with $Y \to V_1$, $Y \to V_2$, $V_2 \to V_3 \in S$. Now

$$\begin{aligned}
Z_3 \to Z_1 \to X \to V_1 \to V_3 &= Z_3 \to Z_2 \to Y \to V_1 \to V_3 \\
&= Z_3 \to Z_2 \to Y \to V_2 \to V_3 \\
&= Z_3 \to Z_1 \to Y \to V_2 \to V_3 \\
&= Z_3 \to Z_1 \to X \to V_2 \to V_3.
\end{aligned}$$

Since the composite $X \to V_2 \to V_3$ is in S by LS 1, by LS 3 there exists a morphism $V_3 \to V_4 \in S$ such that

$$X \to V_1 \to V_3 \to V_4 = X \to V_2 \to V_3 \to V_4.$$

Therefore the following diagram

$$\begin{tikzcd}
X \arrow[dr] \arrow[drrr] & & & & Y \arrow[dlll] \arrow[dl] \\
& V_1 \arrow[dr] & & V_2 \arrow[dl] & \\
& & V_4 & &
\end{tikzcd}$$

shows that the coroofs corresponding to the roofs (s, f) and (t, g), respectively, represent the same morphisms in $\tilde{\mathcal{C}}_S^\vee$. This shows that F is well-defined. Using similar arguments one defines a functor $G : \tilde{\mathcal{C}}_S^\vee \to \tilde{\mathcal{C}}_S$, which is identity on objects and maps coroofs to roofs by using LS 2. Let us show that G is the inverse of F.

On objects $G \circ F$ and $\mathrm{Id}_{\tilde{\mathcal{C}}_S}$ clearly coincide. To show that $G \circ F$ is identity on morphisms, let

$$\begin{tikzcd}
& Z_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& Z_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\
X & & Y
\end{tikzcd}$$

be a roof and its image in $G \circ F$. We show that these represent the same morphism in $\tilde{C}_S$. By definition of F and G, we have the following commutative diagram

$$\begin{tikzcd}
Z_1 \arrow[d] \arrow[drr] & & Z_2 \arrow[d] \arrow[dll] \\
X \arrow[dr] & & Y \arrow[dl] \\
& V_1 &
\end{tikzcd}$$

By LS 2 we get the following commutative square

$$\begin{tikzcd}
Z_3 \arrow[r] \arrow[d] & Z_1 \arrow[d] \\
Z_2 \arrow[r] & X
\end{tikzcd}$$

where $Z_3 \to Z_1 \in S$. Now

$$\begin{aligned} Z_3 \to Z_1 \to Y \to V_1 &= Z_3 \to Z_1 \to X \to V_1 \\ &= Z_3 \to Z_2 \to X \to V_1 \\ &= Z_3 \to Z_2 \to Y \to V_1, \end{aligned}$$

so by LS 3 we get a morphism $Z_4 \to Z_3$ in S, so that the following diagram is an equivalence of roofs

$$\begin{tikzcd}
& & Z_4 \arrow[dl] \arrow[dr] & & \\
& Z_1 \arrow[dl] \arrow[drrr] & & Z_2 \arrow[dlll] \arrow[dr] & \\
X & & & & Y
\end{tikzcd}$$

This shows that $G \circ F = \mathrm{Id}_{\tilde{C}_S}$. Similarly one shows that $F \circ G = \mathrm{Id}_{\tilde{C}_S^{\vee}}$. Therefore F is an isomorphism. $\square$

The following proposition shows that in some cases localization of an additive category is an additive category and the localizing functor is an additive functor.

Proposition 3.2.10. *Let $\mathcal{A}$ be an additive category, S is a localizing class, and $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ the localizing functor. Then $\mathcal{A}[S^{-1}]$ is an additive category, and Q is an additive functor.*

Proof. **Zero object:** Let us first verify that $\mathcal{A}[S^{-1}]$ is an additive category. Consider the zero object 0 in $\mathcal{A}$. If $(0, f_1)$, $(0, f_2)$ are two roofs representing some morphisms from 0 to X, then the following diagram shows that these represent the same morphism.

$$\begin{tikzcd}
& & 0 \arrow[dl, "0"'] \arrow[dr, "0"] & & \\
& Z_1 \arrow[dl, "0"'] \arrow[drrr, "f_1"] & & Z_2 \arrow[dlll, "0"] \arrow[dr, "f_2"] & \\
0 & & & & X
\end{tikzcd}$$

This shows that 0 is the initial object in $\mathcal{A}[S^{-1}]$. Let $(s_1, 0)$ and $(s_2, 0)$ be two roofs which represent some morphisms from X to 0. By LS 2 on obtains the following commutative diagram

$$\begin{tikzcd} W \arrow[r, "t_1"] \arrow[d, "t_2"] & Z_1 \arrow[d, "s_1"] \\ Z_2 \arrow[r, "s_2"] & X \end{tikzcd}$$

with $t_1 \in S$. By LS 1 we have $t_1 s_1 \in S$ and the following diagram

$$\begin{tikzcd} & & W \arrow[dl, "t_1"] \arrow[dr, "t_2"] & & \\ & Z_1 \arrow[dl, "s_1"] \arrow[drrr, "0"] & & Z_2 \arrow[dlll, "s_2"] \arrow[dr, "0"] & \\ X & & & & 0 \end{tikzcd}$$

shows that the roofs represent the same morphism. This shows that 0 is also the terminal object in $\mathcal{A}[S^{-1}]$, and hence the zero object in $\mathcal{A}[S^{-1}]$.

Abelian group $\mathrm{Mor}_{\mathcal{A}[S^{-1}]}(X, Y)$**:** Next, let us define the abelian group structure on the set of morphisms between objects. Let

$$\begin{tikzcd} & Z_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & Z_2 \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd}$$

represent two morphisms between objects of $\mathcal{A}[S^{-1}]$. By LS 2 we can complete $Z_1 \to X \leftarrow Z_2$ to the following commutative square

$$\begin{tikzcd} W \arrow[r, "t_1"] \arrow[d, "t_2"] & Z_1 \arrow[d, "s_1"] \\ Z_2 \arrow[r, "s_2"] & X \end{tikzcd}$$

with $t_1 \in S$. By LS 1, $s_1 t_1 \in S$. Clearly the following two roofs

$$\begin{tikzcd} & W \arrow[dl, "s_1 t_1"'] \arrow[dr, "f_1 t_1"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & W \arrow[dl, "s_1 t_1"'] \arrow[dr, "f_2 t_2"] & \\ X & & Y \end{tikzcd} \tag{3.3}$$

represent the same morphisms as (s_1, f) and (s_2, g), respectively. From now on we will similarly change any two roofs with the same domain to roofs with the same top without mentioning it. We define the sum of the morphisms represented by the roofs (s, f) and (t, g), respectively, to be the morphism represented by the following roof

$$\begin{tikzcd} & W \arrow[dl, "s_1 t_1"'] \arrow[dr, "f_1 t_1 + f_2 t_2"] & \\ X & & Y \end{tikzcd}$$

To see that this definition is well-defined, that is, it does not depend on the choices made in the definition of sum, denote the roofs (3.3) by (s, f_1) and (s, f_2), and let

$$
\begin{tikzcd}
 & V \arrow[dl, "s'"'] \arrow[dr, "f_1'"] & \\
X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V \arrow[dl, "s'"'] \arrow[dr, "f_2'"] & \\
X & & Y
\end{tikzcd}
$$

be roofs such that $(s, f_1) \sim (s', f_1')$ and $(s, f_2) \sim (s', f_2')$. We have the following commutative diagrams

$$
\begin{tikzcd}
 & U_1 \arrow[dl] \arrow[dr] & & \\
W \arrow[d, "s_1"'] \arrow[drr, "f_1"' near end] & & V \arrow[dll, "s_1'" near end] \arrow[d, "f_1'"] & \\
X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U_2 \arrow[dl] \arrow[dr] & & \\
W \arrow[d, "s_2"'] \arrow[drr, "f_2"' near end] & & V \arrow[dll, "s_2'" near end] \arrow[d, "f_2'"] & \\
X & & Y &
\end{tikzcd}
\tag{3.4}
$$

Complete the composites $U_1 \to W \to X$ and $U_2 \to W \to X$ to a square, by LS 2, with $U \to U_1$ in S. From the equation

$$U \to U_1 \to W \to X = U \to U_2 \to W \to X$$

and using the fact that $s : W \to X \in S$, by LS 3, we get a morphism $U' \to U$ such that

$$U' \to U \to U_1 \to W = U' \to U \to U_2 \to W.$$

Since addition is bilinear in $\mathcal{A}$ we see by using the commutativity of the diagrams (3.4) that the following diagram represents equality of the additions

$$
\begin{tikzcd}
 & & U' \arrow[d] & & \\
 & & U \arrow[dl] \arrow[dr] & & \\
 & U_1 \arrow[dl] \arrow[drrr] & & U_2 \arrow[dlll] \arrow[dr] & \\
W \arrow[d, "s"'] \arrow[drrrr, "f_1+f_2" near start] & & & & V \arrow[dllll, "s'" near start] \arrow[d, "f_1'+f_2'"] \\
X & & & & Y
\end{tikzcd}
$$

Therefore addition is well-defined. Using the additive group structure of $\mathcal{A}$ one can easily show that this definition gives an abelian group structure on $\mathrm{Mor}_{\mathcal{A}[S^{-1}]}(X, Y)$.

Bilinear composition: Let us show that $((s_1, f_1) + (s_1, f_2))(s_2, g) = (s_1, f_1)(s_2, g) + (s_1, f_2)(s_2, g)$. Let

$$
\begin{tikzcd}
 & U \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U \arrow[dl, "s_1"'] \arrow[dr, "f_2"] & \\
X & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V \arrow[dl, "s_2"'] \arrow[dr, "g_1"] & \\
Y & & Z
\end{tikzcd}
$$

be roofs, representing some morphisms. We have to show that

$$((s_1, f_1) + (s_1, f_2))(s_2, g) = (s_1, f_1)(s_2, g) + (s_1, f_2)(s_2, g). \tag{3.5}$$

The left-hand side of the equation (3.5) is represented by the following commutative diagram

$$\begin{tikzcd} & & V_1 \arrow[dl] \arrow[dr] & & \\ & U \arrow[dl, "s_1"'] \arrow[dr, "f_1+f_2"] & & V \arrow[dl, "s_2"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd} \tag{3.6}$$

The right-hand side of the equation (3.5) is the morphism which is the sum of the morphisms represented by the following two roofs

$$\begin{tikzcd} & & V_2 \arrow[dl] \arrow[dr] & & \\ & U \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & V \arrow[dl, "s_2"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd} \qquad \begin{tikzcd} & & V_3 \arrow[dl] \arrow[dr] & & \\ & U \arrow[dl, "s_1"'] \arrow[dr, "f_2"] & & V \arrow[dl, "s_2"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd} \tag{3.7}$$

By LS 2 we obtain the following commutative square

$$\begin{tikzcd} V_4 \arrow[r] \arrow[d] & V_2 \arrow[d] \\ V_3 \arrow[r] & X \end{tikzcd}$$

Since $U \to X$ is in S, by LS 3 we get a morphism $V_5 \to V_4$, contained in S, such that

$$V_5 \to V_4 \to V_2 \to U = V_5 \to V_4 \to V_3 \to U. \tag{3.8}$$

Again, by LS 2 we get the following commutative diagram

$$\begin{tikzcd} V_6 \arrow[r] \arrow[d] & V_5 \arrow[d] \\ V_1 \arrow[r] & X \end{tikzcd}$$

and by LS 3 we find a morphism $V_7 \to V_6$ such that

$$V_7 \to V_6 \to V_5 \to V_4 \to V_3 \to U = V_7 \to V_1 \to U. \tag{3.9}$$

Now we have the following equalities

$$
\begin{aligned}
&V_7 \to V_6 \to V_5 \to V_4 \to V_2 \to V \to Y + V_7 \to V_6 \to V_5 \to V_4 \to V_3 \to V \to Y \\
&\qquad = V_7 \to V_6 \to V_5 \to V_4 \to V_2 \to U \to Y + V_7 \to V_6 \to V_5 \to V_4 \to V_3 \to U \to Y && (3.7) \\
&\qquad = (V_7 \to V_6 \to V_5 \to V_4 \to V_2 \to U) \circ (U \xrightarrow{f_1} Y + U \xrightarrow{f_2} Y) && (3.8) \\
&\qquad = V_7 \to V_1 \to U \xrightarrow{f_1+f_2} Y && (3.9) \\
&\qquad = V_7 \to V_1 \to V \to Y && (3.6).
\end{aligned}
$$

Since $V \to Y$ is a morphism in S, by LS 3 we obtain a morphism $V_8 \to V_7$ such that

$$V_8 \to V_7 \to V_6 \to V_5 \to V_4 \to V_2 \to V + V_8 \to V_7 \to V_6 \to V_5 \to V_4 \to V_3 \to V = V_8 \to V_1 \to V.$$

Denote by k the composite $V_1 \to V \to Z$, by k' the following sum

$$V_7 \to V_6 \to V_5 \to V_4 \to V_2 \to V + V_7 \to V_6 \to V_5 \to V_4 \to V_3 \to V,$$

and by s_3 composite $V_7 \to V_6 \to V_5 \to V_4 \to V_2 \to X$. Then commutativity of the following diagram shows that the equation (3.5) holds

$$
\begin{tikzcd}
& & V_8 \arrow[dl] \arrow[dr] & & \\
& V_1 \arrow[dl, "s_1"'] \arrow[drrr, "k"] & & V_7 \arrow[dlll, "s_3"] \arrow[dr, "k'"] & \\
X & & & & Z
\end{tikzcd}
$$

Similary one shows that $\phi(\psi_1 + \psi_2) = \phi\psi_1 + \phi\psi_2$ for any morphisms ϕ, ϕ_1, and ϕ_2 in $\mathcal{A}[S^{-1}]$. Hence we have shown that addition in $\mathcal{A}[S^{-1}]$ is bilinear under composition of morphisms.

Q **additive:** Recall that the localizing functor Q is identity on objects and sends a morphism $f : X \to Y$ of $\mathcal{A}$ to (Id_X, f). By definition of addition in $\mathcal{A}[S^{-1}]$ it is clear that $Q(f+g) = (\mathrm{Id}_X, f+g) = (\mathrm{Id}_X, f) + (\mathrm{Id}_X, g) = Q(f) + Q(g)$ for any morphisms $f, g : X \to Y$ of $\mathcal{A}$. This shows that Q is an additive functor.

□

3.3 Notes

If the reader interested in formal mathematics, he may want to take look at [Sim06], where the Gabriel-Zisman localization is formalized for small categories by using Coq proof assistant.

Chapter 4

Triangulated categories

In this chapter we introduce triangulated categories and show that $K(\mathcal{A})$ has a structure of a triangulated category. We prove that localization of a triangulated category with respect to a localizing class, which respects triangulation, is a triangulated category. Both of these results are needed in the next chapter to prove that derived category is triangulated.

4.1 Triangulated categories

Definition 4.1.1 (Translation functor)**.** A *translation functor* is an additive automorphism of an additive category. Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a translation functor. For any object X of $\mathcal{C}$ we denote by $X[n]$, $n \in \mathbb{Z}$, $n > 0$, the object obtained iterating n times the functor $\mathcal{T}$. More precisely

$$X[n] = \underbrace{\mathcal{T} \circ \ldots \circ \mathcal{T}}_{n}(X).$$

If n is negative, by $X[n]$ we mean the object obtained by iterating $-n$ times the functor $\mathcal{T}^{-1}$. For $n = 0$ we get the identity functor on $\mathcal{C}$. We use the same notation also for morphisms.

Definition 4.1.2 (Triangle)**.** A *triangle* in an additive category $\mathcal{C}$ together with a translation functor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ consists of three objects X, Y, Z and three morphisms $u : X \to Y$, $v : Y \to Z$, and $w : Z \to X[1]$. A triangle is denoted by (X, Y, Z, u, v, w), or just (X, Y, Z) if the morphisms are understood from the context.

A morphism $(f, g, h) : (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ between triangles is a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

It is easy to see that a morphism (f, g, h) of triangles is an isomorphism if and only if f, g, and h are isomorphisms.

Definition 4.1.3 (Triangulated category)**.** An additive category $\mathcal{D}$ together with a translation functor $T : \mathcal{D} \to \mathcal{D}$ and a collection of triangles, called distinguished triangles (also known as exact triangles), is a *triangulated category* if it satisfies the following conditions:

TR 1 **(Trivial triangle)** For any $X \in \operatorname{Ob} \mathcal{D}$, the triangle $(X, X, 0, Id_X, 0, 0)$ is distinguished.

TR 2 **(Isomorphic triangles)** Any triangle which is isomorphic to a distinguished triangle is distinguished.

TR 3 **(Rotation)** A triangle (X, Y, Z, u, v, w) is distinguished if and only if $(Y, Z, X[1], v, w, -u[1])$ is a distinguished triangle.

TR 4 **(Cone)** For any morphism $u : X \to Y$ in $\mathcal{D}$, there exists a distinguished triangle (X, Y, Z, u, v, w) in $\mathcal{D}$, not necessarily unique, and we call Z the mapping cone of u.

TR 5 **(Extension)** Given morphisms $f : X \to X'$, $g : Y \to Y'$, and two distinguished triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') such that the left square in the diagram

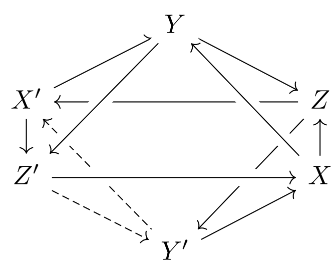
$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

commutes, then the diagram can be completed to a morphism of triangles. Note that the morphism h does not need to be unique.

TR 6 **(Octahedral)** Any upper cap diagram

$$\begin{tikzcd}
X' \arrow[dr, "{[1]}"'] \arrow[dd, "{[1]}"'] \arrow[rr, phantom, bend right=10, "d"'] & & Z \arrow[ll] \\
\circlearrowleft & Y \arrow[ur] \arrow[dl] & \circlearrowleft \\
Z' \arrow[rr, "{[1]}"'] \arrow[rr, phantom, bend left=10, "d"] & & X \arrow[ul] \arrow[uu]
\end{tikzcd}$$

can be completed to a octahedron diagram



such that the two composites from Y to Y' agree, and the two composites from Y' to Y agree, and the lower cap of the octahedron diagram is of the form

$$\begin{tikzcd}
X' \arrow[dd, "{[1]}"'] & \circlearrowleft & Z \arrow[ll] \arrow[dl] \arrow[dd, phantom, "d"] \\
d & Y' \arrow[ul] \arrow[dr, "{[1]}"] & d \\
Z' \arrow[ur] \arrow[rr, "{[1]}"'] & \circlearrowleft & X \arrow[uu]
\end{tikzcd}$$

Note that in the upper and lower cap we use d to mean distinguished triangle and $\circlearrowleft$ to mean a commutative triangle.

According to [GM03, IV.1.2.b], TR6 is equivalent, modulo other axioms, that lower cap can be completed to an upper cap. We will later, in the theory of t-structures, need completion of lower cap to an upper cap, so we have provided a proof of this in appendix (A.1).

We prove some useful elementary results of triangulated categories which show that triangulated categories and abelian categories have some similar properties, when one thinks of distinguished triangles as short exact sequences.

Lemma 4.1.4. *For any distinguished triangle (X, Y, Z, u, v, w) in a triangulated category $\mathcal{D}$, we have $vu = 0$.*

Proof. Consider the following diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}_X"] \arrow[d, equal] & X \arrow[r, "0"] \arrow[d, "u"] & 0 \arrow[r, "0"] \arrow[d, "0"] & X[1] \arrow[d, equal] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] \end{tikzcd}$$

where the first row is a distinguished triangle by TR1. By TR5 the above diagram is a morphism of distinguished triangles, so commutativity of middle square shows that $vu = 0$. ☐

Proposition 4.1.5. *Let $\mathcal{D}$ be a triangulated category and*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

a distinguished triangle in $\mathcal{D}$. For any object T of $\mathcal{D}$ we have the following long exact sequences

$$\dots \xrightarrow{w_*[i-1]} \mathrm{Mor}_{\mathcal{D}}(T, X[i]) \xrightarrow{u_*[i]} \mathrm{Mor}_{\mathcal{D}}(T, Y[i]) \xrightarrow{v_*[i]} \mathrm{Mor}_{\mathcal{D}}(T, Z[i]) \xrightarrow{w_*[i]} \mathrm{Mor}_{\mathcal{D}}(T, X[i+1]) \xrightarrow{u_*[i+1]} \dots$$

$$\dots \xrightarrow{w^*[i+1]} \mathrm{Mor}_{\mathcal{D}}(Z[i], T) \xrightarrow{v^*[i]} \mathrm{Mor}_{\mathcal{D}}(Y[i], T) \xrightarrow{u^*[i]} \mathrm{Mor}_{\mathcal{D}}(X[i], T) \xrightarrow{w^*[i]} \mathrm{Mor}_{\mathcal{D}}(Z[i-1], T) \xrightarrow{v^*[i-1]} \dots$$

Proof. By TR3, it suffices to show that the following sequences are exact

$$\mathrm{Mor}_{\mathcal{D}}(T, X) \xrightarrow{u_*} \mathrm{Mor}_{\mathcal{D}}(T, Y) \xrightarrow{v_*} \mathrm{Mor}_{\mathcal{D}}(T, Z) \qquad \mathrm{Mor}_{\mathcal{D}}(Z, T) \xrightarrow{v^*} \mathrm{Mor}_{\mathcal{D}}(Y, T) \xrightarrow{u^*} \mathrm{Mor}_{\mathcal{D}}(X, T)$$

By Lemma 4.1.4 $\mathrm{Im}\, u_* \subset \ker v_*$ and $\mathrm{Im}\, v^* \subset \ker u^*$. Let $\phi \in \mathrm{Mor}_{\mathcal{D}}(T, Y)$ and $\psi \in \mathrm{Mor}_{\mathcal{D}}(Y, T)$ such that $v\phi = 0$ and $\psi u = 0$. By TR5 we have the following morphisms of distinguished triangles

$$\begin{tikzcd} T \arrow[r] \arrow[d, "\phi"] & 0 \arrow[r] \arrow[d] & T[1] \arrow[r, "-\mathrm{Id}_{T[1]}"] \arrow[d, "g[1]"] & T[1] \arrow[d, "\phi[1]"] \\ Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] \arrow[r, "-u[1]"] & Y[1] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "u"] \arrow[d] & Y \arrow[r, "v"] \arrow[d, "\psi"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d] \\ 0 \arrow[r] & T \arrow[r, "\mathrm{Id}_T"] & T \arrow[r] & 0 \end{tikzcd}$$

Here g is obtained by first taking $g' : T[1] \to X[1]$ by TR5, and then using the fact that translation is an automorphism, so there exists a unique morphism $g : T \to X$ such that $g[1] = g'$. Now $ug = \phi$ and $hv = \psi$, shows that the sequences are exact. ☐

The following corollary can be thought to be the 5-lemma (see lemma 2.6.1) for triangulated categories.

Corollary 4.1.6. *Let $\mathcal{D}$ be a triangulated category and*

$$\begin{tikzcd} X_1 \arrow[r, "u"] \arrow[d, "f"] & Y_1 \arrow[r, "v"] \arrow[d, "g"] & Z_1 \arrow[r, "w"] \arrow[d, "h"] & X_1[1] \arrow[d, "f[1]"] \\ X_2 \arrow[r, "u'"] & Y_2 \arrow[r, "v'"] & Z_2 \arrow[r, "w'"] & X_2[1] \end{tikzcd}$$

a morphism of distinguished triangles. If f and g are isomorphisms, then so is h.

Proof. By proposition 4.1.5 we have the following commutative diagram with exact rows

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{D}}(T,X_1) \arrow[r, "u_*"] \arrow[d, "f_*"] & \mathrm{Mor}_{\mathcal{D}}(T,Y_1) \arrow[r, "v_*"] \arrow[d, "g_*"] & \mathrm{Mor}_{\mathcal{D}}(T,Z_1) \arrow[r, "w_*"] \arrow[d, "h_*"] & \mathrm{Mor}_{\mathcal{D}}(T,X_1[1]) \arrow[r, "u[1]_*"] \arrow[d, "f_*[1]"] & \mathrm{Mor}_{\mathcal{D}}(T,Y_1[1]) \arrow[d, "g_*[1]"] \\ \mathrm{Mor}_{\mathcal{D}}(T,X_2) \arrow[r, "u'_*"] & \mathrm{Mor}_{\mathcal{D}}(T,Y_2) \arrow[r, "v'_*"] & \mathrm{Mor}_{\mathcal{D}}(T,Z_2) \arrow[r, "w'_*"] & \mathrm{Mor}_{\mathcal{D}}(T,X_2[1]) \arrow[r, "u'[1]_*"] & \mathrm{Mor}_{\mathcal{D}}(T,Y_2[1]) \end{tikzcd}$$

Since f_* and g_* are isomorphisms, so are $f_*[1]$ and $g_*[1]$. By lemma 2.6.1 h_* is an isomorphism. Thus when $T = Z_2$ we have $h\phi = \mathrm{Id}_{Z_2}$ for some $\phi : Z_2 \to Z_1$. Similarly, by Lemma 4.1.5, we get the following commutative diagram with exact rows

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{D}}(Y_2[1],T) \arrow[r, "v^*[1]"] \arrow[d, "g^*[1]"] & \mathrm{Mor}_{\mathcal{D}}(X_2[1],T) \arrow[r, "w^*"] \arrow[d, "f^*[1]"] & \mathrm{Mor}_{\mathcal{D}}(Z_2,T) \arrow[r, "v^*"] \arrow[d, "h^*"] & \mathrm{Mor}_{\mathcal{D}}(Y_2,T) \arrow[r, "u^*"] \arrow[d, "g^*"] & \mathrm{Mor}_{\mathcal{D}}(X_2,T) \arrow[d, "f^*"] \\ \mathrm{Mor}_{\mathcal{D}}(Y_1[1],T) \arrow[r, "v'^*[1]"] & \mathrm{Mor}_{\mathcal{D}}(X_1[1],T) \arrow[r, "w'^*"] & \mathrm{Mor}_{\mathcal{D}}(Z_1,T) \arrow[r, "v'^*"] & \mathrm{Mor}_{\mathcal{D}}(Y_1,T) \arrow[r, "u'^*"] & \mathrm{Mor}_{\mathcal{D}}(X_1,T) \end{tikzcd}$$

Again, by Lemma 2.6.1 h^* is an isomorphism, and when $T = Z_1$ we get a morphism ψ such that $\psi h = \mathrm{Id}_{Z_1}$. Now $\phi = \psi h \phi = \psi$, so h is an isomorphism. ☐

We have seen that a triangulated category satisfies some properties similar to an abelian category. Therefore it is natural to ask when a triangulated category is an abelian category? The following proposition shows that the overlap is not large, it consists of semisimple abelian categories.

Proposition 4.1.7. *A triangulated category $\mathcal{D}$ which is an abelian category is semisimple.*

Proof. Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{4.1}$$

be a short exact sequence in $\mathcal{D}$. By TR3 and TR4, we obtain the following distinguished triangles

$$C_1[-1] \xrightarrow{u_1} X \xrightarrow{f} Y \xrightarrow{v_1} C_1 \qquad Y \xrightarrow{g} Z \xrightarrow{v_2} C_2 \xrightarrow{u_2} Y[1]$$

Since f is a monomorphism, and g is an epimorphism, $u_1 = v_2 = 0$ by Lemma 4.1.4. By application of TR1 and TR3, we obtain the following morphisms of distinguished triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "v_1"] \arrow[d, dashed, "\phi_1"] & C_1 \arrow[r, "0"] \arrow[d, "0"] & X[1] \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r, "0"] & 0 \arrow[r, "0"] & X[1] \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, equal] \arrow[d, dashed, "\phi_2"] & Z \arrow[r, "0"] \arrow[d, equal] & 0 \arrow[r, "0"] \arrow[d, "0"] & Z[1] \arrow[d, dashed, "\phi_2[1]"] \\ Y \arrow[r, "g"] & Z \arrow[r, "0"] & C_2 \arrow[r, "v_1"] & Y[1] \end{tikzcd}$$

Commutativity implies that $g\phi_2 = \mathrm{Id}_Z$ and $\phi_1 f = \mathrm{Id}_X$. Thus the short exact sequence (4.1) splits. ☐

4.2 $K(\mathcal{A})$ is triangulated

In this section $\mathcal{A}$ denotes an abelian category if not otherwise mentioned.

Definition 4.2.1. A distinguished triangle in the category $K(\mathcal{A})$ is a triangle isomorphic to a triangle of the form

$$X^\bullet \xrightarrow{i_1} Cyl(f) \xrightarrow{p_2} C(f) \xrightarrow{p_1} X^\bullet[1]$$

for some morphism $f : X^\bullet \to Y^\bullet$ of $K(\mathcal{A})$.

To prove that $K(\mathcal{A})$ is triangulated, we need a few lemmas about the structure of distinguished triangles in $K(\mathcal{A})$. The following lemma shows that for any morphism $f : X^\bullet \to Y^\bullet$ in $K(\mathcal{A})$, $(X^\bullet, Y^\bullet, C(f), f, i_2, p_1)$ is a distinguished triangle.

Lemma 4.2.2. *For any morphism $f : X^\bullet \to Y^\bullet$ in $K(\mathcal{A})$, we have an isomorphism*

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] \arrow[d, equal] & Y^\bullet \arrow[r, "i_2"] \arrow[d, "\alpha"] & C(f) \arrow[r, "p_1"] \arrow[d, equal] & X^\bullet[1] \arrow[d, equal] \\
X^\bullet \arrow[r, "i_1"] & Cyl(f) \arrow[r, "p_2"] & C(f) \arrow[r, "p_1"] & X^\bullet[1]
\end{tikzcd}$$

of triangles in $K(\mathcal{A})$, where $\alpha = i_2 i_2$. In particular, all the distinguished triangles in $K(\mathcal{A})$ are isomorphic to a distinguished triangle of the form of the upper row.

Proof. By Lemma 2.7.9 the morphism α is an isomorphism in $K(\mathcal{A})$. It remains to verify that the left square is commutative in $K(\mathcal{A})$. Let $\chi^i : X^i \to Cyl(f)^{i-1}$ be the morphism $i_2 i_1$. Now

$$\begin{aligned}
\chi^{i+1} d^i_{X^\bullet} + d^{i-1}_{Cyl(f)} \chi^i &= i_2 i_1 d^i_{X^\bullet} + (i_1 d^{i-1}_{X^\bullet} p_1 - i_1 p_1 p_2 + i_2 d^{i-1}_{C(f)} p_2)(i_2 i_1) \\
&= i_2 i_1 d^i_{X^\bullet} - i_1 - i_2 i_1 d^i_{X^\bullet} + i_2 i_2 f^i \\
&= \alpha f^i - i_1,
\end{aligned}$$

which shows that the left square commutes in $K(\mathcal{A})$. Hence the diagram is an isomorphism of triangles. By definition 4.2.1 all the distinguished triangles in $K(\mathcal{A})$ are isomorphic to a triangle of the form of the bottom row, we conclude that they are also isomorphic to some distinguished triangle of the form of the upper row. $\square$

Definition 4.2.3 (Semi-split triangles)**.** A short exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

in $C(\mathcal{A})$ is *semi-split* if for all i the short exact sequence

$$0 \longrightarrow X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \longrightarrow 0$$

splits.

We say that a distinguished triangle (X, Y, Z, u, v, w) in $K(\mathcal{A})$ *comes from a semi-split short exact sequence* if there exists a semi-split short exact sequence

$$0 \longrightarrow X'^\bullet \xrightarrow{f} Y'^\bullet \xrightarrow{g} Z'^\bullet \longrightarrow 0$$

in $C(\mathcal{A})$ and an isomorphism of triangles $(X', Y', Z', f, g, h) \cong (X, Y, Z, u, v, w)$ in $K(\mathcal{A})$ for some morphism $h : Z' \to X'[1]$.

Let

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0 \tag{4.2}$$

be a semi-split short exact sequence in $C(\mathcal{A})$. We show that this short exact sequence is isomorphic to a semi-split short exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{i_1} X^\bullet \oplus Z^\bullet \xrightarrow{p_2} Z^\bullet \longrightarrow 0 \tag{4.3}$$

where the differential of $X^\bullet \oplus Z^\bullet$ is of the form

$$d^i_{X^\bullet \oplus Z^\bullet} = i_1 d^i_{X^\bullet} p_1 + i_1 \psi p_2 + i_2 d^i_{Y^\bullet} p_2 \tag{4.4}$$

for some morphism of complexes $\psi : Z^\bullet \to X^\bullet[1]$.

Fix a splitting $(j_1^i : Y^i \to X^i, j_2^i : Z^i \to Y^i)$ for (4.2). For any i consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & X^i \arrow[r, "f^i"] \arrow[d, equal] & Y^i \arrow[r, "g^i"] \arrow[d, dashed, "\phi^i"] & Z^i \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X^i \arrow[r, "i_1"] & X^i \oplus Z^i \arrow[r, "p_2"] & Z^i \arrow[r] & 0
\end{tikzcd}$$

where $\phi^i = i_1 j_1^i + i_2 g^i$. Then $\phi^i f^i = i_1$ and $p_2 \phi^i = g^i$, so the diagram is commutative. By lemma 2.6.1 ϕ is an isomorphism. Define

$$d^i_{X^\bullet \oplus Z^\bullet} = \phi^{i+1} d^i_{Y^\bullet} (\phi^i)^{-1}.$$

Since $d^i_{Y^\bullet}$ is a differential, so is $d^i_{X^\bullet \oplus Z^\bullet}$. Clearly ϕ is a morphism of complexes. From

$$d^i_{X^\bullet \oplus Z^\bullet} i_1 = \phi^{i+1} d^i_{Y^\bullet} (\phi^i)^{-1} \phi^i f^i = \phi^{i+1} d^i_{Y^\bullet} f^i = \phi^{i+1} f^{i+1} d^i_{X^\bullet} = i_1 d^i_{X^\bullet} \tag{4.5}$$

and

$$p_2 d^i_{X^\bullet \oplus Z^\bullet} = g^{i+1} (\phi^{i+1})^{-1} \phi^{i+1} d^i_{Y^\bullet} (\phi^i)^{-1} = g^{i+1} d^i_{Y^\bullet} (\phi^i)^{-1} = d^{i+1}_{Z^\bullet} g^i (\phi^i)^{-1} = d^{i+1}_{Z^\bullet} p_2 \tag{4.6}$$

we see that $i_1 : X^\bullet \to X^\bullet \oplus Z^\bullet$ and $p_2 : X^\bullet \oplus Z^\bullet \to Z^\bullet$ are morphisms of complexes, so that (4.3) is a semi-split short exact sequence with splitting (p_1, i_2).

Now,

$$\begin{aligned}
d^i_{X^\bullet \oplus Z^\bullet} &= (i_1 p_1 + i_2 p_2) d^i_{X^\bullet \oplus Z^\bullet} (i_1 p_1 + i_2 p_2) \\
&= i_1 p_1 d^i_{X^\bullet \oplus Z^\bullet} i_1 p_1 + i_1 p_1 d^i_{X^\bullet \oplus Z^\bullet} i_2 p_2 + i_2 p_2 d^i_{X^\bullet \oplus Z^\bullet} i_1 p_1 + i_2 p_2 d^i_{X^\bullet \oplus Z^\bullet} i_2 p_2 \\
&= i_1 p_1 i_1 d^i_{X^\bullet} p_1 + i_1 (p_1 d^i_{X^\bullet \oplus Z^\bullet} i_2) p_2 + i_2 d^i_{Z^\bullet} p_2 i_1 p_1 + i_2 d^i_{Z^\bullet} p_2 i_1 p_2 \\
&= i_1 d^i_{X^\bullet} p_1 + i_1 (p_1 d^i_{X^\bullet \oplus Z^\bullet} i_2) p_2 + i_2 d^i_{Z^\bullet} p_2.
\end{aligned}$$

To see that $(p_1 d^i_{X^\bullet \oplus Z^\bullet} i_2) : Z^\bullet \to X^\bullet[1]$ is a morphism of complexes, we have to show that

$$p_1 d^{i+1}_{X^\bullet \oplus Z^\bullet} i_2 d^i_{Z^\bullet} = -d^{i+1}_{X^\bullet} p_1 d^i_{X^\bullet \oplus Z^\bullet} i_2.$$

Since i_1 is a monomorphism and p_2 is an epimorphism, this follows from the following

$$i_1 (p_1 d^{i+1}_{X^\bullet \oplus Z^\bullet} i_2 d^i_{Z^\bullet} + d^{i+1}_{X^\bullet} p_1 d^i_{X^\bullet \oplus Z^\bullet} i_2) p_2 = i_1 p_1 d^{i+1}_{X^\bullet \oplus Z^\bullet} d^i_{X^\bullet \oplus Z^\bullet} + d^{i+1}_{X^\bullet \oplus Z^\bullet} d^i_{X^\bullet \oplus Z^\bullet} i_2 p_2 = 0,$$

where we used the identities (4.5) and (4.6) and the fact that $d_{X^\bullet \oplus Z^\bullet}$ is a differential. From this we conclude that (4.2) is isomorphic to (4.3) with the differential of the form (4.4).

Conversely, it is easy to show that for any morphism of complexes $\psi : Z^\bullet \to X^\bullet[1]$ the short exact sequence (4.3) with differential of the form (4.4) is a semi-split short exact sequence of complexes.

By the definition of distinguished triangles in $K(\mathcal{A})$ every distinguished triangle is isomorphic to a distinguished triangle which comes from a semi-split short exact sequence in $C(\mathcal{A})$. Conversely, the following lemma shows that every semi-split exact sequence in $C(\mathcal{A})$ can be completed to a distinguished triangle in $K(\mathcal{A})$. It implies that there is a bijective correspondence between distinguished triangles in $K(\mathcal{A})$ and semi-split short exact sequences in $C(\mathcal{A})$.

Proposition 4.2.4. *For any semi-split short exact sequence*

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

in $C(\mathcal{A})$ there is a distinguished triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{-\psi} X^\bullet[1]$$

in $K(\mathcal{A})$, where $\psi : Z^\bullet \to X^\bullet[1]$ is the morphism which occurs at the differential of $Y^\bullet$ (see (4.4) and discussion above).

Proof. By the discussion above, we have the following isomorphism of semi-split exact sequences

$$\begin{tikzcd}
0 \arrow[r] & X^\bullet \arrow[r, "f"] \arrow[d, equal] & Y^\bullet \arrow[r, "g"] \arrow[d, "\phi"] & Z^\bullet \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X^\bullet \arrow[r, "i_1"] & X^\bullet \oplus Z^\bullet \arrow[r, "p_2"] & Z^\bullet \arrow[r] & 0
\end{tikzcd}$$

Therefore we have an isomorphism of triangles

$$\begin{tikzcd}
X^\bullet \arrow[r, "f"] \arrow[d, equal] & Y^\bullet \arrow[r, "g"] \arrow[d, "\phi"] & Z^\bullet \arrow[r, "-\psi"] \arrow[d, equal] & X^\bullet[1] \arrow[d, equal] \\
X^\bullet \arrow[r, "i_1"] & X^\bullet \oplus Z^\bullet \arrow[r, "p_2"] & Z^\bullet \arrow[r, "-\psi"] & X[1]
\end{tikzcd} \tag{4.7}$$

By TR2 it suffices to show that the lower triangle of (4.7) is distinguished.

Recall that

$$d^i_{X^\bullet \oplus Z^\bullet} = i_1 d^i_{X^\bullet} p_1 + i_1 \psi^i p_2 + i_2 d^i_{Z^\bullet} p_2.$$

By lemma 4.2.2 it suffices to show that

$$\begin{tikzcd}
X^\bullet \arrow[r, "i_1"] \arrow[d, equal] & X^\bullet \oplus Z^\bullet \arrow[r, "p_2"] \arrow[d, equal] & Z^\bullet \arrow[r, "-\psi"] \arrow[d, "\epsilon"] & X^\bullet[1] \arrow[d, equal] \\
X^\bullet \arrow[r, "i_1"] & X^\bullet \oplus Z^\bullet \arrow[r, "i_2"] & C(i_1) \arrow[r, "p_1"] & X^\bullet[1]
\end{tikzcd}$$

is an isomorphism of triangles, where $\epsilon^i = -i_1\psi^i + i_2 i_2$. The following shows that ϵ is a morphism of complexes

$$\begin{aligned}
\epsilon^{i+1} d^i_{Z^\bullet} &= -i_1 \psi^{i+1} d^i_{Z^\bullet} + i_2 i_2 d^i_{Z^\bullet} \\
&= i_1 d^{i+1}_{X^\bullet} \psi^i - i_2 i_1 \psi^i + i_2 i_1 d^i_{X^\bullet} p_1 i_2 + i_2 i_1 \psi^i + i_2 i_2 d^i_{Z^\bullet} \\
&= (-i_1 d^{i+1}_{X^\bullet} p_1 + i_2 i_1 p_1 + i_2 d^i_{X^\bullet \oplus Y^\bullet} p_2)(-i_1 \psi^i + i_2 i_2) \\
&= d^i_{C(i_1)} \epsilon^i.
\end{aligned}$$

It is easy to see that the left and right squares commute. To show that the middle square commutes, let $\chi^i : X^i \oplus Z^i \to C(i_1)^{i-1}$ be the morphism i_1p_1. Now

$$\begin{aligned} i_2 - \epsilon^i p_2 &= i_2 - (-i_1\psi^i p_2 + i_2 i_2 p_2) \\ &= i_1\psi^i p_2 + i_2(i_1p_1 + i_2p_2) - i_2i_2p_2 \\ &= i_2\psi^i p_2 + i_2 i_1 p_1 \end{aligned}$$

and

$$\begin{aligned} \chi^{i+1} d^i_{X^\bullet \oplus Z^\bullet} + d^{i-1}_{C(i_1)}\chi^i &= i_1p_1(i_1 d^i_{X^\bullet} p_1 + i_1\psi^i p_2 + i_2 d^i_{Z^\bullet} p_2) + (-i_1 d^i_{X^\bullet} p_1 + i_2 i_1 p_1 + i_2 d^{i-1}_{X^\bullet \oplus Z^\bullet} p_2) i_1 p_1 \\ &= i_1 d^i_{X^\bullet} p_1 + i_1 \psi^i p_2 - i_1 d^i_{X^\bullet} p_1 + i_2 i_1 p_1 \\ &= i_1\psi^i p_2 + i_2 i_1 p_1. \end{aligned}$$

Thus the middle square commutes in $K(\mathcal{A})$.

Finally, to show that ϵ is an isomorphism with inverse p_2p_2, let $\chi^i : C(i_1)^i \to C(i_1)^{i-1}$ be the morphism $i_1p_1p_2$. Then $p_2p_2\epsilon^i = \mathrm{Id}_{Z^i}$,

$$\begin{aligned} \mathrm{Id}_{C(i_1)} - \epsilon^i p_2 p_2 &= i_1p_1 + i_2(i_1p_1 + i_2p_2)p_2 - (-i_1\psi^i + i_2 i_2)p_2p_2 \\ &= i_1p_1 + i_2 i_1 p_1 p_2 + i_1\psi^i p_2 p_2, \end{aligned}$$

and

$$\begin{aligned} \chi^{i+1} d^i_{C(i_1)} + d^{i-1}_{C(i_1)}\chi^i &= \chi^{i+1}(-i_1 d^{i+1}_{X^\bullet} p_1 + i_2 i_1 p_1 + i_2 d^i_{X^\bullet \oplus Z^\bullet} p_2) + (-i_1 d^i_{X^\bullet} p_1 + i_2 i_1 p_1 + i_2 d^{i-1}_{X^\bullet \oplus Z^\bullet} p_2)\chi^i \\ &= i_1p_1 + i_1 d^i_{X^\bullet} p_1 p_2 + i_1\psi^i p_2 p_2 - i_1 d^i_{X^\bullet} p_1 p_2 + i_2 i_1 p_1 p_2 \\ &= i_1 p_1 + i_1\psi^i p_2 p_2 + i_2 i_1 p_1 p_2. \end{aligned}$$

This shows that ϵ is an isomorphism, p_2p_2 being the inverse of ϵ. □

We are ready to prove the main result of this section.

Theorem 4.2.5. *Let $\mathcal{A}$ be an abelian category. The category $\mathcal{K}(\mathcal{A})$ is a triangulated category.*

Proof. **TR1** Fix $X^\bullet \in \mathrm{Ob}\,\mathcal{K}(\mathcal{A})$ and consider the diagram

$$\begin{tikzcd} X^\bullet \arrow[r, "\mathrm{Id}_X"] \arrow[d, equal] & X^\bullet \arrow[r, "0"] \arrow[d, equal] & 0 \arrow[r, "0"] \arrow[d, "0"] & X^\bullet[1] \arrow[d, equal] \\ X^\bullet \arrow[r, "\mathrm{Id}_X"] & X^\bullet \arrow[r, "i_2"] & C(id_X) \arrow[r, "p_2"] & X^\bullet[1] \end{tikzcd}$$

We show that this diagram is an isomorphism of triangles, that is, the diagram commutes up to homotopy. Clearly, the left and right squares are commutative. To show that the middle square is commutative up to homotopy, let $\chi^i : X^i \to C(\mathrm{Id}_X)^{i-1}$ be the map i_1. We have

$$\begin{aligned} i_2 &= i_1 d^i_{X^\bullet} + (-i_1 d^i_{X^\bullet} p_1 + i_2 p_1 + i_2 d^{i-1}_{X^\bullet} p_2) i_1 \\ &= \chi^{i+1} d^i_{X^\bullet} + d^{i-1}_{C(\mathrm{Id}_X)}\chi^i. \end{aligned}$$

This shows that $(X^\bullet, X^\bullet, 0, \mathrm{Id}_X, 0, 0)$ is a distinguished triangle in $K(\mathcal{A})$.

TR2 Any triangle isomorphic to a distinguished triangle in $\mathcal{K}(\mathcal{A})$ is distinguished by definition 4.2.1.

TR3 $\Rightarrow$: Let $((X')^\bullet, (Y')^\bullet, (Z')^\bullet, u', v', w')$ be a distinguished triangle. By lemma 4.2.2 this triangle is isomorphic to some distinguished triangle $(X^\bullet, Y^\bullet, C(f), f, i_2, p_1)$. Since rotation to left preserves isomorphisms of triangles it suffices to show that $(Y^\bullet, C(f), X^\bullet[1], i_2, p_1, -f[1])$ is a distinguished triangle. This follows from TR2 if we show that the following diagram is an isomorphism of triangles

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{i_2} & C(f) & \xrightarrow{p_1} & X^\bullet[1] & \xrightarrow{-f[1]} & Y^\bullet[1] \\ \| & & \| & & \downarrow{\scriptstyle \theta} & & \| \\ Y^\bullet & \xrightarrow{i_2} & C(f) & \xrightarrow{i_2} & C(i_2) & \xrightarrow{p_1} & Y^\bullet[1] \end{array} \tag{4.8}$$

where $\theta^i = -i_1 f^{i+1} + i_2 i_1$, and the bottom row is a distinguished triangle by Lemma 4.2.2.

Let us verify that θ is a morphism of complexes. From

$$\theta^{i+1} d^i_{X^\bullet[1]} = i_1 f^{i+2} d^{i+1}_{X^\bullet} - i_2 i_1 d^{i+1}_{X^\bullet}$$

and

$$\begin{aligned} d^i_{C(i_2)} \theta^i &= (-i_1 d^{i+1}_{Y^\bullet} p_1 + i_2 i_2 p_1 + i_2 d^i_{C(f)} p_2)(-i_1 f^{i+1} + i_2 i_1) \\ &= i_1 d^{i+1}_{Y^\bullet} f^{i+1} - i_2 i_2 f^{i+1} - i_2 i_1 d^{i+1}_{X^\bullet} + i_2 i_2 f^{i+1} \\ &= i_1 f^{i+2} d^{i+1}_{X^\bullet} - i_2 i_1 d^{i+1}_{X^\bullet} \end{aligned}$$

one sees that θ is a morphism of complexes.

One can easily see that the left and right squares of the diagram (4.8) are commutative. To show that the middle square is commutative, let $\chi^i : C(f)^i \to C(i_2)^{i-1}$, $\chi^i = i_1 p_2$ for all i. Now

$$\begin{aligned} i_2 - \theta p_1 &= (i_2 i_1 p_1 + i_2 i_2 p_2) - (-i_1 f^{i+1} + i_2 i_1) p_1 \\ &= i_2 i_2 p_2 + i_1 f^{i+1} p_1 \end{aligned}$$

and

$$\begin{aligned} d^{i-1}_{C(i_2)} \chi^i + \chi^{i+1} d^i_{C(f)} = &\ (-i_1 d^i_{Y^\bullet} p_1 + i_2 i_2 p_1 + i_1 d^{i-1}_{C(f)} p_2)(i_1 p_2) \\ &+ (i_1 p_2)(-i_1 d^{i+1}_{X^\bullet} p_1 + i_2 f^{i+1} p_1 + i_2 d^i_{Y^\bullet} p_2) \\ = &\ -i_1 d^i_{Y^\bullet} p_2 + i_2 i_2 p_2 + i_1 f^{i+1} p_1 + i_1 d^i_{Y^\bullet} p_2 \\ = &\ i_2 i_2 p_2 + i_1 f^{i+1} p_1. \end{aligned}$$

Thus the middle square is commutative in $K(\mathcal{A})$.

Finally, we show that θ is an isomorphism with inverse $p_1 p_2$. Let $\chi^i : C(i_2)^i \to C(i_2)^{i-1}$ be the morphism $i_1 p_2 p_2$. Now $p_1 p_2 \theta^i = \mathrm{Id}_{X^\bullet[1]}$,

$$\begin{aligned} \mathrm{Id}_{C(i_2)^i} - \theta^i p_1 p_2 &= i_1 p_1 + i_2 (i_1 p_1 + i_2 p_2) p_2 - (-i_1 f^{i+1} + i_2 i_1) p_1 p_2 \\ &= i_1 f^{i+1} p_1 p_2 + i_1 p_1 + i_2 i_2 p_2 p_2, \end{aligned}$$

and

$$\begin{aligned}\chi^{i+1}d^i_{C(i_2)} + d^{i-1}_{C(i_2)}\chi^i = &\ (i_1p_2p_2)(-i_1d^{i+1}_{Y^\bullet}p_1 + i_2i_2p_1 + i_2d^i_{C(f)}p_2)\\ &+ (-i_1d^i_{Y^\bullet}p_1 + i_2i_2p_1 + i_2d^{i-1}_{C(f)}p_2)(i_1p_2p_2)\\ &= i_1p_1 + i_1f^{i+1}p_1p_1 + i_1d^i_{Y^\bullet}p_2p_2 + -i_1d^i_{Y^\bullet}p_2p_2 + i_2i_2p_2p_2\\ &= i_1f^{i+1}p_1p_1 + i_1p_1 + i_2i_2p_2p_2.\end{aligned}$$

This shows that θ is an isomorphism.

$\Leftarrow$: Conversely, suppose $((Y')^\bullet, (Z')^\bullet, (X')^\bullet[1], v', w', -u'[1])$ is a distinguished triangle. By Lemma 4.2.2 this triangle is isomorphic to a triangle of the form $(X^\bullet, Y^\bullet, C(f), f, i_2, p_1)$. Since rotation to right preserves isomorphisms of triangles, it suffices to show that $(C(f)[-1], X^\bullet, Y^\bullet, -p_1[-1], f, i_2)$ is a distinguished triangle. We do this by using TR2 and showing that the following diagram

$$\begin{tikzcd} C(f)[-1] \arrow[r, "-p_1[-1]"] \arrow[d, equal] & X^\bullet \arrow[r, "i_2"] \arrow[d, equal] & C(-p_1[-1]) \arrow[r, "p_1"] \arrow[d, "\theta"] & C(f) \arrow[d, equal] \\ C(f)[-1] \arrow[r, "-p_1[-1]"] & X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r, "i_2"] & C(f) \end{tikzcd} \tag{4.9}$$

is an isomorphism of triangles, where the top row is a distinguished triangle by Lemma 4.2.2 and the morphism $\theta : Z \to C(u)$ is defined to be $\theta^i = f^ip_2 + p_2p_1$.

The fact that θ is a morphism follows from

$$\begin{aligned}\theta^{i+1}d^i_{C(-p_1[-1])} &= (f^{i+1}p_2 + p_2p_1)(-i_1d^{i+1}_{C(f)[-1]}p_1 - i_2p_1p_1 + i_2d^i_{X^\bullet}p_2)\\ &= -f^{i+1}p_1p_1 + f^{i+1}d^i_Xp_2 + p_2(i_2f^{i+1}p_1 + i_2d^i_{Y^\bullet}p_2)p_1\\ &= -f^{i+1}p_1p_1 + f^{i+1}d^i_{X^\bullet}p_2 + f^{i+1}p_1p_1 + d^i_{Y^\bullet}p_2p_1\\ &= d^i_{Y^\bullet}(f^ip_2 + p_2p_1)\\ &= d^i_{Y^\bullet}\theta^i.\end{aligned}$$

Clearly the left and middle squares of (4.9) are commutative. To show that the right square is commutative, it suffices to show that i_2f is homotopic to 0, because $i_2(f^ip_2 + p_2p_1) - p_1 = i_2f^ip_2$. Let $\chi^i : X^i \to C(f)$ be $\chi^i = i_1$. Then

$$\chi^{i+1}d^i_{X^\bullet} + d^{i-1}_{C(f)}\chi^i = i_1d^i_{X^\bullet} - i_1d^i_{X^\bullet} + i_2f^i = i_2f^i$$

shows that the right square commutes.

It remains to show that θ is an isomorphism with inverse given by i_1i_2. Clearly $\theta i_1i_2 = \mathrm{Id}_Y$. We have to show that $i_1i_2\theta - \mathrm{Id}_{C(-p_1[-1])} = i_1i_2f^ip_2 - i_1i_1p_1p_1 - i_2p_2$ is homotopic to 0. We know already that i_2f^i is homotopic to 0 so it remains to show that $-i_1i_1p_1p_1 - i_2p_2$ is homotopic to 0. Let $\chi^i : C^i(-p_1[-1]) \to C^{i-1}(-p_1[-1])$ be $i_1i_1p_2$. Then

$$\begin{aligned}\chi^{i+1}d^i_{C(-p_1[-1])} + d^{i-1}_{C(-p_1[-1])}\chi^i &= -i_1i_1p_1p_1 + i_1i_1d^{i-1}_Xp_2 - i_1i_1d^{i-1}_Xp_2 - i_2p_2\\ &= -i_1i_1p_1p_1 - i_2p_2.\end{aligned}$$

TR4 Let $f : X^\bullet \to Y^\bullet$ be a morphism in $C(\mathcal{A})$ which represents a morphism $X^\bullet \to Y^\bullet$ in $K(\mathcal{A})$. By lemma 4.2.2

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{i_2} C(f) \xrightarrow{p_1} X^\bullet[1]$$

is a distinguished triangle in $K(\mathcal{A})$.

TR5 By lemma 4.2.2 it suffices to consider the following commutative diagram

$$\begin{tikzcd}
X_1^\bullet \arrow[r, "u_1"] \arrow[d, "f"] & Y_1^\bullet \arrow[r, "i_2"] \arrow[d, "g"] & C(u_1) \arrow[r, "p_1"] \arrow[d, dashed, "h"] & X_1^\bullet[1] \arrow[d, "f[1]"] \\
X_2^\bullet \arrow[r, "u_2"] & Y_2^\bullet \arrow[r, "i_2"] & C(u_2) \arrow[r, "p_1"] & X_2^\bullet[1]
\end{tikzcd}$$

By letting $h^i = i_1 f^{i+1} p_1 + i_2 g^i p_2$, one can easily verify that the diagram commutes. The following shows that h is a morphism of complexes

$$\begin{aligned}
d^i_{C(u_2)} h^i &= (-i_1 d^{i+1}_{X_2^\bullet} p_1 + i_2 u_2^{i+1} p_1 + i_2 d^i_{Y_2^\bullet} p_2)(i_1 f^{i+1} p_1 + i_2 g^i p_2) \\
&= -i_1 d^{i+1}_{X_2^\bullet} f^{i+1} p_1 + i_2 u_2^{i+1} f^{i+1} p_1 + i_2 d^i_{Y_2^\bullet} g^i p_2 \\
&= -i_1 f^{i+2} d^{i+1}_{X_1^\bullet} p_1 + i_2 g^{i+1} u_1^{i+1} p_1 + i_2 g^{i+1} d^i_{Y_1^\bullet} p_2 \\
&= (i_1 f^{i+2} p_1 + i_2 g^{i+1} p_2)(-i_1 d^{i+1}_{X_1^\bullet} p_1 + i_2 u_1^{i+1} p_1 + i_2 d^i_{Y_1^\bullet} p_2) \\
&= h^{i+1} d^i_{C(u_1)}
\end{aligned}$$

This proves TR5.

TR6 Fix an upper cap diagram

$$\begin{tikzcd}
X_2^\bullet \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] & & Z_1^\bullet \arrow[ll] \\
 & Y_1^\bullet \arrow[ur] \arrow[dl] \arrow[u, phantom, "d"] \arrow[d, phantom, "d"] \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
Z_2^\bullet \arrow[rr, "{[1]}"'] & & X_1^\bullet \arrow[ul] \arrow[uu]
\end{tikzcd} \tag{4.10}$$

Denote the morphism $Z_2^\bullet \to X_1^\bullet[1]$ by $-f$. By Proposition 4.2.4 all distinguished triangles in $K(\mathcal{A})$ come from semi-split exact sequences and by discussion before Proposition 4.2.4, the distinguished triangle (X_1, Y_1, Z_1) is isomorphic to a distinguished triangle

$$X_1^\bullet \xrightarrow{i_1} X_1^\bullet \oplus Z_2^\bullet \xrightarrow{p_2} Z_2^\bullet \xrightarrow{-f} X_1^\bullet[1]$$

where the differential of $X_1^\bullet \oplus Z_2^\bullet$ is given by

$$d^i_{X^\bullet \oplus Z^\bullet} = i_1 d^i_{X^\bullet} p_2 + i_1 f^i p_2 + i_2 d^i_{Z^\bullet} p_2.$$

Let us denote the morphism $X_2^\bullet \to (X_1^\bullet \oplus Z_2^\bullet)[1]$ by $-i_1 g - i_2 h$. Then the distinguished triangle (Y_1, Z_1, X_2) is isomorphic to

$$X_1^\bullet \oplus Z_2^\bullet \xrightarrow{i_1} (X_1^\bullet \oplus Z_2^\bullet) \oplus X_2^\bullet \xrightarrow{p_2} X_2^\bullet \xrightarrow{-i_1 g - i_2 h} (X_1^\bullet \oplus Z_2^\bullet)[1]$$

where the differential of $(X_1^\bullet \oplus Z_2^\bullet) \oplus X_2^\bullet$ is given by

$$d^i_{(X^\bullet \oplus Z^\bullet)\oplus X_2^\bullet} = i_1 d^i_{(X^\bullet \oplus Z^\bullet)} p_2 + i_1(i_1 g^i + i_2 h^i) p_2 + i_2 d^i_{X^\bullet} p_2.$$

By commutativity the morphism $X_2^\bullet \to Z_2^\bullet[1]$ is $-h$. Then the upper cap (4.10) is isomorphic to the following upper cap

$$\begin{tikzcd}
X_2^\bullet \arrow[dd, "{[1]}"'] \arrow[dr, "{[1]}"] & & (X_1^\bullet \oplus Z_2^\bullet) \oplus X_2^\bullet \arrow[ll] \\
 & X_1^\bullet \oplus Z_2^\bullet \arrow[ur] \arrow[dl] & \\
Z_2^\bullet \arrow[rr, "{[1]}"'] & & X_1^\bullet \arrow[uu] \arrow[ul]
\end{tikzcd} \tag{4.11}$$

It suffices to prove TR6 for this upper cap, because by using the isomorphism, one also obtains a lower cap for the original upper cap.

Let us construct a lower cap

$$\begin{tikzcd}
X_2^\bullet \arrow[dd, "{[1]}"'] & & (X_1^\bullet \oplus Z_2^\bullet) \oplus X_2^\bullet \arrow[ll] \arrow[dl] \\
 & Z_2^\bullet \oplus X_2^\bullet \arrow[ul] \arrow[dr, "{[1]}"] & \\
Z_2^\bullet \arrow[rr, "{[1]}"'] \arrow[ur] & & X_1^\bullet \arrow[uu]
\end{tikzcd} \tag{4.12}$$

where

$$d^i_{Z_2^\bullet \oplus X_2^\bullet} = i_1 d^i_{Z_2^\bullet} p_1 + i_1 h^i p_2 + i_2 d^i_{X_2^\bullet} p_2.$$

By Proposition 4.2.4, the triangle $(Z_2^\bullet, Z_2^\bullet \oplus X_2^\bullet, X_2^\bullet)$ is a distinguished triangle. Let $Z_2^\bullet \oplus X_2^\bullet \to X_1^\bullet[1]$ be the morphism $-fp_1 - gp_2$. Now, an easy verification shows that

$$((X_1^\bullet \oplus Z_2^\bullet) \oplus X_2^\bullet, i_1 i_1, p_1 p_1, i_1 i_2 p_1 + i_2 p_2, i_1 p_2 p_1 + i_2 p_2) \tag{4.13}$$

is a biproduct of $X_1^\bullet$ and $Z_2^\bullet \oplus X_2^\bullet$. By proposition 4.2.4, to show that

$$X_1^\bullet \xrightarrow{i_1 i_1} (X_1^\bullet \oplus Z_2^\bullet) \oplus X_2^\bullet \xrightarrow{i_1 p_2 p_1 + i_2 p_2} Z_2^\bullet \oplus X_2^\bullet \xrightarrow{-gp_2 - fp_1} X_1^\bullet[1] \tag{4.14}$$

is a distinguished triangle, we need to show that the differential of $(X_1^\bullet \oplus Z_2^\bullet) \oplus X_2^\bullet$ is of the form (4.4), with inclusions and projections of the biproduct (4.13), where $\psi = gp_2 + fp_1$. We have

$$\begin{aligned}
d^i_{(X_1^\bullet \oplus Z_2^\bullet)\oplus X_2^\bullet} &= i_1 d^i_{X_1^\bullet \oplus Z_2^\bullet} p_1 + i_1(i_1 g + i_2 h) p_2 + i_2 d^i_{X_2^\bullet} p_2 \\
&= i_1(i_1 d^i_{X^\bullet} p_1 + i_1 f p_2 + i_2 d^i_{Z_2^\bullet} p_2) p_1 + i_1(i_1 g + i_2 h) p_2 + i_2 d^i_{X_2^\bullet} p_2 \\
&= i_1 i_1 d^i_{X_1^\bullet} p_1 p_1 + i_1 i_1 (g p_2 + f p_1)(i_1 p_2 p_1 + i_2 p_2) + i_1 i_2 d^i_{Z_2^\bullet} p_2 p_1 + i_1 i_2 h^i p_2 + i_2 d^i_{X_2^\bullet} p_2 \\
&= \; i_1 i_1 d^i_{X_1^\bullet} p_1 p_1 + i_1 i_1 (g p_2 + f p_1)(i_1 p_2 p_1 + i_2 p_2) + \\
&\quad (i_1 i_2 p_1 + i_2 p_2)(i_1 d^i_{Z_2^\bullet} p_1 + i_1 h^i p_2 + i_2 d^i_{X_2^\bullet} p_2)(i_1 p_2 p_1 + i_2 p_2) \\
&= i_1 i_1 d^i_{X_1^\bullet} p_1 p_1 + i_1 i_1 (g p_2 + f p_1)(i_1 p_2 p_1 + i_2 p_2) + (i_1 i_2 p_1 + i_2 p_2) d^i_{Z_2^\bullet \oplus X_2^\bullet} (i_1 p_2 p_1 + i_2 p_2).
\end{aligned}$$

This shows that (4.14) is a distinguished triangle.

It remains to verify that the two morphisms from $X_1^\bullet \oplus Z_2^\bullet$ to $Z_2^\bullet \oplus X_2^\bullet$ and the two morphisms from $Z_2^\bullet \oplus X_2^\bullet$ to $X_1^\bullet \oplus Z_2^\bullet$ are equal in (4.12). Clearly the composite

$$X_1^\bullet \oplus Z_2^\bullet \xrightarrow{i_1} (X_1^\bullet \oplus Z_2^\bullet) \oplus X_2^\bullet \xrightarrow{i_1 p_2 p_1 + p_2} Z_2^\bullet \oplus X_2^\bullet$$

equals the composite

$$X_1^\bullet \oplus Z_2^\bullet \xrightarrow{p_2} Z_2^\bullet \xrightarrow{i_1} Z_2^\bullet \oplus X_2^\bullet,$$

so the two morphisms from $X_1^\bullet \oplus Z_2^\bullet$ to $Z_2^\bullet \oplus X_2^\bullet$ are equal.

To show that the two morphisms from $Z_2^\bullet \oplus X_2^\bullet$ to $(X_1^\bullet \oplus Z_2^\bullet)[1]$ are equal, let $\chi^i : Z_2^i \oplus X_2^i \to X_1^i \oplus Z_2^i$ be the morphism $i_2 p_1$. Now, the composite $Z_2^\bullet \oplus X_2^\bullet \to X_1^\bullet[1] \to (X_1^\bullet \oplus Z_2^\bullet)[1]$ is given by $i_1(-f^i p_1 - g^i p_2)$ and the composite $Z_2^\bullet \oplus X_2^\bullet \to X_2^\bullet \to (X_1^\bullet \oplus Z_2^\bullet)[1]$ is given by $(-i_1 g^i - i_2 h^i)p_2$. The difference of these morphisms is $-i_1 f^i p_1 + i_2 h^i p_2$. Since

$$\begin{aligned}
\chi^{i+1} d^i_{Z_2^\bullet \oplus X_2^\bullet} + d^{i-1}_{(X_1^\bullet \oplus Z_2^\bullet)[1]} \chi^i &= (i_2 p_1)(i_1 d^i_{Z_2^\bullet} p_1 + i_1 h^i p_2 + i_2 d^i_{X_2^\bullet} p_2) + (-i_1 d^i_{X_1^\bullet} - i_1 f^i p_2 - i_2 d^i_{Z_2^\bullet} p_2)(i_2 p_1) \\
&= i_1 d^i_{Z_2^\bullet} p_1 + i_2 h^i p_2 - i_1 f^i p_2 - i_2 d^i_{Z_2^\bullet} p_2 \\
&= -i_1 f^i p_1 + i_2 h^i p_2,
\end{aligned}$$

we see that these morphisms are homotopic, hence equal. This completes the proof.

□

Suppose $\mathcal{R}$ is a full additive subcategory of $\mathcal{A}$. Recall the definition of $K^*(\mathcal{R})$, $* = \varnothing, +, -, b$ in definition 2.7.11. From the definition we see that $K^*(\mathcal{R})$ is a full additive subcategory of $K^*(\mathcal{A})$. We define the distinguished triangles of $K^*(\mathcal{R})$, for $* = \varnothing, +, -, b$, to be the distinguished triangles of $K(\mathcal{A})$ where all the objects are contained in $K^*(\mathcal{R})$. Note that the categories $K^*(\mathcal{R})$ contain all the cones and cylinders of their morphisms. The above argument shows that these categories are also triangulated.

Corollary 4.2.6. *Let $\mathcal{A}$ be an abelian category and $\mathcal{R}$ a full subcategory of $\mathcal{A}$ such that $\mathcal{R}$ is an additive subcategory. Then the categories $K^*(\mathcal{R})$, $* = \varnothing, +, -, b$, are well-defined and triangulated. In particular, the categories $K(\mathcal{A})^*$, $* = +, -, b$ are triangulated.*

Proof. In the proof theorem 4.2.5, we used biproducts and mapping cones to produce new objects in the category $K(\mathcal{A})$. By assumption $K^*(\mathcal{R})$ are closed under taking these, so the proof theorem 4.2.5 works also for these categories. □

The following example shows that the category $K(\mathbf{Ab})$ is not abelian.

Example 4.2.7 ($K(\mathbf{Ab})$ not abelian)**.** Suppose $K(\mathbf{Ab})$ is abelian, then by theorem 4.2.5 and proposition 4.1.7 it must be semisimple. Consider the following exact sequence of complexes

$$0 \longrightarrow (\mathbb{Z}/2)[0] \xrightarrow{\ f\ } (\mathbb{Z}/4)[0] \xrightarrow{\ g\ } (\mathbb{Z}/2)[0] \longrightarrow 0 \tag{4.15}$$

Where the only nonzero objects are at index zero and the morphisms f and g are from example 2.2.17. These are complexes with all differentials zero morphisms, so any two morphisms between them are equal in $K(\mathbf{Ab})$ if and only if they are equal in $C(\mathbf{Ab})$. In particular, in example 2.2.17 we have seen that the morphisms in index 0 do not split. This implies that $K(\mathbf{Ab})$ is not an abelian category.

The following example shows that the cone is not functorial in $K(\mathbf{Ab})$.

Example 4.2.8 (Cone not functorial)**.** Consider the homotopy category $K(\mathbf{Ab})$ of abelian groups $\mathbf{Ab}$. Let us denote by $\mathbb{Z}$ the complex with zero differentials where the object at the index 0 is the group of integers and the others are zero objects. By TR1 and TR3 we have the following diagram where both rows are distinguished triangles.

$$\begin{tikzcd}
\mathbb{Z}[-1] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, dashed, "h"] & \mathbb{Z} \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

An easy observation shows that the diagram commutes even in $C(\mathbf{Ab})$ for any group homomorphism $h : \mathbb{Z} \to \mathbb{Z}$. Such a group homomorphism is completely determined by the image of 1 and it can be sent to any integer. Thus there are infinitely many h making the diagram commutative. Since the complexes in consideration are trivial outside index 0 these morphisms represent different morphisms in $K(\mathbf{Ab})$. This shows that the morphism h in the axiom TR5 indeed need not be unique.

4.3 Localization of a triangulated category

Definition 4.3.1. Let $\mathcal{D}$ be a triangulated category, $\mathcal{T}$ the associated translation functor, and S a localizing class for $\mathcal{D}$. We say that S is *compatible with triangulation* if the following conditions hold:

CT 1 For any morphism f in $\mathcal{D}$ we have $f \in S$ if and only if $\mathcal{T}(f) \in S$.

CT 2 In the condition TR5 for $\mathcal{D}$, if $f, g \in S$, then $h \in S$.

If S is a localizing class which is compatible with triangulation in a triangulated category $\mathcal{D}$, with translation functor $\mathcal{T}$, we define a translation functor $\mathcal{T}_S : \mathcal{D}[S^{-1}] \to \mathcal{D}[S^{-1}]$ on $\mathcal{D}[S^{-1}]$ as follows: for any object X of $\mathcal{D}[S^{-1}]$, let $\mathcal{T}_S(X) = \mathcal{T}(X)$. For any morphism $f : X \to Y \in \mathrm{Mor}_{\mathcal{D}[S^{-1}]}(X, Y)$, represented by a roof (s_1, f_1), let $\mathcal{T}_S(f)$ be the morphism represented by the roof $(\mathcal{T}(s_1), \mathcal{T}(f_1))$. By Proposition 3.2.10 $\mathcal{D}[S^{-1}]$ is an additive category. By using the formalism of roofs and the fact that $\mathcal{T}$ is additive one sees that the functor $\mathcal{T}_S$ is additive. The inverse of $\mathcal{T}_S$ is given by the functor $\mathcal{T}_S^{-1}$ which on objects is $\mathcal{T}^{-1}$ and sends a roof (s, t) to $(\mathcal{T}^{-1}(s), \mathcal{T}^{-1}(t))$. Since $\mathcal{T}^{-1}$ is additive, so is $\mathcal{T}_S^{-1}$. Therefore $\mathcal{T}_S$ is an additive automorphism, a translation functor, as in definition 4.1.1.

We say that a triangle in $\mathcal{D}[S^{-1}]$ is a distinguished triangle if it is isomorphic to the image of a distinguished triangle of $\mathcal{D}$ under the localization functor $Q_S : \mathcal{D} \to \mathcal{D}[S^{-1}]$.

Theorem 4.3.2. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{T}$ the associated translation functor, and S a localizing class for $\mathcal{D}$, compatible with triangulation. Then $\mathcal{D}[S^{-1}]$ is a triangulated category with the translation functor $\mathcal{T}_S$ and distinguished triangles described above.*

Proof. **TR1** Since $(X, X, 0, \mathrm{Id}_X, 0, 0)$ is a distinguished triangle in $\mathcal{D}$, it is distinguished in $\mathcal{D}[S^{-1}]$.

TR2 True by definition of distinguished triangles in $\mathcal{D}[S^{-1}]$.

TR3 Since $TR3$ is true in $\mathcal{D}$, and a distinguished triangle in $\mathcal{D}[S^{-1}]$ is isomorphic to image of a distinguished triangle of $\mathcal{D}$, the definition of $\mathcal{T}_S$ implies that TR3 holds in $\mathcal{D}[S^{-1}]$.

TR4 Let $u : X \to Y$ be a morphism in $\mathcal{D}[S^{-1}]$ and let $X \overset{s}{\leftarrow} Z \overset{u'}{\to} Y$ be a roof representing the morphism u. By TR4 for $\mathcal{D}$, we can complete u' to a distinguished triangle

$$Z \xrightarrow{u'} Y \xrightarrow{v} U \xrightarrow{w} Z[1]$$

in $\mathcal{D}$. Now

$$\begin{tikzcd}
Z \arrow[r, "u'"] \arrow[d, "s"] & Y \arrow[r, "v"] \arrow[d, "\mathrm{Id}_Y"] & U \arrow[r, "w"] \arrow[d, "\mathrm{Id}_U"] & Z[1] \arrow[d, "s[1]"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & U \arrow[r, "s[1]w"] & X[1]
\end{tikzcd}$$

is an isomorphism of triangles in $\mathcal{D}[S^{-1}]$, because s is an isomorphism in $\mathcal{D}[S^{-1}]$. Hence u can be completed to a distinguished triangle in $\mathcal{D}_S$.

TR5 Let $f : X \to X'$, $g : Y \to Y'$ be morphisms in $\mathcal{D}[S^{-1}]$ and let $\triangle_1 = (X, Y, Z, u, v, w)$ and $\triangle_2 = (X', Y', Z', u', v', w')$ be two distinguished triangles in $\mathcal{D}[S^{-1}]$ such that the following diagram is commutative in $\mathcal{D}[S^{-1}]$.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

To prove TR5 for $\mathcal{D}[S^{-1}]$, by the definition of distinguished triangles in $\mathcal{D}[S^{-1}]$, we may assume that $\triangle_1$ and $\triangle_2$ are images of distinguished triangles of $\mathcal{D}$. Let $(s_f, \tilde{f})$ and $(s_g, \tilde{g})$ be roofs representing f and g. It suffices to construct the morphisms s_h and $\tilde{h}$ in the following diagram

$$\begin{tikzcd}
& X'' \arrow[rr, dashed, "u''"] \arrow[dl, "s_f"'] \arrow[ddl, "\tilde{f}"] & & Y'' \arrow[rr, dashed, "v''"] \arrow[dl, "s_g"'] \arrow[ddl, "\tilde{g}"] & & Z'' \arrow[rr, dashed, "w''"] \arrow[dl, dashed, "s_h"'] \arrow[ddl, dashed, "\tilde{h}"] & & X''[1] \arrow[dl, "s[1]"'] \arrow[ddl, "\tilde{f}[1]"] \\
X \arrow[rr, "u"] \arrow[d, "f"'] & & Y \arrow[rr, "v"] \arrow[d, "g"'] & & Z \arrow[rr, "w"] \arrow[d, dashed, "h"'] & & X[1] \arrow[d, "f[1]"'] & \\
X' \arrow[rr, "u'"] & & Y' \arrow[rr, "v'"] & & Z' \arrow[rr, "w'"] & & X'[1] &
\end{tikzcd} \tag{4.16}$$

where arrows u'', v'' and w'' are arbitrary morphisms, such that all the front and back squares are commutative. We do this in steps

Step1 First we show that by changing the roof representing f, we obtain the morphism u'' so that the back and front squares of the left square of (4.16) are commutative in $\mathcal{D}$.

By LS 2 we get the following commutative diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r, dashed, "u''"] \arrow[d, dashed, "t"] & Y'' \arrow[d, "s_g"] \\
X'' \arrow[r, "us_f"] & Y
\end{tikzcd}$$

such that $t \in S$. The roof $(s_f t, \tilde{f} t)$ represents the same morphism as $(s_f, \tilde{f})$. The back square

$$\begin{tikzcd}
\tilde{X} \arrow[r, "u''"] \arrow[d, "s_f t"] & Y'' \arrow[d, "s_g"] \\
X \arrow[r, "u"] & Y
\end{tikzcd}$$

commutes in $\mathcal{D}$ and the front square

$$\begin{tikzcd}
\tilde{X} \arrow[r, "u''"] \arrow[d, "\tilde{f} t"] & Y'' \arrow[d, "\tilde{g}"] \\
X' \arrow[r, "u'"] & Y'
\end{tikzcd}$$

is commutative in $\mathcal{D}_S$, because

$$u' \tilde{f} t (s_f t)^{-1} = u' f = g u = \tilde{g} u'' (s_f t)^{-1}$$

and $(s_f t)^{-1}$ is an isomorphism. Hence, there exists an equivalence of roofs

$$\begin{tikzcd}
 & & W \arrow[dl, "q"'] \arrow[dr, "q"] & & \\
 & \tilde{X} \arrow[dl, "\mathrm{Id}_{\tilde{X}}"'] \arrow[drrr, "\tilde{g} u''"' near end] & & \tilde{X} \arrow[dlll, "\mathrm{Id}_{\tilde{X}}" near end] \arrow[dr, "u' \tilde{f} t"] & \\
\tilde{X} & & & & Y'
\end{tikzcd}$$

By commutativity

$$\tilde{g} u'' q = u' \tilde{f} t q.$$

Now both of the following squares, the front and back squares, respectively, are commutative in $\mathcal{D}$.

$$\begin{tikzcd}
W \arrow[r, "u'' q"] \arrow[d, "s_f t q"] & Y'' \arrow[d, "s_g"] \\
X \arrow[r, "u"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "u'' q"] \arrow[d, "\tilde{f} t q"] & Y'' \arrow[d, "\tilde{g}"] \\
X' \arrow[r, "u'"] & Y'
\end{tikzcd}$$

Finally, to keep the notation of the diagram (4.16), we let X'' to denote W, s_f to denote $s_f t q$, $\tilde{f}$ to denote the morphism $\tilde{f} t q$, and u'' to denote $u'' q$.

Step2 By TR4 for $\mathcal{D}$ we can complete $u'' : X'' \to Y''$ to a distinguished triangle and by TR5 we get morphisms s_h and $\tilde{h}$ such that the following diagrams are morphisms of distinguished triangles in $\mathcal{D}$

$$\begin{tikzcd}
X'' \arrow[r, "u''"] \arrow[d, "s_f"] & Y'' \arrow[r, "v''"] \arrow[d, "s_g"] & Z'' \arrow[r, "w''"] \arrow[d, "s_h"] & X''[1] \arrow[d, "s_f[1]"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1]
\end{tikzcd}$$

$$
\begin{tikzcd}
X'' \arrow[r, "u''"] \arrow[d, "\tilde{f}"] & Y'' \arrow[r, "v''"] \arrow[d, "\tilde{g}"] & Z'' \arrow[r, "w''"] \arrow[d, "\tilde{h}"] & X''[1] \arrow[d, "{s_g[1]}"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}
$$

Since $s_f, s_g \in S$, by CT2 we have $s_h \in S$. Let h be the morphism in $\mathcal{D}[S^{-1}]$ represented by the roof $(s_h, \tilde{h})$. Then one can easily check that (f, g, h) is a morphism of distinguished triangles in $\mathcal{D}[S^{-1}]$. This completes the proof of TR5.

TR6 Let

$$
\begin{tikzcd}
X' \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] \arrow[rr, phantom, shift right=4ex, "d"] & & Z \arrow[ll] \\
\arrow[r, phantom, "\circlearrowleft"] & Y \arrow[ur, "g"] \arrow[dl] \arrow[r, phantom, "\circlearrowleft"] & {} \\
Z' \arrow[rr, "{[1]}"'] \arrow[rr, phantom, shift left=4ex, "d"] & & X \arrow[ul, "f"'] \arrow[uu]
\end{tikzcd}
\tag{4.17}
$$

be an upper cap in $\mathcal{D}[S^{-1}]$. Let $(s_f, \tilde{f})$ and $(s_g, \tilde{g})$ be roofs representing f and g, respectively, and let the composite gf be represented by the following commutative diagram

$$
\begin{tikzcd}
 & & W \arrow[dl, "v_1"'] \arrow[dr, "v_2"] & & \\
 & V_1 \arrow[dl, "s_f"'] \arrow[dr, "\tilde{f}"] & & V_2 \arrow[dl, "s_g"'] \arrow[dr, "\tilde{g}"] & \\
X & & Y & & Z
\end{tikzcd}
$$

We have

$$
f = \begin{tikzcd}[column sep=small]
 & W \arrow[dl, "s_f v_1"'] \arrow[dr, "s_g v_2"] & \\
X & & Y
\end{tikzcd}
\qquad
g = \begin{tikzcd}[column sep=small]
 & V_2 \arrow[dl, "s_g"'] \arrow[dr, "\tilde{g}"] & \\
Y & & Z
\end{tikzcd}
\qquad
gf = \begin{tikzcd}[column sep=small]
 & W \arrow[dl, "s_f v_1"'] \arrow[dr, "\tilde{g} v_2"] & \\
X & & Z
\end{tikzcd}
$$

We want to show that in $\mathcal{D}[S^{-1}]$ the upper cap (4.17) is isomorphic to the following upper cap

$$
\begin{tikzcd}
\tilde{X} \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] \arrow[rr, phantom, shift right=4ex, "d"] & & Z \arrow[ll] \\
\arrow[r, phantom, "\circlearrowleft"] & V_2 \arrow[ur, "\tilde{g}"] \arrow[dl] \arrow[r, phantom, "\circlearrowleft"] & {} \\
\tilde{Z} \arrow[rr, "{[1]}"'] \arrow[rr, phantom, shift left=4ex, "d"] & & W \arrow[ul, "v_2"'] \arrow[uu]
\end{tikzcd}
\tag{4.18}
$$

Here, the distinguished triangles $(W, V_2, \tilde{Z})$ and $(V_2, Z, \tilde{W})$ in $\mathcal{D}$ are obtained by applying TR4 to the morphisms $v_2 : W \to Y$ and $\tilde{g} : V_2 \to Z$. The morphisms $W \to Z$ and $\tilde{X} \to \tilde{Z}$ are the obvious composites.

By item TR5 we get the following morphisms of distinguished triangles in $\mathcal{D}[S^{-1}]$.

$$
\begin{tikzcd}
W \arrow[r, "v_2"] \arrow[d, "s_f v_1"] & V_2 \arrow[r] \arrow[d, "s_g"] & \tilde{Z} \arrow[r] \arrow[d, dashed, "r_1"] & W[1] \arrow[d, "(s_f v_1)[1]"] \\
X \arrow[r, "f"] & Y \arrow[r] & Z' \arrow[r] & X[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
V_2 \arrow[r, "\tilde{g}"] \arrow[d, "s_g"] & Z \arrow[r] \arrow[d, equal] & \tilde{X} \arrow[r] \arrow[d, dashed, "r_2"] & V_2[1] \arrow[d, "s_g[1]"] \\
Y \arrow[r, "g"] & Z \arrow[r] & X' \arrow[r] & Y[1]
\end{tikzcd}
$$

By CT2, $r_1, r_2 \in S$. This shows that the upper caps (4.17) and (4.18) are isomorphic in $\mathcal{D}[S^{-1}]$. By TR6 for $\mathcal{D}$ we can complete (4.18) to the following lower cap

$$
\begin{tikzcd}
\tilde{X} \arrow[dd, "{[1]}"'] & {} & Z \arrow[ll] \arrow[dl, "b"'] \arrow[l, phantom, "\circlearrowleft"] \\
{} \arrow[r, phantom, "d"] & Y' \arrow[ul, "a"] \arrow[dr, "{e[1]}"] & {} \arrow[l, phantom, "d"] \\
\tilde{Z} \arrow[ur, "c"] \arrow[rr, "{[1]}"'] & {} & W \arrow[uu] \arrow[l, phantom, "\circlearrowleft"]
\end{tikzcd}
$$

Now

$$
\begin{tikzcd}
X' \arrow[dd, "{[1]}"'] & {} & Z \arrow[ll] \arrow[dl, "b'"'] \arrow[l, phantom, "\circlearrowleft"] \\
{} \arrow[r, phantom, "d"] & Y' \arrow[ul, "a'"] \arrow[dr, "{e'[1]}"] & {} \arrow[l, phantom, "d"] \\
Z' \arrow[ur, "c'"] \arrow[rr, "{[1]}"'] & {} & X \arrow[uu] \arrow[l, phantom, "\circlearrowleft"]
\end{tikzcd}
$$

is a lower cap for (4.17) in $\mathcal{D}[S^{-1}]$, where $a' = r_2 a$, $b' = b$, $c' = c r_1^{-1}$, and $e'[1] = (s_f v_1)[1](e)[1]$. Indeed, the fact that the triangles (Z', Y', X') and (X, Z, W) are distinguished follows by using TR2 and the following isomorphisms of distinguished triangles

$$
\begin{tikzcd}
\tilde{Z} \arrow[r, "c"] \arrow[d, "r_1"] & Y' \arrow[r, "a"] \arrow[d, equal] & \tilde{X} \arrow[r] \arrow[d, "r_2"] & \tilde{Z}[1] \arrow[d, "r_1"] \\
Z' \arrow[r, "c'"] & Y' \arrow[r, "a'"] & X' \arrow[r] & Z'[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "\tilde{g} v_2"] \arrow[d, "s_f v_1"] & Z \arrow[r, "b"] \arrow[d, equal] & Y' \arrow[r, "{e[1]}"] \arrow[d, equal] & W[1] \arrow[d, "(s_f v_1)[1]"] \\
X \arrow[r, "gf"] & Z \arrow[r, "b'"] & Y' \arrow[r, "{e'[1]}"] & X[1]
\end{tikzcd}
$$

One easily checks that the composites $a'b'$ and $e'[1]c'$ equal the morphisms $Z \to X'$ and $Z' \to X$ of the upper cap (4.17), respectively. Equality of the two morphisms from Y' to Y and two morphisms from Y to Y' follows from direct computation. This finishes the proof.

□

4.4 Notes

The reason why triangulated categories are important is the fact that they appear in many branches of mathematics. The following list of triangulated categories is taken mainly from [Orl], with references added.

Representation theory The category $D(\mathbf{RMod})$, the derived category of R-modules, is used in representation theory, see [Zim14]. As we will see in the next chapter this is a triangulated category.

Topology Stable model category is a triangulated category, see [Hov07, Chapter 7]. Model categories are an abstract machinery to do homotopy theory on categories. We will not discuss these categories in this thesis, since these triangulated categories do not come from abelian categories.

Motives The category of mixed motives is triangulated, see [MVW11].

Algebraic geometry The category of coherent sheaves, see [Huy06], and the category of $\mathbb{Q}_\ell$-adic sheaves, see [Fu11], are abelian categories and these categories give rise to derived categories.

Symplectic geometry The derived Fukaya category, see [ABA$^+$09] and [Sei08]. These categories are actually A_∞-categories, but one can extract a triangulated category from such a category. See also [Por] for connection with the derived category of coherent sheaves.

Mathematical physics Dirichlet branes in string theory, see [ABA$^+$09].

Algebraic analysis In algebraic analysis the whole theory uses triangulated categories, see for example [KS90]. For a short introduction, see [HTR10, p.371].

Now, as said in [Orl], the structure of a triangulated category allows us to compare categories of different mathematical objects as triangulated categories. The above triangulated categories are examples of algebraic and topological triangulated categories, see [HTR10, p.389], for definitions and introduction. Not all triangulated categories are abelian or topological, for an example see [MSS07]. In this thesis we only discuss the algebraic triangulated categories.

A standard reference for abstract theory of triangulated categories is [Nee01]. It has to be mentioned that triangulated categories suffer from some problems like nonfunctoriality of cone. For other known problems, see [Ber11, 2.2 (a)-(d)] and [Kaw13, p.279]. Differential graded categories (usually written DG-categories) and A_∞-categories are well-known structures to solve some of the problems of triangulated categories. For DG-categories see [Ber11] and for A_∞-categories see [BLM08].

Chapter 5

Derived categories

In this chapter we define the derived category of an abelian category, and show that it admits a structure of a triangulated category. One of the reasons why we are interested in derived categories is that not all important functors between abelian categories are exact. For example $\mathrm{Mor}_{\mathbf{RMod}}(M,-)$ and $M \otimes_R -$ need not be exact. The formalism of derived categories and derived functors allows one to construct derived counterparts of these functors which are exact functors between triangulated categories.

5.1 Derived category

First we give the definition of derived category which allows us to prove some basic results about derived categories. Then we start proving that a derived category is triangulated, see theorem 5.1.8.

Definition 5.1.1 (Derived category)**.** Let $\mathcal{A}$ be an abelian category and let S be the class consisting of quasi-isomorphisms in $C(\mathcal{A})$. The category $C(\mathcal{A})[S^{-1}]$ is the *derived category* of $\mathcal{A}$ and is denoted by $D(\mathcal{A})$. The categories $C^*(\mathcal{A})[S^{-1}]$, $* = +,-,b$ is the bounded below (resp. bounded above, resp. bounded) derived category of $\mathcal{A}$, where S is the class consisting of quasi-isomorphisms in $C^*(\mathcal{A})$, $* = +,-,b$, and is denoted by $D^*(\mathcal{A})$, $* = +,-,b$.

Recall construction 3.1.1 for the objects and morphism of the derived category and the universal property, theorem 3.1.3, of the derived category. Let $H^\bullet : C^*(\mathcal{A}) \to C^*(\mathcal{A})$ be the cohomology functor presented in proposition 2.5.4. By definition of quasi-isomorphisms, $H^\bullet$ sends quasi-isomorphisms to isomorphisms. Thus, by theorem 3.1.3, the cohomology functor factors though $D^*(\mathcal{A})$. This shows that we have a cohomology functor on derived category. By abuse of notation, we denote this functor also by $H^\bullet$.

We define distinguished triangles on $D^*(\mathcal{A})$ $* = \varnothing,+,-,b$ to be triangles isomorphic to triangles of the form

$$X^\bullet \xrightarrow{i_1} Cyl(f) \xrightarrow{p_2} C(f) \xrightarrow{p_1} X^\bullet[1]$$

where $f : X \to Y$ is some morphism in $C^*(\mathcal{A})$, $* = \varnothing,+,-,b$. We need a few results before we can prove that the derived category of $\mathcal{A}$ is isomorphic to the localization of the homotopy category of $\mathcal{A}$ along quasi-isomorphisms. We start with a lemma, which shows that all short exact sequences in $C(\mathcal{A})$ can be completed to a distinguished triangle in $D(\mathcal{A})$.

Lemma 5.1.2. *An exact sequence*

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0 \tag{5.1}$$

in $C(\mathcal{A})$ is quasi-isomorphic to the following exact sequence of Lemma 2.7.8

$$0 \longrightarrow X^\bullet \xrightarrow{i_1} Cyl(f) \xrightarrow{p_2} C(f) \longrightarrow 0 \tag{5.2}$$

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & X^\bullet \arrow[r, "i_1"] \arrow[d, equal] & Cyl(f) \arrow[r, "p_2"] \arrow[d, "\beta"] & C(f) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r, "g"] & Z^\bullet \arrow[r] & 0
\end{tikzcd}$$

where $\beta^i = f^i p_1 + p_2 p_2$ and $\gamma^i = g^i p_2$. From Lemma 2.7.9 we get that β is a quasi-isomorphism and that the left hand square commutes. The following shows that γ is a morphism of complexes

$$\gamma^{i+1} d^i_{C(f)} = g^{i+1} f^{i+1} p_1 + g^{i+1} d^i_{Y^\bullet} p_2 = d^i_{Z^\bullet} g^i p_2 = d^i_{Z^\bullet} \gamma^i,$$

and the following that the right hand square is commutative

$$\gamma^i p_2 = g^i p_2 p_2 = g^i (f^i p_1 + p_2 p_2) = g^i \beta^i.$$

It remains to show that γ is a quasi-isomorphism. Since p_2 and $g\beta$ are epimorphisms of complexes, so is γ. Hence, we can consider the following short exact sequence in $C(\mathcal{A})$

$$0 \longrightarrow \ker\gamma \xrightarrow{u} C(f) \xrightarrow{\gamma} Z^\bullet \longrightarrow 0$$

We show that $\ker\gamma$ is quasi-isomorphic to 0, so that $H^i(\ker\gamma) = 0$. Using this and the long exact sequence of theorem 2.6.6 associated to the above short exact sequence we get the following long exact sequence

$$\dots \xrightarrow{H^i(u)} H^i(C(f)) \xrightarrow{H^i(\gamma)} H^i(Z^\bullet) \xrightarrow{\delta^i} H^{i+1}(\ker\gamma) \xrightarrow{H^i(u)} H^{i+1}(C(f)) \xrightarrow{H^{i+1}(\gamma)} \dots$$

Thus if $H^i(\ker\gamma) = 0$ for all i, then γ is a quasi-isomorphism.

Let us construct the kernel of γ. By exactness of the short exact sequence (5.1) and the universal property of kernel we have

$$\begin{aligned}
(\ker\gamma)^i &= \ker(X^{i+1} \oplus Y^i \xrightarrow{g^i p_2} Z^i) \\
&\cong \ker(X^{i+1} \xrightarrow{0} Z^i) \oplus \ker(Y^i \xrightarrow{g^i} Z^i) \\
&\cong X^{i+1} \oplus X^i.
\end{aligned}$$

Let $K^\bullet$ be the complex where $K^i = X^{i+1} \oplus X^i$ and $d^i_{K^\bullet} = -i_1 d^{i+1}_{X^\bullet} p_1 + i_2 p_1 + i_2 d^i_{X^\bullet} p_2$. It is easy to see that this is indeed a complex. Let $u : K^\bullet \to C(f)$ be the morphism $u^i = i_1 p_1 + i_2 f^i p_2$. The following computation shows that

u is indeed a morphisms of complexes

$$\begin{aligned}
u^{i+1}d^i_{K^\bullet} &= (i_1p_1 + i_2f^{i+1}p_2)(-i_1d^{i+1}_{X^\bullet}p_1 + i_2p_1 + i_2d^i_{X^\bullet}p_2) \\
&= -i_1d^i_{X^\bullet}p_1 + i_2f^{i+1}p_1 + i_2f^{i+1}d^i_{X^\bullet}p_2 \\
&= -i_1d^i_{X^\bullet}p_1 + i_2f^{i+1}p_1 + i_2d^i_{Y^\bullet}f^ip_2 \\
&= (-i_1d^i_{X^\bullet}p_1 + i_2f^{i+1}p_1 + i_2d^i_{Y^\bullet}p_2)(i_1p_1 + i_2f^ip_2) \\
&= d^i_{C(f)}u^i.
\end{aligned}$$

To show that the pair $(K^\bullet, u)$ is the kernel of γ, we need to show that it satisfies the universal property of kernel. A direct computation shows that $\gamma u = 0$. Let $\phi : W^\bullet \to C(f)$ be any morphism such that $\gamma\phi = 0$. Since f is the kernel of g, there exists a unique morphism $\psi : W^\bullet \to X^\bullet$ such that $f\psi = p_2\phi$. In particular, by construction of kernels in the category of complexes, see the proof of Theorem 2.5.7, for all $i \in \mathbb{Z}$, $\phi^i : W^i \to X^i$ is the unique morphism given by the kernel property of f^i. Now, for all $i \in \mathbb{Z}$, we have the following commutative diagram

$$\begin{tikzcd}
 & X^{i+1} \arrow[r, "\mathrm{Id}_{X^{i+1}}"] \arrow[d, "i_1"] & X^{i+1} \arrow[d, "i_1"] \arrow[dr, "0"] & \\
W^i \arrow[ur, "p_1\phi^i"] \arrow[r, "i_1p_1\phi^i+i_2\psi^i"] \arrow[dr, "\psi^i"'] & K^i \arrow[r, "u^i"] & C(f)^i \arrow[r, "g^ip_2"] & Z^i \\
 & X^i \arrow[u, "i_2"] \arrow[r, "f^i"] & Y^i \arrow[u, "i_2"'] \arrow[ur, "g^i"'] &
\end{tikzcd}$$

A direct computation and commutativity of the above diagram shows that $u^i(i_1p_1\phi^i + i_2\psi^i) = \phi^i$. Now

$$\begin{aligned}
(i_1p_1\phi^{i+1} + i_2\psi^{i+1})d^i_{W^\bullet} &= i_1p_1d^i_{C(f)}\phi^i + i_2d^i_{X^\bullet}\psi^i \\
&= i_1p_1(-i_1d^{i+1}_{X^\bullet}p_1 + i_2f^{i+1}p_1 + i_2d^i_{Y^\bullet}p_2)\phi^i + i_2d^i_{X^\bullet}\psi^i \\
&= -i_1d^{i+1}_{X^\bullet}p_1\phi^i + i_2d^i_{X^\bullet}\psi^i \\
&= (-i_1d^{i+1}_{X^\bullet}p_1 + i_2p_1 + i_2d^i_{X^\bullet}p_2)(i_1p_1\phi^i + i_2\psi^i) \\
&= d^i_{K^\bullet}(i_1p_1\phi^i + i_2\psi^i)
\end{aligned}$$

shows that $(i_1p_1\phi^i + i_2\psi^i)$ is a morphism of complexes. Uniqueness of this morphism follows from uniqueness of $p_1\phi^i$ and ψ^i. Hence we have shown that $(K^\bullet, u)$ is the kernel of γ.

To show that $K^\bullet$ is quasi-isomorphic to 0, let $\chi^i : K^{i+1} \to K^i$ be the family of morphisms i_1p_2. Then

$$\begin{aligned}
\chi^{i+1}d^i_{K^\bullet} + d^{i-1}_{K^\bullet}\chi^i &= i_1p_2(-i_1d^{i+1}_{X^\bullet}p_1 + i_2p_1 + i_2d^i_{X^\bullet}p_2) + (-i_1d^i_{X^\bullet}p_1 + i_2p_1 + i_2d^{i-1}_{X^\bullet}p_2)i_1p_2 \\
&= i_1p_1 + i_1d^i_{X^\bullet}p_2 - i_1d^i_{X^\bullet}p_2 + i_2p_2 \\
&= \mathrm{Id}_{K^\bullet}.
\end{aligned}$$

This shows that the identity morphism $\mathrm{Id}_{K^\bullet}$ is homotopic to 0. By lemma 2.7.3, $H^i(\mathrm{Id}_{K^\bullet}) = 0$ for all i. Hence the induced morphism $0 : H^\bullet(K^\bullet) \to H^\bullet(0^\bullet) = 0^\bullet$ is an isomorphism. This shows that γ is a quasi-isomorphism and finishes the proof. $\square$

The following theorem is a version of theorem 2.6.6 for derived categories.

Theorem 5.1.3. *A distinguished triangle*

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1]$$

in $D^*(\mathcal{A})$, $* = \varnothing, +, -, b$ *induces a long exact sequence*

$$\dots \xrightarrow{H^{i-1}(w)} H^i(X^\bullet) \xrightarrow{H^i(u)} H^i(Y^\bullet) \xrightarrow{H^i(v)} H^i(Z^\bullet) \xrightarrow{H^i(w)} H^{i+1}(X^\bullet) \xrightarrow{H^{i+1}(u)} \dots \tag{5.3}$$

Moreover, this long exact sequence is functorial in the sense that if

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

is a morphism of distinguished triangles, then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{H^i(u)} & H^i(Y^\bullet) & \xrightarrow{H^i(v)} & H^i(Z^\bullet) & \xrightarrow{H^i(w)} & H^{i+1}(X^\bullet) & \xrightarrow{H^{i+1}(u)} & H^{i+1}(Y^\bullet) & \xrightarrow{H^{i+1}(v)} & \dots \\ & & \downarrow H^i(g) & & \downarrow H^i(h) & & \downarrow H^{i+1}(f) & & \downarrow H^{i+1}(g) & & \\ \dots & \xrightarrow{H^i(u')} & H^i(Y'^\bullet) & \xrightarrow{H^i(v')} & H^i(Z'^\bullet) & \xrightarrow{H^i(w')} & H^{i+1}(X'^\bullet) & \xrightarrow{H^{i+1}(u')} & H^{i+1}(Y'^\bullet) & \xrightarrow{H^{i+1}(v')} & \dots \end{array}$$

Proof. We will prove the theorem for $D(\mathcal{A})$, but the same proof works for all the $D^*(\mathcal{A})$ because these contain cones and cylinders of morphisms.

By lemma 5.1.2 and definition of distinguished triangles in $D(\mathcal{A})$, we have the following commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{u} & Y^\bullet & \xrightarrow{v} & Z^\bullet & \xrightarrow{w} & X^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X^\bullet & \xrightarrow{i_1} & Cyl(u) & \xrightarrow{p_2} & C(u) & \xrightarrow{p_1} & X^\bullet[1] \end{array} \tag{5.4}$$

where the vertical morphisms are isomorphisms. Thus for all $i \in \mathbb{Z}$ we have the following commutative diagram

$$\begin{array}{ccccccc} H^i(X^\bullet) & \xrightarrow{H^i(u)} & H^i(Y^\bullet) & \xrightarrow{H^i(v)} & H^i(Z^\bullet) & \xrightarrow{H^i(w)} & H^{i+1}(X^\bullet) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ H^i(X^\bullet) & \xrightarrow{H^i(i_1)} & H^i(Cyl(u)) & \xrightarrow{H^i(p_2)} & H^i(C(u)) & \xrightarrow{H^i(p_1)} & H^{i+1}(X^\bullet) \end{array}$$

If the statement holds for the lower distinguished triangle in the diagram (5.4), then it also holds for the upper distinguished triangle by using the isomorphism.

By lemma 2.7.8 the following is a short exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{i_1} Cyl(u) \xrightarrow{p_2} C(u) \longrightarrow 0 \tag{5.5}$$

By theorem 2.6.6 for abelian categories, we have a long exact sequence

$$\dots \xrightarrow{H^i(i_1)} H^i(Cyl(u)) \xrightarrow{H^i(p_2)} H^i(C(u)) \xrightarrow{\delta^i} H^{i+1}(X^\bullet) \xrightarrow{H^{i+1}(i_1)} H^{i+1}(Cyl(u)) \xrightarrow{H^{i+1}(p_2)} \dots$$

We will show that $H^i(p_1) = -\delta^i$, which implies the exactness of (5.3).

Recall from (2.6) and (2.8) and lemma 2.6.5 that we have the following commutative diagrams which uniquely determine the morphisms $\ker p_2$, $\operatorname{coker} p_2$, $H^i(p_2)$

$$
\begin{tikzcd}
Cyl(u)^{i+1} \arrow[r, "p_2"] & C(u)^{i+1} \\
\ker d^{i+1}_{Cyl(u)} \arrow[u] \arrow[r, "\ker p_2"] \arrow[d] & \ker d^{i+1}_{C(u)} \arrow[u] \arrow[d] \\
H^i(Cyl(u)) \arrow[r, "H^i(p_2)"] & H^i(C(u))
\end{tikzcd}
\qquad
\begin{tikzcd}
Cyl(u)^{i} \arrow[r, "p_2"] \arrow[d] & C(u)^{i} \arrow[d] \\
\operatorname{coker} d^{i-1}_{Cyl(u)} \arrow[r, "\operatorname{coker} p_2"] & \operatorname{coker} d^{i-1}_{C(u)} \\
H^i(Cyl(u)) \arrow[u] \arrow[r, "H^i(p_2)"] & H^i(C(u)) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{coker} d^{i-1}_{Cyl(u)} \arrow[r] \arrow[d] & \operatorname{coker} d^{i-1}_{C(u)} \arrow[d] \\
\ker d^{i+1}_{Cyl(u)} \arrow[r] & \ker d^{i+1}_{C(u)}
\end{tikzcd}
$$

Similar commutative diagrams hold for the morphism $i_1 : X^\bullet \to Cyl(u)^\bullet$. By (2.4) and lemma 2.5.3 the following diagrams are commutative

$$
\begin{tikzcd}
\operatorname{coker} d^{i-1}_{C(u)} \arrow[dr] & \\
C(u)^i \arrow[u] \arrow[r] & C(u)^{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C(u)^{i+1} \\
\operatorname{coker} d^{i-1}_{C(u)} \arrow[ur] \arrow[r, dashed] & \ker d^{i+1}_{C(u)} \arrow[u]
\end{tikzcd}
$$

and we have similar commutative diagrams for $Cyl(u)^{i+1}$ and X^{i+1}. We have the following commutative diagram

$$
\begin{tikzcd}
 & H^{i-1}(X^\bullet) \arrow[rr] \arrow[dd] & & H^{i-1}(Cyl(u)) \arrow[rr] \arrow[dd] & & H^{i-1}(C(u)) \arrow[dd] \\
 & & & & \ker d^i_{C(u)} \arrow[ur, dashed] \arrow[dd] & \\
 & \operatorname{coker} d^{i-1}_{X^\bullet} \arrow[rr] \arrow[dd] \arrow[dddl] & & \operatorname{coker} d^{i-1}_{Cyl(u)} \arrow[rr] \arrow[dd] \arrow[dddl] & & \operatorname{coker} d^{i-1}_{C(u)} \arrow[dd] \arrow[dddl] \\
X^i \arrow[ur] \arrow[rr] \arrow[dd] & & Cyl(u)^i \arrow[ur] \arrow[rr] \arrow[dd] & & C(u)^i \arrow[ur] \arrow[dd] & \\
 & \ker d^{i+1}_{X^\bullet} \arrow[rr] \arrow[dl] \arrow[dd] & & \ker d^{i+1}_{Cyl(u)} \arrow[rr] \arrow[dl] \arrow[dd] & & \ker d^{i+1}_{C(u)} \arrow[dl] \arrow[dd] \\
X^{i+1} \arrow[rr] & & Cyl(u)^{i+1} \arrow[rr] & & C(u)^{i+1} & \\
 & H^i(X^\bullet) \arrow[rr] & & H^i(Cyl(u)) \arrow[rr] & & H^i(C(u))
\end{tikzcd}
\tag{5.6}
$$

By commutativity

$$
\ker d^i_{C(u)} \to C(u)^i \to \operatorname{coker} d^{i-1}_{C(u)} \to \ker d^{i+1}_{C(u)} \to C(u)^{i+1} = \ker d^i_{C(u)} \to C(u)^i \to C(u)^{i+1} = 0
$$

and since $\ker d^{i+1}_{C(u)} \to C(u)^{i+1}$ is a monomorphism, and $H^{i-1}(C(u)) \to \operatorname{coker} d^{i-1}_{C(u)}$ is the kernel of $\operatorname{coker} d^{i-1}_{C(u)} \to \ker d^{i+1}_{C(u)}$, the morphism $\ker d^i_{C(u)} \to H^{i-1}(C(u))$ is the unique morphism such that

$$\ker d^i_{C(u)} \to H^i(C(u)) \to \operatorname{coker} d^{i-1}_{C(u)} = \ker d^i_{C(u)} \to C(u)^i \to \operatorname{coker} d^{i-1}_{C(u)}.$$

A simple computation shows that

$$C(u)^i \overset{i_2}{\to} Cyl(u)^i \overset{d^i_{Cyl(u)}}{\to} Cyl(u)^{i+1} \overset{p_1}{\to} X^{i+1} = -p_1.$$

By commutativity of the diagram (5.6)

$$\begin{aligned}
\ker d^i_{C(u)} &\overset{-\ker p_1}{\to} \ker d^i_{X^\bullet} \to X^{i+1} \\
&= \ker d^i_{C(u)} \to C(u)^i \to Cyl(u)^i \to Cyl(u)^{i+1} \to X^{i+1} \\
&= \ker d^i_{C(u)} \to H^i(C(u)) \to \operatorname{coker} d^{i-1}_{C(u)} \to \operatorname{coker} d^{i-1}_{Cyl(u)} \to \ker d^{i+1}_{Cyl(u)} \to \ker d^{i+1}_{X^\bullet} \to X^{i+1}.
\end{aligned}$$

Since $\ker d^{i+1}_{X^\bullet} \to X^{i+1}$ is a monomorphism, we have

$$\ker(-p_1) = \ker d^i_{C(u)} \to H^i(C(u)) \to \operatorname{coker} d^{i-1}_{C(u)} \to \operatorname{coker} d^{i-1}_{Cyl(u)} \to \ker d^{i+1}_{Cyl(u)} \to \ker d^{i+1}_{X^\bullet}. \tag{5.7}$$

From the proof of the snake lemma, corollary 2.6.4, we obtain the following commutative diagram

$$\begin{tikzcd}
 & & & & & \ker d^i_{C(u)} \arrow[dll] \arrow[d] & \\
 & & & X_1 \arrow[r] \arrow[d] & X'_1 \arrow[r, "\beta_2"] \arrow[dr, "\eta"] & H^i(C(u)) \arrow[d, "\epsilon"] & \\
 & \operatorname{coker} d^{i-1}_{X^\bullet} \arrow[r] \arrow[d] & A'_1 \arrow[r] \arrow[ur, dashed] \arrow[dl] & \operatorname{coker} d^{i-1}_{Cyl(u)} \arrow[rr] \arrow[d] & & \operatorname{coker} d^{i-1}_{C(u)} \arrow[r] \arrow[d, "h"] \arrow[dl, "\gamma"] & 0 \\
0 \arrow[r] & \ker d^{i+1}_{X^\bullet} \arrow[rr] \arrow[d] \arrow[dr] & & \ker d^{i+1}_{Cyl(u)} \arrow[r] \arrow[d] & B'_2 \arrow[r, "\eta_2"] & \ker d^{i+1}_{C(u)} & \\
 & H^{i+1}(X^\bullet) \arrow[r, "\tau_1"] & X'_2 \arrow[r] & X_2 \arrow[ur, dashed] & & &
\end{tikzcd} \tag{5.8}$$

where the morphism $\ker d^i_{C(u)} \to X_1$ is the unique morphism obtained by using the universality of the pullback X_1 and satisfies the following equalities

$$\begin{cases}
\ker d^i_{C(u)} \to X_1 \to X'_1 = \ker d^i_{C(u)} \to H^i(C(u)) \to X'_1 \\
\ker d^i_{C(u)} \to X_1 \to \operatorname{coker} d^{i-1}_{Cyl(u)} = \ker d^i_{C(u)} \to H^i(C(u)) \to \operatorname{coker} d^{i-1}_{C(u)} \to \operatorname{coker} d^{i-1}_{Cyl(u)}
\end{cases}$$

Recall from the proof of corollary 2.6.4 that we have the following equality

$$X_1 \to X'_1 \to X_2 = X_1 \to \operatorname{coker} d^{i-1}_{Cyl(u)} \to \ker d^{i+1}_{Cyl(u)} \to X_2.$$

Using the equalities and diagrams we have introduced, one obtains the following equalities

$$\begin{aligned}
&\ker d^i_{C(u)} \overset{\ker(-p_1)}{\to} \ker d^{i+1}_{X^\bullet} \to H^{i+1}(X^\bullet) \to X_2' \to X_2 \\
&\quad = \ker d^i_{C(u)} \to H^i(C(u)) \to \operatorname{coker} d^{i-1}_{C(u)} \to \operatorname{coker} d^{i-1}_{Cyl(u)} && (5.7)\\
&\qquad \to \ker d^{i+1}_{Cyl(u)} \to \ker d^{i+1}_{X^\bullet} \to H^{i+1}(X^\bullet) \to X_2' \to X_2 \\
&\quad = \ker d^i_{C(u)} \to X_1 \to \operatorname{coker} d^{i-1}_{Cyl(u)} \to \ker d^{i+1}_{Cyl(u)} \to X_2 && (5.8)\\
&\quad = \ker d^i_{C(u)} \to H^i(C(u)) \to H^{i+1}(X^\bullet) \to X_2' \to X_2 && (5.8)
\end{aligned}$$

The composite $H^{i+1}(X^\bullet) \to X_2' \to X_2$ is a monomorphism, so

$$\ker d^i_{C(u)} \overset{\ker(-p_1)}{\to} \ker d^{i+1}_{X^\bullet} \to H^{i+1}(X^\bullet) = \ker d^i_{C(u)} \to H^i(C(u)) \xrightarrow{\delta^i} H^{i+1}(X^\bullet).$$

The morphism $H^i(-p_1)$ is the unique morphism which satisfies this equation. Hence $H^i(p_1) = -\delta^i$.

To show that this long exact sequence is functorial, it suffices to show functoriality for distinguished triangles of the form in definition 4.2.1. Let

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "f"] & Cyl(u) \arrow[r, "p_2"] \arrow[d, "g"] & C(u) \arrow[r, "p_1"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "i_1"] & Cyl(u') \arrow[r, "p_2"] & C(u') \arrow[r, "p_1"] & X'[1]
\end{tikzcd}$$

be a morphism of distinguished triangles in $D^*(\mathcal{A})$. Since the cohomology functor $H^\bullet$ factors through $D^*(\mathcal{A})$, the following diagram is commutative.

$$\begin{tikzcd}
\dots \arrow[r, "H^i(i_1)"] & H^i(Cyl(u)) \arrow[r, "H^i(p_2)"] \arrow[d, "H^i(g)"] & H^i(C(u)) \arrow[r, "H^i(p_1)"] \arrow[d, "H^i(h)"] & H^{i+1}(X) \arrow[r, "H^{i+1}(i_1)"] \arrow[d, "H^i(f)"] & H^{i+1}(Cyl(u)) \arrow[r, "H^{i+1}(p_2)"] \arrow[d, "H^i(g)"] & \dots \\
\dots \arrow[r, "H^i(i_1)"] & H^i(Cyl(u')) \arrow[r, "H^i(p_2)"] & H^i(C(u')) \arrow[r, "H^i(p_1)"] & H^{i+1}(X') \arrow[r, "H^{i+1}(i_1)"] & H^{i+1}(Cyl(u')) \arrow[r, "H^{i+1}(p_2)"] & \dots
\end{tikzcd}$$

This completes the proof. □

We are ready to prove that the derived category of an abelian category $\mathcal{A}$ can be obtained from the homotopy category of $\mathcal{A}$, up to isomorphism.

Theorem 5.1.4. *For an abelian category $\mathcal{A}$ we have an isomorphism of categories $G : D^*(\mathcal{A}) \overset{\cong}{\to} K(\mathcal{A})^*[S^{-1}]$, $* = \varnothing, +, -, b$, where S is the class of quasi-isomorphisms in $K^*(\mathcal{A})$, $* = \varnothing, +, -, b$.*

Proof. **Construction of G:** The composite

$$C(\mathcal{A}) \longrightarrow K(\mathcal{A}) \xrightarrow{Q_S} K(\mathcal{A})[S^{-1}]$$

sends quasi-isomorphsims to isomorphisms and is the identity map on objects. By theorem 3.1.3 there exists a unique functor G making the following diagram commutative

$$\begin{tikzcd}
C(\mathcal{A}) \arrow[r, "Q_T"] \arrow[d] & D(\mathcal{A}) \arrow[d, "G"] \\
K(\mathcal{A}) \arrow[r, "Q_S"] & K(\mathcal{A})[S^{-1}]
\end{tikzcd} \tag{5.9}$$

By construction 3.1.1 the localizing functor $Q_T : C(\mathcal{A}) \to D(\mathcal{A})$, where T is the class of quasi-isomorphisms in $C(\mathcal{A})$, is identity on objects. Hence, by commutativity, G is identity on objects.

To see how G is defined on morphisms, note that

$$G((f_1, \ldots, f_n)) = G((f_n)) \circ \ldots \circ G((f_1)),$$

where $(f_1, \ldots, f_n)$ is an arbitrary morphism in $D(\mathcal{A})$. By commutativity, we must have

$$G((f_j)) = \begin{cases} (f_j) & , \text{ if } f_j \neq s^{-1} \\ (s^{-1}) & , \text{ if } f_j = s^{-1} \end{cases}$$

for any $j = 1, \ldots, n$. This determines G on morphisms.

G **is surjective on morphisms:** To show that G is surjective on morphisms, it suffices to show that for any string $((f_1))$ of length 1 in $K(\mathcal{A})[S^{-1}]$ there exists a string $((g_1))$ of length 1 in $D(\mathcal{A})$ such that

$$G((g_1)) = ((f_1)).$$

If f_1 is a morphism of $K(\mathcal{A})$ there exists a morphism f_1' of $C(\mathcal{A})$ which represents this morphism and by commutativity we get $(G \circ Q_T)(f_1') = G((f_1')) = (f_1)$. If $f_1 = s^{-1}$ for some quasi-isomorphism s in $K(\mathcal{A})$, then we choose a morphism s' in $C(\mathcal{A})$, which also is a quasi-isomorphism by Lemma 2.7.3 (c), because it is homotopic to s. Now, by definition of G, $G((s')) = (f_1)$. This shows that G is surjective on morphisms.

G **is injective on morphisms:** To show that G is injective on morphisms, le us make a few important notes. Since any functor preserve isomorphisms, and isomorphisms are both monomorphisms and epimorphisms, we see that for any two parallel morphisms f and g in $C(A)$ all the following equations hold

$$G((t_1, f)) = G((t_1, g)) \qquad G((t_1^{-1}, f)) = G((t_1^{-1}, g)) \qquad G((f, t_2)) = G((g, t_2)) \qquad G((f, t_2^{-1})) = G((g, t_2^{-1})),$$

for any quasi-isomorphisms t_1 and t_2 of $C(A)$, if and only if $(f) = (g)$ in $K(\mathcal{A})$. By using the equivalence relation on strings in $D(\mathcal{A})$, it is easy to see that any string is equivalent to a composition of strings of the form in the above equations. Hence it suffices to show that Q_T maps homotopic morphisms to same morphism in $D(\mathcal{A})$, that is

$$f \sim g \text{ in } C(\mathcal{A}) \qquad \Rightarrow \qquad ((f)) = ((g)) \text{ in } D(\mathcal{A}).$$

Let $f, g : X^\bullet \to Y^\bullet$ be two parallel homotopic morphisms in $C(\mathcal{A})$ and let $\chi^i : X^i \to Y^{i-1}$ be a homotopy between them. Define $c(\chi) : C(f) \to C(g)$ and $cyl(\chi) : Cyl(f) \to Cyl(g)$ by

$$\begin{aligned} c(\chi) &= i_1 p_1 + i_2 p_2 + i_2 \chi^{i+1} p_1 \\ cyl(\chi) &= i_1 p_1 + i_2 i_1 p_1 p_2 + i_2 i_2 p_2 p_2 + i_2 i_2 \chi^{i+1} p_1 p_2 \end{aligned}$$

These are morphisms of complexes since they commute with differentials. Indeed, from

$$\begin{aligned} c(\chi)^{i+1} d^i_{C(f)} &= (i_1 p_1 + i_2 p_2 + i_2 \chi^{i+2} p_1)(-i_1 d^{i+1}_{X^\bullet} p_1 + i_2 f^{i+1} p_1 + i_2 d^i_{Y^\bullet} p_2) \\ &= -i_1 d^{i+1}_{X^\bullet} p_1 + i_2 f^{i+1} p_1 + i_2 d^i_{Y^\bullet} p_2 - i_2 \chi^{i+2} d^{i+1}_{X^\bullet} p_2 \end{aligned}$$

and

$$\begin{aligned}
d^i_{C(g)}c(\chi)^i &= (-i_1 d^{i+1}_{X^\bullet} p_1 + i_2 g^{i+1} p_1 + i_2 d^i_{Y^\bullet} p_2)(i_1 p_1 + i_2 p_2 + i_2 \chi^{i+1} p_1) \\
&= -i_1 d^{i+1}_{X^\bullet} p_1 + i_1 g^{i+1} p_2 + i_2 d^i_{Y^\bullet} p_2 + i_2 d^i_{Y^\bullet} \chi^{i+1} p_1
\end{aligned}$$

we get

$$\begin{aligned}
c(\chi)^{i+1} d^i_{C(f)} - d^i_{c(g)} c(\chi)^i &= i_2 f^{i+1} p_1 - i_2 g^{i+1} p_1 - i_2 \chi^{i+2} d^{i+1}_{X^\bullet} p_2 - i_2 d^i_{Y^\bullet} \chi^{i+1} p_1 \\
&= i_2 \chi^{i+1} d^{i+1}_{X^\bullet} p_2 + i_2 d^i_{Y^\bullet} \chi^{i+1} p_1 - i_2 \chi^{i+2} d^{i+1}_{X^\bullet} p_2 - i_2 d^i_{Y^\bullet} \chi^{i+1} p_1 \\
&= 0.
\end{aligned}$$

Similarly, from

$$\begin{aligned}
cyl(\chi)^{i+1} d^i_{Cyl(f)} = \ & (i_1 p_1 + i_2 i_1 p_1 p_2 + i_2 i_2 p_2 p_2 + i_2 i_2 \chi^{i+2} p_1 p_2) \\
& (i_1 d^i_{X^\bullet} p_1 - i_1 p_1 p_2 - i_2 i_1 d^{i+1}_{X^\bullet} p_1 p_2 + i_2 i_2 f^{i+1} p_1 p_2 + i_2 i_2 d^i_{Y^\bullet} p_2 p_2) \\
= \ & i_1 d^i_{X^\bullet} p_1 - i_1 p_1 p_2 - i_2 i_1 d^{i+1}_{X^\bullet} p_1 p_2 + i_2 i_2 f^{i+1} p_1 p_2 + i_2 i_2 d^i_{Y^\bullet} p_2 p_2 - i_2 i_2 \chi^{i+2} d^{i+2}_{X^\bullet} p_1 p_2
\end{aligned}$$

and

$$\begin{aligned}
d^i_{Cyl(g)} cyl(\chi)^i = \ & (i_1 d^i_{X^\bullet} p_1 - i_1 p_1 p_2 - i_2 i_1 d^{i+1}_{X^\bullet} p_1 p_2 + i_2 i_2 g^{i+1} p_1 p_2 + i_2 i_2 d^i_{Y^\bullet} p_2 p_2) \\
& (i_1 p_1 + i_2 i_1 p_1 p_2 + i_2 i_2 p_2 p_2 + i_2 i_2 \chi^{i+1} p_1 p_2) \\
= \ & i_1 d^i_{X^\bullet} p_1 - i_1 p_1 p_2 - i_2 i_1 d^{i+1}_{X^\bullet} p_1 p_2 + i_2 i_2 g^{i+1} p_1 p_2 + i_2 i_2 d^i_{Y^\bullet} p_2 p_2 + i_2 i_2 d^i_{Y^\bullet} \chi^{i+1} p_1 p_2
\end{aligned}$$

we get

$$\begin{aligned}
& cyl(\chi)^{i+1} d^i_{Cyl(f)} - d^i_{Cyl(g)} cyl(\chi)^i \\
&= i_2 i_2 f^{i+1} p_1 p_2 - i_2 i_2 g^{i+1} p_1 p_2 - i_2 i_2 \chi^{i+2} d^{i+2}_{X^\bullet} p_1 p_2 - i_2 i_2 d^i_{Y^\bullet} \chi^{i+1} p_1 p_2 \\
&= i_2 i_2 \chi^{i+2} d^{i+1}_{X^\bullet} p_1 p_2 + i_2 i_2 d^i_{Y^\bullet} \chi^{i+1} p_1 p_2 - i_2 i_2 \chi^{i+2} d^{i+1}_{X^\bullet} p_2 p_2 - i_2 i_2 d^i_{Y^\bullet} \chi^{i+1} p_1 p_2 \\
&= 0
\end{aligned}$$

so $cyl(\chi)$ is a morphism.

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y^\bullet \arrow[r, "i_2"] \arrow[d, equal] & C(f) \arrow[r, "p_1"] \arrow[d, "c(\chi)"] & X^\bullet[1] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y^\bullet \arrow[r, "i_2"] & C(g) \arrow[r, "p_1"] & X^\bullet[1] \arrow[r] & 0
\end{tikzcd}$$

Applying theorem 2.6.6 to the above morphism of short exact sequences, we get for all i the following commutative diagram with exact rows

$$\begin{tikzcd}
H^{i-1}(X^\bullet) \arrow[r, "\delta^{i-1}"] \arrow[d, equal] & H^i(Y^\bullet) \arrow[r, "H^i(i_2)"] \arrow[d, equal] & H^i(C(f)) \arrow[r, "H^i(p_1)"] \arrow[d, "H^i(c(h))"] & H^i(X^\bullet) \arrow[r, "\delta^i"] \arrow[d, equal] & H^{i+1}(Y^\bullet) \arrow[d, equal] \\
H^{i-1}(X^\bullet) \arrow[r, "\delta^{i-1}"] & H^i(Y^\bullet) \arrow[r, "H^i(i_2)"] & H^i(C(g)) \arrow[r, "H^i(p_1)"] & H^i(X^\bullet) \arrow[r, "\delta^i"] & H^{i+1}(Y^\bullet)
\end{tikzcd}$$

By lemma 2.6.1, $H^i(c(\chi))$ is an isomorphism for all i, so the morphism $c(\chi)$ is a quasi-isomorphism.

Similarly, the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X^\bullet \arrow[r, "i_1"] \arrow[d, equal] & Cyl(f) \arrow[r, "p_2"] \arrow[d, "cyl(\chi)"] & C(f) \arrow[r] \arrow[d, "c(\chi)"] & 0 \\
0 \arrow[r] & X^\bullet \arrow[r, "i_1"] & Cyl(g) \arrow[r, "p_2"] & C(g) \arrow[r] & 0
\end{tikzcd}$$

gives rise to a morphism of the associated long exact sequences, and by applying the 5-lemma, we get that $H^i(cyl(\chi))$ is an isomorphism for all i. Hence $cyl(\chi)$ is a quasi-isomorphism.

Consider the following diagram

$$\begin{tikzcd}
 & Y^\bullet \arrow[d, "\alpha_f"] \\
X^\bullet \arrow[ur, "f"] \arrow[r, "i_1"] \arrow[d, equal] & Cyl(f) \arrow[d, "cyl(\chi)"] \\
X^\bullet \arrow[r, "i_1"] \arrow[dr, "g"] & Cyl(g) \arrow[d, "\beta_g"] \\
 & Y^\bullet
\end{tikzcd} \tag{5.10}$$

where the morphisms $\alpha_f = i_2 i_2$, $\beta_g = g^i p_1 + p_2 p_2$, are the quasi-isomorphisms from lemma 2.7.9 such that $\beta_f \alpha_f = \mathrm{Id}_{Y^\bullet}$ and $\alpha_f \beta_f$ is homotopic to $\mathrm{Id}_{Cyl(f)}$. In $C(\mathcal{A})$ we have $\alpha_f \beta_f \neq \mathrm{Id}_{Cyl(f)}$, but the images in $D(\mathcal{A})$ are equal. Indeed, we have $Q_T(\alpha_f) = Q_T(\beta_f)^{-1} Q_T(\beta_f \alpha_f) = Q_T(\beta_f)^{-1}$, so

$$Q_T(\alpha_f \beta_f) = Q_T(\beta_f)^{-1} Q_T(\beta_f) = \mathrm{Id}_{Q_T(Cyl(f))}\,.$$

Clearly $Q_T(\beta_f) = Q_T(\alpha_f)^{-1}$. Similar equations hold for the morphisms α_g and β_g.

Now, from commutativity in Lemma 2.7.9, it follows that $Q_T(i_1) = Q_T(\alpha_f) Q_T(\beta_f) Q(i_1) = Q_T(\alpha_f) Q_T(f)$, so the upper triangle in the diagram (5.10) is commutative. A simple computation shows that the middle square and the lower triangle of (5.10) are commutative in $C(\mathcal{A})$, hence also in $D(\mathcal{A})$. From

$$(\beta_g cyl(\chi) \alpha_f)^i = (g^i p_1 + p_2 p_2) cyl(\chi) i_2 i_2 = \mathrm{Id}_{Y^i}$$

we conclude that

$$Q(g) = Q(f) Q(\beta_g cyl(h) \alpha_f) = Q(f).$$

This completes the proof.

□

The theorem gives us a criterion for a morphism to by a quasi-isomorphism by studying its mapping cone.

Corollary 5.1.5. *A morphism $f : X \to Y$ in $C(\mathcal{A})$ is a quasi-isomorphism if and only if $C(f)$ is exact, that is the cohomology of $C(f)$ is the zero complex.*

Proof. By the proof of the previous theorem, homotopic morphisms in $C(\mathcal{A})$ are mapped to the same morphism in $D(\mathcal{A})$. Therefore, by the proof of Lemma 4.2.2

$$X \xrightarrow{f} Y \xrightarrow{i_2} C(f) \xrightarrow{p_1} X[1]$$

is a distinguished triangle in $D(\mathcal{A})$. By theorem 5.1.3 we have the long exact sequence

$$\dots \longrightarrow H^{i-1}(C(f)) \xrightarrow{H^{i-1}(p_1)} H^i(X) \xrightarrow{H^i(f)} H^i(Y) \xrightarrow{H^i(i_2)} H^i(C(f)) \xrightarrow{H^i(p_1)} \dots$$

associated to the distinguished triangle. From assumption and exactness it follows that that $H^i(C(f)) = 0$ for all $i \in \mathbb{Z}$. Therefore $C(f)$ is exact.

Conversely, suppose $C(f)$ is exact. From the same long exact sequence it follows that $H^i(f)$ is an isomorphism for all i. Thus f is a quasi-isomorphism. □

Next, we want to show that the localization of the homotopy category of $\mathcal{A}$ admits the formalism of roofs. To show this, we need to show that the quasi-isomorphisms in $K(\mathcal{A})$ form a localizing class.

Proposition 5.1.6. *Let $\mathcal{A}$ be an abelian category. Then the family of quasi-isomorphisms in the category $\mathcal{K}(\mathcal{A})$ is a localizing class compatible with the triangulation of $\mathcal{K}(\mathcal{A})$.*

Proof. **(LS 1):** For any object $X^\bullet \in \mathcal{K}(\mathcal{A})$ the identity morphism $Id_{X^\bullet} : X^\bullet \to X^\bullet$ induces identity morphism $H^i(Id_{X^\bullet}) : H^i(X^\bullet) \to H^i(X^\bullet)$ for all i, and so the identity morphisms are quasi-isomorphisms.

Let $f : X^\bullet \to Y^\bullet$ and $g : Y^\bullet \to Z^\bullet$ be quasi-isomorphisms. For any $i \in \mathbb{Z}$, we have $H^i(f) : H^i(X^\bullet) \cong H^i(Y^\bullet)$ and $H^i(g) : H^i(Y^\bullet) \cong H^i(Z^\bullet)$. Hence $H^i(gf) = H^i(g)H^i(f) : H^i(X^\bullet) \cong H^i(Z^\bullet)$ for all i, and thus gf is a quasi-isomorphism.

(LS 2): Step 1: Let

$$\begin{tikzcd} X^\bullet \arrow[r, "f"] \arrow[d, "g"] & Y^\bullet \\ Z^\bullet & \end{tikzcd}$$

be a diagram in $\mathcal{A}$ where f is a quasi-isomorphism. Consider the following diagram

$$\begin{tikzcd} C(f)[-1] \arrow[r, "-p_1"] \arrow[d, equal] & X^\bullet \arrow[r, "f"] \arrow[d, "g"] & Y^\bullet \arrow[r, "i_2"] \arrow[d, "h"] & C(f) \arrow[d, equal] \\ C(f)[-1] \arrow[r, "-gp_1"] & Z^\bullet \arrow[r, "i_2"] & C(-gp_1) \arrow[r, "p_1"] & C(f) \end{tikzcd}$$

where both of the rows are distinguished triangles in $K(\mathcal{A})$ by Lemma 4.2.2 and the morphism h is given by TR5 for $K(\mathcal{A})$. Since f is a quasi-isomorphism, $C(f)$ is exact by Corollary 5.1.5. By lemma 4.2.2 the bottom triangle in the previous diagram is distinguished in $D(\mathcal{A})$. Thus, by theorem 5.1.3 applied to the bottom distinguished triangle and the fact that homotopic maps in $C(\mathcal{A})$ are mapped to equal morphisms in $D(\mathcal{A})$, we get the following long exact sequence

$$\dots \xrightarrow{H^{i-1}(p_1)} H^i(C(f)[-1]) \xrightarrow{H^i(gp_1)} H^i(Z^\bullet) \xrightarrow{H^i(i_2)} H^i(C(gp_1)) \xrightarrow{H^i(p_1)} H^i(C(f)[-1]) \xrightarrow{H^{i+1}(p_1)} \dots$$

By exactness i_2 is a quasi-isomorphism. Thus we have obtained the following commutative square

$$\begin{tikzcd} X^\bullet \arrow[r, "f"] \arrow[d, "g"] & Y^\bullet \arrow[d, "h"] \\ Z^\bullet \arrow[r, "i_2"] & C(-gp_1) \end{tikzcd}$$

Step 2: Let

$$\begin{tikzcd} & Z^\bullet \arrow[d, "g"] \\ X^\bullet \arrow[r, "f"] & Y^\bullet \end{tikzcd}$$

be a diagram in $\mathcal{K}(\mathcal{A})$ with f a quasi-isomorphism. Consider the following diagram

$$\begin{tikzcd} C(i_2g)[-1] \arrow[r, "-p_1[-1]"] \arrow[d, "h"] & Z^\bullet \arrow[r, "i_2g"] \arrow[d, "g"] & C(f) \arrow[r, "i_2"] \arrow[d, equal] & C(i_2g) \arrow[d, "h[1]"] \\ X^\bullet \arrow[r, "f"] & Y^\bullet \arrow[r, "i_2"] & C(f) \arrow[r, "p_1"] & X^\bullet[1] \end{tikzcd}$$

where both of the rows are distinguished triangles in $K(\mathcal{A})$ and the morphism h is given by TR5 for $K(\mathcal{A})$. Now

$$Z^\bullet \xrightarrow{i_2g} C(f) \xrightarrow{i_2} C(i_2g) \xrightarrow{p_1} Z^\bullet[1]$$

is a distinguished triangle in $D(\mathcal{A})$ by Lemma 4.2.2 and the fact that homotopies in $C(\mathcal{A})$ map to equal morphism in $D(\mathcal{A})$. Thus by Theorem 5.1.3 we have the following long exact sequence

$$\ldots \xrightarrow{H^{i-1}(gp_1)} H^{i-1}(C(f)) \xrightarrow{H^{i-1}(i_2)} H^{i-1}(C(i_2g)[-1]) \xrightarrow{H^{i-1}(p_1)} H^i(Z^\bullet) \xrightarrow{H^i(i_2g)} H^i(C(f)) \xrightarrow{H^i(i_2)} \ldots$$

Since f is a quasi-isomorphism, $C(f)$ is exact by Lemma 5.1.5, so from exactness it follows that $-p_1[-1]$ is a quasi-isomorphism. Hence we have obtained the following commutative diagram in $K(\mathcal{A})$

$$\begin{tikzcd} C(i_2g)[-1] \arrow[r, "-p_1[-1]"] \arrow[d, "h"] & Z^\bullet \arrow[d, "g"] \\ X^\bullet \arrow[r, "f"] & Y^\bullet \end{tikzcd}$$

(LS 3): Since the category $\mathcal{K}(\mathcal{A})$ is additive, this condition is equivalent to following: for any morphism $f : X^\bullet \to Y^\bullet$ in $\mathcal{K}(\mathcal{A})$, there exists a quasi-isomorphism $s : \bar{X}^\bullet \to X^\bullet$ such that $fs = 0$ if and only if there exists a quasi-isomorphism $t : Y^\bullet \to \bar{Y}^\bullet$ such that $tf = 0$.

$\Rightarrow$: Let $s : Y^\bullet \to \bar{Y}^\bullet$ be a quasi-isomorphism such that $sf = 0$. Let $h^i : X^i \to \bar{Y}^{i-1}$ be the homotopy between sf and the zero morphism. Consider the following commutative diagram

$$\begin{tikzcd} C(g)[-1] \arrow[r, "-p_1[-1]"] & X^\bullet \arrow[r, "g"] \arrow[dr, "f"] & C(s)[-1] \arrow[r, "i_2"] \arrow[d, "p_1"] & C(g) \\ & & Y^\bullet \arrow[d, "s"] & \\ & & \bar{Y}^\bullet & \end{tikzcd}$$

where $g^i : X^i \to C(s)[-1]^i$ is the morphism $i_1 f^i - i_2 h^i$. From

$$\begin{aligned}
g^{i+1} d^i_{X^\bullet} &= i_1 f^{i+1} d^i_{X^\bullet} - i_2 h^{i+1} d^i_{X^\bullet} \\
&= i_1 d^i_{Y^\bullet} f^i - i_2 h^{i+1} d^i_{X^\bullet} \\
&= i_1 d^i_{Y^\bullet} f^i - i_2 d^{i-1}_{\bar{Y}^\bullet} h^i - i_2 h^{i+1} d^i_{X^\bullet} + i_2 d^{i-1}_{\bar{Y}^\bullet} h^i \\
&= i_1 d^i_{Y^\bullet} f^i - i_2 s^i f^i + i_2 d^{i-1}_{\bar{Y}^\bullet} h^i \\
&= (i_1 d^i_{Y^\bullet} p_1 - i_2 s^i p_1 - i_2 d^{i-1}_{\bar{Y}^\bullet} p_2)(i_1 f^i - i_2 h^i) \\
&= d^i_{C(s)[-1]} g^i
\end{aligned}$$

we see that g is a morphism. The row in the diagram is a distinguished triangle by lemma 4.2.2. Therefore, from lemma 4.1.4 we get that $g(-p_1[-1]) = 0$, so $f(-p_1[-1]) = 0$. Since s is a quasi-isomorphism, $H^i(C(s)) = 0$, for all i, by Corollary 5.1.5. From the definition of translation and cohomology, $H^i(C(s)) = H^{i+1}(C(s)[-1])$. Therefore, from the long exact sequence

$$\dots \xrightarrow{H^{i-1}(g)} H^{i-1}(C(s)[-1]) \xrightarrow{H^{i-1}(i_2)} H^{i-1}(C(g)) \xrightarrow{H^i(p_1)} H^i(X^\bullet) \xrightarrow{H^i(g)} H^i(C(s)[-1]) \xrightarrow{H^i(i_2)} \dots$$

associated to the distinguished triangle

$$X^\bullet \xrightarrow{g} C(s)[-1] \xrightarrow{i_2} C(g) \xrightarrow{p_2} X^\bullet[1]$$

we get that $-p_1[-1]$ is a quasi-isomorphism. Thus we can take $t = -p_1[-1]$.

$\Leftarrow$: Let $t : \bar{X}^\bullet \to X^\bullet$ be a quasi-isomorphism such that ft is homotopic to 0 and let $h^i : \bar{X}^\bullet \to Y^\bullet$ be a set of morphisms such that $f^i t^i = h^{i+1} d^i_{\bar{X}^\bullet} + d^{i-1}_{Y^\bullet} h^i$ for all i. Consider the following commutative diagram

$$\begin{tikzcd}
 & \bar{X}^\bullet \arrow[d, "t"] & & \\
 & X^\bullet \arrow[d, "i_2"] \arrow[dr, "f"] & & \\
C(g)[-1] \arrow[r, "-p_1[-1]"] & C(t) \arrow[r, "g"] & Y^\bullet \arrow[r, "i_2"] & C(g)
\end{tikzcd}$$

where the row is a distinguished triangle and $g^i = f^i p_2 + h^{i+1} p_1$. Then

$$\begin{aligned}
g^{i+1} d^i_{C(t)} &= (f^{i+1} p_2 + h^{i+2} p_1)(-i_1 d^{i+1}_{\bar{X}^\bullet} p_1 + i_2 t^{i+1} p_1 + i_2 d^i_{X^\bullet} p_2) \\
&= f^{i+1} t^{i+1} p_1 + f^{i+1} d^i_{X^\bullet} p_2 - h^{i+2} d^{i+1}_{\bar{X}^\bullet} p_1 \\
&= d^i_{Y^\bullet} f^i p_2 + d^i_{Y^\bullet} h^{i+1} p_1 \\
&= d^i_{Y^\bullet} g^i
\end{aligned}$$

shows that g is a morphism of complexes. Now, $i_2 f = i_2 g i_2 = 0$, because $i_2 g = 0$ by Lemma 4.1.4. Since t is a quasi-isomorphism, $H^i(C(t)) = 0$ for all i. From the long exact sequence

$$\dots \xrightarrow{H^{i-1}(p_1)} H^i(C(t)) \xrightarrow{H^i(g)} H^i(Y^\bullet) \xrightarrow{H^i(i_2)} H^{i+1}(C(g)) \xrightarrow{H^{i+1}(p_1)} H^{i+1}(C(t)) \xrightarrow{H^{i+1}(g)} \dots$$

associated to the distinguished triangle in the diagram

$$C(t) \xrightarrow{g} Y^{\bullet} \xrightarrow{i_2} C(g) \xrightarrow{p_1} C(t)[1]$$

we get that $H^i(i_2)$ is an isomorphism for all i. Thus i_2 is a quasi-isomorphism, and we can choose $s = i_2$. □

The above proposition also proves the following after a simple observation

Corollary 5.1.7. *The class S consisting of quasi-isomorphisms in $K^*(\mathcal{A})$, $* = +, -, b$, is a localizing class.*

Proof. The proof of proposition 5.1.6 works for these categories also, because the mapping cone of a morphism of $K^*(\mathcal{A})$, is contained in $K^*(\mathcal{A})$. □

Finally we are able to prove that derived categories are triangulated.

Theorem 5.1.8. *The category $D^*(\mathcal{A})$, $* = \varnothing, +, -, b$, is a triangulated category.*

Proof. By Theorem 5.1.4 $D^*(\mathcal{A})$ is isomorphic to $K^*(\mathcal{A})[S^{-1}]$, where S is the class consisting of quasi-isomorphisms in $K^*(\mathcal{A})$. Since $K^*(\mathcal{A})$ is a triangulated category, Theorem 4.2.5, by Theorem 4.3.2 it suffices to show that S is compatible with triangulation.

Let f be a quasi-isomorphism. Now $H^i(f[1]) = H^{i+1}(f)$, so $H^i(f[1])$ is an isomorphism. Thus CT1 holds for S.

To prove property CT2 for S, consider the following morphism of distinguished triangles in $K^*(\mathcal{A})$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

where f and g are quasi-isomorphisms. By functoriality of the long exact sequence for distinguished triangles, theorem 5.1.3, we obtain for all $i \in \mathbb{Z}$ the following commutative diagram with exact rows

$$\begin{tikzcd}
H^i(X) \arrow[r, "H^i(u)"] \arrow[d, "H^i(f)"] & H^i(Y) \arrow[r, "H^i(v)"] \arrow[d, "H^i(g)"] & H^i(Z) \arrow[r, "H^i(w)"] \arrow[d, "H^i(h)"] & H^{i+1}(X) \arrow[r, "H^{i+1}(u)"] \arrow[d, "H^{i+1}(f)"] & H^{i+1}(Y) \arrow[d, "H^{i+1}(g)"] \\
H^i(X') \arrow[r, "H^i(u')"] & H^i(Y') \arrow[r, "H^i(v')"] & H^i(Z') \arrow[r, "H^i(w')"] & H^{i+1}(X') \arrow[r, "H^{i+1}(u')"] & H^{i+1}(Y')
\end{tikzcd}$$

Now $H^i(f), H^i(g), H^{i+1}(f)$, and $H^{i+1}(g)$ are isomorphisms, because f and g are quasi-isomorphisms, so by Lemma 2.6.1 $H^i(h)$ is an isomorphism. Therefore h is a quasi-isomorphism, and S satisfies CT2. □

Let us show that the derived category $D(\mathbf{RMod})$ exists, when the underlying class of the ring R is a set. One can also show that the derived categories of sheaves and presheaves of R-modules exists. The following lemma is from [Wei95, Proposition 10.4.4].

Lemma 5.1.9. *The category $D^*(\mathbf{RMod})$, $* = \varnothing, +, -, b$ exists.*

Proof. We prove the claim for $D(\mathbf{RMod})$. The other cases are proved similarly. By lemma 3.2.6 it suffices to show that the class of quasi-isomorphisms is locally small, see definition 3.2.5. In the following we use results about cardinals. For introduction to cardinals see [Jec13, Chapter 3].

Let R be a ring and consider an object $X^\bullet \in \mathbf{RMod}$. Fix a cardinal k which is greater than the cardinality of R and any of X^i. We call a complex $Y^\bullet$ of R-modules k-petite, if $|Y^i| < k$ for all i. One can see that the class of isomorphism classes of k-petite complexes is a set. Therefore, for the complex $X^\bullet$, we can choose the set $S_{X^\bullet}$ by axiom of choice by picking one morphism from each of the isomorphism classes of quasi-isomorphisms $s : Y^\bullet \to X^\bullet$ with codomain $X^\bullet$, such that $Y^\bullet$ is k-petite.

Now, given a quasi-isomorphism $Z^\bullet \to X^\bullet$, it suffices to show that $Z^\bullet$ contains a k-petite subcomplex $W^\bullet$ quasi-isomorphic to $X^\bullet$. Clearly $|H^i(X^\bullet)| < k$ for all $i \in \mathbb{Z}$. For any $i \in \mathbb{Z}$ and any $x_j \in H^i(X^\bullet)$ choose an element $z^i_j \in Z^i$ such that the image of z^i_j in $H^i(X^\bullet)$ is x_j. Let $W^\bullet_0$ be the complex where each W^i is a submodule of Z^i generated by all the objects z^i_j. Then $|W^i| < k$ for each i and with the induced differential, this is a subcomplex of $X^\bullet$ such that $f_0 : H^i(W^\bullet) \to H^i(X^\bullet)$ is surjective for all i.

By induction we can enlarge $W^\bullet_n$ to a subcomplex $W^\bullet_{n+1}$ such that kernel of $f_n : H^\bullet(W^\bullet_n) \to H^\bullet(X^\bullet)$ vanishes in $H^\bullet(W^\bullet_{1+1})$. Indeed, for all $w^i_{n,j} \in (ker f_n)^i$, choose an element $z^{i-1}_{n,j} \in Z^{i-1}$ such that $d^i_{Z^\bullet}(z^{i-1}_{n,j}) = w^i_{n,j}$. Let W^i_{n+1} be the submodule of Z^i generated by the elements of W^i_n and the elements $w^i_{n,j}$. With the induced differential it is easy to see that $W^\bullet_{n+1}$ is a k-petite subcomplex of $Z^\bullet$ such that $f_{n+1} : H^\bullet(W^\bullet_{n+1}) \to H^\bullet(X^\bullet)$ is surjective and the kernel of f_n vanishes on $H^\bullet(W^\bullet_{n+1})$. The union $W^\bullet = \cup_n W^\bullet_n$ is a subcomplex of $Z^\bullet$ and we have

$$H^\bullet(W^\bullet) \cong \varinjlim_n H^\bullet(W^\bullet_n) \cong H^\bullet(X^\bullet).$$

This can be seen by noting that if $w \in H^\bullet(W^\bullet)$ is mapped to 0 in $H^\bullet(X^\bullet)$, then $w \in H^\bullet(W^\bullet_n)$ for some n and by construction w vanishes in $H^\bullet(W^\bullet_{n+1})$. Thus by definition of direct limit (colimit), we have that w represent the zero element in $H^\bullet(W^\bullet)$. This completes the proof, because the choice of $X^\bullet$ and the quasi-isomorphism $Z^\bullet \to X^\bullet$ were arbitrary. ☐

In the following we study the relationship of the underlying abelian category and the corresponding derived category. The following proposition shows that the derived category contains a subcategory which is equivalent to the original abelian category. We use notation $A[i]$ to denote the complex such that $(A[i])^j = 0$ for $i \neq j$ and $(A[i])^i = A$.

Proposition 5.1.10. *Let $\mathcal{A}$ be an abelian category, $D^*(\mathcal{A})$, $* = \varnothing, +, -, b$, the derived category of $\mathcal{A}$, and $F : \mathcal{A} \to K^*(\mathcal{A})$ the inclusion functor which maps A to $A[0]$. Then the composite functor $G := (Q \circ F) : \mathcal{A} \to D^*(\mathcal{A})$ induces an equivalence of categories between $\mathcal{A}$ and the full subcategory $\mathcal{C}$ of $D^*(\mathcal{A})$, consisting of complexes $X^\bullet \in \operatorname{Ob} D^*(\mathcal{A})$ such that $H^i(X^\bullet) = 0$ for $i \neq 0$.*

Proof. First, let us show that the functor $F : \mathcal{A} \to K^*(\mathcal{A})$, which sends an object A to the complex $A[0]$, is fully faithful. Indeed, clearly the only homotopy between $A[0]$ and $B[0]$ is the zero homotopy, i.e., all the $\chi^i : (A[0])^i \to (B[0])^{i-1}$ are 0 morphisms. This implies that F is faithful. The fact that F is full follows from the fact that any morphism ϕ from $A[0]$ to $B[0]$ is determined by ϕ^0. Thus F is fully faithful.

By Theorem 1.1.10 it suffices to show that $Q \circ F$ is fully faithful.

$Q \circ F$ **fully faithful:** To show that $Q \circ F$ is fully faithful, fix two objects $A, B \in \mathcal{A}$. We have to show that the following map

$$(Q \circ F) : \operatorname{Mor}_{\mathcal{A}}(A, B) \to \operatorname{Mor}_{D*(\mathcal{A})}(A[0], B[0])$$

is bijective. We show this by constructing an inverse for this map. Clearly $H^0(A[0]) \cong A$ and $H^0(B[0]) \cong B$. Let $\phi_1 : H^0(A[0]) \to A$ and $\phi_2 : H^0(B[0]) \to B$ be isomorphisms. We claim that the map $\psi := \phi_2 \circ H^0(-) \circ \phi_1^{-1}$ is the inverse of the above map. Let $f : A \to B$ be a morphism. To see that $(\psi \circ (Q \circ F))(f) = f$, note that we can choose $\ker d^0_{A[0]} = A$ and $\ker d^0_{B[0]} = B$ so that the cohomology functor is given by the following commutative diagram

$$\begin{tikzcd}
H^0(A[0]) \arrow[rr, "H^0(Q(F(f)))=H^0(F(f))"] & & H^0(B[0]) \\
A \arrow[u, "\phi_1^{-1}"] \arrow[rr, "F(f)"] \arrow[d, equal] & & B \arrow[u, "\phi_2^{-1}"'] \arrow[d, equal] \\
A[0]^0 = A \arrow[rr, "f"] & & B[0]^0 = B
\end{tikzcd}$$

Here we have abused notation and written H^0 for the cohomology functor for both $K^*(\mathcal{A}) \to \mathcal{A}$ and $D^*(\mathcal{A}) \to \mathcal{A}$. Recall that the cohomology functor factors through $D^*(\mathcal{A})$ so that we indeed have $H^0 \circ Q = H^0$.

It remains to show that $((Q \circ F) \circ \psi)(g) = g$, where g is a morphism of $\mathcal{C}$ represented by the following roof

$$\begin{tikzcd}
 & Z^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
A[0] & & B[0]
\end{tikzcd} \tag{5.11}$$

A direct computation shows that the morphism $((Q \circ F) \circ \psi)(g)$ is represented by the following roof

$$\begin{tikzcd}
 & A[0] \arrow[dl, "\mathrm{Id}_{A[0]}"'] \arrow[dr, "\phi_2 H^0(f) H^0(s)^{-1} \phi_1^{-1}"] & \\
A[0] & & B[0]
\end{tikzcd} \tag{5.12}$$

We show that this roof represents the same morphism as the one above. Let $V^\bullet$ be the complex defined as follows

$$V^i = \begin{cases} Z^i & ,i < 0 \\ \ker d^0_{Z^\bullet} & ,i = 0 \\ 0 & ,i > 0 \end{cases} \qquad d^i_{V^\bullet} = \begin{cases} d^i_{Z^\bullet} & ,i < -1 \\ \beta & ,i = -1 \\ 0 & ,i > -1 \end{cases} \tag{5.13}$$

where β is the unique morphism such that the following diagram commutes

$$\begin{tikzcd}
Z^{-1} \arrow[r, "d^{-1}_{Z^\bullet}"] \arrow[dr, "\beta"] & Z^0 \\
 & \ker d^0_{Z^\bullet} \arrow[u, "\alpha"']
\end{tikzcd} \tag{5.14}$$

Let $i : V^\bullet \to Z^\bullet$ be the natural inclusion morphism, where $i^0 = \alpha$ and let $h : V^\bullet \to A[0]$ be the natural morphism $h^0 := \ker s^0 : \ker d^0_{Z^\bullet} \to \ker d^0_{A[0]} = A[0]^0 = A$. Both morphisms i and h are easily seen to be quasi-isomorphisms. Then the following commutative diagram shows the equivalence of the roofs (5.11) and (5.12).

$$\begin{tikzcd}
 & & V^\bullet \arrow[dl, "i"'] \arrow[dr, "h"] & & \\
 & Z^\bullet \arrow[dl, "s"'] \arrow[drrr, "f"] & & A[0] \arrow[dlll, "\mathrm{Id}_{A[0]}"] \arrow[dr, "\phi_2 H^0(f) H^0(s)^{-1}\phi_1^{-1}"] & \\
A[0] & & & & B[0]
\end{tikzcd}$$

Indeed, commutativity follows from commutativity of the following diagram

$$\begin{tikzcd}
B & H^0(B[0]) \arrow[l, "\phi_2"'] & H^0(Z^\bullet) \arrow[l, "H^0(f)"'] \arrow[r, "H^0(s)"] & H^0(A[0]) \arrow[r, "\phi_1"] & A \\
B \arrow[u, equal] \arrow[r, equal] & \ker d^0_{B[0]} = B \arrow[u, equal, "\phi_2^{-1}"] & \ker d^0_{Z^\bullet} \arrow[l, "\ker f^0"'] \arrow[r, "\ker s^0"] \arrow[u] \arrow[d, "\alpha"] & \ker d^0_{A[0]} = A \arrow[u, "\phi_1^{-1}"] \arrow[r, equal] & A \arrow[u, equal] \\
B \arrow[u, equal] \arrow[r, equal] & B[0]^0 = B \arrow[u, equal] & Z^0 \arrow[l, "f^0"'] \arrow[r, "s^0"] & A[0]^0 = A \arrow[u, equal] \arrow[r, equal] & A \arrow[u, equal]
\end{tikzcd}$$

$Q \circ F$ **is essentially surjective:** Let $Z^\bullet \in \mathrm{Ob}\,\mathcal{C}$ be any object and let $V^\bullet$ be the complex (5.13). Denote by $i : V^\bullet \to Z^\bullet$ the natural inclusion morphism, which is a quasi-isomorphism, and let $h : V^\bullet \to (H^0(V^\bullet))[0]$ be the morphism given by $h^0 : V^0 = \ker d^0_{V^\bullet} \to \ker d^0_{(H^0(V^\bullet))[0]} = H^0(V^\bullet)$, given by diagram of the form (5.14). This morphism is clearly a quasi-isomorphism. Then

$$ih^{-1} : H^0(V^\bullet)[0] \to Z^\bullet$$

is an isomorphism in $D^*(\mathcal{A})$. This shows that $Q \circ F$ is essentially surjective and completes the proof.

□

In proposition 4.2.4 we have seen that the distinguished triangles in $K^*(\mathcal{A})$ come from semi-split short exact sequences. In the case of derived categories, the distinguished triangles come from all short exact sequences.

Proposition 5.1.11. *Given a short exact sequence*

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0 \tag{5.15}$$

in $C^*(\mathcal{A})$, there exists a morphism $h : Z^\bullet \to X^\bullet[1]$ such that

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X^\bullet[1]$$

is a distinguished triangle in $D^*(\mathcal{A})$. Conversely, any distinguished triangle in $D^*(\mathcal{A})$ is isomorphic to a distinguished triangle $(X^\bullet, Y^\bullet, Z^\bullet, f, g, h)$ in $D^*(\mathcal{A})$ such that

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

is an exact sequence in $C^*(\mathcal{A})$.

Proof. By lemma 5.1.2, every short exact sequence of the form (5.15) in $C^*(\mathcal{A})$ is quasi-isomorphic to a short exact sequence

$$0 \longrightarrow X^\bullet \xrightarrow{i_1} Cyl(f) \xrightarrow{p_2} C(f) \longrightarrow 0 \tag{5.16}$$

and by definition $(X^\bullet, Cyl(f), C(f), i_1, p_2, p_1)$ is a distinguished triangle in $D^*(\mathcal{A})$.

Conversely, by definition, the distinguished triangles of $D^*(\mathcal{A})$ are isomorphic to $(X^\bullet, Cyl(f), C(f), i_1, p_2, p_1)$. Now (5.16) is exact in $C^*(\mathcal{A})$. This finishes the proof. □

5.2 Examples

As an example we study the derived category of finite dimensional vector spaces over a field.

Example 5.2.1 ($D(\mathbf{kvect})$)**.** Let **kvect** be the category of finite dimensional vector spaces over a field k. It is well known to be a semisimple abelian category. This implies that for any morphism $f : V_1 \to V_2$ of finite dimensional k-vector spaces we have $V_1 \cong \ker(f) \oplus \operatorname{Im}(f)$ and $V_2 \cong V_2/\operatorname{Im}(f) \oplus \operatorname{Im}(f)$. Hence any complex $X^\bullet$ in $C(\mathbf{kvect})$ is isomorphic to a complex of the form

$$Y^\bullet : \ldots \stackrel{i_1p_3}{\longrightarrow} \operatorname{Im} d_{X^\bullet}^{i-2} \oplus (\ker d_{X^\bullet}^{i-1}/\operatorname{Im} d_{X^\bullet}^{i-1}) \oplus \operatorname{Im} d_{X^\bullet}^{i-1} \stackrel{i_1p_3}{\longrightarrow} \operatorname{Im} d_{X^\bullet}^{i-1} \oplus (\ker d_{X^\bullet}^{i}/\operatorname{Im} d_{X^\bullet}^{i-1}) \oplus \operatorname{Im} d_{X^\bullet}^{i} \stackrel{i_1p_3}{\longrightarrow} \ldots$$

Consider the complex

$$Z^\bullet : \ldots \stackrel{0}{\longrightarrow} (\ker d_{X^\bullet}^{i-1}/\operatorname{Im} d_{X^\bullet}^{i-1}) \stackrel{0}{\longrightarrow} (\ker d_{X^\bullet}^{i}/\operatorname{Im} d_{X^\bullet}^{i-1}) \stackrel{0}{\longrightarrow} \ldots$$

We show that the morphism $p_2 : Y^\bullet \to Z^\bullet$ is an isomorphism in $K(\mathbb{Q}\text{-}\mathbf{vect})$ with inverse given by $i_2 : Z^\bullet \to Y^\bullet$. By definition of biproduct $p_2 i_2 = \mathrm{Id}_{Z^\bullet}$. We show that the morphism $i_2p_2 : Y^\bullet \to Y^\bullet$ is homotopic to $\mathrm{Id}_{Y^\bullet}$. Let $\chi^i : Y^i \to Y^{i-1}$ be the morphism $\chi^i = i_3p_1$. Then $\mathrm{Id}_{Y^i} - i_2p_2 = i_1p_1 + i_3p_3 = i_1p_3i_3p_1 + i_3p_1i_1p_3$, which shows that $\mathrm{Id}_{Y^\bullet} \sim i_2p_2$. Thus $Y^\bullet$, and hence $X^\bullet$, is isomorphic to the complex $Z^\bullet$ in $K(\mathbf{kvect})$. Since $H^\bullet(Z^\bullet) \cong Z^\bullet$ it is easy to see that the all the quasi-isomorphisms are already invertible in $K(\mathbf{kvect})$. Thus $K(\mathbf{kvect}) \cong D(\mathbf{kvect})$, and $D(\mathbf{kvect})$ is equivalent to the full abelian subcategory of $C(\mathbf{kvect})$ consisting of complexes with zero differentials.

In the above example the derived category turned out to be an abelian category. This is because **kvect** is semisimple. Furthermore, the derived category $D(\mathcal{A})$ is an abelian category if an only if $\mathcal{A}$ is semisimple, [GM03, Exercise IV.1.1]. We have already seen in example 2.2.17 that **Ab** is not semisimple. Thus the derived category of an abelian category is not in general an abelian category.

The idea following example is taken from [HTR10, p.191 4.15]. In particular, it gives an example where localization of a category is not a category.

Example 5.2.2 (Derived category not a category)**.** In this example we show that the derived category of an abelian category is not necessarily a category because the collection of morphisms may fail to be a set. This also shows that localization of categories is not well-defined in general, if one does not use the axiom of strongly inaccessible cardinals.

Let $\mathscr{U}$ denote the class of all small cardinals and let $\mathcal{A}$ be the category of all finite dimensional $(\mathbb{Z}/2)[\mathscr{U}]$-$\mathbb{Z}/2$-bimodules, where the dimension is taken as $\mathbb{Z}/2$ vector space. One can verify that this category is an abelian category. We show that $D(\mathcal{A})$ is not a category by deriving a contradiction when assuming its existence.

Suppose that $D(\mathcal{A})$ is a category. For any small cardinal λ, let V_λ be the $(\mathbb{Z}/2)[\mathscr{U}]$-$\mathbb{Z}/2$-bimodule $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ with the action

$$\alpha.(z_1, z_2) = \begin{cases} (z_2, 0) & \alpha = \lambda \\ (0, 0) & \alpha \neq \lambda \end{cases}$$

Consider the following two morphisms in $C(\mathcal{A})$

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "i_1"] \arrow[d] & V_\lambda \arrow[r] \arrow[d, "p_2"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "i_1"] \arrow[d, "\mathrm{Id}_{\mathbb{Z}/2}"] & V_\lambda \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

where the morphisms i_1 are differentials at index 0, the first morphism is a quasi-isomorphism and the bimodules $\mathbb{Z}/2$ have trivial action of $(\mathbb{Z}/2)[\mathscr{U}]$. Denote the first morphism by s_λ and the second by f_λ. Denote the domain complex of s_λ by $Z^\bullet_\lambda$, the codomain complex by $X^\bullet$ and the codomain complex of the morphism f_λ by $Y^\bullet = X^\bullet[1]$.

Consider morphisms from $X^\bullet$ to $Y^\bullet$. For any small cardinal λ the following roof represents such a morphism

$$\begin{tikzcd}
& Z^\bullet_\lambda \arrow[dl, "s_\lambda"'] \arrow[dr, "f_\lambda"] & \\
X^\bullet & & Y^\bullet
\end{tikzcd}$$

We show that for two different cardinals $\alpha \neq \lambda$ the corresponding roofs represent different morphisms, so that $\mathrm{Mor}_{D(\mathcal{A})}(X^\bullet, Y^\bullet)$ cannot be a set, because the class of all small cardinals is not a set.

Let $\alpha \neq \lambda$ be different small cardinals. If the corresponding roofs would be equal, we would have a commutative diagram of the form in $K(\mathcal{A})$

$$\begin{tikzcd}
& & W^\bullet \arrow[dl, "s'_\lambda"'] \arrow[dr, "s'_\alpha"] & & \\
& Z^\bullet_\lambda \arrow[dl, "s_\lambda"'] \arrow[drrr, "f_\lambda"] & & Z^\bullet_\alpha \arrow[dlll, "s_\alpha"'] \arrow[dr, "f_\alpha"] & \\
X^\bullet & & & & Y^\bullet
\end{tikzcd}$$

Now $s_\lambda s'_\lambda : W^\bullet \to X^\bullet$ is a quasi-isomorphism, and $H^0(X^\bullet) = \mathbb{Z}/2$, so W^0 must be nonzero. By definition of a morphism of $(\mathbb{Z}/2)[\mathscr{U}]$-$\mathbb{Z}/2$-bimodules, for a nonzero element $w \in W^0$ we have

$$(s'_\lambda)^0(\lambda.(w)) = \lambda.((s'_\lambda)^0(w)) \neq 0$$
$$(s'_\alpha)^0(\lambda.(w)) = \lambda.((s'_\alpha)^0(w)) = 0.$$

This shows that the image of $\lambda.w$ under $H^0(s_\lambda s'_\lambda) : H^0(W^\bullet) \to H^0(X^\bullet)$ is zero, but the image of $\lambda.w$ under $H^0(s_\alpha s'_\alpha) : H^0(W^\bullet) \to H^0(X^\bullet)$ is nonzero. Hence the diagram cannot be commutative in $K(\mathcal{A})$.

5.3 Notes

Derived categories were initially developed by Grothendieck and Verdier to generalize Serre duality to relative case. In this it was needed that the direct image functor f_* to have a right adjoint, which is impossible in the category of schemes over a field k, because f_* is not right exact. When one passes to the derived category, the functor f_* induces a derived functor between derived categories, see the next chapter, which preserves distinguished triangles and this functor has a right adjoint.

We have already noted that triangulated categories arise in many branches of mathematics, see section 4.4. This allows one to study connections between different branches of mathematics, by using the formalism of triangulated categories. In some sense, homological mirror symmetry can be seen to be an example of such connection.

Chapter 6

Derived functors

In this chapter we introduce right and left derived functors. We prove the existence of right derived functors and state the corresponding results for left derived functors. A right (resp. left) derived functor is an exact functor between derived categories which satisfies a certain universal property and is obtained from a left (resp. right) exact functor between the underlying abelian categories. This construction agrees with the classical derived functors. At the end of this chapter we give examples of derived functors.

Classically, to form the ith right derived functor R^iF (resp. left derived functor L^iF) of a left (resp. right) exact functor $F : \mathcal{A} \to \mathcal{B}$, the idea is to assign an injective (resp. projective) resolution to each object of $\mathcal{A}$, apply the functor F pointwise to this resolution, and then take the ith cohomology of the resulting complex. Here we generalize this idea and construct the right (resp. left) derived functor by using an adapted class $\mathcal{R}$ of objects of $\mathcal{A}$, with respect to the functor F, and then we take RF (resp. LF) to be the composite of an inverse of $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^+(\mathcal{A})$ (resp. $K^-(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^-(\mathcal{A})$) given by the universal property of localization of categories, followed by the unique functor $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^+(\mathcal{B})$ (resp. $K^-(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^-(\mathcal{B})$) given by the universal property of localization of categories.

$$\begin{tikzcd}
K^+(\mathcal{A}) \arrow[d, "Q_{\mathcal{A}}"] & K^+(\mathcal{R}) \arrow[l, "K^+(I)"'] \arrow[r, "K^+(F\circ I)"] \arrow[d, "Q_{\mathcal{R}}"] & K^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
D^+(\mathcal{A}) \arrow[rr, bend right, "RF"'] & K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \arrow[l] \arrow[r] & D^+(\mathcal{B})
\end{tikzcd}
\quad \left(\text{resp.} \quad
\begin{tikzcd}
K^-(\mathcal{A}) \arrow[d, "Q_{\mathcal{A}}"] & K^-(\mathcal{R}) \arrow[l, "K^-(I)"'] \arrow[r, "K^-(F\circ I)"] \arrow[d, "Q_{\mathcal{R}}"] & K^-(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
D^-(\mathcal{A}) \arrow[rr, bend right, "LF"'] & K^-(\mathcal{R})[S_{\mathcal{R}}^{-1}] \arrow[l] \arrow[r] & D^-(\mathcal{B})
\end{tikzcd} \right)$$

Note that in this chapter, as in the diagrams above, we occasionally abuse notation and identify the categories $K^*(\mathcal{A})[S^{-1}]$ and $D^*(\mathcal{A})$, $* = \emptyset, +, -, b$, where $\mathcal{A}$ is an abelian category and S is the class of quasi-isomorphisms in $K^*(\mathcal{A})$. This identification is justified by the fact that it is easier to manipulate morphisms by using roofs and coroofs than strings of morphisms, and that these categories are isomorphic by theorem 5.1.4 and all the properties we are interested in are true in isomorphic categories. If one is not satisfied with this approach, one can add the isomorphism G, or its inverse G^{-1}, of theorem 5.1.4 to appropriate places. In particular in most of the places where one might want add G or G^{-1}, the resulting functors are uniquely determined by theorem 3.1.3.

6.1 Construction of derived functors

We have defined exact sequences, but we have not defined exact (resp. left exact, resp. right exact) functors. These functors are important because one can view derived functors as a kind of machinery to measure how much a left (resp. right) exact functor fails to be exact. Let us define these.

Definition 6.1.1 (Exact functor)**.** Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. If for any short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{6.1}$$

in $\mathcal{A}$, the sequence

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is exact, then F is said to be *exact.* Similarly, if for any short exact sequence (6.1) the following sequence is exact

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

then F is *left exact.* Dually, if for any short exact sequence (6.1) the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

is exact, then F is *right exact.*

Consider an additive functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories. This induces a functor $C^*(F)$, $* = \varnothing, +, -, b$, between the abelian categories $C^*(\mathcal{A})$ and $C^*(\mathcal{B})$, by pointwise application of F. The functor $C^*(F)$ maps homotopies to homotopies. Indeed, if $f - g = d_{Y^\bullet}\chi + \chi d_{X^\bullet}$, then

$$C^*(F)(f - g) = (F(d^{i-1}_{Y^\bullet})F(\chi^i) + F(\chi^{i+1})F(d^i_{X^\bullet}))_{i\in\mathbb{Z}}.$$

Thus it induces an additive functor $K^*(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B})$.

We say that a complex in $C^*(\mathcal{A})$ (or $K^*(\mathcal{A})$) is *exact*, if the cohomology complex is the zero complex.

Definition 6.1.2 (Adapted class)**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left (resp. right) exact functor between abelian categories. A class of objects $\mathcal{R} \subset \operatorname{Ob}\mathcal{A}$ containing the zero object and stable under biproducts is said to be *adapted* to F if any object A of $\mathcal{A}$ is a subobject (resp. quotient) of an object of $\mathcal{R}$ and $C^+(F)$ (resp. $C^-(F)$) preserves exact complexes from $C^+(\mathcal{R})$ (resp. $C^-(\mathcal{R})$).

More precisely, a class of objects $\mathcal{R}$ is adapted to a left exact functor when it contains the zero object and satisfies the following properties

AC 1 For any $R_1, R_2 \in \mathcal{R}$, we have $R_1 \oplus R_2 \in \mathcal{R}$.

AC 2 For any object $X \in \mathcal{A}$, there exists a monomorphism $X \to R$ with $R \in \mathcal{R}$.

AC 3 For any exact complex $R^\bullet \in C^+(\mathcal{R})$, $C^+(F)(R^\bullet)$ is an exact complex.

By abuse of notation, we will write $\mathcal{R}$ also for the full subcategory of $\mathcal{A}$ consisting of the objects of $\mathcal{R}$. Also, the category $K^*(\mathcal{R})$ is the full subcategory of $K^*(\mathcal{A})$ consisting of all complexes $X^\bullet$ such that $X^i \in \mathcal{R}$ for all i.

Lemma 6.1.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, $\mathcal{R}$ a class of objects adapted to F, and $S_\mathcal{R}$ the class of quasi-isomorphisms in $K^+(\mathcal{R})$. Then $S_\mathcal{R}$ is a localizing class of morphisms and the category $K^+(\mathcal{R})$ is a triangulated category.*

Proof. By corollary 5.1.7 the class of quasi-isomorphisms is a localizing class in $K^+(\mathcal{A})$. The same proof works for $K^+(R)$ because $\mathcal{R}$ is stable under direct sums, so mapping cones of any morphism is contained in $K^+(\mathcal{R})$. The inclusion $K^+(\mathcal{R}) \to K^+(\mathcal{A})$ induces a triangulated category structure on $K^+(\mathcal{R})$ and one can easily verify the axioms of triangulated category for $K^+(\mathcal{R})$ using the axioms for $K^+(\mathcal{A})$ and corollary 4.1.6. ☐

We also have the dual version.

Lemma 6.1.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories, $\mathcal{R}$ a class of objects adapted to F, and $S_{\mathcal{R}}$ the class of quasi-isomorphisms in $K^-(\mathcal{R})$. Then $S_{\mathcal{R}}$ is a localizing class and $K^-(\mathcal{R})$ is a triangulated category.*

Lemma 6.1.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive exact functor between abelian categories. Then $C^*(F)$ (resp. $K^*(F)$), $* = \varnothing, +, -, b$, preserves exact complexes.*

Proof. Let $X^\bullet$ be an exact complex in $C^*(\mathcal{A})$. The functor C^*F maps the following diagram of lemma 2.2.15

$$\begin{tikzcd}
X^{i-1} \arrow[rr, "d^{i-1}_{X^\bullet}"] \arrow[dr, "e^{i-1}"] & & X^i \arrow[rr, "d^i_{X^\bullet}"] \arrow[dr, "e^i"] & & X^{i+1} \\
0 \arrow[r] & \ker d^i_{X^\bullet} \arrow[ur, "m^{i-1}"] & & \ker d^{i+1}_{X^\bullet} \arrow[ur, "m^i"] \arrow[r] & 0
\end{tikzcd}$$

to the following diagram

$$\begin{tikzcd}
F(X^{i-1}) \arrow[rr, "F(d^{i-1}_{X^\bullet})"] \arrow[dr, "F(e^{i-1})"] & & F(X^i) \arrow[rr, "F(d^i_{X^\bullet})"] \arrow[dr, "F(e^i)"] & & X^{i+1} \\
0 \arrow[r] & F(\ker d^i_{X^\bullet}) \arrow[ur, "F(m^{i-1})"] & & F(\ker d^{i+1}_{X^\bullet}) \arrow[ur, "F(m^i)"] \arrow[r] & 0
\end{tikzcd}$$

By lemma 2.2.15 it suffices to show that $(F(m^{i-1}), F(e^i))$ is an exact sequence. But this follows from the fact that F is exact. ☐

Lemma 6.1.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories and $\mathcal{R}$ a class of objects adapted to F. For any complex $X^\bullet \in K^+(\mathcal{A})$ there is a quasi-isomorphism $t : X^\bullet \to R^\bullet$ with $R^\bullet \in K^+(\mathcal{R})$.*

Proof. **Construction of $R^\bullet$ and t:** By application of the translation functor for $X^\bullet$ we may assume that $X^\bullet$ is of the form

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^0 \xrightarrow{d^0_{X^\bullet}} X^1 \xrightarrow{d^1_{X^\bullet}} X^2 \xrightarrow{d^2_{X^\bullet}} \ldots$$

By assumption, we can find a monomorphism $t^0 : X^0 \to R^0$. Take the pushout of $d^0_{X^\bullet}$ and t^0 to get the following commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, "d^0_{X^\bullet}"] \arrow[d, "t^0"] & X^1 \arrow[d, "j^0"] \\
R^0 \arrow[r, "i^0"] & Y^0
\end{tikzcd}$$

where i^0 is a monomorphism by corollary 2.3.5. Choose a monomorphism $k^0 : Y^0 \to R^1$, by AC2, and define $d^0_{R^\bullet} = k^0 i^0$ and $t^1 = k^0 j^0$. We have obtained the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X^0 \arrow[r, "d^0_{X^\bullet}"] \arrow[d, "t^0"] & X^1 \arrow[d, "t^1"] \\
0 \arrow[r] & R^0 \arrow[r, "d^0_{R^\bullet}"] & R^1
\end{tikzcd}$$

Suppose that we have already chosen the objects $R^0, \dots, R^{n-1}$ and differentials between them. Then we can obtain the following commutative diagram

$$\begin{tikzcd}
X^{n-1} \arrow[rrr, "d_{X^\bullet}^{n-1}"] \arrow[d, "t^{n-1}"] & & & X^n \arrow[d, "j^{n-1}"] \\
R^{n-1} \arrow[r, "a"] & \operatorname{coker} d_{R^\bullet}^{n-2} \arrow[rr, "b"] & & Y^{n-1} \arrow[r, "k^{n-1}"] & R^n
\end{tikzcd} \tag{6.2}$$

by taking the pushout Y^{n-1} of the morphisms at^{n-1} and $d_{X^\bullet}^{n-1}$ and then the monomorphism $k^{n-1} : Y^{n-1} \to R^n$, by AC2, for some object R^n of $\mathcal{R}$. Let $d_{R^\bullet}^{n-1} = k^{n-1}ba$ and $t^n = k^{n-1}j^{n-1}$. By induction we have obtained a complex $R^\bullet$ and a morphism of complexes $t : X^\bullet \to R^\bullet$.

$H^i(t)$ **is an isomorphism:** Since $\mathcal{A}$ is an abelian category, to show that $H^i(t)$ are isomorphisms for all i, it suffices to check that $H^i(t)$ are monomorphisms and epimorphisms. Recall from proposition 2.5.5 that the pseudo-elements of $H^i(X^\bullet)$ are in one-to-one correspondence with the equivalence classes of pseudo-elements of X^i which are sent to the zero-morphism by $d_{X^\bullet}^i$ modulo the relation $x_1 : Y_1 \to X^i \sim x_2 : Y_2 \to X^i$ if and only if there exists epimorphisms $h_1 : V \to Y_1$ and $h_2 : V \to Y_2$ and a morphism $k : V \to X^{i-1}$ such that $x_1h_1 - x_2h_2 = d_{X^\bullet}^i k$.

$H^i(t)$ **is an epimorphism:** Let us use proposition 2.5.5 to show that $H^i(t)$ is an epimorphism. Let $r \in^* R^i$ such that $d_{R^\bullet}^i h = 0$. Since k^i is a monomorphism, by proposition 2.4.3 (ii) we get $(ci_1a)(r) = 0$, where c is the morphism $\operatorname{coker} d_{R^\bullet}^{i-1} \oplus X^i \to Y^i$ like in corollary 2.3.4. By corollary 2.3.4 the following sequence is exact

$$X^i \xrightarrow{i_1at^i - i_2d_{X^\bullet}^i} \operatorname{coker} d_{R^\bullet}^{i-1} \oplus X^{i+1} \xrightarrow{\quad c \quad} Y^i \xrightarrow{\qquad} 0 \tag{6.3}$$

By proposition 2.4.3 (iv) there exists a pseudo-element $x \in^* X^i$ such that $(i_1at^i - i_2d_{X^\bullet}^i)(x) =^* (i_1a)(r)$. Hence $at^ixv = aru$ for some epimorphisms $u : A \to X^i$ and $v : A \to R^i$. We have the following diagram

$$\begin{tikzcd}
 & A \arrow[d, shift right, "ru"'] \arrow[d, shift left, "t^ixv"] & \\
R^{i-1} \arrow[r, "d_{R^\bullet}^{i-1}"] & R^i \arrow[r, "a"] & \operatorname{coker} d_{R^\bullet}^{i-1}
\end{tikzcd}$$

The row is exact by proposition 2.2.14, so by proposition 2.4.3 (iv) there exists a pseudo-element $r' \in^* R^{i-1}$ such that $d_{R^\bullet}^{i-1} r' =^* t^ixv - ru$. Therefore for some epimorphisms w_1 and w_2 we have $d_{R^\bullet}^{i-1} r^\bullet w_1 = t^ixvw_2 - ruw_2$. By the equivalence relation of proposition 2.5.5 this means that t^ix and r represent the same pseudo-element in $H^i(R^\bullet)$. This shows by proposition 2.4.3 (iii) that $H^i(R^\bullet)$ is an epimorphism.

$H^i(t)$ **is a monomorphism:** We use proposition 2.5.5. By proposition 2.4.3 (ii) it suffices to show that a pseudo-element of $H^i(X^\bullet)$ which is mapped by $H^i(t)$ to $0 \in^* H^i(R^\bullet)$ is pseudo-equal to 0. Let $x \in^* X^i$ correspond to a pseudo-element $x' \in^* H^i(X^\bullet)$ such that $H^i(t)(x') =^* 0$. By commutativity of the following diagram

$$\begin{tikzcd}
X^i \arrow[d, "t^i"] & \ker d_{X^\bullet}^{i-1} \arrow[l] \arrow[r] \arrow[d, "\ker t^i"] & H^i(X^\bullet) \arrow[d, "H^i(t)"] \\
R^i & \ker d_{R^\bullet}^{i-1} \arrow[l] \arrow[r] & H^i(R^\bullet)
\end{tikzcd}$$

and the correspondence of the pseudo-elements of R^i with $H^i(R^\bullet)$ given by proposition 2.5.5, we get that $t^i x \in^* R^i$ corresponds to the pseudo-element $0 \in^* H^i(R^\bullet)$. Therefore there exists an epimorphism v and morphism u such that $d^{i-1}_{R^\bullet} u = t^i x v$. Since the morphism k^{i-1} in (6.2) is a monomorphism, we have $ci_1 au = ci_2 xv$. By corollary 2.3.5 the pushout and pullback diagrams coincide, so there exists a unique morphism ϕ such that $d^{i-1}_{X^\bullet}\phi = xv$. Therefore x corresponds to $0 \in^* H^i(X^\bullet)$ by proposition 2.5.5. This completes the proof.

□

Similarly one proves similar lemma for right exact functors.

Lemma 6.1.7. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories and $\mathcal{R}$ a class of objects adapted to F. For any complex $X^\bullet \in K^+(\mathcal{A})$ there is a quasi-isomorphism $t : R^\bullet \to X^\bullet$ with $R^\bullet \in K^+(\mathcal{R})$.*

The following proposition is important in the construction of the right derived functor.

Proposition 6.1.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, $\mathcal{R}$ a class of objects adapted to F, and $S_{\mathcal{R}}$ the class of quasi-isomorphisms in $K^+(\mathcal{R})$. The canonical functor*

$$D^+(I_{S_{\mathcal{R}}}) : K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^+(\mathcal{A})$$

is an equivalence of categories and commutes with the translation functor.

Proof. By theorem 5.1.4 we have an isomorphism of categories $G^{-1} : K^+(\mathcal{A})[S^{-1}] \to D^+(\mathcal{A})$, where S is the class of quasi-isomorphisms in $K^+(\mathcal{A})$. We let $D^+(I_{S_{\mathcal{R}}}) = G^{-1} \circ D^+(I)$, where $D^+(I)$ is the unique morphism, obtained by theorem 3.1.3, such that the following diagram is commutative

$$\begin{tikzcd} K^+(\mathcal{R}) \arrow[r, "K^+(I)"] \arrow[d, "Q_{\mathcal{R}}"] & K^+(\mathcal{A}) \arrow[d, "Q_{\mathcal{A}}"] \\ K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \arrow[r, "D^+(I)"] & K^+(\mathcal{A})[S^{-1}] \end{tikzcd}$$

Thus it suffices to show that $D^+(I)$ is an equivalence of categories. By commutativity $D^+(I)$ is identity on objects and morphisms. By theorem 1.1.10 we have to show that $D^+(I)$ is fully faithful and essentially surjective. By lemma 6.1.6, any object of $K^+(\mathcal{A})$ is quasi-isomorphic to an object of $K^+(\mathcal{R})$. Thus $D^+(I)$ is essentially surjective. The lemma 6.1.6 show that the condition (ii) of proposition 3.2.7 holds, so $K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ is a full subcategory of $K^+(\mathcal{A})[S^{-1}]$. Hence the functor $D^+(I)$ is fully faithful.

The inclusion functor I commutes with the translation functor. We know that the localization functor $Q_{\mathcal{A}}$, and thus $Q_{S_{\mathcal{R}}}$, commute with the traslation functor. Therefore, the functor $D^+(I_{S_{\mathcal{R}}})$ commutes with the translation functor. □

The following is a version of the above proposition for right exact functors.

Proposition 6.1.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories, $\mathcal{R}$ a class of objects adapted to F, and $S_{\mathcal{R}}$ the class of quasi-isomorphisms in $K^-(\mathcal{R})$. The canonical functor*

$$D^-(I_{S_{\mathcal{R}}}) : K^-(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^-(\mathcal{A})$$

is an equivalence of categories and commutes with the translation functor.

The following shows that an inverse for the equivalence in proposition 6.1.8 also commutes with translations.

Corollary 6.1.10. *Let $D^+(I_{S_\mathcal{R}}) : K^+(\mathcal{R})[S_\mathcal{R}^{-1}] \to D^+(\mathcal{A})$ be the functor in proposition 6.1.8. There exists a functor $\Phi : D^+(\mathcal{A}) \to K^+(\mathcal{R})[S_\mathcal{R}^{-1}]$ such that*

$$\mathrm{Id}_{K^+(\mathcal{R})[S_\mathcal{R}^{-1}]} = \Phi \circ D^+(I_{S_\mathcal{R}}), \qquad D^+(I_{S_\mathcal{R}}) \circ \Phi \cong \mathrm{Id}_{D^+(\mathcal{A})}$$

and the functor Φ commutes with the translation functor.

Proof. First we construct Φ on objects. Let $\Phi(0^\bullet) = 0^\bullet$. For any object $A^\bullet \neq 0^\bullet$ of $D^+(\mathcal{A})$ such that $A^\bullet \notin \mathrm{Ob}\, K^+(\mathcal{R})[S_\mathcal{R}^{-1}]$, $A^0 \neq 0$ and $A^i = 0$ for $i < 0$, fix by lemma 6.1.6 a quasi-isomorphism $q_{A^\bullet} : A^\bullet \to R^\bullet$ to some object $R^\bullet \in K^+(\mathcal{R})[S_\mathcal{R}^{-1}]$. For any $R^\bullet \in K^+(\mathcal{R})[S_\mathcal{R}^{-1}]$, let $q_{R^\bullet} = \mathrm{Id}_{R^\bullet}$. In general, for an object $A^\bullet \neq 0^\bullet$ of $D^+(\mathcal{A})$, let $n \in \mathbb{Z}$ such that $A^n \neq 0$ and $A^i = 0$ for $i < n$. Then, we define $q_{A^\bullet} = q_{A^\bullet[-n]}[n]$.

For any object $A^\bullet \in D^+(\mathcal{A})$ define $\Phi(A^\bullet) = \mathrm{Cod}(q_{A^\bullet})$. We see that this definition implies that Φ commutes with the translation functors on objects. Indeed, let $A^\bullet \in D^+(\mathcal{A})$ and $n \in \mathbb{Z}$ such that $A^n \neq 0$ and $A^i = 0$ for $i < n$. Then

$$\Phi(A^\bullet[1]) = \mathrm{Cod}(q_{A^\bullet[1]}) = \mathrm{Cod}(q_{A^\bullet[1][-n-1]})[n+1] = \mathrm{Cod}(q_{A^\bullet[-n]})[n][1] = \Phi(A^\bullet)[1].$$

To define the map on morphisms of the functor Φ, let $f : A_1^\bullet \to A_2^\bullet$ be a morphism in $D^+(\mathcal{A})$. Let $\Phi(f) = q_{A_2^\bullet} f (q_{A_1^\bullet})^{-1}$. Since the subcategory $K^+(\mathcal{R})[S_\mathcal{R}^{-1}]$ is a full subcategory of $D^+(\mathcal{A})$ this is well defined. A simple computation shows that Φ is a functor such that $\mathrm{Id}_{K^+(\mathcal{R})[S_\mathcal{R}^{-1}]} = \Phi \circ D^+(I_{S_\mathcal{R}})$. To see that $\mathrm{Id}_{D^+(\mathcal{A})}$ is isomorphic to $D^+(I_{S_\mathcal{R}}) \circ \Phi$, let $\tau : \mathrm{Id}_{D^+(\mathcal{A})} \to D^+(I_{S_\mathcal{R}}) \circ \Phi$ be the natural transformation which sends an object $A^\bullet$ to $q_{A^\bullet}$. Then for any morphism $(s, f) : A_1^\bullet \to A_2^\bullet$ the following square commutes

$$\begin{array}{ccc} A_1^\bullet & \xrightarrow{q_{A_1^\bullet}} & \mathrm{Cod}(q_{A_1^\bullet}) \\ \Big\downarrow{\scriptstyle (s,f)} & & \Big\downarrow{\scriptstyle \Psi(s,f) = q_{A_2^\bullet} \circ (s,f) \circ (q_{A_1^\bullet})^{-1}} \\ A_2^\bullet & \xrightarrow{q_{A_2^\bullet}} & \mathrm{Cod}(q_{A_2^\bullet}) \end{array}$$

This shows that τ is an isomorphism of functors and finishes the proof. $\square$

A similar corollary holds for right exact functors.

Corollary 6.1.11. *Let $D^+(I_{S_\mathcal{R}}) : K^-(\mathcal{R})[S_\mathcal{R}^{-1}] \to D^-(\mathcal{A})$ be the functor in proposition 6.1.9. There exists a functor $\Phi : D^-(\mathcal{A}) \to K^-(\mathcal{R})[S_\mathcal{R}^{-1}]$ such that*

$$\mathrm{Id}_{K^-(\mathcal{R})[S_\mathcal{R}^{-1}]} = \Phi \circ D^-(I_{S_\mathcal{R}}), \qquad D^-(I_{S_\mathcal{R}}) \circ \Phi \cong \mathrm{Id}_{D^-(\mathcal{A})}$$

and the functor Φ commutes with the translation functor.

Lemma 6.1.12. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and $\mathcal{R}$ a full subcategory of $\mathcal{A}$ stable under biproducts such that the functor $K^*(F \circ I) = K^*(F) \circ K^*(I) : K^*(\mathcal{R}) \to K^*(\mathcal{A}) \to K^*(\mathcal{B})$ preserves exact complexes, where $I : \mathcal{R} \to \mathcal{A}$ is the inclusion functor. Then $K^*(F) \circ K^*(I)$ preserves mapping cones, mapping cylinders, and quasi-isomorphisms. Moreover, $F \circ I$ induces an exact functor $D^*(F \circ I) : K^*(\mathcal{R})[S_\mathcal{R}^{-1}] \to D^*(\mathcal{B})$.*

Proof. Let $f : X^\bullet \to Y^\bullet$ be a representative of a morphism of $K^*(\mathcal{R})$. Since $F \circ I$ is additive, by proposition 2.1.5 it preserves biproducts. A direct computation shows that

$$K^*(F \circ I)(d^i_{C(f)}) = i_1 (F \circ I)(d^{i+1}_{X^\bullet}) p_1 + i_2 (F \circ I)(f^{i+1}) p_1 + i_2 (F \circ I)(d^i_{Y^\bullet}) p_2$$

and

$$\begin{aligned}K^*(F\circ I)(C(f)) &= K^*(F\circ I)(X^\bullet[1]\oplus Y^\bullet)\\ &= (K^*(F\circ I)(X^\bullet[1]))\oplus(K^*(F\circ I)Y^\bullet)\\ &= (K^*(F)X^\bullet)[1]\oplus(K^*(F\circ I)Y^\bullet)\\ &= C(K^*(F\circ I)(f)),\end{aligned}$$

so $F\circ I$ preserves the mapping cones. Similarly, from

$$\begin{aligned}K^*(F\circ I)(d^i_{Cyl(f)}) = {} & i_1(F\circ I)(d^i_{X^\bullet})p_1 - i_1p_1p_2\\ & - i_2i_1(F\circ I)(d^{i+1}_{X^\bullet})p_1p_2 + i_2i_2(F\circ I)(f^{i+1})p_1p_2 + i_2i_2(F\circ I)(d^i_{Y^\bullet})p_2p_2.\end{aligned}$$

and

$$\begin{aligned}K^*(F\circ I)(Cyl(f)) &= K^*(F\circ I)(X^\bullet\oplus C(f))\\ &= (K^*(F\circ I)(X^\bullet))\oplus(K^*(F\circ I)C(f))\\ &= (K^*(F\circ I)(X^\bullet))\oplus C(K^*(F\circ I)(f))\\ &= Cyl(K^*(F\circ I)(f))\end{aligned}$$

we see that $K^*(F\circ I)$ preserves cylinders. To see that $K^*(F\circ I)$ preserves quasi-isomorhisms, let f be a quasi-isomorphism in $\mathcal{R}$. Then by lemma 6.1.6 applied in $K^*(\mathcal{A})$, $C(f)$ is an exact complex, so $K^*(F\circ I)(C(f)) = C(K^*(F\circ I)(f))$ is exact by assumption. Therefore lemma 6.1.6 shows that $K^*(F\circ I)(f)$ is a quasi-isomorphism.

Since $K^*(F\circ I)$ preserves quasi-isomorphisms, by theorem 3.1.3 we have the following commutative diagram

$$\begin{tikzcd} K^*(\mathcal{R}) \arrow[r, "K^*(F\circ I)"] \arrow[d, "Q_{S_\mathcal{R}}"] & K^*(\mathcal{B}) \arrow[d, "Q_\mathcal{B}"] \\ K^*(\mathcal{R})[S_\mathcal{R}^{-1}] \arrow[r, "D^*(F\circ I)"] & D^*(\mathcal{B}) \end{tikzcd} \tag{6.4}$$

where $S_\mathcal{R}$ is the class of quasi-isomorphisms in $K^*(\mathcal{R})$. The functor $K^*(F\circ I)$ preserves biproducts, cones and cylinders, so it maps any triangle of the form

$$X^\bullet \xrightarrow{i_1} Cyl(f) \xrightarrow{p_2} C(f) \xrightarrow{p_1} X^\bullet[1]$$

to a triangle of the same form in $K^*(\mathcal{B})$. This shows that $K^*(F\circ I)$ preserves distinguished triangles. Since the localizing functors $Q_{S_\mathcal{R}}$ and $Q_\mathcal{B}$ preserve distinguished triangles, by commutativity of (6.4) shows that $D^*(F\circ I)$ preserves distinguished triangles. Hence it is an exact functor. □

By lemma 6.1.5 $C^*(F)$ of an exact functor F preserves exact complexes. Hence, by above $D^*(F) : D^*(\mathcal{A}) \to D^*(\mathcal{B})$ is an exact functor.

We come now to the definition of derived functors.

Definition 6.1.13. The *right derived functor* of a left exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories is a pair consisting of an exact functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and a natural transformation $\epsilon_F : Q_\mathcal{B}\circ K^+(F) \to RF\circ Q_\mathcal{B}$

such that for any exact functor $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and any natural transformation $\epsilon_G : Q_{\mathcal{B}} \circ K^+(F) \to G \circ Q_{\mathcal{A}}$ there exists a unique natural transformation $\eta : RF \to G$ such that the following diagram

$$\begin{tikzcd} Q_{\mathcal{B}} \circ K^+(F) \arrow[r, "\epsilon_F"] \arrow[dr, "\epsilon_G"'] & RF \circ Q_{\mathcal{A}} \arrow[d, "\eta \circ Q_{\mathcal{A}}"] \\ & G \circ Q_{\mathcal{A}} \end{tikzcd}$$

is commutative.

The *left derived functor* of a right exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories is a pair consisting of an exact functor $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ and a morphism of functors $\epsilon_F : Q_{\mathcal{B}} \circ K^-(F) \to LF \circ Q_{\mathcal{A}}$ such that for any exact functor $G : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ and any natural transformation $\epsilon_G : Q_{\mathcal{B}} \circ K^-(F) \to G \circ Q_{\mathcal{A}}$ there exists a unique natural transformation $\eta : G \to LF$ such that the following diagram

$$\begin{tikzcd} G \circ Q_{\mathcal{A}} \arrow[r, "\epsilon_G"] \arrow[d, "\epsilon_F"] & Q_{\mathcal{B}} \circ K^-(F) \\ LF \circ Q_{\mathcal{A}} \arrow[ur, "\eta"] & \end{tikzcd}$$

is commutative.

We are ready to prove the main result of this section, the existence of derived functors. To prove this theorem we need the formalism of coroofs, see proposition 3.2.9.

Theorem 6.1.14. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left (resp. right) exact functor of abelian categories, $\mathcal{R}$ a class of objects adapted to F, and $S_{\mathcal{R}}$ the class of quasi-isomorphisms in $K^+(\mathcal{R})$ (resp. $K^-(\mathcal{R})$). Then the right (resp. left) derived functor RF (resp. LF) exists.*

Proof. We will prove only the existence of the right derived functor RF. Existence of the left derived functor LF is proved similarly.

Construction of RF: By corollary 6.1.10 we have a functor $\Phi : D^+(\mathcal{A}) \to K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ which commutes with the translation functors and we have the following equality and isomorphism of functors

$$\mathrm{Id}_{K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]} = \Phi \circ D^+(I_{S_{\mathcal{R}}}),$$

$$\beta : \; D^+(I_{S_{\mathcal{R}}}) \circ \Phi \xrightarrow{\cong} \mathrm{Id}_{D^+(\mathcal{A})} \; .$$

By proposition 6.1.8 and (6.4) the following diagram is commutative

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[d, "Q_{\mathcal{A}}"] & K^+(\mathcal{R}) \arrow[l, "K^+(I)"'] \arrow[r, "K^+(F\circ I)"] \arrow[d, "Q_{S_{\mathcal{R}}}"] & K^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ D^+(\mathcal{A}) \arrow[r, shift left, "\Phi"] & K^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] \arrow[l, shift left, "D^+(I_{S_{\mathcal{R}}})"] \arrow[r, "D^+(F\circ I)"] & D^+(\mathcal{B}) \end{tikzcd} \tag{6.5}$$

We define

$$RF = D^+(F \circ I) \circ \Phi.$$

Exactness of RF: Since $D^+(F \circ I)$ preserves distinguished triangles by lemma 6.1.12, it remains to show that Φ preserves distinguished triangles. Let $(X_1^\bullet, Y_1^\bullet, Z_1^\bullet, f_1, g_1, h_1)$ be a distinguished triangle in $D^+(\mathcal{A})$. By lemma 4.2.2 it is isomorphic to a triangle of the form $(X_2^\bullet, Y_2^\bullet, C(f_2), f_2, i_2, p_1)$. By lemma 6.1.6, we find isomorphisms $X_2^\bullet \to R_0^\bullet$ and $Y_2^\bullet \to R_1^\bullet$ in $D^+(\mathcal{A})$ such that $R_0^\bullet, R_1^\bullet \in K^+(\mathcal{R})$. Let us denote by ϕ_1 the composite $X_1^\bullet \to X_2^\bullet \to R_0^\bullet$ and by ϕ_2 the composite $Y_1^\bullet \to Y_2^\bullet \to R_1^\bullet$. Since ϕ_1 and ϕ_2 are isomorphisms, by TR5 and corollary 4.1.6, we obtain the following isomorphism of distinguished triangles in $D^+(\mathcal{A})$.

$$\begin{tikzcd}
X_1^\bullet \arrow[r, "f_1"] \arrow[d, "\phi_1"] & Y_1^\bullet \arrow[r, "g_1"] \arrow[d, "\phi_2"] & Z_1^\bullet \arrow[r, "h_1"] \arrow[d, dashed, "\phi_3"] & X_1^\bullet[1] \arrow[d, "\phi_1[1]"] \\
R_0^\bullet \arrow[r, "\phi_2 f_1 (\phi_1)^{-1}"] & R_1^\bullet \arrow[r, "i_2"] & C(\phi_2 f_1 (\phi_1)^{-1}) \arrow[r, "p_1"] & R_0^\bullet[1]
\end{tikzcd}$$

Now, $D^+(I_{S_\mathcal{R}})$ is identity on objects, so by the fact that $D^+(I_{S_\mathcal{R}})$ is fully faithful, the morphisms between the elements of the bottom row are morphisms of $K^+(\mathcal{R})[S_\mathcal{R}^{-1}]$. From the equality $\Phi \circ D^+(\mathcal{R})[I_{S_\mathcal{R}}] = \mathrm{Id}_{K^+(\mathcal{R})[S_\mathcal{R}^{-1}]}$ we find that the following diagram is an isomorphism of distinguished triangles

$$\begin{tikzcd}
\Phi(X_1^\bullet) \arrow[r, "\Phi(f_1)"] \arrow[d, "\Phi(\phi_1)"] & \Phi(Y_1^\bullet) \arrow[r, "\Phi(g_1)"] \arrow[d, "\Phi(\phi_2)"] & \Phi(Z_1^\bullet) \arrow[r, "\Phi(h_1)"] \arrow[d, "\Phi(\phi_3)"] & \Phi(X_1^\bullet)[1] \arrow[d, "\Phi(\phi_1)[1]"] \\
\Phi(R_0^\bullet) \arrow[r, "\Phi(\phi_2 f_1 (\phi_1)^{-1})"] \arrow[d, equal] & \Phi(R_1^\bullet) \arrow[r, "\Phi(i_2)"] \arrow[d, equal] & \Phi(C(\phi_2 f_1 (\phi_1)^{-1})) \arrow[r, "\Phi(p_1)"] \arrow[d, equal] & \Phi(R_0^\bullet)[1] \arrow[d, equal] \\
R_0^\bullet \arrow[r, "\Phi(\phi_2 f_1 (\phi_1)^{-1})"] & R_1^\bullet \arrow[r, "i_2"] & C(\phi_2 f_1 (\phi_1)^{-1}) \arrow[r, "p_1"] & R_0^\bullet[1]
\end{tikzcd}$$

Since the bottom triangle of this diagram is a distinguished triangle in $K^+(\mathcal{R})$, by lemma 4.2.2, we have shown that Φ preserves distinguished triangles.

Definition of ϵ_F: Let us construct the natural transformation $\epsilon_F : Q_\mathcal{B} \circ K^+(F) \to RF \circ Q_\mathcal{A}$ in the definition of the right derived functor. Let $X^\bullet \in K^+(\mathcal{A})$ and let $Y^\bullet \in K^+(\mathcal{R})$ such that $(\Phi \circ Q_\mathcal{A})(X^\bullet) = Q_{S_\mathcal{R}}(Y^\bullet)$. We want to construct the following morphism

$$\begin{aligned}
\epsilon_F(X^\bullet) : (Q_\mathcal{B} \circ K^+(F))(X^\bullet) \to & (RF \circ Q_\mathcal{A})(X^\bullet) \\
&= (D^+(F \circ I) \circ \Phi \circ Q_\mathcal{A})(X^\bullet) \\
&= (D^+(F \circ I) \circ Q_{S_\mathcal{R}})(Y^\bullet) \\
&= (Q_\mathcal{B} \circ K^+(F \circ I))(Y^\bullet).
\end{aligned}$$

The objects $K^+(I)(Y^\bullet)$ and $X^\bullet$ are not necessarily isomorphic in $K^+(\mathcal{A})$, but in $D^+(\mathcal{A})$ they are. Let

$$\begin{tikzcd}
X^\bullet \arrow[dr, "f"] & & K^+(I)(Y^\bullet) \arrow[dl, "s"'] \\
& Z^\bullet &
\end{tikzcd}$$

be a coroof which represents the isomorphism $\beta(X)$ in $D^+(\mathcal{A})$. By lemma 6.1.6 we have a quasi-isomorphism $r : Z^\bullet \to K^+(I)(R^\bullet)$ for some $R^\bullet \in K^+(\mathcal{R})$. Since $K^+(\mathcal{R})$ is a full subcategory of $K^+(\mathcal{A})$ the composite

$K^+(I)(Y^\bullet) \to Z^\bullet \to K^+(I)(R^\bullet)$ equals $K^+(I)(g)$ for some quasi-isomorphism $g : Y^\bullet \to R^\bullet$ of $K^+(\mathcal{R})$. Application of $K^+(F)$ yields the following diagram

$$\begin{tikzcd}
K^+(F)(X^\bullet) \arrow[dr, "K^+(F)(rf)"] & & K^+(F\circ I)(Y^\bullet) \arrow[dl, "K^+(F\circ I)(g)"'] \\
& K^+(F\circ I)(R^\bullet) &
\end{tikzcd}$$

where $K^+(F\circ I)(g)$ is a quasi-isomorphism by lemma 6.1.12. We define $\epsilon_F(X^\bullet)$ to be the morphism in $D^+(\mathcal{B})$ represented by this coroof.

$\epsilon_F(X)$ **well-defined:** To show that ϵ_F is well-defined, we have to show that the morphism $\epsilon_F(X^\bullet)$ does not depend on the choice of the coroof in $D^+(\mathcal{A})$. Let

$$\begin{tikzcd}
X^\bullet \arrow[dr, "f"] \arrow[drrr, "g"] & & & K^+(I)(Y^\bullet) \arrow[dlll, "t"'] \arrow[dl, "s"] \\
& Z_1^\bullet \arrow[dr, "t'"] & & Z_2^\bullet \arrow[dl, "s'"] \\
& & Z_3^\bullet &
\end{tikzcd}$$

be an equivalence of two coroofs which represent the isomorphism $\beta(X^\bullet)$ of $Q_{\mathcal{A}}(X^\bullet)$ and $Q_{\mathcal{A}}(K^+(I)(Y^\bullet))$ in $D^+(\mathcal{A})$. By lemma 6.1.6 we have quasi-isomorphisms $a_1 : Z_1^\bullet \to K^+(I)(R_1^\bullet)$ and $a_2 : Z_2^\bullet \to K^+(I)(R_2^\bullet)$ and by transitivity of equivalence of coroofs, the coroofs

$$\begin{tikzcd}
X^\bullet \arrow[dr, "a_1 f"] & & K^+(I)(Y^\bullet) \arrow[dl, "K^+(I)(a_1 t)"'] \\
& K^+(I)(R_1^\bullet) &
\end{tikzcd}
\qquad
\begin{tikzcd}
X^\bullet \arrow[dr, "a_2 g"] & & K^+(I)(Y^\bullet) \arrow[dl, "K^+(I)(a_2 s)"'] \\
& K^+(I)(R_2^\bullet) &
\end{tikzcd}$$

are equivalent. Therefore we can find morphisms $b_1 : K^+(I)(R_1^\bullet) \to Z_4^\bullet$ and $b_2 : K^+(I)(R_2^\bullet) \to Z_4^\bullet$ such that the following diagram is an equivalence of coroofs

$$\begin{tikzcd}
X^\bullet \arrow[dr, "a_1 f"] \arrow[drrr, "a_2 g"] & & & K^+(I)(Y^\bullet) \arrow[dlll, "a_1 K^+(I)(t)"'] \arrow[dl, "a_2 K^+(I)(s)"] \\
& K^+(I)(R_1^\bullet) \arrow[dr, "b_1"] & & K^+(I)(R_2^\bullet) \arrow[dl, "b_2"] \\
& & Z_4^\bullet &
\end{tikzcd}$$

Again, by lemma 6.1.6, we can find a quasi-isomorphism $a_3 : Z_4^\bullet \to K^+(I)(R_3^\bullet)$. Thus we have the following commutative diagram

$$\begin{tikzcd}
X^\bullet \arrow[dr, "a_1 f"] \arrow[drrr, "a_2 g"] & & & & K^+(I)(Y^\bullet) \arrow[dlll, "a_1 K^+(I)(t)"'] \arrow[dl, "a_2 K^+(I)(s)"] \\
& K^+(I)(R_1^\bullet) \arrow[dr, "a_3 b_1"] & & K^+(I)(R_2^\bullet) \arrow[dl, "a_3 K^+(I)(b_2)"] & \\
& & K^+(I)(R_3^\bullet) & &
\end{tikzcd}$$

Application of $K^+(F)$ gives the following diagram

$$\begin{tikzcd}
K^+(F)(X^\bullet) \arrow[dr, "K^+(F)(a_1 f)"] \arrow[drrr, "K^+(F)(a_2 g)"] & & & & K^+(F \circ I)(Y^\bullet) \arrow[dlll, "k^+(F)(a_1)K^+(F \circ I)(t)"'] \arrow[dl, "K^+(F)(a_2)K^+(F \circ I)(s)"] \\
& K^+(F \circ I)(R_1^\bullet) \arrow[dr, "K^+(F)(a_3 b_1)"] & & K^+(F \circ I)(R_2^\bullet) \arrow[dl, "K^+(F)(a_3)K^+(F \circ I)(b_2)"] & \\
& & K^+(F \circ I)(R_3^\bullet) & &
\end{tikzcd}$$

which is an equivalence of coroofs in $D^+(\mathcal{B})$. This shows that $\epsilon_F(X^\bullet)$ is well-defined.

$\epsilon_F(X)$ **is a natural transformation:** To show that $\epsilon_F(X^\bullet)$ is a natural transformation, let $\phi : X_1^\bullet \to X_2^\bullet$ be a morphism in $K^+(\mathcal{A})$. We have to show that the following diagram is commutative in $D^+(\mathcal{B})$

$$\begin{tikzcd}
(Q_\mathcal{B} \circ K^+(F))(X_1^\bullet) \arrow[r, "\epsilon_F(X_1^\bullet)"] \arrow[d, "(Q_\mathcal{B} \circ K^+(F))(\phi)"] & (RF \circ Q_\mathcal{A})(X_1^\bullet) \arrow[d, "(RF \circ Q_\mathcal{A})(\phi)"] \\
(Q_\mathcal{B} \circ K^+(F))(X_2^\bullet) \arrow[r, "\epsilon_F(X_2^\bullet)"] & (RF \circ Q_\mathcal{A})(X_2^\bullet)
\end{tikzcd}$$

Let $Y_1^\bullet, Y_2^\bullet \in \operatorname{Ob} K^+(\mathcal{R})$ be such that $(Q_{S_\mathcal{R}})(Y_1^\bullet) = (\Phi \circ Q_\mathcal{A})(X_1^\bullet)$ and $(Q_{S_\mathcal{R}})(Y_2^\bullet) = (\Phi \circ Q_\mathcal{A})(X_2^\bullet)$. Let ψ be a morphism of $K^+(\mathcal{R})$ such that $Q_{S_\mathcal{R}}(\psi) = (\Phi \circ Q_\mathcal{A})(\phi)$. By definition of RF we have

$$(RF \circ Q_\mathcal{A})(\phi) = (D^+(F \circ I) \circ \Phi \circ Q_\mathcal{A})(\phi) = (D^+(F \circ I) \circ Q_{S_\mathcal{R}})(\psi) = (Q_\mathcal{B} \circ K^+(F \circ I))(\psi),$$

where the last equation follows from commutativity of proposition 6.1.8. Therefore we have to show that

$$\epsilon_F(X_2) \circ (Q_\mathcal{B} \circ K^+(F))(\phi) = (Q_\mathcal{B} \circ K^+(F \circ I))(\psi) \circ \epsilon_F(X_1). \tag{6.6}$$

Since β is a natural transformation, the following diagram

$$\begin{tikzcd}
Q_\mathcal{A}(X_1^\bullet) \arrow[r, "\beta(X_1^\bullet)"] \arrow[d, "Q_\mathcal{A}(\phi)"] & (D^+(I_{S_\mathcal{R}}) \circ Q_{S_\mathcal{R}})(Y_1^\bullet) \arrow[d, "(D^+(I_{S_\mathcal{R}}) \circ Q_{S_\mathcal{R}})(\psi)"] \\
Q_\mathcal{A}(X_2^\bullet) \arrow[r, "\beta(X_2^\bullet)"] & (D^+(I_{S_\mathcal{R}}) \circ Q_{S_\mathcal{R}})(Y_2^\bullet)
\end{tikzcd} \tag{6.7}$$

is commutative in $D^+(\mathcal{A})$. We fix the following coroofs to represent the morphisms

$$\beta(X_1^\bullet) := \begin{tikzcd} X_1^\bullet \arrow[dr] & & K^+(I)(Y_1^\bullet) \arrow[dl] \\ & R_1^\bullet & \end{tikzcd} \qquad \beta(X_2^\bullet) := \begin{tikzcd} X_2^\bullet \arrow[dr] & & K^+(I)(Y_2^\bullet) \arrow[dl] \\ & R_2^\bullet & \end{tikzcd}$$

$$(D^+(I_{S_\mathcal{R}}) \circ Q_{S_\mathcal{R}})(\psi) := \begin{tikzcd} K^+(I)(Y_1^\bullet) \arrow[dr, "K^+(I)(\psi)"] & & K^+(I)(Y_2^\bullet) \arrow[dl, "\mathrm{Id}_{K^+(I)(Y_2^\bullet)}"] \\ & K^+(I)(Y_2^\bullet) & \end{tikzcd}$$

where we have assumed by using lemma 6.1.6 that the bottom objects are objects coming from $K^+(\mathcal{R})$. By commutativity of (6.7), the following compositions of coroofs represent the same morphisms in $D^+(\mathcal{A})$

$$\begin{tikzcd} X_1^\bullet \arrow[dr] & & K^+(I)(Y_1^\bullet) \arrow[dl] \arrow[dr] & & K^+(I)(Y_2^\bullet) \arrow[dl] \\ & K^+(I)(R_1^\bullet) \arrow[dr] & & K^+(I)(Y_2^\bullet) \arrow[dl] & \\ & & K^+(I)(R_3^\bullet) & & \end{tikzcd}$$

$$\begin{tikzcd} X_1^\bullet \arrow[dr, "\phi"] & & X_2^\bullet \arrow[dl, "\mathrm{Id}_{X_2^\bullet}"] \arrow[dr] & & K^+(I)(Y_2^\bullet) \arrow[dl] \\ & X_2^\bullet \arrow[dr] & & K^+(I)(R_2^\bullet) \arrow[dl] & \\ & & K^+(I)(R_4^\bullet) & & \end{tikzcd}$$

where the objects $K^+(I)(R_3^\bullet)$ and $K^+(I)(R_4^\bullet)$ and morphisms to them are constructed by using lemma 6.1.6. An application of $K^+(F)$ to these two coroofs yields two coroofs of $D^+(\mathcal{B})$, by lemma 6.1.12, which represent the same morphism. Since $K^+(F)(\beta(X_1^\bullet)) = \epsilon_F(X_1^\bullet)$ $K^+(F)(\beta(X_2^\bullet)) = \epsilon_F(X_2^\bullet)$, and $(K^+(F) \circ D^+(I_{S_\mathcal{R}}) \circ Q_{S_\mathcal{R}})(\psi) = K^+(F \circ I)(\psi)$, we have obtained the equation (6.6). This shows that ϵ_F is a natural transformation.

Universal property of RF: Let $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be an exact functor and $\epsilon_G : Q_\mathcal{B} \circ K^+(F) \to G \circ Q_\mathcal{A}$ a natural transformation. For any $X^\bullet \in K^+(\mathcal{A})$, let

$$\begin{tikzcd} X^\bullet \arrow[dr, "f"] & & K^+(I)(Y^\bullet) \arrow[dl, "K^+(I)(s)"] \\ & K^+(I)(R^\bullet) & \end{tikzcd}$$

be a coroof, obtained by using lemma 6.1.6, which represents the isomorphism $\beta(X^\bullet)$ in $D^+(\mathcal{A})$, where

$Y^\bullet \in K^+(\mathcal{R}^\bullet)$ such that $Q_{S_\mathcal{R}}(Y^\bullet) = (\Phi \circ Q_\mathcal{A})(X^\bullet)$. We have the following commutative diagram

$$\begin{tikzcd}
(Q_\mathcal{B} \circ K^+(F)(X^\bullet)) \arrow[r, "(Q_\mathcal{B}\circ K^+(F))(f)"] \arrow[d, "\epsilon_G(X^\bullet)"] & (Q_\mathcal{B} \circ K^+(F\circ I))(R^\bullet) \arrow[d, "\epsilon_G(Q_\mathcal{B}\circ K^+(I))(R^\bullet)"] & (Q_\mathcal{B} \circ K^+(F\circ I))(Y^\bullet) \arrow[l, "(Q_\mathcal{B}\circ K^+(F\circ I))(s)"'] \arrow[d, "\epsilon_G(K^+(I)(Y^\bullet))"] \\
(G\circ Q_\mathcal{A})(X^\bullet) \arrow[r, "(G\circ Q_\mathcal{A})(f)"] & (G\circ Q_\mathcal{A} \circ K^+(I))(R^\bullet) & (G\circ Q_\mathcal{A} \circ K^+(I))(Y^\bullet) \arrow[l, "(G\circ Q_\mathcal{A}\circ K^+(I))(s)"']
\end{tikzcd}$$

By lemma 6.1.12, $(Q_\mathcal{B} \circ K^+(F\circ I))(s)$ is an isomorphism and $(G\circ Q_\mathcal{A} \circ K^+(I))(s)$ is an isomorphism, because s is a quasi-isomorphism, $Q_\mathcal{A}$ turns it into isomorphism, and any functor preserves isomorphisms. Therefore these two morphisms have inverses in $D^+(\mathcal{B})$ and we obtain the following commutative diagram

$$\begin{tikzcd}
(Q_\mathcal{B} \circ K^+(F))(X^\bullet) \arrow[r, "\epsilon_G(X^\bullet)"] \arrow[d, "\epsilon_F(X^\bullet)"] & (G\circ Q_\mathcal{A})(X^\bullet) \arrow[d, "G(\beta(X^\bullet))"] \\
(RF\circ Q_\mathcal{A})(X^\bullet) \arrow[r, "\epsilon_G(K^+(I)(Y^\bullet))"] & (G\circ Q_\mathcal{A} \circ K^+(I))(Y^\bullet)
\end{tikzcd} \tag{6.8}$$

Note, that here $(RF\circ Q_\mathcal{A})(X^\bullet) = (Q_\mathcal{B} \circ K^+(F\circ I))(Y^\bullet)$, so that the diagram makes sense. Now $\beta(X^\bullet)$ is an isomorphism in $D^+(\mathcal{A})$, so we can define $\eta : RF \to G$ to be

$$\eta(Q_\mathcal{A}(X^\bullet)) = G(\beta(X^\bullet))^{-1} \circ \epsilon_G(K^+(I)(Y^\bullet)).$$

η **a natural transformation:** Let $\phi : X_1^\bullet \to X_2^\bullet$ be a morphism in $K^+(\mathcal{A})$. To show that η is a natural transformation, we have to show that the following diagram is commutative

$$\begin{tikzcd}
(RF\circ Q_\mathcal{A})(X_1^\bullet) \arrow[r, "\epsilon_G(K^+(I)(Y_1^\bullet))"] \arrow[d, "(RF\circ Q_\mathcal{A})(\phi)"] & (G\circ Q_\mathcal{A})(K^+(I)(Y_1^\bullet)) \arrow[r, "(G(\beta(X_1^\bullet)))^{-1}"] \arrow[d, "(G\circ Q_\mathcal{A})(K^+(I)(\psi))"] & (G\circ Q_\mathcal{A})(X_1^\bullet) \arrow[d, "(G\circ Q_\mathcal{A})(\phi)"] \\
(RF\circ Q_\mathcal{A})(X_2^\bullet) \arrow[r, "\epsilon_G(K^+(I)(Y_2^\bullet))"] & (G\circ Q_\mathcal{A})(K^+(I)(Y_2^\bullet)) \arrow[r, "(G(\beta(X_2^\bullet)))^{-1}"] & (G\circ Q_\mathcal{A})(X_2^\bullet)
\end{tikzcd}$$

where $(RF\circ Q_\mathcal{A})(X_1^\bullet) = (Q_\mathcal{B} \circ K^+(F))(K^+(I)(Y_1^\bullet))$, $(RF\circ Q_\mathcal{A})(X_2^\bullet) = (Q_\mathcal{B} \circ K^+(F))(K^+(I)(Y_2^\bullet))$, and $(RF\circ Q_\mathcal{A})(\phi) = (Q_\mathcal{B} \circ K^+(F))(K^+(I)(\psi))$. The left square of this diagram is commutative because ϵ_G is a natural transformation. Since β is a natural transformation, the following diagram

$$\begin{tikzcd}
(G\circ Q_\mathcal{A})(X_1^\bullet) \arrow[r, "G(\beta(X_1^\bullet))"] \arrow[d, "(G\circ Q_\mathcal{A})(\phi)"] & (G\circ Q_\mathcal{A})(K^+(I)(Y_1^\bullet)) \arrow[d, "(G\circ Q_\mathcal{A})(K^+(I)(\psi))"] \\
(G\circ Q_\mathcal{A})(X_2^\bullet) \arrow[r, "G(\beta(X_2^\bullet))"] & (G\circ Q_\mathcal{A})(K^+(I)(Y_2^\bullet))
\end{tikzcd}$$

is commutative. Hence

$$\begin{aligned}
&(G\circ Q_\mathcal{A})(K^+(I)(Y_1^\bullet)) \to (G\circ Q_\mathcal{A})(X_1^\bullet) \to (G\circ Q_\mathcal{A})(X_2^\bullet) \\
&\quad= (G\circ Q_\mathcal{A})(K^+(I)(Y_1^\bullet)) \to (G\circ Q_\mathcal{A})(X_1^\bullet) \to (G\circ Q_\mathcal{A})(X_2^\bullet) \to \\
&\qquad\quad (G\circ Q_\mathcal{A})(K^+(I)(Y_2^\bullet)) \to (G\circ Q_\mathcal{A})(X_2^\bullet) \\
&\quad= (G\circ Q_\mathcal{A})(K^+(I)(Y_1^\bullet)) \to (G\circ Q_\mathcal{A})(X_1^\bullet) \to (G\circ Q_\mathcal{A})(K^+(I)(Y_1^\bullet)) \to \\
&\qquad\quad (G\circ Q_\mathcal{A})(K^+(I)(Y_2^\bullet)) \to (G\circ Q_\mathcal{A})(X_2^\bullet) \\
&\quad= (G\circ Q_\mathcal{A})(K^+(I)(Y_1^\bullet)) \to (G\circ Q_\mathcal{A})(K^+(I)(Y_2^\bullet)) \to (G\circ Q_\mathcal{A})(X_2^\bullet)
\end{aligned}$$

This shows that the right square of 6.1 is commutative. Therefore η is a natural transformation.

Uniqueness of η: To show that η is unique, let $X^\bullet \in K^+(\mathcal{A})$ and let $Y^\bullet \in K^+(\mathcal{R})$ such that $(Q_\mathcal{A})(X^\bullet) \cong (D^+(I_{S_\mathcal{R}}) \circ Q_\mathcal{R})(Y^\bullet)$. Then by definition of η, (6.8), we have the following commutative diagram

$$\begin{tikzcd}
(Q_\mathcal{B} \circ K^+(F))(X^\bullet) \arrow[r, "\epsilon_G(X^\bullet)"] \arrow[d, "\epsilon_F(X^\bullet)"] & (G \circ Q_\mathcal{A})(X^\bullet) \arrow[d, "G(\beta(X^\bullet))"] \\
(RF \circ Q_\mathcal{A})(X^\bullet) \arrow[ur, "\eta(Q_\mathcal{A})(X^\bullet)"] \arrow[r, "\epsilon_G(K^+(I)(Y^\bullet))"] \arrow[d, "\epsilon_F(K^+(I)(Y^\bullet))"] & (G \circ Q_\mathcal{A})(Y^\bullet) \\
(RF \circ Q_\mathcal{A})(K^+(I)(Y^\bullet)) \arrow[ur, "\eta(Q_\mathcal{A})(Y^\bullet)"'] &
\end{tikzcd}$$

where $\epsilon_F(K^+(I)(Y^\bullet))$ is the identity morphism by the definition of ϵ_F. Hence we obtain that

$$\eta(Q_\mathcal{A})(Y^\bullet) = \epsilon_G(K^+(I)(Y^\bullet)),$$

so the morphism $\eta(Q_\mathcal{A})(Y^\bullet)$ is uniquely determined, and by the fact that η is a natural transformation, we get

$$G(\beta(X^\bullet)) \circ \eta(Q_\mathcal{A})(X^\bullet) = \eta(Q_\mathcal{A})(Y^\bullet).$$

Since $G(\beta(X^\bullet))$ is an isomorphism, the morphism $\eta(Q\mathcal{A})(X^\bullet)$ is uniquely determined. This shows that η is unique.

□

The following theorem shows that taking the derived functor behaves well under composition of functors.

Theorem 6.1.15. *Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ be abelian categories, $F_1 : \mathcal{A}_1 \to \mathcal{A}_2$ and $F_2 : \mathcal{A}_2 \to \mathcal{A}_3$ left exact functors, and $\mathcal{R}_1$ and $\mathcal{R}_2$ classes of objects adapted to the functors F_1 and F_2, respectively, such that $F_1(\mathcal{R}_1) \subset \mathcal{R}_2$. Then we have an isomorphism of functors*

$$R(G \circ F) \cong RG \circ RF. \tag{6.9}$$

Proof. The composite

$$\begin{aligned}(Q_{\mathcal{A}_3} \circ K^+(G)) \circ K^+(F) &\overset{\epsilon_{F_2}}{\to} RG \circ (Q_{\mathcal{A}_2} \circ K^+(F)) \\ &\overset{\epsilon_{F_1}}{\to} RG \circ RF \circ Q_{\mathcal{A}_1}\end{aligned}$$

is a natural transformation. Denote this by E. By the universal property of right derived functor there exists a natural transformation $\delta : R(G \circ F) \to RG \circ RF$ such that for any object $X^\bullet \in K^+(\mathcal{A}_1)$ the following diagram is commutative

$$\begin{tikzcd}
(Q_\mathcal{B} \circ K^+(G \circ F))(X^\bullet) \arrow[r, "E"] \arrow[d, "\epsilon_{G \circ F}(X^\bullet)"] & (RG \circ RF \circ Q_{\mathcal{A}_1})(X^\bullet) \\
(R(G \circ F) \circ Q_{\mathcal{A}_1})(X^\bullet) \arrow[ur, dashed, "(\delta \circ Q_{\mathcal{A}_1})(X^\bullet)"'] &
\end{tikzcd}$$

By definition of ϵ_F, ϵ_G, and $\epsilon_{G \circ F}$, and by assumption, for any object $K^+(I)(Y^\bullet) \in \operatorname{Ob} K^+(\mathcal{A}_1)$, with $Y^\bullet \in \operatorname{Ob} K^+(\mathcal{R}_1)$, the morphisms $\epsilon_F(K^+(I)(Y^\bullet))$, $\epsilon_G(K^+(F_1 \circ I)(Y^\bullet))$, and $\epsilon_{G \circ F}(K^+(I)(Y^\bullet))$ are identity morhisms.

This means that $E(Q_{\mathcal{A}_\infty} \circ K^+(I))(Y^\bullet)$ is an isomorphism, and because all the objects of $D^+(\mathcal{A}_\infty)$ are isomorphic to an object of $K^+(\mathcal{R}_1)$, given by β, we find that E is an isomorphism for all objects. By commutativity δ is an isomorphism. This proves the isomorphism (6.9). □

The following is a version of the above for left derived functors

Theorem 6.1.16. *Let $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ be abelian categories, $F_1 : \mathcal{A}_1 \to \mathcal{A}_2$ and $F_2 : \mathcal{A}_2 \to \mathcal{A}_3$ right exact functors, and $\mathcal{R}_1$ and $\mathcal{R}_2$ adapted classes of object to the functors F_1 and F_2, respectively, such that $F_1(\mathcal{R}_1) \subset \mathcal{R}_2$. Then we have an isomorphism of functors*

$$L(G \circ F) \cong LG \circ LF. \tag{6.10}$$

6.2 Examples

In this section we show that injective objects form an adapted class for a left exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories, with $\mathcal{A}$ having enough injective objects, and give an example of one such functor. Also, we show that the functor $Ext^i(Y) := \mathrm{Mor}_{D^+(\mathcal{A})}(X, Y[i])$ coincides with the functor $R^i \mathrm{Mor}_{\mathcal{A}}(X, -) := H^i(R\,\mathrm{Mor}_{\mathcal{A}}(X, -))$. We use this characterization to show that the derived category of $\mathbb{Z}$-modules is not an abelian category. For more derived functors and further references, see the notes.

Definition 6.2.1 (Injective and projective objects)**.** Let $\mathcal{A}$ be an abelian category. An object $I \in \mathrm{Ob}\,\mathcal{A}$ is an *injective object*, if $\mathrm{Mor}_{\mathcal{A}}(-, I)$ is an exact functor. We say $\mathcal{A}$ has *enough injective* objects if for any object $X \in \mathrm{Ob}\,\mathcal{A}$ there exists a monomorphism $X \to I$ for some injective object I of $\mathcal{A}$.

Similarly, an object P of $\mathcal{A}$ is called *projective* if the functor $\mathrm{Mor}_{\mathcal{A}}(P, -)$ is exact. The category $\mathcal{A}$ is said to have *enough projective* objects, if for any object $X \in \mathrm{Ob}\,\mathcal{A}$ there exists an epimorphism $P \to X$ for some projective object P of $\mathcal{A}$.

To show that injective objects form an adapted class, we need the following lemma. Note that this lemma has a dual version for projective objects. We do not prove, or even state this, but the reader can try to state and prove it on his own.

Lemma 6.2.2. *Let $\mathcal{A}$ be an abelian category, $I^\bullet \in C^+(\mathcal{A})$ a complex of injective objects of $\mathcal{A}$, $X^\bullet \in C^+(\mathcal{A})$, and $f : X^\bullet \to I^\bullet$ a morphism of complexes. Then f is homotopic to the zero morphism.*

Proof. It is clear that we can assume the morphism f and the complexes $X^\bullet$ and $I^\bullet$ to be of the form as in the following diagram

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r] \arrow[d, "f^0"] & C^1 \arrow[r] \arrow[d, "f^1"] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \dots \end{tikzcd}$$

Let $\chi^i = 0$ for $i \leqslant 0$. For $i = 1$, χ^1 is given by definition of the injective object I^0 for the following diagram

$$\begin{tikzcd} 0 \arrow[r] & C^0 \arrow[r, "d^0_{X^\bullet}"] \arrow[d, "f^0"] & C^1 \arrow[dl, "\chi^1"] \\ & I^0 & \end{tikzcd}$$

By induction, suppose we have constructed the morphism χ^n such that $f^{n-1} = \chi^n d_{X^\bullet}^{n-1}$. We have

$$\begin{aligned}(f^n - d_{I^\bullet}^{n-1}\chi^n)d_{X^\bullet}^{n-1} &= d_{I^\bullet}^{n-1}f^{n-1} - d_{I^\bullet}^{n-1}\chi^n d_{X^\bullet}^{n-1} \\ &= d_{I^\bullet}^{n-1}(f^{n-1} - \chi^n d_{X^\bullet}^{n-1}) = 0.\end{aligned}$$

By the cokernel property of $d_{X^\bullet}^{n-1}$ there exists a unique morphism $\psi : \operatorname{coker} d_{X^\bullet}^{n-1} \to I^n$ such that $\psi\phi_2 = f^n - d_{I^\bullet}^{n-1}\chi^n$, where ϕ_2 is the epimorphism in the following epimorphism monomorphism factorization of $d_{X^\bullet}^n$

$$\begin{tikzcd} \operatorname{coker} d_{X^\bullet}^{n-1} \arrow[dr, "\beta"] & \\ X^n \arrow[u, "\phi_2"] \arrow[r, "d_{X^\bullet}^n"] & X^{n+1} \end{tikzcd}$$

Then we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker} d_{X^\bullet}^n \arrow[r, "\beta"] \arrow[d, "\psi"] & X^{n+1} \arrow[dl, dashed, "\chi^{n+1}"] \\ & I^n & \end{tikzcd}$$

where the first row is exact. Thus there exists a morphism $\chi^{n+1} : X^{n+1} \to I^n$ such that the diagram is commutative. We have

$$\begin{aligned} f^n &= f^n - d_{I^\bullet}^{n-1}\chi^n + d_{I^\bullet}^{n-1}\chi^n \\ &= \psi\phi_2 + d_{I^\bullet}^{n-1}\chi^n \\ &= \chi^{n+1}d_{X^\bullet}^n + d_{I^\bullet}^{n-1}\chi^n. \end{aligned}$$

This shows that f^n is homotopic to the zero morphism. $\square$

We need the following lemma to show that the biproduct of injective objects is injective.

Lemma 6.2.3. *Let $\mathcal{A}$ be an abelian category and let*

$$0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0 \qquad 0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \longrightarrow 0$$

be two short exact sequences in $\mathcal{A}$. Then the induced sequence

$$0 \longrightarrow M_1 \oplus N_1 \xrightarrow{i_1f_1p_1 + i_2g_1p_2} M_2 \oplus N_2 \xrightarrow{i_1f_2p_1 + i_2g_2p_2} M_3 \oplus N_3 \longrightarrow 0$$

is exact.

Proof. By proposition 2.2.14 (ii) and (iii), it suffices show that $i_1f_2p_1 + i_2g_2p_2$ is an epimorphism and that $\ker(i_1f_2p_1 + i_2g_2p_2) = i_1f_1p_1 + i_2g_1p_2$. Recall that the biproduct of two objects is both the product and the coproduct of the objects by proposition 2.1.3.

To prove that $i_1f_2p_1 + i_2g_2p_2$ is an epimorphism, let $h : M_3 \oplus N_3 \to K$ be any morphism such that $h(i_1f_2p_1 + i_2g_2p_2) = 0$. Now

$$h(i_1f_2p_1 + i_2g_2p_2)i_1 = hi_1f_2 = 0$$

and

$$h(i_1f_2p_1 + i_2g_2p_2)i_2 = hi_2g_2 = 0.$$

Since f_2 and g_2 are epimorphisms, $hi_1 = hi_2 = 0$. By the uniqueness property of biproduct, we must have $h = 0$. This shows that $i_1f_2p_1 + i_2g_2p_2$ is an epimorphism.

To prove that $\ker(i_1f_2p_1 + i_2g_2p_2) = i_1f_1p_1 + i_2g_1p_2$, consider the following commutative diagram

$$\begin{tikzcd}
 & M_1 \oplus N_1 \arrow[ddl, "f_1p_1"'] \arrow[ddr, "g_1p_2"] & \\
 & \ker(i_1f_2p_1 + i_2g_2p_2) \arrow[d, "k_{i_1f_2p_1+i_2g_2p_2}"] & \\
M_2 \arrow[r, "i_1"] \arrow[d, "f_2"] & M_2 \oplus N_2 \arrow[d, "i_1f_2p_1+i_2g_2p_2"] & N_2 \arrow[l, "i_2"'] \arrow[d, "g_2"] \\
M_3 \arrow[r, "i_1"] & M_3 \oplus N_3 & N_3 \arrow[l, "i_2"']
\end{tikzcd}$$

By the fact that f_1 and g_1 are kernels of f_2 and g_2, respectively, and by commutativity, we have $(i_1f_2p_1 + i_2g_2p_2)(i_1f_1p_1 + i_2g_1p_2) = 0$. Thus we obtain a unique morphism $\phi : M_1 \oplus N_1 \to \ker(i_1f_2p_1 + i_2g_2p_2)$ such that

$$k_{i_1f_2p_1+i_2g_2p_2}\phi = i_1f_1p_1 + i_2g_1p_2.$$

Again, by the fact that f_1 and g_1 are kernels of f_2 and g_2, respectively, and by commutativity, we have $p_1(i_1f_2p_1 + i_2g_2p_2)k_{i_1f_2p_1+i_2g_2p_2} = 0 = p_2(i_1f_2p_1 + i_2g_2p_2)k_{i_1f_2p_1+i_2g_2p_2}$. Therefore we get a unique morphism $\psi : \ker(i_1f_2i_2 + i_2g_2p_2) \to M_1 \oplus N_1$ such that

$$p_1k_{i_1f_2p_1+i_2g_2p_2} = f_1p_1\psi \qquad \text{and} \qquad p_2k_{i_1f_2p_1+i_2g_2p_2} = g_1p_2\psi.$$

From these equations for ψ and ϕ we get $(i_1p_1+i_2p_2)k_{i_1f_2p_1+i_2g_2p_2} = i_1f_1p_1\psi+i_2g_1p_2\psi = k_{i_1f_2p_1+i_2g_2p_2}\phi\psi$. Since $(i_1p_1+i_2p_2)$ is the identity morphism and $k_{i_1f_2p_1+i_2g_2p_2}$ is a monomorphism, we deduce that $\mathrm{Id}_{\ker(i_1f_2i_2+i_2g_2p_2)} = \phi\psi$.

To show that $\psi\phi = \mathrm{Id}_{M_1\oplus N_1}$, consider the following equations

$$f_1p_1 = p_1k_{i_1f_2p_1+i_2g_2p_2}\phi = f_1p_1\psi\phi \qquad \text{and} \qquad g_1p_2 = p_2k_{i_2f_2p_1+i_2g_2p_2}\phi = g_1p_2\psi\phi.$$

By the universal property of $M_1 \oplus N_1$ as a product, we get $\psi\phi = \mathrm{Id}_{M_1\oplus N_1}$. This completes the proof. $\square$

The following example shows that injective objects form an adapted class. We also give examples of left and right exact functors.

Example 6.2.4. Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and $C^+(F) : C^+(\mathcal{A}) \to C^+(\mathcal{B})$ the induced left exact functor. To see that the class $\mathcal{I}$ of all injective objects of $\mathcal{A}$ form an adapted class, let I_1 and I_2 be two injective objects. By definition of biproduct we have

$$\mathrm{Mor}_{\mathcal{A}}(Y, I_1 \oplus I_2) \cong \mathrm{Mor}_{\mathcal{A}}(Y, I_1) \oplus \mathrm{Mor}_{\mathcal{A}}(Y, I_2),$$

where Y is an arbitrary object of $\mathcal{A}$. Thus, by lemma 6.2.3, the biproduct of two injective objects is an injective object, and so condition AC1 holds. The condition AC2 holds because $\mathcal{A}$ has enough injectives. It remains to verify AC3, that is to show that for any injective exact complex $I^\bullet$ of $K^+(\mathcal{I})$ the complex $K(F)^+(I^\bullet)$ is exact. To show this it suffices to show that $\mathrm{Id}_{I^\bullet}$ is homotopic to the zero morphism. But by 6.2.2 $\mathrm{Id}_{I^\bullet}$ is homotopic to 0 morphism. Thus $K(F)^+(I^\bullet)$ is an exact complex. This shows that the class of injective objects is adapted to F.

Similarly one can show that if an abelian category $\mathcal{A}$ has enough projective objects, then the class of projective objects is adapted to any right exact functor $F : \mathcal{A} \to \mathcal{B}$.

Now, for any R-module M, the functor $\mathrm{Mor}_{\mathbf{RMod}}(M, -) : \mathbf{RMod} \to \mathbf{Ab}$ is a left exact functor. It is well known that the category **RMod** has enough injectives, see for example [Lan02, p.784 Theorem 4.1]. Thus by theorem 6.1.14 there exists a derived functor $R\,\mathrm{Mor}_{\mathbf{RMod}}(M, -) : D^+(\mathbf{RMod}) \to D^+(\mathbf{Ab})$.

To give an example of a left derived functor, let M be an R-module. By [Sch, p.65 Example 4.3.6.(i)], **RMod** has enough projective objects. The functor $- \otimes M : \mathbf{RMod} \to \mathbf{RMod}$ is right exact, so by theorem 6.1.14 there exists a left derived functor $- \otimes^L M := L(- \otimes M) : D^-(\mathbf{RMod}) \to D^-(\mathbf{RMod})$.

We will need the following two lemmas in an example about Ext.

Lemma 6.2.5. *Let $\mathcal{A}$ be an abelian category with enough injectives, $X^\bullet \in \mathrm{Ob}\, K^+(\mathcal{A})$ and $s : I^\bullet \to X^\bullet$ a quasi-isomorphism from a complex consisting of injective objects. Then there exists a quasi-isomorphism $t : X^\bullet \to I^\bullet$ such that $t \circ s$ is homotopic to $\mathrm{Id}_{I^\bullet}$.*

Proof. Complete s to the following distinguished triangle

$$I^\bullet \xrightarrow{s} X^\bullet \xrightarrow{i_2} C(s) \xrightarrow{p_1} I^\bullet[1]$$

Now $C(s)$ is exact, and by the proof of lemma 6.2.2 we have the homotopy $\chi^i : I^{i+1} \oplus X^i \to I^i$ such that $p_1 = \chi^{i+1} d^i_{C(s)} + d^{i-1}_{I^\bullet} \chi^i$. We show that $\chi^i i_2 s^i \sim \mathrm{Id}_{I^\bullet}$. By abuse of notation, we use notation i_1 and p_1 for the inclusion $I^\bullet \to C(s)[-1]$ and projection $C(s)[-1] \to I^\bullet$, respectively. We have

$$\begin{aligned}
\mathrm{Id}_{I^\bullet} &= p_1 i_1 \\
&= (\chi^i d^i_{C(s)[-1]} + d^{i-1}_{I^\bullet} \chi^{i-1}) i_1 \\
&= (\chi^i (i_1 d^i_{I^\bullet} p_1 + i_2 s^i p_1 - i_2 d^i_{X^\bullet} p_2) + d^{i-2}_{I^\bullet} \chi^{i-1}) i_1 \\
&= (\chi^i i_1) d^i_{I^\bullet} + \chi^i i_2 s^i + d^{i-2}_{I^\bullet} (\chi^{i-1} i_1)
\end{aligned}$$

This shows that we can choose $t := \chi^i i_2 s^i$. ☐

Lemma 6.2.6. *Let $\mathcal{A}$ be an abelian category with enough injective objects and $I^\bullet \in K^+(\mathcal{A})$ an object consisting of injective objects. Then the localization functor $Q_{\mathcal{A}}$ induces a bijection*

$$\mathrm{Mor}_{K+(\mathcal{A})}(X^\bullet, I^\bullet) \to \mathrm{Mor}_{D^+(\mathcal{A})}(X^\bullet, I^\bullet).$$

Proof. To show that the map is surjective, let

$$\begin{tikzcd}
X^\bullet \arrow[dr, "f"] & & I^\bullet \arrow[dl, "s"'] \\
& Z^\bullet &
\end{tikzcd}$$

be a coroof which represents some morphism from $X^\bullet$ to $I^\bullet$ in $D^+(\mathcal{A})$. By lemma 6.2.5 there exists a quasi-isomorphism $t : Z^\bullet \to I^\bullet$ such that $t \circ s = \mathrm{Id}_{I^\bullet}$. The following coroof thus represents the same morphism as the one above

$$\begin{tikzcd} X^\bullet \arrow[dr, "tf"] & & I^\bullet \arrow[dl, "\mathrm{Id}_{I^\bullet}"] \\ & I^\bullet & \end{tikzcd}$$

It is clear that this coroof corresponds the following roof

$$\begin{tikzcd} & X^\bullet \arrow[dl, "\mathrm{Id}_{X^\bullet}"'] \arrow[dr, "tf"] & \\ X^\bullet & & I^\bullet \end{tikzcd}$$

This morphism is the image of tf. Hence the map is surjective.

To show that the map is injective, we recall from proposition 3.2.10 that the functor is additive, and hence the map is an abelian group homomorphism. Thus is suffices to show that if the following roof

$$\begin{tikzcd} & X^\bullet \arrow[dl, "\mathrm{Id}_{X^\bullet}"'] \arrow[dr, "\phi"] & \\ X^\bullet & & I^\bullet \end{tikzcd}$$

represents the zero morphism from $X^\bullet$ to $I^\bullet$ in $D^+(\mathcal{A})$, then $\phi = 0$. But this follows directly from lemma 6.2.2. □

The following example is taken from [GM03, p.194 14].

Example 6.2.7 (Ext functor)**.** Let $\mathcal{A}$ be an abelian category with enough injective objects. In this example we study the right derived functor of the left exact functor $\mathrm{Mor}_{\mathcal{A}}(X, -)$.

For any object $X \in \mathcal{A}$ let us denote by $X[i]$ the complex where the ith object is X and others are zero. For any object $Y \in \mathcal{A}$ fix an injective resolution with a quasi-isomorphism $r : Y \to I_Y^\bullet$, by lemma 6.1.6. Then we define the functor $Ext^i_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathbf{Ab}$, $i \geqslant 0$, by

$$Ext^i_{\mathcal{A}}(X, Y) = \mathrm{Mor}_{D^+(\mathcal{A})}(X[0], Y[i]).$$

Note that since $r[i] : Y[i] \to I_Y^\bullet[i]$ is a quasi-isomorphism, we have

$$\mathrm{Mor}_{D^+(\mathcal{A})}(X[0], Y[i]) \cong \mathrm{Mor}_{D^+(\mathcal{A})}(X[0], I_Y^\bullet[i]).$$

Since $I_Y^\bullet$ is a complex consisting of injective complexes, by lemma 6.2.6, we have the following isomorphism

$$\mathrm{Mor}_{D^+(\mathcal{A})}(X[0], I_Y^\bullet[i]) \cong \mathrm{Mor}_{K^+(\mathcal{A})}(X[0], I_Y^\bullet[i]).$$

Construction of right derived functor implies that

$$\begin{aligned} R^i \mathrm{Mor}_{\mathcal{A}}(X, Y) &= H^i(R\,\mathrm{Mor}_{\mathcal{A}}(X, -)(Y)) \\ &= H^i((D^+(\mathrm{Mor}_{\mathcal{A}}(X, -)) \circ I)(I_Y^\bullet)) \\ &= H^i(\mathrm{Mor}_{K^+(\mathcal{A})}(X[0], I_Y^\bullet)). \end{aligned}$$

Since the only homotopy between objects of $X[0]$ and $I_Y^\bullet$ is the zero homotopy, we have

$$H^i(\mathrm{Mor}_{K^+(\mathcal{A})}(X[0], I_Y^\bullet)) = H^i(\mathrm{Mor}_{C^+(\mathcal{A})}(X[0], I_Y^\bullet)).$$

Thus it remains to show that we have the following isomorphism

$$H^i(\mathrm{Mor}_{C^+(\mathcal{A})}(X[0], I_Y^\bullet)) \cong \mathrm{Mor}_{K^+(\mathcal{A})}(X[0], I_Y^\bullet[i]).$$

To construct this isomorphism between these objects in the category **Ab**, we define a structure of complexes on the set of morphisms in $C^+(\mathcal{A})$. These are also called "inner Mor" in $C^+(\mathcal{A})$. Let $A^\bullet, B^\bullet \in \mathrm{Ob}\, C^+(\mathcal{A})$,

$$\mathrm{Mor}^n_{C^+(\mathcal{A})}(A^\bullet, B^\bullet) = \prod_{i \in \mathbb{Z}} \mathrm{Mor}_{\mathcal{A}}(A^i, B^{i+n})$$

and

$$d^i_{\mathrm{Mor}(A^\bullet, B^\bullet)}(f) = d^i_{B^\bullet} f^i - (-1)^n f^{i+1} d^i_{A^\bullet}$$

for $f \in \mathrm{Mor}(A^\bullet, B^\bullet)$. It is easy to check that this defines a structure of a complex on $\mathrm{Mor}_{C^+(\mathcal{A})}(A^\bullet, B^\bullet)$. Hence we get the following isomorphism

$$H^i(\mathrm{Mor}_{C^+(\mathcal{A})}(A^\bullet, B^\bullet)) \cong \mathrm{Mor}_{K^+(\mathcal{A})}(A^\bullet, B^\bullet[1]).$$

This shows that

$$H^i(\mathrm{Mor}_{C^+(\mathcal{A})}(X[0], I_Y^\bullet)) \cong \mathrm{Mor}_{K^0(\mathcal{A})}(X[0], I_Y^\bullet[i]).$$

It is left to reader to check that the isomorphism induces a natural transformation which is an isomorphism.

The following example is taken from [Ber11, 2.2(a)]

Example 6.2.8 ($D^+(\mathbb{Z}-mod)$ is not abelian)**.** We show this by assuming that $D^+(\mathbb{Z}-mod)$ is abelian and then we derive a contradiction. Consider the morphism $f : \mathbb{Z}/2[0] \to \mathbb{Z}/2[1]$ which corresponds the nonzero morphism of $Ext^1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathrm{Mor}_{\mathcal{D}^+(\mathbb{Z}-mod)}(\mathbb{Z}/2[0], \mathbb{Z}/2[1])$, given by the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{1 \mapsto 2} \mathbb{Z}/4 \xrightarrow{1 \mapsto 1, 2 \mapsto 0} \mathbb{Z}/2 \longrightarrow 0 \tag{6.11}$$

This correspondence between morphisms in $Ext^1(\mathbb{Z}/2, \mathbb{Z}/2)$ and short exact sequences of the form (6.11) is given as an exercise at [Lan02, p.831 Ex.27].

Let $g : M^\bullet \to \mathbb{Z}/2[0]$ be the kernel of f. Now we have the following short exact sequence for any $i \in \mathbb{Z}$

$$0 \to \mathrm{Mor}_{D^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], M^\bullet[i]) \overset{g_*}{\to} \mathrm{Mor}_{D^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], (\mathbb{Z}/2)[i]) \overset{f_*}{\to} \mathrm{Mor}_{D^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], (\mathbb{Z}/2)[i+1]) \tag{6.12}$$

The category $\mathbb{Z}-mod$ has enough injective objects so for any complex $N^\bullet \in D^+(\mathbb{Z}-mod)$, by lemma 6.2.6, we have

$$\mathrm{Mor}_{D^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], N^\bullet) \cong \mathrm{Mor}_{D^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], I^\bullet_{N^\bullet}) \cong \mathrm{Mor}_{K^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], I^\bullet_{N^\bullet}).$$

We have the following isomorphisms

$$\mathrm{Mor}_{K^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], I^\bullet_{N^\bullet}[i]) \cong H^i(I^\bullet_{N^\bullet}) \cong H^i(N^\bullet). \tag{6.13}$$

Indeed, the second isomorphism follows from the fact that $I^\bullet_{N^\bullet}$ is quasi-isomorphic to $N^\bullet$ and the first isomorphism follows from the following.

Let $\phi^\bullet, (\phi^\bullet)' : \mathbb{Z}[0] \to I^\bullet_{N^\bullet}[i]$ be morphisms of chain complexes as in the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\phi^0,(\phi')^0"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & I^{i-1} \arrow[r, "d^{-1}"] & I^i \arrow[r, "d^0"] & I^{i+1} \arrow[r] & \dots
\end{tikzcd}$$

Suppose that these morphisms are homotopic, that is $\phi - \phi' = d^{-1}\chi^0$, for some morphism χ^0. By application of the cohomology functor we get the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(\mathbb{Z}[0]) \cong \mathbb{Z} \arrow[r] \arrow[d, "H^0(\phi^\bullet),H^0((\phi')^\bullet)"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^{i-1}(I^\bullet_{N^\bullet}) \arrow[r, "0"] & H^i(I^\bullet_{N^\bullet}) \arrow[r, "0"] & H^{i+1}(I^\bullet_{N^\bullet}) \arrow[r] & \dots
\end{tikzcd}$$

The morphisms $H^0(\phi^\bullet)$ and $H^0((\phi')^\bullet)$ are equal by commutativity and the equation $\phi - \phi' = d^{-1}\chi^0$. Since $\mathrm{Mor}_{\mathbb{Z}-mod}(\mathbb{Z}, H^i(I^\bullet_{N^\bullet})) \cong H^i(I^\bullet_{N^\bullet})$ we have shown that the map $\mathrm{Mor}_{K^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], I^\bullet_{N^\bullet}[i]) \to H^i(I^\bullet_{N^\bullet})$ is injective.

To show that the map is surjective, pick any element of $H^i(I^\bullet_{N^\bullet})$ and let $\psi : \mathbb{Z} \to H^i(I^\bullet_{N^\bullet})$ to be the corresponding morphism of $\mathbb{Z}$-modules. By picking any element of I^i which is mapped to $\psi(1)$ by the cohomology functor, we get the corresponding morphism $\phi : \mathbb{Z}[0] \to I^\bullet_{N^\bullet}$. It is clear that the morphism ϕ is mapped to ψ by the cohomology functor. Therefore the map $\mathrm{Mor}_{K^+(\mathbb{Z}-mod)}(\mathbb{Z}[0], I^\bullet_{N^\bullet}[i]) \to H^i(I^\bullet_{N^\bullet})$ is also surjective, and hence an isomorphism.

Using the isomorphisms (6.13) to the exact sequence (6.12), we get the following exact sequence for any $i \in \mathbb{Z}$

$$0 \longrightarrow H^i(M^\bullet) \xrightarrow{H^i(g)} H^i(\mathbb{Z}/2[0]) \xrightarrow{H^i(f)} H^i(\mathbb{Z}/2[1])$$

The map $H^i(f) : H^i(\mathbb{Z}/2[0]) \to H^i(\mathbb{Z}/2[1])$ is the zero morphism for all i. Thus the morphisms $H^i(g) : H^i(M^\bullet) \to H^i(\mathbb{Z}/2[0])$ are isomorphisms for all i, so g is an isomorphism in $D^+(\mathbb{Z}-mod)$. By lemma 2.2.6 and corollary 2.2.11 this means that f is the zero morphism. This contradicts the assumption on f. Therefore we have obtained a proof of the fact that the category $D^+(\mathbb{Z} - mod)$ is not an abelian category.

6.3 Notes

For the following derived functors Tor^i, $\otimes^L$, Rf_*, $Rf_!$ on sheaves of abelian groups on topological spaces, see [GM03].

In the formalism of ℓ-adic sheaves one has the following six derived functors (Rf_*, Lf^*, $f_!$, $f^!$, $\otimes^L$, $RHom^L$). These are sometimes called Grothendieck's six functors. See [Fu11] for more information about these functors.

Chapter 7

T-structures

In this chapter we introduce t-structures on triangulated categories, following [HTT08, Chapter 8, Section 1] and [GM03, Chapter 3, IV.4]. T-structures allow one to do cohomology on triangulated categories, see theorem 7.3.4.

7.1 T-structures

In this section we define t-structures for triangulated categories. We say that a subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is *strictly full* if it is full, and for $C \in \operatorname{Ob}\mathcal{C}$, $C \cong B$, for some $B \in \operatorname{Ob}\mathcal{B}$, implies $C \in \operatorname{Ob}\mathcal{B}$.

Definition 7.1.1 (T-structure)**.** Let $\mathcal{D}$ be a triangulated category. A *t-structure* on $\mathcal{D}$ is a pair $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ where both $\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 0}$ are strictly full subcategories of $\mathcal{D}$ satisfying the following conditions

T1 We have $\operatorname{Ob}\mathcal{D}^{\leqslant 0} \subset \operatorname{Ob}\mathcal{D}^{\leqslant 1}$ and $\operatorname{Ob}\mathcal{D}^{\geqslant 1} \subset \operatorname{Ob}\mathcal{D}^{\geqslant 0}$.

T2 $\operatorname{Mor}_{\mathcal{D}}(X, Y) = 0$ for any $X \in \operatorname{Ob} D^{\leqslant 0}$ and $Y \in \operatorname{Ob}\mathcal{D}^{\geqslant 1}$.

T3 For any $X \in \operatorname{Ob} D$ there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \operatorname{Ob} D^{\leqslant 0}$, $B \in \operatorname{Ob} D^{\geqslant 1}$.

Here we use the notation $\mathcal{D}^{\leqslant n}$ (resp. $\mathcal{D}^{\geqslant n}$) for $\mathcal{D}^{\leqslant 0}[-n]$ (resp. $\mathcal{D}^{\geqslant 0}[-n]$) for any $n \in \mathbb{Z}$. Let $i_{\leqslant n} : \mathcal{D}^{\leqslant n} \to \mathcal{D}$ and $i_{\geqslant m+1} : \mathcal{D}^{\geqslant m+1} \to \mathcal{D}$ denote the inclusion functors for any $n, m \in \mathbb{Z}$.

Here we prove a useful lemma which generalizes T1.

Lemma 7.1.2. *Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ a t-structure on $\mathcal{D}$. Then for any $m \leqslant n$ we have*

$$\operatorname{Ob}\mathcal{D}^{\leqslant m} \subset \operatorname{Ob}\mathcal{D}^{\leqslant n} \qquad \textit{and} \qquad \operatorname{Ob}\mathcal{D}^{\geqslant n} \subset \operatorname{Ob}\mathcal{D}^{\geqslant m}.$$

Proof. Proof by induction on $n - m$. The case $n - m = 0$ is clear. Note that

$$\operatorname{Ob}\mathcal{D}^{\leqslant m} = \operatorname{Ob}\mathcal{D}^{\leqslant 0}[-m] \subset \operatorname{Ob}\mathcal{D}^{\leqslant 1}[-m] = \operatorname{Ob}\mathcal{D}^{\leqslant m+1}$$

and

$$\operatorname{Ob}\mathcal{D}^{\geqslant n} = \operatorname{Ob}\mathcal{D}^{\geqslant 1}[-n+1] \subset \operatorname{Ob}\mathcal{D}^{\geqslant 0}[-n+1] = \operatorname{Ob}\mathcal{D}^{\leqslant n-1}$$

by T1. Thus the result follows by induction hypothesis. □

The following lemma generalizes T2.

Lemma 7.1.3. *Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ a t-structure on $\mathcal{D}$. Then for any $m < n$, $X \in \operatorname{Ob}\mathcal{D}^{\leqslant m}$ and $Y \in \operatorname{Ob}\mathcal{D}^{\geqslant n}$ we have* $\operatorname{Mor}_{\mathcal{D}}(X, Y) = 0$.

Proof. By induction on $n-m$. The case $n-m=1$ follows from the fact that $\operatorname{Mor}_{\mathcal{D}}(X[m], Y[m]) = 0$ by T2 and that translation is an automorphism. In general, we have $X \in \operatorname{Ob}\mathcal{D}^{\leqslant m} \subset \operatorname{Ob}\mathcal{D}^{\leqslant m+1}$, by lemma 7.1.2, so the result follows from induction hypothesis. □

We will need the following lemma to construct unique isomorphisms between distinguished triangles of the form T3.

Lemma 7.1.4. *Consider the following diagram in $\mathcal{D}$*

$$\begin{tikzcd}
X_1 \arrow[r, "u_1"] & Y_1 \arrow[r, "v_1"] \arrow[d, "g"] & Z_1 \arrow[r, "w_1"] & X_1[1] \\
X_2 \arrow[r, "u_2"] & Y_2 \arrow[r, "v_2"] & Z_2 \arrow[r, "w_2"] & X_2[1]
\end{tikzcd}$$

where both rows are distinguished triangles. If $v_2 g u_1 = 0$, then the diagram can be completed to a morphisms of triangles. Moreover, if $\operatorname{Mor}_{\mathcal{D}}(X_1, Z_2[-1]) = 0$, then the morphism of triangles is unique.

Proof. By proposition 4.1.5 we have the following exact sequence

$$\operatorname{Mor}_{\mathcal{D}}(X_1, Z_2[-1]) \xrightarrow{-w_2[-1]_*} \operatorname{Mor}_{\mathcal{D}}(X_1, X_2) \xrightarrow{u_{2*}} \operatorname{Mor}_{\mathcal{D}}(X_1, Y_2) \xrightarrow{v_{2*}} \operatorname{Mor}_{\mathcal{D}}(X_1, Z_2)$$

From exactness it follows that there exists a morphism $f : X_1 \to X_2$ such that $u_2 f = g u_1$ and that this morphism is unique up to the image of $\operatorname{Mor}_{\mathcal{D}}(X_1, Z_2[-1])$. Hence, by TR5, there we obtain a morphism of triangles.

Suppose $\operatorname{Mor}_{\mathcal{D}}(X_1, Z_2[-1]) = 0$. Then the morphism $f : X_1 \to X_2$ is unique by above. To see that the morphism $Z_1 \to Z_2$, in the morphism of distinguished triangles, is unique, consider the exact sequence

$$\operatorname{Mor}_{\mathcal{D}}(X_1[1], Z_2) \xrightarrow{w_1^*} \operatorname{Mor}_{\mathcal{D}}(Z_1, Z_2) \xrightarrow{v_1^*} \operatorname{Mor}_{\mathcal{D}}(Y_1, Z_2)$$

Now $\operatorname{Mor}_{\mathcal{D}}(X_1[1], Z_2) \cong \operatorname{Mor}_{\mathcal{D}}(X_1, Z_2[-1]) = 0$, because translation is an additive automorphism, and by exactness there exists a unique morphism $h : Z_1 \to Z_2$ such that $h v_1 = v_2 g$. This shows uniqueness of the morphism of triangles. □

7.2 Abstract truncations

In this section we introduce abstract truncation functors for a t-structure. These will be needed in proving that the core of a t-structure is an abelian category. In this section $\mathcal{D}$ denotes a triangulated category and $t = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ a fixed t-structure on $\mathcal{D}$.

Let us construct the abstract truncation functors. For any $X \in \operatorname{Ob} \mathcal{D}$ fix a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

where $A \in \mathcal{D}^{\leqslant 0}$ and $B \in \mathcal{D}^{\geqslant 1}$, given by T3, and let $\tau_{\leqslant 0} X = A$ and $\tau_{\geqslant 1} X = B$. For any integer $n \in \mathbb{Z}$, we define

$$\tau_{\leqslant n} X = (\tau_{\leqslant 0}(X[n]))[-n] \in \mathcal{D}^{\leqslant n} \qquad \text{and} \qquad \tau_{\geqslant n+1} = (\tau_{\geqslant 1}(X[n]))[-n] \in \mathcal{D}^{\geqslant n+1}.$$

Using these definitions, and by application of rotation TR3, one obtains that for any $n \in \mathbb{Z}$ the following is a distinguished triangle

$$\tau_{\leqslant n} X \longrightarrow X \longrightarrow \tau_{\geqslant n+1} X \longrightarrow (\tau_{\leqslant n} X)[1] \tag{7.1}$$

Indeed, by definition of $\tau_{\leqslant 0}$ and $\tau_{\geqslant 1}$ the following is a distinguished triangle

$$\tau_{\leqslant 0}(X[n]) \longrightarrow X[n] \longrightarrow \tau_{\geqslant 1}(X[n]) \longrightarrow (\tau_{\leqslant 0}(X[n]))[1]$$

By rotation, TR3, we get that

$$\tau_{\leqslant 0}(X[n])[-n] \longrightarrow X[n][-n] \longrightarrow \tau_{\geqslant 1}(X[n])[-n] \longrightarrow (\tau_{\leqslant 0}(X[n]))[-n+1]$$

is a distinguished triangle. Since translation is an additive automorphism, $X[n][-n] = X$, and we have obtained that (7.1) is a distinguished triangle.

Let $f : X \to Y$ be any morphism in $\mathcal{D}$. Then by lemma 7.1.3 and lemma 7.1.4, the following diagram is a unique morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant n} X \arrow[r] \arrow[d, "\tau_{\leqslant n} f"] & X \arrow[r] \arrow[d, "f"] & \tau_{\geqslant n+1} X \arrow[r] \arrow[d, "\tau_{\geqslant n+1} f"] & (\tau_{\leqslant n} X)[1] \arrow[d, "f[1]"] \\
\tau_{\leqslant n} Y \arrow[r] & Y \arrow[r] & \tau_{\geqslant n+1} Y \arrow[r] & (\tau_{\leqslant n} X)[1]
\end{tikzcd}$$

for any $n \in \mathbb{Z}$. We define the morphism $\tau_{\leqslant n} f$ and $\tau_{\geqslant n+1} f$ as in the diagram. By using uniqueness, one easily verifies that $\tau_{\leqslant n} : \mathcal{D} \to \mathcal{D}^{\leqslant n}$ and $\tau_{\geqslant n+1} : \mathcal{D} \to \mathcal{D}^{\geqslant n+1}$ are functors.

The following proposition shows that the distinguished triangles, given by T3, are isomorphic up to unique isomorphism. In particular, the functors $\tau_{\leqslant n}$ and $\tau_{\geqslant n+1}$ are unique up to unique isomorphism.

Proposition 7.2.1. *Suppose*

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1] \tag{7.2}$$

is a distinguished triangle, with $A \in \operatorname{Ob} \mathcal{D}^{\leqslant n}$ and $B \in \operatorname{Ob} \mathcal{D}^{\geqslant n+1}$, then it is uniquely isomorphic to the distinguished triangle (7.1)*.*

Proof. Since $\operatorname{Mor}_{\mathcal{D}}(\tau_{\leqslant n} X, B) = 0$ and $\operatorname{Mor}_{\mathcal{D}}(A, \tau_{\geqslant n+1} X) = 0$ by lemma 7.1.3, lemma 7.1.4 implies that there exists a unique morphisms ϕ and ψ between (7.1) and (7.2), because $\operatorname{Mor}_{\mathcal{D}}(\tau_{\leqslant n} X, B[-1]) = 0$ and $\operatorname{Mor}_{\mathcal{D}}(A, (\tau_{\geqslant n+1} X)[-1])$ $= 0$ by lemma 7.1.3. By lemma 7.1.4 and lemma 7.1.3 the only automorphisms of distinguished triangles (7.1) and (7.2) are the identity morphisms. This shows that $\psi\phi$ and $\phi\psi$ are identity morphisms, hence $\psi = \phi^{-1}$, and the distinguished triangles are canonically isomorphic. $\square$

Lemma 7.1.4 and lemma 7.1.3 imply that for any $m \leqslant n$ there are unique morphisms $\tau_{\leqslant m}X \to \tau_{\leqslant n}X$ and $\tau_{\geqslant m}X \to \tau_{\geqslant n}X$, because of the following unique morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant m}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & \tau_{\geqslant m+1}X \arrow[r] \arrow[d] & (\tau_{\leqslant m}X)[1] \arrow[d] \\
\tau_{\leqslant n}X \arrow[r] & X \arrow[r] & \tau_{\geqslant n+1}X \arrow[r] & (\tau_{\leqslant n}X)[1]
\end{tikzcd} \tag{7.3}$$

Lemma 7.2.2. *The following conditions are equivalent for any object X of $\mathcal{D}$ and for any integer $n \in \mathbb{Z}$.*

(i) $X \in \mathcal{D}^{\leqslant n}$.

(ii) The morphism $\tau_{\leqslant n}X \to X$ is an isomorphism.

(iii) $\tau_{\geqslant n+1}X = 0$.

Proof. $(i) \Rightarrow (ii)$: By lemma 7.1.3, the morphism $X \to \tau_{\geqslant n+1}X$ is the zero morphism. Thus by TR3 and TR5 we have the following unique, by lemma 7.1.4, morphism of distinguished triangles

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, dashed, "\psi"] & X \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, dashed, "{\psi[1]}"] \\
\tau_{\leqslant n}X \arrow[r] & X \arrow[r] & \tau_{\geqslant n+1}X \arrow[r] & (\tau_{\leqslant n-1}X)[1]
\end{tikzcd}$$

Also, we have the following unique, by lemma 7.1.4, morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant n}X \arrow[r] \arrow[d, dashed, "\psi'"] & X \arrow[r] \arrow[d, equal] & \tau_{\geqslant n+1}X \arrow[r] \arrow[d] & (\tau_{\leqslant n-1}X)[1] \arrow[d, "{\phi[1]}"] \\
X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & X[1]
\end{tikzcd}$$

where ψ' is the morphism $\tau_{\leqslant n}X \to X$ by commutativity. Composite of these morphisms, in both order, is a unique morphism of distinguished triangles, by lemma 7.1.4, so we have $\psi'\psi = \mathrm{Id}_X$ and $\psi\psi' = \mathrm{Id}_{\tau_{\leqslant n}X}$. Hence $X \cong \tau_{\leqslant n}X$.

$(ii) \Rightarrow (iii)$: This follows from 4.1.6 applied to the following morphism of distinguished triangles obtained by TR5

$$\begin{tikzcd}
\tau_{\leqslant n}X \arrow[r, equal] \arrow[d, equal] & \tau_{\leqslant n}X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, dashed] & (\tau_{\leqslant n}X)[1] \arrow[d] \\
\tau_{\leqslant n}X \arrow[r] & X \arrow[r] & \tau_{\geqslant n+1}X \arrow[r] & (\tau_{\leqslant n}X)[1]
\end{tikzcd}$$

$(iii) \Rightarrow (i)$: Apply 4.1.6 to the following morphism of distinguished triangles obtained by TR3 and TR5

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, dashed] & X \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & (\tau_{\leqslant 0}X)[1] \arrow[d] \\
\tau_{\leqslant n}X \arrow[r] & X \arrow[r] & 0 \arrow[r] & (\tau_{\leqslant n}X)[1]
\end{tikzcd}$$

□

We have also the dual version of the previous lemma.

Proposition 7.2.3. *For any object $X \in \mathcal{D}$ and $n \in \mathbb{Z}$ the following conditions are equivalent.*

(i) $X \in \mathcal{D}^{\geqslant n+1}$.

(ii) The morphism $X \to \tau_{\geqslant n+1}X$ *is an isomorphism.*

(iii) $\tau_{\leqslant n}X = 0$.

Proof. $(i) \Rightarrow (ii)$: The morphism $\tau_{\leqslant n}X \to X$ is 0 by lemma 7.1.3. Hence by TR5 we have the following morphisms of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant n}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & \tau_{\geqslant n+1}X \arrow[r] \arrow[d, "\psi"] & (\tau_{\leqslant n}X)[1] \arrow[d] \\
0 \arrow[r] & X \arrow[r, equal] & X \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "\psi'"] & 0 \arrow[d] \\
\tau_{\leqslant n}X \arrow[r] & X \arrow[r] & \tau_{\geqslant n+1}X \arrow[r] & (\tau_{\leqslant n}X)[1]
\end{tikzcd}$$

where ψ' is the morphism $X \to \tau_{\geqslant n+1}X$ by commutativity. The composites, in both order, of the two morphisms are unique by lemma 7.1.4, so $\mathrm{Id}_X = \psi\psi'$ and $\mathrm{Id}_{\tau_{\geqslant n+1}X} = \psi'\psi$. This shows that $X \cong \tau_{\geqslant n+1}X$.

$(ii) \Rightarrow (iii)$: This follows from corollary 4.1.6 applied to the following morphism of distinguished triangles obtained by TR3 and TR5

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, dashed] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d] & 0 \arrow[d, dashed] \\
\tau_{\leqslant n}X \arrow[r] & X \arrow[r] & \tau_{\geqslant n+1}X \arrow[r] & (\tau_{\leqslant n}X)[1]
\end{tikzcd}$$

$(iii) \Rightarrow (i)$: Apply 4.1.6 to the following morphism of distinguished triangles obtained by TR5

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, dashed] & (\tau_{\leqslant 0}X)[1] \arrow[d] \\
0 \arrow[r] & X \arrow[r] & \tau_{\geqslant n+1}X \arrow[r] & 0
\end{tikzcd}$$

□

We are ready to prove that $\tau_{\leqslant n}$ and $\tau_{\geqslant n+1}$ are adjoints to the corresponding inclusion functors $i_{\leqslant n}$ and $i_{\geqslant n+1}$, respectively.

Proposition 7.2.4 (Adjoints)**.** *The functors $\tau_{\leqslant n}$ and $\tau_{\geqslant n+1}$ are the right and left adjoints to the inclusion functors $i_{\leqslant n} : \mathcal{D}^{\leqslant n} \to \mathcal{D}$ and $i_{\geqslant n+1} : \mathcal{D}^{\geqslant n+1} \to \mathcal{D}$ for all $n \in \mathbb{Z}$.*

Proof. To prove the theorem, we will need the following isomorphisms

$$\Phi^n_{X,Y} : \mathrm{Mor}_{\mathcal{D}}(X, Y) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{D}^{\leqslant n}}(X, \tau_{\leqslant n}Y) \qquad \Psi^{n+1}_{Z,W} : \mathrm{Mor}_{\mathcal{D}^{\geqslant n+1}}(\tau_{\geqslant n+1}Z, W) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{D}}(Z, W)$$

for any $X \in \mathrm{Ob}\,\mathcal{D}^{\leqslant n}$, $W \in \mathrm{Ob}\,\mathcal{D}^{\geqslant n+1}$, and $Y, Z \in \mathrm{Ob}\,\mathcal{D}$. To construct these maps, let $h : X \to Y$ be a morphism in $\mathcal{D}$ and let $k' : \tau_{\geqslant n+1}Z \to W$ be a morphism in $\mathcal{D}^{\geqslant n+1}$. By TR1 to TR3 and TR5 and lemma 7.1.4 we have the following unique morphisms of distinguished triangles in $\mathcal{D}$

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "h'"] & X \arrow[r] \arrow[d, "h"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "h[1]"] \\
\tau_{\leqslant n}Y \arrow[r, "\phi_Y"] & Y \arrow[r] & \tau_{\geqslant n+1}Y \arrow[r] & (\tau_{\leqslant n}Y)[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau_{\leqslant n}Z \arrow[r] \arrow[d] & Z \arrow[r, "\psi_Z"] \arrow[d, "k"] & \tau_{\geqslant n+1}Z \arrow[r] \arrow[d, "k'"] & (\tau_{\leqslant n}Z)[1] \arrow[d] \\
0 \arrow[r] & W \arrow[r, equal] & W \arrow[r] & 0
\end{tikzcd}$$

We define $\Phi^n_{X,Y}(h) = h'$ and $\Psi^{n+1}_{Z,W}(k') = k$. By proposition 4.1.5, applied to the distinguished triangle (7.1) for Y and Z, we have the following exact sequences

$$\mathrm{Mor}_{\mathcal{D}}(X, (\tau_{\geqslant n+1}Y)[-1]) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(X, \tau_{\leqslant n}Y) \xrightarrow{(\phi_Y)_*} \mathrm{Mor}_{\mathcal{D}}(X, Y) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(X, \tau_{\geqslant n+1}Y)$$

and

$$\mathrm{Mor}_{\mathcal{D}}((\tau_{\leqslant n}Z)[1], W) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\tau_{\geqslant n+1}Z, W) \xrightarrow{(\psi_Z)^*} \mathrm{Mor}_{\mathcal{D}}(Z, W) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}Z, W)$$

where $\mathrm{Mor}_{\mathcal{D}}(X, (\tau_{\geqslant n+1}Y)[-1]) = \mathrm{Mor}_{\mathcal{D}}(X, \tau_{\geqslant n+1}Y) = 0$ and $\mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}Z, W) = \mathrm{Mor}_{\mathcal{D}}((\tau_{\leqslant n}Z)[1], W) = 0$, by lemma 7.1.3. Hence from exactness it follows that $\Phi^n_{X,Y}$ and $\Psi^{n+1}_{Z,W}$ are isomorphisms.

$\tau_{\leqslant n}$ **right adjoint to** $i_{\leqslant n}$**:** To show that $\tau_{\leqslant n}$ is right adjoint to $i_{\leqslant n}$, by theorem 1.3.4, it suffices to show that for any morphisms $f : X' \to X$ of $\mathcal{D}^{\leqslant n}$ and $g : Y \to Y'$ of $\mathcal{D}$ the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}}(X, Y) \arrow[r, "\Phi^n_{X,Y}"] \arrow[d, "\mathrm{Mor}_{\mathcal{D}}(f,g)"] & \mathrm{Mor}_{\mathcal{D}^{\leqslant n}}(X, \tau_{\leqslant n}Y) \arrow[d, "\mathrm{Mor}_{\mathcal{D}}(f,\tau_{\leqslant n}g)"] \\
\mathrm{Mor}_{\mathcal{D}}(X', Y') \arrow[r, "\Phi^n_{X',Y'}"] & \mathrm{Mor}_{\mathcal{D}^{\leqslant n}}(X', \tau_{\leqslant n}Y')
\end{tikzcd} \tag{7.4}$$

Let $h \in \mathrm{Mor}_{\mathcal{D}}(X, Y)$. Then $\mathrm{Mor}_{\mathcal{D}}(f, g)(h) = ghf$. Now $\Phi^n_{X',Y'}(ghf)$ is the unique morphism $k : X' \to \tau_{\leqslant n}Y'$ such that $\phi' k = ghf$, by lemma 7.1.4. This means that we have the following unique morphism of distinguished triangles

$$\begin{tikzcd}
X' \arrow[r, equal] \arrow[d, "k"] & X' \arrow[r] \arrow[d, "ghf"] & 0 \arrow[r] \arrow[d] & X'[1] \arrow[d] \\
\tau_{\leqslant n}Y' \arrow[r, "\phi'"] & Y' \arrow[r] & \tau_{\geqslant n+1}Y' \arrow[r] & (\tau_{\geqslant n+1}Y')[1]
\end{tikzcd}$$

The map $\Phi_{X,Y}$ sends h to the unique morphism $l : X \to \tau_{\leqslant n}Y$ such that $\phi l = h$. Now $\mathrm{Mor}_{\mathcal{D}}(f, \tau_{\leqslant n}g)(l)$ is the composite $(\tau_{\leqslant n}g)lf$. Hence we have the following unique morphism of distinguished triangles

$$\begin{tikzcd}
X' \arrow[r, equal] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & X'[1] \arrow[d] \\
X \arrow[r, equal] \arrow[d, "l"] & X \arrow[r] \arrow[d, "h"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d] \\
\tau_{\leqslant n}Y \arrow[r, "\phi"] \arrow[d, "\tau_{\leqslant n}g"] & Y \arrow[r] \arrow[d, "g"] & \tau_{\geqslant n+1}Y \arrow[r] \arrow[d] & (\tau_{\geqslant n+1}Y)[1] \arrow[d, "(\tau_{\leqslant n}g)[1]"] \\
\tau_{\leqslant n}Y' \arrow[r, "\phi'"] & Y' \arrow[r] & \tau_{\geqslant n+1}Y' \arrow[r] & (\tau_{\geqslant n+1}Y')[1]
\end{tikzcd}$$

By lemma 7.1.4, the morphism of distinguished triangles $(X', X', 0) \to (\tau_{\leqslant n}Y', Y', \tau_{\geqslant n+1}Y')$, is unique. Therefore $k = \tau_{\leqslant 0}(g)lf$. This shows that the diagram (7.4) is commutative.

$\tau_{\geqslant n+1}$ **is left adjoint to** $i_{\geqslant n+1}$**:** By theorem 1.3.4 we have to show that for all morphisms $f : X \to X'$ of $\mathcal{D}^{\geqslant n+1}$ and $g : Y' \to Y$ of $\mathcal{D}$, the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}^{\geqslant n+1}}(\tau_{\geqslant n+1}(Y), X) \arrow[r, "\Psi_{Y,X}"] \arrow[d, "\mathrm{Mor}(\tau_{\geqslant n+1}(g), f)"] & \mathrm{Mor}_{\mathcal{D}}(Y, X) \arrow[d, "\mathrm{Mor}(g,f)"] \\
\mathrm{Mor}_{\mathcal{D}^{\geqslant n+1}}(\tau_{\geqslant n+1}(Y'), X') \arrow[r, "\Psi_{Y',X'}"] & \mathrm{Mor}_{\mathcal{D}}(Y', X')
\end{tikzcd} \tag{7.5}$$

Let $h \in \mathrm{Mor}_{\mathcal{D}^{\geqslant n+1}}(\tau_{\geqslant n+1}(Y), X)$. Then $\Psi_{Y',X'}(fh\tau_{\geqslant n+1}(g))$ is the unique morphism $k : Y' \to X'$ such that the following diagram is a unique morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant n}(Y') \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d, "k"] & \tau_{\geqslant n+1}(Y') \arrow[r] \arrow[d, "fh\tau_{\geqslant n+1}(g)"] & (\tau_{\leqslant n}(Y'))[1] \arrow[d] \\
0 \arrow[r] & X' \arrow[r, equal] & X' \arrow[r] & 0
\end{tikzcd}$$

The morphism $\Psi_{Y,X}(h)$ is the unique morphism $l : Y \to X$ such that the following is a morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant n} Y \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "l"] & \tau_{\geqslant n+1} Y \arrow[r] \arrow[d, "h"] & (\tau_{\leqslant n+1} Y)[1] \arrow[d] \\
0 \arrow[r] & X \arrow[r, equal] & X \arrow[r] & 0
\end{tikzcd}$$

Now $\mathrm{Mor}_{\mathcal{D}}(f, g)(l)$ is the unique morphism such that the following diagram is a morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant n}(Y') \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d, "flg"] & \tau_{\geqslant n+1}(Y') \arrow[r] \arrow[d, "fh\tau_{\geqslant n+1}(g)"] & (\tau_{\leqslant n}(Y')) \arrow[d] \\
0 \arrow[r] & X' \arrow[r, equal] & X' \arrow[r] & 0
\end{tikzcd}$$

By uniqueness of proposition 7.2.1, $k = flg$. This shows that the diagram (7.5) is commutative.

□

We have the following isomorphisms for abstract truncations.

Proposition 7.2.5. *For any integers $m \leqslant n$ we have canonical isomorphisms of functors*

$$\tau_{\leqslant m}\tau_{\leqslant n} \cong \tau_{\leqslant m} \cong \tau_{\leqslant n}\tau_{\leqslant m} \qquad \tau_{\geqslant m}\tau_{\geqslant n} \cong \tau_{\geqslant n} \cong \tau_{\geqslant n}\tau_{\geqslant m}.$$

Proof. $\tau_{\leqslant m}\tau_{\leqslant n} \cong \tau_{\leqslant m}$**:** By proposition 7.2.4 and lemma 1.3.2 it suffices to show that $\tau_{\leqslant m}\tau_{\leqslant n}$ is right adjoint to the inclusion functor $i_{\leqslant m} : \mathcal{D}^{\leqslant m} \to \mathcal{D}$. Recall from the proof of proposition 7.2.4 that we have the following isomorphism

$$\Phi^n_{X,Y} : \mathrm{Mor}_{\mathcal{D}}(X, Y) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{D}^{\leqslant n}}(X, \tau_{\leqslant n} Y),$$

for any $n \in \mathbb{Z}$. Let $\Phi'_{X,Y} = \Phi^m_{X,\tau_{\leqslant n} Y} \circ \Phi^n_{X,Y}$. By theorem 1.3.4, it suffices to show that for all morphisms $f : X' \to X$ of $\mathcal{D}^{\leqslant m}$ and $g : Y \to Y'$ of $\mathcal{D}$ the following diagram commutes

$$\begin{tikzcd}
\mathrm{Mor}_{\mathcal{D}}(X, Y) \arrow[r, "\Phi'_{X,Y}"] \arrow[d, "\mathrm{Mor}_{\mathcal{D}}(f,g)"] & \mathrm{Mor}_{\mathcal{D}^{\leqslant m}}(X, \tau_{\leqslant m}\tau_{\leqslant n} Y) \arrow[d, "\mathrm{Mor}_{\mathcal{D}^{\leqslant m}}(f, \tau_{\leqslant m}\tau_{\leqslant n} g)"] \\
\mathrm{Mor}_{\mathcal{D}}(X', Y') \arrow[r, "\Phi'_{X',Y'}"] & \mathrm{Mor}_{\mathcal{D}^{\leqslant m}}(X', \tau_{\leqslant m}\tau_{\leqslant n} Y')
\end{tikzcd}$$

But by the proof of proposition 7.2.4 the left and right squares of the following diagram are commutative

$$\begin{array}{ccccc}
\mathrm{Mor}_{\mathcal{D}}(X,Y) & \xrightarrow{\Phi^n_{X,Y}} & \mathrm{Mor}_{\mathcal{D}^{\leqslant n}}(X,\tau_{\leqslant n}Y) & \xrightarrow{\Phi^m_{X,\tau_{\leqslant n}Y}} & \mathrm{Mor}_{\mathcal{D}^{\leqslant m}}(X,\tau_{\leqslant m}\tau_{\leqslant n}Y) \\
\downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{D}}(f,g)} & & \downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{D}^{\leqslant n}}(f,\tau_{\leqslant n}g)} & & \downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{D}^{\leqslant m}}(f,\tau_{\leqslant m}\tau_{\leqslant n}g)} \\
\mathrm{Mor}_{\mathcal{D}}(X',Y') & \xrightarrow{\Phi^n_{X',Y'}} & \mathrm{Mor}_{\mathcal{D}^{\leqslant n}}(X',\tau_{\leqslant n}Y') & \xrightarrow{\Phi^m_{X',\tau_{\leqslant n}Y'}} & \mathrm{Mor}_{\mathcal{D}^{\leqslant m}}(X',\tau_{\leqslant m}\tau_{\leqslant n}Y')
\end{array}$$

Thus the diagram 7.2 is commutative.

To construct the isomorphism $\tau_{\leqslant m}\tau_{\leqslant n}X \to \tau_{\leqslant m}X$, note that the pairs $(\tau_{\leqslant m}\tau_{\leqslant n}X, \tau_{\leqslant m}\tau_{\leqslant n}X \to X)$ and $(\tau_{\leqslant m}X, \tau_{\leqslant m}X \to X)$ are universal objects from $i_{\leqslant m}$ to X, respectively, by the proof of theorem 1.3.4. Thus there exists the following unique morphism of distinguished triangles

$$\begin{array}{ccccccc}
\tau_{\leqslant m}\tau_{\leqslant n}X & = & \tau_{\leqslant m}\tau_{\leqslant n}X & \longrightarrow & 0 & \longrightarrow & (\tau_{\leqslant m}\tau_{\leqslant n}X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_{\leqslant m}X & \longrightarrow & X & \longrightarrow & \tau_{\geqslant m+1}X & \longrightarrow & (\tau_{\leqslant m}X)[1]
\end{array}$$

where $\tau_{\leqslant m}\tau_{\leqslant n}X \to \tau_{\leqslant m}X$ is the isomorphism.

$\tau_{\geqslant n}\tau_{\geqslant m} \simeq \tau_{\geqslant n}$: The argument is similar as in the previous case. Let $\Psi'_{X,Y} = \Psi^n_{\tau_{\geqslant m}X,Y} \circ \Psi^m_{X,Y}$. These functors are defined in the proof of proposition 7.2.4. By lemma 1.3.2 we need to show that $\tau_{\geqslant n}\tau_{\geqslant m}$ is the left adjoint of $i_{\geqslant n}$. By theorem 1.3.4, it suffices to show that for all morphisms $f : X' \to X$ of $\mathcal{D}$ and $g : Y \to Y'$ of $\mathcal{D}^{\geqslant n}$ the following diagram commutes

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathcal{D}^{\geqslant n}}(\tau_{\geqslant n}\tau_{\geqslant m}X,Y) & \xrightarrow{\Psi'_{X,Y}} & \mathrm{Mor}_{\mathcal{D}}(X,Y) \\
\downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{D}^{\geqslant n}}(\tau_{\geqslant n}\tau_{\geqslant m}f,g)} & & \downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{D}}(f,g)} \\
\mathrm{Mor}_{\mathcal{D}^{\geqslant n}}(\tau_{\geqslant n}\tau_{\geqslant m}X',Y') & \xrightarrow{\Psi'_{X',Y'}} & \mathrm{Mor}_{\mathcal{D}}(X',Y')
\end{array} \tag{7.6}$$

But by the proof of proposition 7.2.4 both of the squares in the following diagram are commutative

$$\begin{array}{ccccc}
\mathrm{Mor}_{\mathcal{D}^{\geqslant n}}(\tau_{\geqslant n}\tau_{\geqslant m}X,Y) & \xrightarrow{\Psi^n_{\tau_{\geqslant m}X,Y}} & \mathrm{Mor}_{\mathcal{D}^{\geqslant m}}(\tau_{\geqslant m}X,Y) & \xrightarrow{\Psi^m_{X,Y}} & \mathrm{Mor}_{\mathcal{D}}(X,Y) \\
\downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{D}^{\geqslant n}}(\tau_{\geqslant n}\tau_{\geqslant m}f,g)} & & \downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{D}^{\geqslant m}}(\tau_{\geqslant m}f,g)} & & \downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{D}}(f,g)} \\
\mathrm{Mor}_{\mathcal{D}^{\geqslant n}}(\tau_{\geqslant n}\tau_{\geqslant m}X',Y') & \xrightarrow{\Psi^n_{\tau_{\geqslant m}X',Y'}} & \mathrm{Mor}_{\mathcal{D}^{\geqslant m}}(\tau_{\geqslant m}X',Y') & \xrightarrow{\Psi^m_{X',Y'}} & \mathrm{Mor}_{\mathcal{D}}(X',Y')
\end{array}$$

This shows that (7.6) is commutative.

To construct the isomorphism $\tau_{\geqslant n}X \to \tau_{\geqslant n}\tau_{\geqslant m}X$, by the proof of theorem 1.3.4 the pairs $(\tau_{\geqslant n}X, X \to \tau_{\geqslant n}X = \Psi^{-1}(\mathrm{Id}_{\tau_{\geqslant n}X}))$ and $(\tau_{\geqslant n}\tau_{\geqslant m}X, X \to \tau_{\geqslant n}\tau_{\geqslant m}X = (\Phi'_{X,X})^{-1}(\mathrm{Id}_{\tau_{\geqslant n}\tau_{\geqslant m}X})$ are universal objects from X to $i_{\geqslant n}$. Hence we have the following unique morphism of distinguished triangles where $\tau_{\geqslant n}X \to \tau_{\geqslant n}\tau_{\geqslant m}X$ is the isomorphism

$$\begin{array}{ccccccc}
\tau_{\leqslant n-1}X & \longrightarrow & X & \longrightarrow & \tau_{\geqslant n}X & \longrightarrow & (\tau_{\geqslant n-1}X)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tau_{\geqslant n}\tau_{\geqslant m}X & = & \tau_{\geqslant n}\tau_{\geqslant m}X & \longrightarrow & 0
\end{array}$$

$\tau_{\leqslant n}\tau_{\leqslant m} \cong \tau_{\leqslant m}$**:** The morphism $\tau_{\leqslant n}\tau_{\leqslant m}X \to \tau_{\leqslant m}X$ is an isomorphism by lemma 7.2.2, because $\tau_{\leqslant m}X \in \mathcal{D}^{\leqslant m} \subset \mathcal{D}^{\leqslant n}$, by lemma 7.1.2. Let $f : X \to Y$ be a morphism in $\mathcal{D}$. Then commutativity of the left square of the following morphism of distinguished triangles shows that the functors are isomorphic.

$$\begin{tikzcd}
\tau_{\leqslant n}\tau_{\leqslant m}X \arrow[r, "\cong"] \arrow[d, "\tau_{\leqslant n}\tau_{\leqslant m}f"] & \tau_{\leqslant m}X \arrow[r] \arrow[d, "\tau_{\leqslant m}f"] & \tau_{\geqslant n+1}\tau_{\leqslant m}X \arrow[r] \arrow[d] & (\tau_{\leqslant n}\tau_{\leqslant m}X)[1] \arrow[d] \\
\tau_{\leqslant n}\tau_{\leqslant m}Y \arrow[r, "\cong"] & \tau_{\leqslant m}Y \arrow[r] & \tau_{\geqslant n+1}\tau_{\leqslant m}Y \arrow[r] & (\tau_{\leqslant n}\tau_{\leqslant m}Y)[1]
\end{tikzcd}$$

$\tau_{\geqslant m}\tau_{\geqslant n} \cong \tau_{\geqslant n}$**:** By proposition 7.2.3, the morphism $\tau_{\geqslant m}\tau_{\geqslant n}X \to \tau_{\geqslant n}X$ is an isomorphism, because $\tau_{\geqslant n}X \in \mathcal{D}^{\geqslant n} \subset \mathcal{D}^{\geqslant m}$ by lemma 7.1.2. Let $f : X \to Y$ be a morphism in $\mathcal{D}$. Then commutativity of the middle square of the following diagram shows that $\tau_{\geqslant m}\tau_{\geqslant n} \simeq \tau_{\geqslant n}$.

$$\begin{tikzcd}
\tau_{\leqslant m-1}\tau_{\geqslant n}X \arrow[r] \arrow[d] & \tau_{\geqslant n}X \arrow[r, "\cong"] \arrow[d, "\tau_{\geqslant n}f"] & \tau_{\geqslant m}\tau_{\geqslant n}X \arrow[r] \arrow[d, "\tau_{\geqslant m}\tau_{\geqslant n}f"] & (\tau_{\leqslant m-1}\tau_{\geqslant n}X)[1] \arrow[d] \\
\tau_{\leqslant m-1}\tau_{\geqslant n}Y \arrow[r] & \tau_{\geqslant n}Y \arrow[r, "\cong"] & \tau_{\geqslant m}\tau_{\geqslant n}Y \arrow[r] & (\tau_{\leqslant m-1}\tau_{\geqslant n}Y)[1]
\end{tikzcd}$$

□

Lemma 7.2.6. *Let*

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

be a distinguished triangle with $A, B \in \mathcal{D}^{\leqslant n}$. Then $X \in \mathcal{D}^{\leqslant n}$. Similarly, if $A, B \in \mathcal{D}^{\geqslant n+1}$, then $X \in \mathcal{D}^{\geqslant n+1}$.

Proof. Let $A, B \in \mathcal{D}^{\leqslant n}$. By lemma 7.2.2 it suffices to show that $\tau_{\geqslant n+1}X = 0$. Consider the following exact sequence

$$\mathrm{Mor}_{\mathcal{D}}(B, \tau_{\geqslant n+1}X) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(X, \tau_{\geqslant n+1}X) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(A, \tau_{\geqslant n+1}X)$$

given by proposition 4.1.5 applied to the given distinguished triangle. Then

$$\mathrm{Mor}_{\mathcal{D}}(B, \tau_{\geqslant n+1}X) = \mathrm{Mor}_{\mathcal{D}}(A, \tau_{\geqslant n+1}X) = 0$$

by lemma 7.1.3. Recall that $\Psi^{n+1}_{X,X} : \mathrm{Mor}_{\mathcal{D}^{\geqslant n+1}}(\tau_{\geqslant n+1}X, \tau_{\geqslant n+1}X) \to \mathrm{Mor}_{\mathcal{D}}(X, \tau_{\geqslant n+1}X)$, defined in the proof of proposition 7.2.4, is an isomorphism. By exactness $\mathrm{Mor}_{\mathcal{D}}(X, \tau_{\geqslant n+1}X) = 0$. Hence

$$\mathrm{Mor}_{\mathcal{D}^{\geqslant n+1}}(\tau_{\geqslant n+1}X, \tau_{\geqslant n+1}X) = 0$$

Now $\mathrm{Id}_{\tau_{\geqslant n+1}X} = 0$, and so $\tau_{\geqslant n+1}X = 0$.

Suppose that $A, B \in \mathcal{D}^{\geqslant n+1}$. By proposition 7.2.3 it suffices to show that $\tau_{\leqslant n}X = 0$. By proposition 4.1.5 we have the following exact sequence

$$\mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, A) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, X) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, B)$$

where $\mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, A) = \mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, B) = 0$ by lemma 7.1.3. Now $\Phi^n_{X,\tau_{\leqslant n}X} : \mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, X) \to \mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, \tau_{\leqslant n}X)$ is an isomorphism. Thus by exactness and $\Phi^n_{X,\tau_{\leqslant n}X}$, $\mathrm{Id}_{\tau_{\leqslant n}X} = 0$. This completes the proof. □

Proposition 7.2.7. *For any $n, m \in \mathbb{Z}$ we have*

$$\tau_{\geqslant m}\tau_{\leqslant n} \cong \tau_{\leqslant n}\tau_{\geqslant m}.$$

Proof. Suppose $m > n$ and let $X \in \operatorname{Ob}\mathcal{D}$. Then by lemma 7.1.2 $\tau_{\leqslant n}X \in \mathcal{D}^{\leqslant m-1}$, so $\tau_{\geqslant m}\tau_{\leqslant n}X = 0$ by lemma 7.2.2. Similarly, by lemma 7.1.2, $\tau_{\geqslant m}X \in \mathcal{D}^{\geqslant n+1}$. Hence $\tau_{\leqslant n}\tau_{\geqslant m}X = 0$ by proposition 7.2.3. Therefore we may assume that $m \leqslant n$.

Let $X \in \operatorname{Ob}\mathcal{D}$ and consider the distinguished triangle (7.1) for $\tau_{\geqslant m}X$

$$\tau_{\leqslant n}\tau_{\geqslant m}X \longrightarrow \tau_{\geqslant m}X \longrightarrow \tau_{\geqslant n+1}\tau_{\geqslant m}X \longrightarrow (\tau_{\leqslant n}\tau_{\geqslant m}X)[1]$$

By TR3 and lemma 7.2.6, $\tau_{\leqslant n}\tau_{\geqslant m}X \in \mathcal{D}^{\geqslant m}$, because $(\tau_{\geqslant n}\tau_{\geqslant m}X)[-1] \cong (\tau_{\geqslant m}\tau_{\geqslant n+1}X)[-1] \in D^{\geqslant m+1} \subset D^{\geqslant m}$, by lemma 7.1.2 and proposition 7.2.5, and $\tau_{\geqslant n+1}\tau_{\geqslant m}X \cong \tau_{\geqslant m}\tau_{\geqslant n+1}X \in \mathcal{D}^{\geqslant m}$, by proposition 7.2.5.

Recall the definition of the maps $\Phi^n_{X,Y}$ and $\Psi^n_{X,Y}$ in the proof of proposition 7.2.4. We have the following composition of isomorphisms

$$\operatorname{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, \tau_{\geqslant m}X) \overset{\Phi^n_{\tau_{\leqslant n}X,\tau_{\geqslant m}Y}}{\longrightarrow} \operatorname{Mor}_{\mathcal{D}}(\tau_{\leqslant n}X, \tau_{\leqslant n}\tau_{\geqslant m}X) \overset{(\Psi^m_{\tau_{\leqslant n}X,\tau_{\leqslant n}\tau_{\geqslant m}Y})^{-1}}{\longrightarrow} \operatorname{Mor}_{\mathcal{D}}(\tau_{\geqslant m}\tau_{\leqslant n}X, \tau_{\leqslant n}\tau_{\geqslant m}X)$$

The image of the composite $c : \tau_{\leqslant n}X \to X \to \tau_{\geqslant m}X$ under $(\Psi^m_{\tau_{\leqslant n}X,\tau_{\leqslant n}\tau_{\geqslant m}Y})^{-1} \circ \Phi^n_{\tau_{\leqslant n}X,\tau_{\geqslant m}Y}$ is the morphism $\phi : \tau_{\geqslant m}\tau_{\leqslant n}X \to \tau_{\leqslant n}\tau_{\geqslant m}X$ obtained by first taking by lemma 7.1.4 the following unique morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant n}X \arrow[r, equal] \arrow[d, "l"] & \tau_{\leqslant n}X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & (\tau_{\leqslant n}X)[1] \arrow[d] \\
\tau_{\leqslant n}\tau_{\geqslant m}X \arrow[r] & \tau_{\geqslant m}X \arrow[r] & \tau_{\geqslant n+1}\tau_{\geqslant m}X \arrow[r] & (\tau_{\leqslant n}\tau_{\geqslant m}X)[1]
\end{tikzcd} \tag{7.7}$$

and then use the morphism l to get the following unique, by lemma 7.1.4, morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant m-1}\tau_{\leqslant n}X \arrow[r] \arrow[d] & \tau_{\leqslant n}X \arrow[r, "k"] \arrow[d, "l"] & \tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r] \arrow[d, "\phi"] & (\tau_{\leqslant m-1}\tau_{\leqslant n}X)[1] \arrow[d] \\
0 \arrow[r] & \tau_{\leqslant n}\tau_{\geqslant m}X \arrow[r, equal] & \tau_{\leqslant n}\tau_{\geqslant m}X \arrow[r] & 0
\end{tikzcd} \tag{7.8}$$

To show that ϕ is an isomorphism, consider the following upper cap diagram

$$\begin{tikzcd}
\tau_{\geqslant n+1}X \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] & & X \arrow[ll] \\
\arrow[r, phantom, "\circlearrowleft"] & \tau_{\leqslant n}X \arrow[ur] \arrow[dl] \arrow[u, phantom, "d"] \arrow[d, phantom, "d"] & \arrow[l, phantom, "\circlearrowleft"] \\
\tau_{\geqslant m}\tau_{\leqslant n}X \arrow[rr, "{[1]}"'] & & \tau_{\leqslant m-1}X \arrow[ul] \arrow[uu]
\end{tikzcd}$$

The lower triangle in the diagram is distinguished because by proposition 7.2.5 we have the following isomorphism of triangles

$$\begin{tikzcd}
\tau_{\leqslant m-1}\tau_{\leqslant n}X \arrow[r] \arrow[d, "\cong"] & \tau_{\leqslant n}X \arrow[r] \arrow[d, equal] & \tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r] \arrow[d, equal] & (\tau_{\leqslant m-1}\tau_{\leqslant n}X)[1] \arrow[d, "\cong"] \\
\tau_{\leqslant m-1}X \arrow[r] & \tau_{\leqslant n}X \arrow[r] & \tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r] & (\tau_{\leqslant m-1}X)[1]
\end{tikzcd}$$

In particular, the morphism $\tau_{\leqslant m-1}X \to \tau_{\leqslant n}X$ in the diagram is the unique morphism given in (7.3) by the proof of proposition 7.2.5. Hence the right triangle in the upper cap is commutative.

Complete the upper cap to the following lower cap

$$\begin{tikzcd}
\tau_{\geqslant n+1}X \arrow[d, "{[1]}"'] & \arrow[d, phantom, "\circlearrowleft"] & X \arrow[ll] \arrow[dl] \\
\arrow[r, phantom, "d"] & X' \arrow[ul] \arrow[dr, "{[1]}"] \arrow[d, phantom, "\circlearrowleft"] & \arrow[l, phantom, "d"] \\
\tau_{\geqslant m}\tau_{\leqslant n}X \arrow[ur] \arrow[rr, "{[1]}"'] & & \tau_{\leqslant m-1}X \arrow[uu]
\end{tikzcd}$$

By proposition 7.2.1, $X' \cong \tau_{\geqslant m}X$. By commutativity of the upper triangle, $\tau_{\geqslant m}X \to \tau_{\geqslant n+1}X$ is the unique morphism given by (7.3). We have the following unique morphism of distinguished triangles, obtained by lemma 7.1.4, where the morphism $\tau_{\geqslant n+1}X \to \tau_{\geqslant n+1}\tau_{\geqslant m}X$ is an isomorphism by proposition 7.2.5

$$\begin{tikzcd}
\tau_{\leqslant n}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \tau_{\geqslant n+1}X \arrow[r] \arrow[d, "\cong"] & (\tau_{\leqslant n}X)[1] \arrow[d] \\
0 \arrow[r] & \tau_{\geqslant n+1}\tau_{\geqslant m}X \arrow[r, equal] & \tau_{\geqslant n+1}\tau_{\geqslant m}X \arrow[r] & 0
\end{tikzcd}$$

Here both of the morphisms $X \to \tau_{\geqslant n+1}X$ and $X \to \tau_{\geqslant n+1}\tau_{\geqslant m}X$ factor through the morphism $X \to \tau_{\geqslant m}X$ by (7.3). From this we get the following isomorphism of triangles

$$\begin{tikzcd}
\tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r] \arrow[d, equal] & \tau_{\geqslant m}X \arrow[r] \arrow[d, equal] & \tau_{\geqslant n+1}X \arrow[r] \arrow[d, "\cong"] & (\tau_{\geqslant m}\tau_{\leqslant n}X)[1] \arrow[d, equal] \\
\tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r] & \tau_{\geqslant m}X \arrow[r] & \tau_{\geqslant n+1}\tau_{\geqslant m}X \arrow[r] & (\tau_{\geqslant m}\tau_{\leqslant n}X)[1]
\end{tikzcd}$$

where $\tau_{\geqslant n+1}\tau_{\geqslant m}X \to (\tau_{\geqslant m}\tau_{\leqslant n}X)[1]$ is the composite

$$\tau_{\geqslant n+1}\tau_{\geqslant m}X \to \tau_{\geqslant n+1}X \to (\tau_{\geqslant m}\tau_{\leqslant n}X)[1]. \tag{7.9}$$

In particular, $(\tau_{\geqslant m}\tau_{\leqslant n}X, \tau_{\geqslant m}X, \tau_{\geqslant n+1}\tau_{\geqslant m}X)$ is a distinguished tringle. Therefore by TR3 and TR5 and corollary 4.1.6, we have a unique isomorphism $\delta(X)$ which makes the following diagram an isomorphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r] \arrow[d, "\delta(X)"] & \tau_{\geqslant m}X \arrow[r] \arrow[d, equal] & \tau_{\geqslant n+1}\tau_{\geqslant m}X \arrow[r] \arrow[d, equal] & (\tau_{\geqslant m}\tau_{\leqslant n}X)[1] \arrow[d, "{\delta(X)[1]}"] \\
\tau_{\leqslant n}\tau_{\geqslant m}X \arrow[r] & \tau_{\geqslant m}X \arrow[r] & \tau_{\geqslant n+1}\tau_{\geqslant m}X \arrow[r] & (\tau_{\leqslant n}\tau_{\geqslant m}X)[1]
\end{tikzcd} \tag{7.10}$$

To verify that $\phi = \delta(X)$, it suffices, by lemma 7.1.4, to show that $l = \delta(X) \circ k$. By commutativity of the octahedra and (7.10) we have the following equalities

$$\begin{aligned}
\tau_{\leqslant n}X \xrightarrow{k} \tau_{\geqslant m}\tau_{\leqslant n}X \xrightarrow{\delta(X)} \tau_{\leqslant n}\tau_{\geqslant m}X \to \tau_{\geqslant m}X &= \tau_{\leqslant n}X \xrightarrow{k} \tau_{\geqslant m}\tau_{\leqslant n}X \to \tau_{\geqslant m}X \\
&= \tau_{\leqslant n}X \to X \to \tau_{\geqslant m}X.
\end{aligned}$$

Since $\Psi^n_{\tau_{\leqslant n}X, \tau_{\geqslant m}X}$ is an isomorphism, we have $\delta(X) \circ k = l$. Therefore $\delta(X) = \phi$.

To show that δ is an isomorphism of functors, we need to verify that for any morphism $f : X \to Y$ the following diagram is commutative

$$\begin{tikzcd}
\tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r, "\delta(X)"] \arrow[d, "\tau_{\geqslant m}\tau_{\leqslant n}(f)"] & \tau_{\leqslant n}\tau_{\geqslant m}X \arrow[d, "\tau_{\leqslant n}\tau_{\geqslant m}(f)"] \\
\tau_{\geqslant m}\tau_{\leqslant n}Y \arrow[r, "\delta(Y)"] & \tau_{\leqslant n}\tau_{\geqslant m}Y
\end{tikzcd}$$

By lemma 7.1.4 we have the following unique morphism of distinguished triangles.

$$\begin{tikzcd}
\tau_{\leqslant n}X \arrow[r] \arrow[d, "\tau_{\leqslant n}f"] & X \arrow[r] \arrow[d, "f"] & \tau_{\geqslant n+1}X \arrow[r] \arrow[d, "\tau_{\geqslant n+1}f"] & (\tau_{\leqslant n}X)[1] \arrow[d, "(\tau_{\leqslant n}f)[1]"] \\
\tau_{\leqslant n}Y \arrow[r] & Y \arrow[r] & \tau_{\geqslant n+1}Y \arrow[r] & (\tau_{\leqslant n}Y)[1]
\end{tikzcd} \tag{7.11}$$

The composite $\tau_{\leqslant n}\tau_{\geqslant m}(f) \circ \delta(X)$ is the unique, by lemma 7.1.4, morphism such that the following composite is a morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r] \arrow[d, "\delta(X)"] & \tau_{\geqslant m}X \arrow[r] \arrow[d, equal] & \tau_{\geqslant n+1}\tau_{\geqslant m}X \arrow[r] \arrow[d, equal] & (\tau_{\geqslant m}\tau_{\leqslant n}X)[1] \arrow[d] \\
\tau_{\leqslant n}\tau_{\geqslant m}X \arrow[r] \arrow[d, "\tau_{\leqslant n}\tau_{\geqslant m}(f)"] & \tau_{\geqslant m}X \arrow[r] \arrow[d, "\tau_{\geqslant m}(f)"] & \tau_{\geqslant n+1}\tau_{\geqslant m}X \arrow[r] \arrow[d, equal] & (\tau_{\leqslant n}\tau_{\geqslant m}X)[1] \arrow[d] \\
\tau_{\leqslant n}\tau_{\geqslant m}Y \arrow[r] & \tau_{\geqslant m}Y \arrow[r] & \tau_{\geqslant n+1}\tau_{\geqslant m}Y \arrow[r] & (\tau_{\leqslant n}\tau_{\geqslant m}Y)[1]
\end{tikzcd} \tag{7.12}$$

The composite $\delta(Y) \circ \tau_{\geqslant m}\tau_{\leqslant n}(f)$ is the unique morphism such that the following composite is the unique, by lemma 7.1.4, morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant m-1}\tau_{\leqslant n}X \arrow[r] \arrow[d, "\tau_{\leqslant m-1}\tau_{\leqslant n}(f)"] & \tau_{\leqslant n}X \arrow[r] \arrow[d, "\tau_{\leqslant n}(f)"] & \tau_{\geqslant m}\tau_{\leqslant n}X \arrow[r] \arrow[d, "\tau_{\geqslant m}\tau_{\leqslant n}(f)"] & (\tau_{\leqslant m-1}\tau_{\leqslant n}X)[1] \arrow[d] \\
\tau_{\leqslant m-1}\tau_{\leqslant n}Y \arrow[r] \arrow[d] & \tau_{\leqslant n}Y \arrow[r] \arrow[d, "l"] & \tau_{\geqslant m}\tau_{\leqslant n}Y \arrow[r] \arrow[d, "\delta(Y)=\phi"] & (\tau_{\leqslant m-1}\tau_{\leqslant n}Y)[1] \arrow[d] \\
0 \arrow[r] & \tau_{\leqslant n}\tau_{\geqslant m}Y \arrow[r, equal] & \tau_{\leqslant n}\tau_{\geqslant m}Y \arrow[r] & 0
\end{tikzcd} \tag{7.13}$$

To show that $\tau_{\leqslant n}\tau_{\geqslant m}(f) \circ \delta(X) = \delta(Y) \circ \tau_{\geqslant m}\tau_{\leqslant n}(f)$, it suffices, by the fact that $(\Psi^m_{X,\tau_{\leqslant n}\tau_{\geqslant m}Y})^{-1}$ is an isomorphism, to show that

$$\begin{aligned}
&\tau_{\leqslant n}X \to \tau_{\geqslant m}\tau_{\leqslant n}X \overset{\delta(X)}{\to} \tau_{\leqslant n}\tau_{\geqslant m}X \overset{\tau_{\leqslant n}\tau_{\geqslant m}(f)}{\to} \tau_{\leqslant n}\tau_{\geqslant m}Y \\
&\quad = \tau_{\leqslant n}X \to \tau_{\geqslant m}\tau_{\leqslant n}X \overset{\tau_{\geqslant m}\tau_{\leqslant n}(f)}{\to} \tau_{\geqslant m}\tau_{\leqslant n}Y \overset{\delta(Y)=\phi}{\to} \tau_{\leqslant n}\tau_{\geqslant m}Y.
\end{aligned} \tag{7.14}$$

By commutativity of (7.13) we have

$$\tau_{\leqslant n}X \to \tau_{\geqslant m}\tau_{\leqslant n}X \overset{\tau_{\geqslant m}\tau_{\leqslant n}(f)}{\to} \tau_{\geqslant m}\tau_{\leqslant n}Y \overset{\delta(Y)=\phi}{\to} \tau_{\leqslant n}\tau_{\geqslant m}Y = \tau_{\leqslant n}X \overset{\tau_{\leqslant n}(f)}{\to} \tau_{\leqslant n}Y \overset{l}{\to} \tau_{\leqslant n}\tau_{\geqslant m}Y.$$

To show equality (7.14), by the fact that $(\Phi^m_{\tau_{\leqslant n}X,\tau_{\leqslant n}\tau_{\geqslant m}Y})^{-1}$ is an isomorphism, it suffices to show that

$$\tau_{\leqslant n}X \to \tau_{\geqslant m}\tau_{\leqslant n}X \overset{\delta(X)}{\to} \tau_{\leqslant n}\tau_{\geqslant m}X \overset{\tau_{\leqslant n}\tau_{\geqslant m}(f)}{\to} \tau_{\leqslant n}\tau_{\geqslant m}Y \to \tau_{\geqslant m}Y = \tau_{\leqslant n}X \overset{\tau_{\leqslant n}(f)}{\to} \tau_{\leqslant n}Y \overset{l}{\to} \tau_{\leqslant n}\tau_{\geqslant m}Y \to \tau_{\geqslant m}Y.$$

By commutativity of (7.7) for Y and (7.11), we have

$$\begin{aligned}\tau_{\leqslant n}X \overset{\tau_{\leqslant n}(f)}{\to} \tau_{\leqslant n}Y \overset{l}{\to} \tau_{\leqslant n}\tau_{\geqslant m}Y \to \tau_{\geqslant m}Y &= \tau_{\leqslant n}X \overset{\tau_{\leqslant n}(f)}{\to} \tau_{\leqslant n}Y \to Y \to \tau_{\geqslant m}Y \\ &= \tau_{\leqslant n}X \to X \overset{f}{\to} Y \to \tau_{\geqslant m}Y.\end{aligned}$$

Now

$$\begin{aligned}&\tau_{\leqslant n}X \to \tau_{\geqslant m}\tau_{\leqslant n}X \overset{\delta(X)}{\to} \tau_{\leqslant n}\tau_{\geqslant m}X \overset{\tau_{\leqslant n}\tau_{\geqslant m}(f)}{\to} \tau_{\leqslant n}\tau_{\geqslant m}Y \to \tau_{\geqslant m}Y \\ &\quad = \tau_{\leqslant n}X \to \tau_{\geqslant m}\tau_{\leqslant n}X \overset{\delta(X)}{\to} \tau_{\leqslant n}\tau_{\geqslant m}X \to \tau_{\geqslant m}X \overset{\tau_{\geqslant m}(f)}{\to} \tau_{\geqslant m}Y && (7.12) \\ &\quad = \tau_{\leqslant n}X \to \tau_{\geqslant m}\tau_{\leqslant n}X \to \tau_{\geqslant m}X \overset{\tau_{\geqslant m}(f)}{\to} \tau_{\geqslant m}Y && (7.12) \\ &\quad = \tau_{\leqslant n}X \to X \to \tau_{\geqslant m}X \overset{\tau_{\geqslant m}(f)}{\to} \tau_{\geqslant m}Y && (7.7) \\ &\quad = \tau_{\leqslant n}X \to X \overset{f}{\to} Y \to \tau_{\geqslant m}Y. && (7.11)\end{aligned}$$

This shows that $\tau_{\geqslant m}\tau_{\geqslant n}(f) \circ \delta(X) = \delta(Y) \circ \tau_{\geqslant m}\tau_{\leqslant n}(f)$ and completes the proof. ☐

7.3 Core and cohomology

In this section we use the abstract adjoint functors introduced in the previous section to prove that the core of a t-structure is an abelian category. Then we show that one can do cohomology on the triangulated category with values in the core.

Definition 7.3.1 (Core)**.** Let $\mathcal{D}$ be a triangulated category and $t = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ a t-structure on $\mathcal{D}$. The *core*, $\mathrm{Core}(t)$, of the t-structure t, is the full subcategory of $\mathcal{D}$ consisting of the objects in $\mathrm{Ob}\,\mathcal{D}^{\leqslant 0} \cap \mathrm{Ob}\,\mathcal{D}^{\geqslant 0}$.

Theorem 7.3.2 (Core is abelian)**.** *Let $\mathcal{D}$ be a triangulated category and $t = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ a t-structure on $\mathcal{D}$. Then $\mathrm{Core}(t)$ is an abelian category.*

Proof. **AB1:** Since $\mathcal{D}$ is an additive category, by T3 we have the following distinguished triangle

$$\tau_{\leqslant 0}0 \longrightarrow 0 \longrightarrow \tau_{\geqslant 1}0 \longrightarrow (\tau_{\leqslant 0}0)[1]$$

Consider the following distinguished triangle given by T3

$$\tau_{\leqslant 0}\tau_{\leqslant 0}0 \longrightarrow \tau_{\leqslant 0}0 \longrightarrow \tau_{\geqslant 1}\tau_{\leqslant 0}0 \longrightarrow (\tau_{\leqslant 0}\tau_{\leqslant 0}0)[1]$$

By lemma 7.2.2, $\tau_{\geqslant 1}\tau_{\leqslant 0}0 = 0$. The translation functor of $\mathcal{D}$ is an additive automorphism, so by the proof of proposition 2.1.5 it preserves the zero object. Hence $0[1] = 0$ and $0 \in \mathcal{D}^{\geqslant 0}$.

Similarly, consider the following distinguished triangle given by T3

$$\tau_{\leqslant 0}\tau_{\geqslant 1}0 \longrightarrow \tau_{\geqslant 1}0 \longrightarrow \tau_{\geqslant 1}\tau_{\geqslant 1}0 \longrightarrow (\tau_{\leqslant 0}\tau_{\geqslant 1}0)[1]$$

We have $\tau_{\leqslant 0}\tau_{\geqslant 1}0 = 0$ by proposition 7.2.3. This shows that $0 \in \mathcal{D}^{\leqslant 0}$. We conclude that $0 \in \mathrm{Ob}\,\mathrm{Core}(t)$.

AB2: By proposition 2.1.3 it suffices to show that Core(t) has all biproducts. Let $X, Y \in$ Core(t). By lemma 7.2.6, it suffices to show that

$$X \xrightarrow{i_1} X \oplus Y \xrightarrow{p_2} Y \xrightarrow{0} X[1] \tag{7.15}$$

is a distinguished triangle in $\mathcal{D}$. It is easy to verify that from properties of a biproduct it follows that the following sequence is exact for any $U \in \operatorname{Ob} \mathcal{D}$

$$\operatorname{Mor}_{\mathcal{D}}((X \oplus Y)[1], U) \xrightarrow{(i_1)^*} \operatorname{Mor}_{\mathcal{D}}(X[1], U) \xrightarrow{0} \operatorname{Mor}_{\mathcal{D}}(Y, U) \xrightarrow{(p_2)^*} \operatorname{Mor}_{\mathcal{D}}(X \oplus Y, U) \xrightarrow{(i_1)^*} \operatorname{Mor}_{\mathcal{D}}(X, U)$$

Complete the morphism i_1 by TR4 to the following distinguished triangle

$$X \xrightarrow{i_1} X \oplus Y \xrightarrow{g} C \xrightarrow{\psi} X[1]\ .$$

We have the following exact sequence

$$\operatorname{Mor}_{\mathcal{D}}((X \oplus Y)[1], C) \xrightarrow{(i_1)^*} \operatorname{Mor}_{\mathcal{D}}(X[1], C) \xrightarrow{0} \operatorname{Mor}_{\mathcal{D}}(Y, C) \xrightarrow{(p_2)^*} \operatorname{Mor}_{\mathcal{D}}(X \oplus Y, C) \xrightarrow{(i_1)^*} \operatorname{Mor}_{\mathcal{D}}(X, C)$$

By exactness and the fact that $gi_1 = 0$, by lemma 4.1.4, there exists a unique morphism $h : Y \to C$ such that $hp_2 = g$. Now

$$\begin{aligned} Y \xrightarrow{h} C \xrightarrow{\psi} X[1] &= Y \xrightarrow{i_2} X \oplus Y \xrightarrow{p_2} Y \xrightarrow{h} C \xrightarrow{\psi} X[1] \\ &= Y \xrightarrow{i_2} X \oplus Y \xrightarrow{g} C \xrightarrow{\psi} X[1] \\ &= 0, \end{aligned}$$

because $\psi g = 0$ by lemma 4.1.4. Hence we have the following morphisms of triangles

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, equal] & X \oplus Y \arrow[r, "p_2"] \arrow[d, equal] & Y \arrow[r, "0"] \arrow[d, "h"] & X[1] \arrow[d, equal] \\ X \arrow[r, "i_1"] & X \oplus Y \arrow[r, "g"] & C \arrow[r, "\psi"] & X[1] \end{tikzcd} \tag{7.16}$$

Therefore, by commutativity of (7.16) and proposition 4.1.5, we have the following commutative diagram with exact rows

$$\begin{tikzcd} \operatorname{Mor}_{\mathcal{D}}(X[1] \oplus Y[1], U) \arrow[r, "(i_1[1])^*"] \arrow[d, equal] & \operatorname{Mor}_{\mathcal{D}}(X[1], U) \arrow[r, "\psi^*"] \arrow[d, equal] & \operatorname{Mor}_{\mathcal{D}}(C, U) \arrow[r, "g^*"] \arrow[d, "h^*"] & \operatorname{Mor}_{\mathcal{D}}(X \oplus Y, U) \arrow[r, "(i_1)^*"] \arrow[d, equal] & \operatorname{Mor}_{\mathcal{D}}(X, U) \arrow[d, equal] \\ \operatorname{Mor}_{\mathcal{D}}(X[1] \oplus Y[1], U) \arrow[r, "(i_1[1])^*"] & \operatorname{Mor}_{\mathcal{D}}(X[1], U) \arrow[r, "0"] & \operatorname{Mor}_{\mathcal{D}}(Y, U) \arrow[r, "(p_2)^*"] & \operatorname{Mor}_{\mathcal{D}}(X \oplus Y, U) \arrow[r, "(i_1)^*"] & \operatorname{Mor}_{\mathcal{D}}(X, U) \end{tikzcd}$$

By lemma 2.6.1 the morphism $h^* : \operatorname{Mor}_{\mathcal{D}}(C, U) \to \operatorname{Mor}_{\mathcal{D}}(Y, U)$ is an isomorphism. Since this holds for every object U, by example 2.1.6 and proposition 2.1.7, $h^* : \operatorname{Mor}_{\mathcal{D}}(C, -) \to \operatorname{Mor}_{\mathcal{D}}(Y, -)$ is a natural transformation which is an isomorphism of additive functors. By corollary 1.1.8 the morphism $h : Y \to C$ is an isomorphism. This shows that the triangle (7.15) is a distinguished triangle.

AB3: Let $f : X \to Y$ be a morphism in Core(t). Complete it to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

We show that the composites $\phi_1 : \tau_{\leqslant 0}(Z[-1]) \to Z[-1] \overset{-h[-1]}{\to} X$ and $\phi_2 : Y \overset{g}{\to} Z \to \tau_{\geqslant 0}Z$ can be used to define the kernel and the cokernel of f in Core(t), respectively. Thus we need to verify that $\tau_{\leqslant 0}(Z[-1]) \in \operatorname{Ob}\mathcal{D}^{\leqslant 0}$ and $\tau_{\geqslant 0}Z \in \operatorname{Ob}\mathcal{D}^{\geqslant 0}$ are objects of Core(t).

First we show that $Z \in \operatorname{Ob}\mathcal{D}^{\leqslant 0} \cap \operatorname{Ob}\mathcal{D}^{\geqslant -1}$. To show that $Z \in \operatorname{Ob}\mathcal{D}^{\leqslant 0}$, it suffices to show that $X[1] \in \operatorname{Ob}\mathcal{D}^{\leqslant 0}$, by lemma 7.2.6. This is equivalent to showing that $X \in \operatorname{Ob}\mathcal{D}^{\leqslant 1}$. But this follows from lemma 7.1.2, because $X \in \operatorname{Ob}\mathcal{D}^{\leqslant 0}$. To show that $Z \in \operatorname{Ob}\mathcal{D}^{\geqslant -1}$, by TR3 and lemma 7.2.6, it suffices to show that $Y \in \operatorname{Ob}\mathcal{D}^{\geqslant -1}$. But this follows from lemma 7.1.2 because $Y \in \operatorname{Ob}\mathcal{D}^{\geqslant 0}$.

To show that $\tau_{\leqslant 0}(Z[-1]) \in \operatorname{Ob}\mathcal{D}^{\geqslant 0}$, note that $Z[-1] \in \mathcal{D}^{\geqslant 0}$. Therefore $\tau_{\leqslant 0}(Z[-1]) \cong \tau_{\leqslant 0}\tau_{\geqslant 0}(Z[-1]) \cong \tau_{\geqslant 0}\tau_{\leqslant 0}(Z[-1]) \in \mathcal{D}^{\geqslant 0}$ by proposition 7.2.3 and proposition 7.2.7. By lemma 7.2.2, proposition 7.2.7, and the fact that $Z \in \mathcal{D}^{\leqslant 0}$, we have $\tau_{\geqslant 0}Z \cong \tau_{\geqslant 0}\tau_{\leqslant 0}Z \cong \tau_{\leqslant 0}\tau_{\geqslant 0}Z \in \operatorname{Ob}\mathcal{D}^{\leqslant 0}$. These show that $\tau_{\leqslant 0}(Z[-1]), \tau_{\geqslant 0}Z \in \operatorname{Ob}\operatorname{Core}(t)$.

$(\tau_{\geqslant 0}Z, \phi_2)$ **is the cokernel of** f**:** Let us show that ϕ_2 is the cokernel of f. Let $\phi : Y \to T$ be a morphism in Core(t) such that $\phi f = 0$ and consider the following distinguished triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

By proposition 4.1.5 we have the following exact sequence

$$\operatorname{Mor}_{\mathcal{D}}(X[1], T) \xrightarrow{h^*} \operatorname{Mor}_{\mathcal{D}}(Z, T) \xrightarrow{g^*} \operatorname{Mor}_{\mathcal{D}}(Y, T) \xrightarrow{f^*} \operatorname{Mor}_{\mathcal{D}}(X, T) \tag{7.17}$$

Since $X[1] \in \mathcal{D}^{\leqslant -1}$, $\operatorname{Mor}_{\mathcal{D}}(X[1], T) = 0$, by lemma 7.1.3. By assumption the morphism ϕ is sent to 0 by f^*, so by exactness there exists a unique morphism $\delta_1 : Z \to T$ such that $\delta_1 g = \phi$. Since $\Psi^0_{Z,T}$ is an isomorphism, we have $\operatorname{Mor}_{\mathcal{D}}(\tau_{\geqslant 0}Z, T) \cong \operatorname{Mor}_{\mathcal{D}}(Z, T)$. Thus there exists a unique morphism δ_2 such that $\delta_2\phi_2 = \phi$. This shows that ϕ_2 is the cokernel of f.

$(\tau_{\leqslant 0}(Z[-1]), \phi_1)$ **is the kernel of** f**:** To show that ϕ_1 is the kernel of f, let $\phi : T \to X$ be a morphism in Core(t) such that $f\phi = 0$, and consider the following distinguished triangle

$$Y[-1] \xrightarrow{-g[-1]} Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y$$

By proposition 4.1.5 we have the following exact sequence

$$\operatorname{Mor}_{\mathcal{D}}(T, Y[-1]) \xrightarrow{(-g[-1])_*} \operatorname{Mor}_{\mathcal{D}}(T, Z[-1]) \xrightarrow{(-h[-1])_*} \operatorname{Mor}_{\mathcal{D}}(T, X) \xrightarrow{f_*} \operatorname{Mor}_{\mathcal{D}}(T, Y) \xrightarrow{g_*} \operatorname{Mor}_{\mathcal{D}}(T, Z)$$

By lemma 7.1.3, $\operatorname{Mor}_{\mathcal{D}}(T, Y[-1]) = 0$, and $f\phi = 0$, by assumption, so by exactness there exists a unique morphism $\delta_1 : T \to Z[-1]$ such that $(-h[-1])\delta_1 = \phi$. Since $\Phi^0_{T,Z[-1]}$ is an isomorphism, we have $\operatorname{Mor}_{\mathcal{D}}(T, Z[-1]) \cong \operatorname{Mor}_{\mathcal{D}}(T, \tau_{\leqslant 0}(Z[-1]))$. Therefore there exists a unique morphism δ_2 such that $\phi_1\delta_2 = \phi$. This shows that ϕ_1 is the kernel of f.

AB4: Let $f : X \to Y$ be an epimorphism in $\mathrm{Core}(t)$. By the proof of AB3 we have the following distinguished triangle in $\mathcal{D}$

$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$$

and the composite $\tau_{\leqslant -1}Z[-1] \to Z[-1] \overset{-h[-1]}{\to} X$ is the kernel of f. By proposition 4.1.5 we have the following exact sequence

$$\mathrm{Mor}_{\mathcal{D}}(Y,T) \xrightarrow{f^*} \mathrm{Mor}_{\mathcal{D}}(X,T) \xrightarrow{(-h[-1])^*} \mathrm{Mor}_{\mathcal{D}}(Z[-1],T) \tag{7.18}$$

for any $T \in \mathrm{Core}(t)$. If $\phi : X \to T$ is a morphism in $\mathrm{Core}(t)$ such that $\phi(-h[-1]) = 0$, then by exactness of (7.18) there exists a morphism $\psi : Y \to T$ such that $\psi f = \phi$. Since f is an epimorphism, the morphism ψ is unique with this property. This shows that f is the cokernel of $-h[-1]$.

Now, we show that $(\tau_{\leqslant 0}(Z[-1]) \to Z[-1])^* : \mathrm{Mor}_{\mathcal{D}}(Z[-1],T) \to \mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant 0}(Z[-1]),T)$ is an isomorphism, from which it follows that f is the cokernel of the composite $\tau_{\leqslant 0}(Z[-1]) \to Z[-1] \to X$. Consider the following morphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant 0}(Z[-1]) \arrow[r] \arrow[d] & Z[-1] \arrow[r] \arrow[d] & \tau_{\geqslant 1}(Z[-1]) \arrow[r] \arrow[d] & (\tau_{\leqslant 0}(Z[-1]))[1] \arrow[d] \\
T \arrow[r, equal] & T \arrow[r] & 0 \arrow[r] & T[1]
\end{tikzcd}$$

It follows from TR3 and TR5 and lemma 7.1.4 that if we have a morphism $Z[-1] \to T$ then we get a unique morphism of distinguished triangles as above and if we have a morphism $\tau_{\leqslant 0}(Z[-1]) \to T$, then we get a unique morphism of distinguished triangles as above. Therefore we have a bijection from $\mathrm{Mor}_{\mathcal{D}}(Z[-1],T)$ to $\mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant 0}(Z[-1]),T)$. Since $\tau_{\leqslant 0}(Z[-1]) \in \mathrm{Core}(t)$ by the proof of AB3, this shows that every epimorphism is the cokernel of some morphism of $\mathrm{Core}(t)$.

To show that every monomorphism is a kernel, let $f : X \to Y$ be a monomorphism in $\mathrm{Core}(t)$. We have the following distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

and the composite $Y \overset{g}{\to} Z \to \tau_{\geqslant 0}Z$ is the cokernel of f by the previous step. By proposition 4.1.5 we have the following exact sequence

$$\mathrm{Mor}_{\mathcal{D}}(T,X) \xrightarrow{f_*} \mathrm{Mor}_{\mathcal{D}}(T,Y) \xrightarrow{g_*} \mathrm{Mor}_{\mathcal{D}}(T,Z) \tag{7.19}$$

for any $T \in \mathrm{Ob}\,\mathrm{Core}(t)$. If $\phi : T \to Y$ is a morphism such that $g\phi = 0$, then by exactness of (7.19) there exists a morphism $\psi : T \to X$ such that $\phi = f\psi$. Since f is a monomorphism, ψ is unique with this property. Hence f is the kernel of g.

We show that $(Z \to \tau_{\geqslant 0}Z)_* : \mathrm{Mor}_{\mathcal{D}}(T,Z) \to \mathrm{Mor}_{\mathcal{D}}(T,\tau_{\geqslant 0}Z)$ is an isomorphism. Consider a morphism of distinguished triangles of the form

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T \arrow[r, equal] \arrow[d] & T \arrow[r] \arrow[d] & 0[1] \arrow[d] \\
\tau_{\leqslant -1}Z \arrow[r] & Z \arrow[r] & \tau_{\geqslant 0}Z \arrow[r] & (\tau_{\leqslant -1}Z)[1]
\end{tikzcd}$$

If we are given any morphism $T \to Z$, then by lemma 7.1.4 there exists a unique morphism of distinguished triangles as above. By TR3, TR5, and lemma 7.1.4, if we are given any morphism $T \to \tau_{\geqslant 0} Z$, then there exists a unique morphism of distinguished triangles as above. These together show that $(Z \to \tau_{\geqslant 0} Z)_*$ is an isomorphism. Therefore f is the kernel of the composite $Y \xrightarrow{g} Z \to \tau_{\geqslant 0} Z$.

□

Next we show that t-structures allow one to define cohomology on triangulated categories, with values in the core.

Definition 7.3.3 (Cohomology functor)**.** Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ which maps any distinguished triangle (X, Y, Z, u, v, w) to an exact sequence

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

in $\mathcal{A}$ is *cohomological.*

Let $\mathcal{D}$ be a triangulated category and $t = (D^{\leqslant 0}, D^{\geqslant 0})$ a t-structure on $\mathcal{D}$. Let

$$H^0 := \tau_{\leqslant 0}\tau_{\geqslant 0} : \mathcal{D} \to \mathrm{Core}(t) \qquad \text{and} \qquad H^n(X) = H^0(X[n]).$$

This functor is well-defined, because $\tau_{\leqslant 0}\tau_{\geqslant 0} \cong \tau_{\geqslant 0}\tau_{\leqslant 0}$, by proposition 7.2.7. It is easy to see from the definition that if the functor H^0 is cohomological, so are all the H^n, $n \in \mathbb{Z}$, by TR3.

Note that for all $n \in \mathbb{Z}$ we have the following by definition of abstract truncation functors

$$H^n(X) = \tau_{\leqslant 0}\tau_{\geqslant 0}(X[n]) = \tau_{\leqslant 0}((\tau_{\geqslant n} X)[n]) = (\tau_{\leqslant n}\tau_{\geqslant n} X)[n].$$

Theorem 7.3.4 (Cohomology functor)**.** *The functor H^0 is a cohomology functor.*

Proof. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{7.20}$$

be a distinguished triangle in $\mathcal{D}$. We prove the theorem in several steps.

Step 1, $X, Y, Z \in \mathrm{Ob}\,\mathcal{D}^{\leqslant 0}$: We show that the sequence

$$H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z) \xrightarrow{H^0(h)} H^0(X[1]) \cong 0 \tag{7.21}$$

is exact, where the isomorphism $H^0(X[1]) \cong 0$ follows from lemma 7.2.2 because $X[1] \in \mathcal{D}^{\leqslant -1}$. Since $\tau_{\leqslant 0}$ and $\tau_{\geqslant 0}$ are adjoints to inclusions, for any objects $U \in \mathcal{D}^{\leqslant 0}$ and $V \in \mathcal{D}^{\geqslant 0}$ we have

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{D}}(H^0(U), H^0(V)) &= \mathrm{Mor}_{\mathcal{D}}(\tau_{\leqslant 0}\tau_{\geqslant 0} U, \tau_{\leqslant 0}\tau_{\geqslant 0} V) & & \\
&\cong \mathrm{Mor}_{\mathcal{D}}(\tau_{\geqslant 0}\tau_{\leqslant 0} U, \tau_{\leqslant 0}\tau_{\geqslant 0} V) & & 7.2.7 \\
&\cong \mathrm{Mor}_{\mathcal{D}}(\tau_{\geqslant 0} U, \tau_{\leqslant 0} V) & & 7.2.2 \text{ and } 7.2.3 \\
&\cong \mathrm{Mor}_{\mathcal{D}}(U, \tau_{\leqslant 0} V) & & \Psi^0_{U, \tau_{\leqslant 0} V} \\
&\cong \mathrm{Mor}_{\mathcal{D}}(U, V) & & (\Phi^0_{U,V})^{-1}.
\end{aligned} \tag{7.22}$$

Using these isomorphisms, by lemma 7.1.3, lemma 7.2.2 and proposition 7.2.3, for any object $W \in \mathrm{Ob}\,\mathrm{Core}(t)$ we have

$$\mathrm{Mor}_{\mathcal{D}}(H^0(X[1]), W) \cong \mathrm{Mor}_{\mathcal{D}}(X[1], W) = 0. \tag{7.23}$$

The following exact sequence is obtained by proposition 4.1.5 applied to (7.20) and using (7.23)

$$\operatorname{Mor}_{\mathcal{D}}(H^0(X[1]),W) \xrightarrow{H^0(h)^*} \operatorname{Mor}_{\mathcal{D}}(H^0(Z),W) \xrightarrow{H^0(g)^*} \operatorname{Mor}_{\mathcal{D}}(H^0(Y),W) \xrightarrow{H^0(f)^*} \operatorname{Mor}_{\mathcal{D}}(H^0(X),W)\ . \tag{7.24}$$

To show that the sequence (7.21) is exact, by proposition 2.2.14 it suffices to show that $H^0(g)$ is the cokernel of $H^0(f)$. Let $\phi : H^0(Y) \to W$ be a morphism in $\operatorname{Core}(t)$ such that $\phi H^0(f) = 0$. By exactness of (7.24), there exists a unique morphism $\psi : H^0(Z) \to W$ such that $\phi = \psi H^0(g)$. This shows that $H^0(g)$ is the cokernel of $H^0(f)$ and the sequence (7.21) is exact.

Step 2, $X \in \operatorname{Ob}\mathcal{D}^{\leqslant 0}$**:** In this step we show that the sequence (7.21) is exact also in this case. First we show that $\tau_{\geqslant 1}g : \tau_{\geqslant 1}Y \to \tau_{\geqslant 1}Z$ is an isomorphism. Let $W \in \operatorname{Ob}\mathcal{D}^{\geqslant 1}$. Then by proposition 4.1.5, we have the following exact sequence

$$\operatorname{Mor}_{\mathcal{D}}(X[1],W) \xrightarrow{h^*} \operatorname{Mor}_{\mathcal{D}}(Z,W) \xrightarrow{g^*} \operatorname{Mor}_{\mathcal{D}}(Y,W) \xrightarrow{f^*} \operatorname{Mor}_{\mathcal{D}}(X,W) \tag{7.25}$$

associated to (7.20). Now $\operatorname{Mor}_{\mathcal{D}}(X,W) = 0$ and $\operatorname{Mor}_{\mathcal{D}}(X[1],W) = 0$ by lemma 7.1.3. By exactness we obtain that g^* is an isomorphism. To show that $\tau_{\geqslant 1}g : \tau_{\geqslant 1}Y \to \tau_{\geqslant 1}Z$ is an isomorphism, recall that by theorem 1.3.4 the following diagram is commutative

$$\begin{tikzcd}
\operatorname{Mor}_{\mathcal{D}^{\geqslant 1}}(\tau_{\geqslant 1}Z,W) \arrow[r, "\Psi^1_{Z,W}"] \arrow[d, "\operatorname{Mor}_{\mathcal{D}^{\geqslant 1}}(\tau_{\geqslant 1}g,\operatorname{Id}_W)"] & \operatorname{Mor}_{\mathcal{D}}(Z,W) \arrow[d, "\operatorname{Mor}_{\mathcal{D}}(g,\operatorname{Id}_W)"] \\
\operatorname{Mor}_{\mathcal{D}^{\geqslant 1}}(\tau_{\geqslant 1}Y,W) \arrow[r, "\Psi^1_{Y,W}"] & \operatorname{Mor}_{\mathcal{D}}(Y,W)
\end{tikzcd}$$

Since $g^* = \operatorname{Mor}_{\mathcal{D}}(g,\operatorname{Id}_W)$ is an isomorphism, this shows that $\operatorname{Mor}_{\mathcal{D}^{\geqslant 1}}(\tau_{\geqslant 1}g,\operatorname{Id}_W)$ is an isomorphism. Hence the representable functors $\operatorname{Mor}_{\mathcal{D}^{\geqslant 1}}(\tau_{\geqslant 1}Z,-)$ and $\operatorname{Mor}_{\mathcal{D}^{\geqslant 1}}(\tau_{\geqslant 1}Y,-)$ are isomorphic. Thus, by corollary 1.1.8, $\tau_{\geqslant 1}g : \tau_{\geqslant 1}Y \to \tau_{\geqslant 1}Z$ is an isomorphism.

To show that the sequence (7.21) is exact, we complete a lower cap to octahedra. See appendix (A.1) for a proof. Let

$$\begin{tikzcd}
\tau_{\geqslant 1}Z \arrow[dd, "{[1]}"'] \arrow[dr, phantom, "d" near end, shift right=6] & & Y \arrow[ll] \arrow[dl] \\
 & Z \arrow[ul] \arrow[dr, "{[1]}"] \arrow[u, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] \arrow[r, phantom, "d"] & \\
\tau_{\leqslant 0}Z \arrow[ur] \arrow[rr, "{[1]}"'] & & X \arrow[uu]
\end{tikzcd}$$

be a lower cap and complete it to the following upper cap

$$\begin{tikzcd}
\tau_{\geqslant 1}Z \arrow[dd, "{[1]}"'] \arrow[dr, "{[1]}"] & & Y \arrow[ll] \\
 & V \arrow[ur] \arrow[dl] \arrow[u, phantom, "d"] \arrow[d, phantom, "d"] \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
\tau_{\leqslant 0}Z \arrow[rr, "{[1]}"'] & & X \arrow[uu] \arrow[ul]
\end{tikzcd}$$

By assumption $X \in \mathcal{D}^{\leqslant 0}$, and because $(X, V, \tau_{\leqslant 0} Z)$ is a distinguished triangle, by lemma 7.2.6 $V \in \operatorname{Ob} \mathcal{D}^{\leqslant 0}$. Therefore by proposition 7.2.1 there exists a unique isomorphism of distinguished triangles from $(V, Y, \tau_{\geqslant 1} Z)$ to $(\tau_{\leqslant 0} Y, Y, \tau_{\geqslant 1} Y)$. In particular, $V \cong \tau_{\leqslant 0} Y$, and we get that $(X, \tau_{\leqslant 0} Y, \tau_{\leqslant 0} Z)$ is a distinguished triangle. Now $H^0(Y) = \tau_{\leqslant 0}\tau_{\geqslant 0} Y \cong \tau_{\geqslant 0}\tau_{\leqslant 0} Y \cong \tau_{\geqslant 0}\tau_{\leqslant 0}\tau_{\leqslant 0} Y \cong \tau_{\leqslant 0}\tau_{\geqslant 0}\tau_{\leqslant 0} Y = H^0(\tau_{\leqslant 0} Y)$ and $H^0(Z) = \tau_{\leqslant 0}\tau_{\geqslant 0} Z \cong \tau_{\geqslant 0}\tau_{\leqslant 0} Z \cong \tau_{\geqslant 0}\tau_{\leqslant 0}\tau_{\leqslant 0} Z \cong \tau_{\leqslant 0}\tau_{\geqslant 0}\tau_{\leqslant 0} Z = H^0(\tau_{\leqslant 0} Z)$ by lemma 7.2.2 and propositions 7.2.5 and 7.2.7. Therefore the sequence (7.21) is exact by the previous step applied to the distinguished triangle $(X, \tau_{\leqslant 0} Y, \tau_{\leqslant 0} Z)$.

Step 3, $X, Y, Z \in \operatorname{Ob} \mathcal{D}^{\geqslant 0}$: We show that the sequence

$$H^0(Z[-1]) \cong 0 \xrightarrow{H^0(-h[-1])} H^0(X) \xrightarrow{H^0(f)} H^0(Y) \xrightarrow{H^0(g)} H^0(Z) \tag{7.26}$$

is exact. Consider the following exact sequence given by proposition 4.1.5 and (7.22) for any $W \in \operatorname{Ob} \operatorname{Core}(t)$

$$\operatorname{Mor}_{\mathcal{D}}(W, H^0(Z[-1])) \xrightarrow{H^0(-h[-1])_*} \operatorname{Mor}_{\mathcal{D}}(W, H^0(X)) \xrightarrow{H^0(f)_*} \operatorname{Mor}_{\mathcal{D}}(W, H^0(Y)) \xrightarrow{H^0(g)_*} \operatorname{Mor}_{\mathcal{D}}(W, H^0(Z))$$

where $\operatorname{Mor}_{\mathcal{D}}(W, H^0(Z[-1])) \cong 0$, because $H^0(Z[-1]) = \tau_{\leqslant 0}\tau_{\geqslant 0}(Z[-1]) \cong 0$ by lemma 7.2.2. To show that (7.26) is exact, by proposition 2.2.14 it suffices to show that $H^0(f)$ is the kernel of $H^0(g)$. Let $\phi : W \to H^0(Y)$ be a morphism in $\operatorname{Core}(t)$ such that $H^0(g)\phi = 0$. By exactness of the above exact sequence, there exists a unique morphism $\phi : W \to H^0(X)$ such that $\phi = H^0(f)\psi$. This shows that $H^0(f)$ is the kernel of $H^0(g)$.

Step 4, $Z \in \operatorname{Ob} \mathcal{D}^{\geqslant 0}$: Let us first show that the morphism $\tau_{\leqslant -1} f : \tau_{\leqslant -1} X \to \tau_{\leqslant -1} Y$ is an isomorphism. For any object $W \in \operatorname{Ob} \mathcal{D}^{\leqslant -1}$ we have the following exact sequence

$$\operatorname{Mor}_{\mathcal{D}}(W, Z[-1]) \xrightarrow{(-h[-1])_*} \operatorname{Mor}_{\mathcal{D}}(W, X) \xrightarrow{f_*} \operatorname{Mor}_{\mathcal{D}}(W, Y) \xrightarrow{g_*} \operatorname{Mor}_{\mathcal{D}}(W, Z)$$

By lemma 7.1.3 $\operatorname{Mor}_{\mathcal{D}}(W, Z[-1]) \cong \operatorname{Mor}_{\mathcal{D}}(W, Z) \cong 0$. Therefore f_* is an isomorphism. Consider the following commutative diagram for any $W \in \operatorname{Ob} \mathcal{D}^{\leqslant -1}$

$$\begin{tikzcd}
\operatorname{Mor}_{\mathcal{D}}(W, X) \arrow[r, "\Phi_{W,Y}"] \arrow[d, "\operatorname{Mor}_{\mathcal{D}}(\operatorname{Id}_W, f)"] & \operatorname{Mor}_{\mathcal{D}^{\leqslant -1}}(W, \tau_{\leqslant -1} X) \arrow[d, "\operatorname{Mor}_{\mathcal{D}^{\leqslant -1}}(\operatorname{Id}_W, \tau_{\leqslant -1} f)"] \\
\operatorname{Mor}_{\mathcal{D}}(W, Y) \arrow[r, "\Phi_{W,X}"] & \operatorname{Mor}_{\mathcal{D}^{\leqslant -1}}(W, \tau_{\leqslant -1} Y)
\end{tikzcd}$$

Since $f_* = \operatorname{Mor}_{\mathcal{D}}(\operatorname{Id}_W, f)$ is an isomorphism, by commutativity $\operatorname{Mor}_{\mathcal{D}^{\leqslant -1}}(\operatorname{Id}_W, \tau_{\leqslant -1} f)$ is an isomorphism. Hence the functors $\operatorname{Mor}_{\mathcal{D}^{\leqslant -1}}(-, \tau_{\leqslant -1} X)$ and $\operatorname{Mor}_{\mathcal{D}^{\leqslant -1}}(-, \tau_{\leqslant -1} Y)$ from $(\mathcal{D}^{\leqslant -1})^{op}$ to $\mathbf{Ab}$ are isomorphic, so by corollary 1.1.8 we get that $\tau_{\leqslant -1} f : \tau_{\leqslant -1} X \to \tau_{\leqslant -1} Y$ is an isomorphism.

Consider the following upper cap

$$\begin{tikzcd}
Z \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] \arrow[rr, phantom, "d" near end, shift right=4] & & Y \arrow[ll] \\
\arrow[r, phantom, "\circlearrowleft"] & X \arrow[ur] \arrow[dl] \arrow[r, phantom, "\circlearrowleft"] & {} \\
\tau_{\geqslant 0} X \arrow[rr, "{[1]}"'] \arrow[rr, phantom, "d", shift left=4] & & \tau_{\leqslant -1} X \arrow[ul] \arrow[uu]
\end{tikzcd}$$

and by TR6 complete it to the following lower cap

$$\begin{tikzcd}
Z \arrow[dd, "{[1]}"'] \arrow[rr, phantom, "\circlearrowleft"', shift right=4ex] \arrow[dd, phantom, "d", shift left=5ex] & & Y \arrow[ll] \arrow[dl] \arrow[dd, phantom, "d", shift right=5ex] \\
& V \arrow[ul] \arrow[dr, "{[1]}"] & \\
\tau_{\geqslant 0}X \arrow[ur] \arrow[rr, "{[1]}"'] \arrow[rr, phantom, "\circlearrowleft", shift left=4ex] & & \tau_{\leqslant -1}X \arrow[uu]
\end{tikzcd}$$

By lemma 7.2.6 $V \in \operatorname{Ob}\mathcal{D}^{\geqslant 0}$. Since $\tau_{\leqslant -1}f : \tau_{\leqslant -1}X \to \tau_{\leqslant -1}Y$ is an isomorphism, $(\tau_{\leqslant -1}Y, Y, V)$ is a distinguished triangle, canonically isomorphic to $(\tau_{\leqslant -1}Y, Y, \tau_{\geqslant 0}Y)$, by proposition 7.2.1. Hence $V \cong \tau_{\geqslant 0}Y$, so $(\tau_{\geqslant 0}X, \tau_{\geqslant 0}Y, Z)$ is a distinguished triangle. Now the fact that (7.26) is an exact sequence follows from the isomorphisms $H^0(X) = \tau_{\leqslant 0}\tau_{\geqslant 0}X \cong \tau_{\leqslant 0}\tau_{\geqslant 0}\tau_{\geqslant 0}X = H^0(\tau_{\geqslant 0}X)$, $H^0(Y) = \tau_{\leqslant 0}\tau_{\geqslant 0}Y \cong \tau_{\leqslant 0}\tau_{\geqslant 0}\tau_{\geqslant 0}Y = H^0(\tau_{\geqslant 0}Y)$ and Step 3 applied to $(\tau_{\geqslant 0}X, \tau_{\geqslant 0}Y, Z)$.

General case: Consider the following upper cap

$$\begin{tikzcd}
Z \arrow[dd, "{[1]}"'] \arrow[dr, "{[1]}"] \arrow[rr, phantom, "d"', shift right=3ex] \arrow[dd, phantom, "\circlearrowleft", shift left=5ex] & & Y \arrow[ll] \arrow[dd, phantom, "\circlearrowleft", shift right=5ex] \\
& X \arrow[ur] \arrow[dl] & \\
\tau_{\geqslant 1}X \arrow[rr, "{[1]}"'] \arrow[rr, phantom, "d", shift left=3ex] & & \tau_{\leqslant 0}X \arrow[uu] \arrow[ul]
\end{tikzcd}$$

By TR6 we get the lower cap

$$\begin{tikzcd}
Z \arrow[dd, "{[1]}"'] \arrow[rr, phantom, "\circlearrowleft"', shift right=4ex] \arrow[dd, phantom, "d", shift left=5ex] & & Y \arrow[ll] \arrow[dl] \arrow[dd, phantom, "d", shift right=5ex] \\
& U \arrow[ul] \arrow[dr, "{[1]}"] & \\
\tau_{\geqslant 1}X \arrow[ur] \arrow[rr, "{[1]}"'] \arrow[rr, phantom, "\circlearrowleft", shift left=4ex] & & \tau_{\leqslant 0}X \arrow[uu]
\end{tikzcd}$$

By step 2 applied to $(\tau_{\leqslant 0}X, Y, U)$ we get an exact sequence

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(U) \longrightarrow 0$$

By applying step 4 to $(U, Z, (\tau_{\geqslant 1}X)[1])$, obtained by TR3, we get an exact sequence

$$0 \longrightarrow H^0(U) \longrightarrow H^0(Z) \longrightarrow H^0((\tau_{\geqslant 1}X)[1])$$

From these exact sequences one gets that the composite $H^0(Y) \to H^0(U) \to H^0(Z)$ is the epimorphism monomorphism factorization of $H^0(Y) \to H^0(Z)$, see theorem 2.2.10. By corollary 2.2.12 the kernel of $H^0(Y) \to H^0(Z)$ is the kernel of $H^0(Y) \to H^0(U)$. By proposition 2.2.14, using the first exact sequence,

$H^0(Y) \to H^0(U)$ is the cokernel of $H^0(X) \to H^0(Y)$. Thus

$$\begin{aligned}\operatorname{Im}(H^0(X) \to H^0(Y)) &= \ker(\operatorname{coker}(H^0(X) \to H^0(Y))) \\ &= \ker(H^0(Y) \to H^0(U)) \\ &= \ker(H^0(Y) \to H^0(Z)).\end{aligned}$$

This shows that

$$H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z)$$

is an exact sequence. Therefore H^0 is a cohomological functor.

□

We give two corollaries which apply to particular kind of t-structures. These corollaries show that one can use cohomology to identify isomorphisms in the triangulated category and that cohomology identifies the categories $\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 0}$.

Definition 7.3.5 (Bounded t-structure)**.** Let $t = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant} 0)$ be a t-structure on a triangulated category $\mathcal{D}$. We say that t is *bounded* if

$$\bigcap_{n \in \mathbb{Z}} \operatorname{Ob} \mathcal{D}^{\leqslant n} = \bigcap_{n \in \mathbb{Z}} \operatorname{Ob} D^{\geqslant n} = \{0\}$$

and for any object X of $\mathcal{D}$, $H^i(X)$ is nonzero for only a finite number of $i \in \mathbb{Z}$.

Corollary 7.3.6. *Let $t = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ be a bounded t-structure on a triangulated category $\mathcal{D}$. Then a morphism $f : X \to Y$ is an isomorphism in $\mathcal{D}$ if and only if $H^n(f)$ are isomorphisms in* $\operatorname{Core}(t)$ *for all $n \in \mathbb{Z}$. In particular, if $H^n(X) = 0$ for all $n \in \mathbb{Z}$, then $X = 0$.*

Proof. $\Rightarrow$: First let us show that if $H^n(X) = 0$ for all n then $X = 0$. Suppose that $X \in \operatorname{Ob} \mathcal{D}^{\geqslant 0}$. Then $H^0(X) \cong \tau_{\leqslant 0}\tau_{\geqslant 0}X \cong \tau_{\leqslant 0}X = 0$, by proposition 7.2.3, so $X \cong \tau_{\geqslant 0}X \cong \tau_{\geqslant 1}X \in \operatorname{Ob} \mathcal{D}^{\geqslant 1}$ by proposition 7.2.3. By induction, suppose $X \in \operatorname{Ob} \mathcal{D}^{\geqslant n}$. Then $H^n(X) \cong (\tau_{\leqslant n}\tau_{\geqslant n}X)[n] \cong (\tau_{\leqslant n}X)[n] = 0$, so $X \cong \tau_{\geqslant n}X \cong \tau_{\geqslant n+1}X \in \operatorname{Ob} \mathcal{D}^{\geqslant n+1}$, by proposition 7.2.3 and the fact that translation functor is an additive automorphism, see proposition 2.1.5. By lemma 7.1.2, $X \in \operatorname{Ob} \mathcal{D}^{\geqslant n}$ for any $n < 0$. This shows that $X \in \cap_{n \in \mathbb{Z}} \operatorname{Ob} \mathcal{D}^{\geqslant n} = \{0\}$. Hence $X = 0$.

Suppose that $X \in \operatorname{Ob} \mathcal{D}^{\leqslant 0}$. Then $H^0(X) = \tau_{\leqslant 0}\tau_{\geqslant 0}X \cong \tau_{\geqslant 0}\tau_{\leqslant 0}X \cong \tau_{\geqslant 0}X = 0$, so $X \cong \tau_{\leqslant 0}X \cong \tau_{\leqslant -1}X \in \operatorname{Ob} \mathcal{D}^{\leqslant -1}$, by lemma 7.2.2. By induction, suppose $X \in \operatorname{Ob} \mathcal{D}^{\leqslant n}$. Then $H^n(X) = (\tau_{\leqslant n}\tau_{\geqslant n}X)[n] \cong (\tau_{\geqslant n}\tau_{\leqslant n}X)[n] \cong (\tau_{\geqslant n}X)[n] = 0$, by lemma 7.2.2 and proposition 7.2.7, so $X \cong \tau_{\leqslant n}X \cong \tau_{\leqslant n-1}X \in \operatorname{Ob} \mathcal{D}^{\leqslant n-1}$, by lemma 7.2.2 and the fact that translation functor is an additive automorphism, see proposition 2.1.5. By lemma 7.1.2, $X \in \operatorname{Ob} \mathcal{D}^{\leqslant n}$ for all $n > 0$. Therefore $X \in \cap_{n \in \mathbb{Z}} \operatorname{Ob} \mathcal{D}^{\leqslant n} = \{0\}$. Thus $X = 0$.

Now, for $n \leqslant 0$

$$\begin{aligned}H^n(X) &= (\tau_{\leqslant n}\tau_{\geqslant n}X)[n] \cong (\tau_{\geqslant n}\tau_{\leqslant n}X)[n] \\ &\cong (\tau_{\geqslant n}\tau_{\leqslant n}\tau_{\leqslant 0}X)[n] \cong (\tau_{\leqslant n}\tau_{\geqslant n}\tau_{\leqslant 0}X)[n] = H^n(\tau_{\leqslant 0}X)\end{aligned}$$

and for $n \geqslant 1$

$$H^n(X) = (\tau_{\leqslant n}\tau_{\geqslant n}X)[n] \cong (\tau_{\leqslant n}\tau_{\geqslant n}\tau_{\geqslant 1}X)[n] = H^n(\tau_{\geqslant 1}X).$$

For $n > 0$ we have

$$H^n(\tau_{\leqslant 0}X) = (\tau_{\leqslant n}\tau_{\geqslant n}\tau_{\leqslant 0}X)[n] = 0,$$

because $\tau_{\leqslant 0}X \in \operatorname{Ob}\mathcal{D}^{\leqslant n-1}$, by lemma 7.1.2. For $n < 1$ we have

$$H^n(\tau_{\geqslant 1}X) = (\tau_{\leqslant n}\tau_{\geqslant n}\tau_{\geqslant 1}X)[n] \cong (\tau_{\geqslant n}\tau_{\leqslant n}\tau_{\geqslant 1}X)[n] = 0,$$

because $\tau_{\geqslant 1}X \in \operatorname{Ob}\mathcal{D}^{\geqslant n+1}$, by lemma 7.1.2. These show that $H^n(X) = H^n(\tau_{\geqslant 1}X) = H^n(\tau_{\leqslant 0}X) = 0$ for all n. Because $\tau_{\geqslant 1}X \in \operatorname{Ob}\mathcal{D}^{\geqslant 0}$, by lemma 7.1.2, we have that $\tau_{\geqslant 1}X = \tau_{\leqslant 0}X = 0$, by what we have already shown. From the distinguished triangle (7.1) we get $X = 0$ by using TR1, TR3, TR5 and corollary 4.1.6.

Let $f : X \to Y$ be a morphism such that $H^n(f)$ is an isomorphism for all n, and complete f to a distinguished triangle (X, Y, Z, f, g, h). Since H^n are cohomological functors, for any n the sequence

$$H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(g)} H^n(Z)$$

is exact. Hence $H^n(Z) = 0$ for all n, so $Z = 0$. If we apply corollary 4.1.6 to rotation of the following morphism of distinguished triangles

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, equal] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r] & 0 \arrow[r] & X[1]
\end{tikzcd}$$

we get that f is an isomorphism.

$\Leftarrow$: If f is an isomorphism, then because any functor preserves isomorphisms, $H^n(f)$ is an isomorphism for all n. $\square$

Corollary 7.3.7. *Let $t = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant}0)$ be a bounded t-structure on a triangulated category $\mathcal{D}$. Then we have*

$$\operatorname{Ob}\mathcal{D}^{\leqslant n} = \left\{X \in \operatorname{Ob}\mathcal{D} \mid H^i(X) = 0 \quad \forall i > n\right\} \qquad \operatorname{Ob}\mathcal{D}^{\geqslant n} = \left\{X \in \operatorname{Ob}\mathcal{D} \mid H^i(X) = 0 \quad \forall i < n\right\}.$$

Proof. $\operatorname{Ob}\mathcal{D}^{\leqslant n} = \left\{X \in \operatorname{Ob}\mathcal{D} \mid H^i(X) = 0 \quad \forall i > n\right\}$**:** $\subset$: Let $n \in \mathbb{Z}$ and suppose $X \in \operatorname{Ob}\mathcal{D}^{\leqslant n}$. Then for $i > n$ we have

$$H^i(X) = (\tau_{\leqslant i}\tau_{\geqslant i}X)[i] = 0,$$

because $X \in \mathcal{D}^{\leqslant i-1}$ by lemma 7.1.2. This shows that $X \in \left\{X \in \operatorname{Ob}\mathcal{D} \mid H^i(X) = 0 \quad \forall i > n\right\}$.

$\supset$: Let $n \in \mathbb{Z}$ and suppose $X \in \left\{X \in \operatorname{Ob}\mathcal{D} \mid H^i(X) = 0 \quad \forall i > n\right\}$. For all $i \leqslant n$ we have

$$H^i(\tau_{\geqslant n+1}X) = (\tau_{\leqslant i}\tau_{\geqslant i}\tau_{\geqslant n+1}X)[i] \cong (\tau_{\leqslant i}\tau_{\geqslant n+1}\tau_{\geqslant i}X)[i] = 0,$$

by propositions 7.2.3 and 7.2.7, because $\tau_{\geqslant n+1}\tau_{\geqslant i}X \in \mathcal{D}^{\geqslant i+1}$. For $i > n$ we have

$$H^i(\tau_{\geqslant n+1}X) = (\tau_{\leqslant i}\tau_{\geqslant i}\tau_{\geqslant n+1}X)[i] \cong (\tau_{\leqslant i}\tau_{\geqslant i}X)[i] = H^i(X) = 0,$$

by proposition 7.2.7, proposition 7.2.5 and assumption. Therefore, by corollary 7.3.6, $\tau_{\geqslant n+1}X = 0$. By lemma 7.2.2 $X \in \mathcal{D}^{\leqslant n}$.

$\operatorname{Ob}\mathcal{D}^{\geqslant n} = \left\{X \in \operatorname{Ob}\mathcal{D} \mid H^i(X) = 0 \quad \forall i < n\right\}$**:** $\subset$: Let $n \in \mathbb{Z}$ and suppose $X \in \operatorname{Ob}\mathcal{D}^{\geqslant n}$. For $i < n$ we have

$$H^i(X) = (\tau_{\leqslant i}\tau_{\geqslant i}X)[i] \cong (\tau_{\geqslant i}\tau_{\leqslant i}X)[i] = 0,$$

by propositions 7.2.3 and 7.2.7, because $X \in \operatorname{Ob} \mathcal{D}^{\geqslant i+1}$ by lemma 7.1.2. This shows that

$$X \in \left\{X \in \operatorname{Ob} \mathcal{D} \mid H^i(X) = 0 \quad \forall i < n\right\}.$$

$\supset$: Let $n \in \mathbb{Z}$ and let $X \in \left\{X \in \operatorname{Ob} \mathcal{D} \mid H^i(X) = 0 \quad \forall i < n\right\}$. Then for all $i \geqslant n$ we have

$$H^i(\tau_{\leqslant n-1}X) = (\tau_{\leqslant i}\tau_{\geqslant i}\tau_{\leqslant n-1}X)[i] = 0,$$

by lemma 7.2.2, because $\tau_{\leqslant n-1}X \in \mathcal{D}^{\leqslant i-1}$ by lemma 7.1.2. For $i < n$ we have

$$H^i(\tau_{\leqslant n-1}X) = (\tau_{\leqslant i}\tau_{\geqslant i}\tau_{\leqslant n-1}X)[i] \cong (\tau_{\leqslant i}\tau_{\geqslant i}X)[i] = H^i(X) = 0,$$

by proposition 7.2.5, lemma 7.2.2, and assumption. This implies $\tau_{\leqslant n-1}X = 0$ by corollary 7.3.6. By proposition 7.2.3, $X \in \operatorname{Ob} \mathcal{D}^{\geqslant n}$.

□

7.4 Examples

For any abelian category $\mathcal{A}$, we can define the standard t-structure on $D^b(\mathcal{A})$ as follows. Let $D^{\leqslant 0}$ be the full subcategory of $D^b(\mathcal{A})$ consisting of complexes $X^\bullet \in D^b(\mathcal{A})$ such that $H^i(X^\bullet) = 0$ for all $i > 0$ and let $D^{\geqslant 0}$ be the full subcategory of $D^b(\mathcal{A})$ consisting of complexes $Y^\bullet \in D^b(\mathcal{A})$ such that $H^i(Y^\bullet) = 0$ for all $i < 0$. It is clear that T1 holds for the pair $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$.

For any object $X^\bullet \in D^b(\mathcal{A})$, there exists the following exact sequence in $C^b(\mathcal{A})$

$$0 \longrightarrow \tau_{\leqslant 0}X^\bullet \longrightarrow X^\bullet \longrightarrow \tau_{\geqslant 1}X^\bullet \longrightarrow 0$$

where $\tau_{\leqslant 0}X^\bullet$ is the complex

$$\dots \xrightarrow{d_{X^\bullet}^{-2}} X^{-1} \xrightarrow{\alpha} \ker d_{X^\bullet}^0 \xrightarrow{0} 0 \longrightarrow \dots$$

and $\tau_{\geqslant 1}X^\bullet$ is the complex

$$\dots \xrightarrow{0} 0 \xrightarrow{0} \operatorname{coker} d_{X^\bullet}^0 \xrightarrow{\beta} X^2 \xrightarrow{d_{X^\bullet}^2} \dots$$

Here the morphisms α and β are the unique morphisms from the diagram (2.4). It is clear that the complexes $\tau_{\leqslant 0}X^\bullet$ and $\tau_{\geqslant 1}X^\bullet$ are objects of $D^b(\mathcal{A})$. By proposition 5.1.11 the exact sequence corresponds to the following distinguished triangle

$$\tau_{\leqslant 0}X^\bullet \longrightarrow X^\bullet \longrightarrow \tau_{\geqslant 1}X^\bullet \longrightarrow (\tau_{\leqslant 0}X^\bullet)[1]$$

This shows that condition T3 holds for the pair $(D^{\leqslant 0}, D^{\geqslant 0})$.

To show that the condition T2 holds for the pair, let $f' : X^\bullet \to Y^\bullet$ be a morphism in $D^b(\mathcal{A})$ with $X^\bullet \in \operatorname{Ob} D^{\leqslant 0}$ and $Y^\bullet \in \operatorname{Ob} D^{\geqslant n}$, $n \geqslant 1$. Let

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

be a roof which represents the morphism f'. One can easily check that the morphism $\phi : Y^\bullet \to \tau_{\geqslant 1} Y^\bullet$ is a quasi-isomorphism because $Y^\bullet \in \operatorname{Ob} D^{\geqslant 1}$. Hence $\operatorname{Mor}_{D^b(\mathcal{A})}(X^\bullet, Y^\bullet) \cong \operatorname{Mor}_{D^b(\mathcal{A})}(X^\bullet, \tau_{\geqslant 1} Y^\bullet)$, so it suffices to show that any roof

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "s"'] \arrow[dr, "\phi f"] & \\ X^\bullet & & \tau_{\geqslant 1} Y^\bullet \end{tikzcd}$$

represents the zero morphism from $X^\bullet$ to $\tau_{\geqslant 1} Y^\bullet$. Since $Z^\bullet$ is quasi-isomorphic to $X^\bullet$, which is an object of $D^{\leqslant 0}$, so is $Z^\bullet$ an object of $D^{\leqslant 0}$. Hence it is easy to see that the inclusion $\tau_{\leqslant 0} Z^\bullet \to Z^\bullet$ is a quasi-isomorphism, and thus the above roof is equivalent to the following roof

$$\begin{tikzcd} & \tau_{\leqslant 0} Z^\bullet \arrow[dl, "sr"'] \arrow[dr, "\phi f r"] & \\ X^\bullet & & \tau_{\geqslant 1} Y^\bullet \end{tikzcd}$$

Clearly, it suffices to show that $\phi f r$ is the zero morphism, but this follows from the fact that $\phi^i f^i r^i = 0$ for all $i \in \mathbb{Z}$. This shows that $\operatorname{Mor}_{D^b(\mathcal{A})}(X^\bullet, Y^\bullet) = 0$. We have shown that $(D^{\leqslant 0}, D^{\geqslant 0})$ is a t-structure on $D^b(\mathcal{A})$.

The cohomology functor $H^\bullet$ associated to this t-structure coincides with the classical cohomology on the bounded derived category. Indeed, $\mathcal{A}$ is equivalent to $\operatorname{Core}(t)$, by proposition 5.1.10 and corollary 7.3.7. Now

$$\tau_{\leqslant 0} \tau_{\geqslant 0}(X^\bullet[n]) = H^n(X^\bullet),$$

so cohomology on triangulated categories can be seen as a generalization of classical cohomology.

7.5 Notes

For an example of nonstandard t-structures on triangulated categories one has perverse t-structures, which are used to construct perverse sheaves. Perverse sheaves [HTT08, Definition 8.1.28] are the objects of the core of perverse t-structure on $D^b_c(X)$, the derived bounded category of $\mathbb{C}_X$-modules on an analytic space X with constructible cohomology. For basic properties of perverse sheaves see [HTT08, Chapter 8]. The construction of perverse sheaves also works for ℓ-adic sheaves, see [KW13, Chapter 3]. For coherent sheaves on an algebraic stack, perverse t-structure is constructed in [AB10].

Appendix A

Octahedral axiom

Here we prove that by the octahedral axiom, every lower cap can be completed to an upper cap. Fix notation by the following octahedron

$$
\begin{tikzcd}
 & Y_1 \arrow[dr] \arrow[ddl] & \\
X_2 \arrow[ur] \arrow[d] & & Y_2 \arrow[ll] \arrow[ddl] \\
Z_1 \arrow[rr] \arrow[dr, dashed] & & X_1 \arrow[uul] \arrow[u] \\
 & Z_2 \arrow[ur] \arrow[uul, dashed] &
\end{tikzcd}
$$

Let us first study the octahedral axiom in detail. To give an upper cap is equivalent, up to TR3 to TR5, to giving three distinguished triangles (X_1, Y_1, Z_1), (Y_1, Y_2, X_2), and (X_1, Y_2, Z_2), where the first morphism $X_1 \to Y_2$ of the third triangle is the composite $X_1 \to Y_1 \to Y_2$ of the first morphisms of the first two triangles. By TR5 we obtain a morphism $(X_1, Y_1, Z_1) \to (X_1, Y_2, Z_2)$ of distinguished triangles. Now the axiom TR6 is equivalent to that we get the following commutative diagram

$$
\begin{tikzcd}
X_1 \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & X_1[1] \arrow[d, equal] \\
X_1 \arrow[r] & Y_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & X_1[1] \arrow[d] \\
 & X_2 \arrow[r, equal] \arrow[d] & X_2 \arrow[r, "{[1]}"] \arrow[d] & Y_1 \\
 & Y_1[1] \arrow[r] & Z_1[1] &
\end{tikzcd}
\tag{A.1}
$$

where (Z_1, Z_2, X_2) is a distinguished triangle.

Now we prove that a lower cap can be completed to octahedra. If we are given a lower cap, it is equivalent, up to TR3 to TR5, to give the following three distinguished triangles (Z_1, Z_2, X_2), (X_1, Y_2, Z_2), and (X_1, Y_1, Z_1), where $Z_1 \to X_1[1]$, the last morphism of the third triangle, is the composite $Z_1 \to Z_2 \to X_1[1]$, where the first morphism is the second morphism of the first triangle and the second morphism is the last morphism of the second triangle. By TR3 and TR5 we get a morphism $(X_1, Y_1, Z_1) \to (X_1, Y_2, Z_2)$ of distinguished triangles. This means that we have the following commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & X_1[1] \arrow[d, equal] \\
X_1 \arrow[r] & Y_2 \arrow[r] & Z_2 \arrow[r] \arrow[d] & X_1[1] \\
 & & X_2 \arrow[d] & \\
 & & Z_1[1] &
\end{tikzcd}$$

By TR4 we can complete the morphism $Y_1 \to Z_2$ to a distinguished triangle (Y_1, Z_2, A). Hence we have obtained three distinguished triangles (X_1, Y_1, Z_1), (Y_1, Y_2, A), and (X_1, Y_2, Z_2), such that the first morphism $X_1 \to Y_2$ of the third triangle is the composite $X_1 \to Y_1 \to Y_2$ of the first two morphisms of the first two triangles. Hence we can apply TR6 to get the following commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & X_1[1] \arrow[d, equal] \\
X_1 \arrow[r] & Y_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & X_1[1] \arrow[d] \\
 & A \arrow[r, equal] \arrow[d] & A \arrow[r, "{[1]}"] \arrow[d] & Y_1 \\
 & Y_1[1] \arrow[r] & Z_1[1] &
\end{tikzcd}$$

But now, by item TR5 and corollary 4.1.6, we have the following isomorphism of distinguished triangles

$$\begin{tikzcd}
Z_1 \arrow[r] \arrow[d, equal] & Z_2 \arrow[r] \arrow[d, equal] & X_2 \arrow[r] \arrow[d] & Z_1[1] \arrow[d, equal] \\
Z_1 \arrow[r] & Z_2 \arrow[r] & A \arrow[r] & Z_1[1]
\end{tikzcd}$$

Hence, by using the fact that X_2 and A are isomorphic, we have the following commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & X_1[1] \arrow[d, equal] \\
X_1 \arrow[r] & Y_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & X_1[1] \arrow[d] \\
 & X_2 \arrow[r, equal] \arrow[d] & X_2 \arrow[r, "{[1]}"] \arrow[d] & Y_1 \\
 & Y_1[1] \arrow[r] & Z_1[1] &
\end{tikzcd}$$

which represents octahedron. This completes our proof that a lower cap can be completed to an octahedron.

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