“Exact present solution” in continuous time

Tamás K. Papp*

March 28, 2017

Abstract

A small experiment that I performed to understand the method of Den Haan, Kobielarz, and Rendahl (2015), which I wrote up as a tutorial with a small program in Julia (Bezanson et al. 2017).

See the associated repository at https://github.com/tpapp/exact-present for the code.

Setup. Consider a deterministic Ramsey model in continuous time, with CRRA utility function (IES θ), discount rate ρ, production function

$$F(k) = Ak^{\alpha} - \delta k$$

and capital accumulation equation

$$\dot{k}_t = F(k_t) - c_t \tag{1}$$

The most frequently used form of the Euler equation is[1]

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\big(F'(k_t) - \rho\big) \tag{2}$$

First, we rewrite this into a recursive form. We are solving for the policy function $c(k)$, and thus

$$\dot{c} = c'(k)\dot{k} = c'(k)\big(F(k) - c(k)\big)$$

where we have used (1) and dropped time indices. Plugging into (2), we obtain

$$\frac{c'(k)}{c(k)}\big(F(k) - c(k)\big) = \frac{1}{\theta}\big(F'(k) - \rho\big) \tag{3}$$

We cast this into the form

$$c'(k)\big(F(k) - c(k)\big) = \frac{1}{\theta}\big(F'(k) - \rho\big)c(k) \tag{4}$$

which should be easier to manipulate. We are looking for the solution $c(k)$ to (4).

*tkpapp@gmail.com

[1]Eg see Acemoglu (2008, Chapter 8).

Methodology. The key to the method outlined in Den Haan, Kobielarz, and Rendahl (2015) is the following:

1. fix k,
2. assume a functional form for $c(k)$ around this point,
3. solve (4) by imposing that this form holds locally.

For simplicity, we choose a linear form

$$c(\tilde{k}) = c_0(k) + c_1(k)(k - \tilde{k}) \tag{5}$$

First, from (4) and (5) we obtain

$$c_1(k)\big(F(k) - c_0(k)\big) = \frac{1}{\theta}\big(F'(k) - \rho\big)c_0(k) \tag{6}$$

Implicit differentiation by k yields

$$c_1'(k)\big(F(k) - c_0(k)\big) + c_1(k)\big(F'(k) - c_0'(k)\big) = \frac{1}{\theta}\Big[F''(k)c_0(k) + \big(F'(k) - \rho\big)c_0'(k)\Big] \tag{7}$$

Then we impose that the approximation is valid locally around k:

$$c_0'(k) = c_1(k) \qquad c_1'(k) = 0 \tag{8}$$

The first equation in (8) imposes that when we change k a bit, we move along the tangent, while the second states that the tangent remains the same. *This is a crucial assumption*, as it basically imposes no curvature. The performance of the method will depend on how good an approximation this is.

Using (8), (7) becomes

$$c_1(k)\big(F'(k) - c_1(k)\big) = \frac{1}{\theta}\Big[F''(k)c_0(k) + \big(F'(k) - \rho\big)c_1(k)\Big] \tag{9}$$

For each k, we solve the system of (5) and (9). Note that (9) still takes the curvature of F into account.

Practicalities. Note that the system (6) and (9) is quadratic. Using random starting points, I mostly found *three* solutions: the right one ($c_1(k), c_0(k) > 0$), a trivial one $c_1(k) = c_0(k) = 0$, and a bogus one (with $c_1(k) < 0$). Consequently, reformulating the problem and using initial guesses is beneficial. Using (3) rules out the zero solution.

Let k_s, c_s denote steady state capital and consumption. Initial guesses for the nonlinear optimizer are chosen as follows:

1. for a k near k_s, $c_0(k), c_1(k) = c_s, F'(k_s)$.
2. for other k, find a nearby $\tilde{k}$ for which we have solved the problem, and use $c_0(\tilde{k}), c_1(\tilde{k})$.

Solution. Figure 1 shows the policy function. Figure 2 shows the same function, but with tangents drawn using $c_0(k)$ and $c_1(k)$. Note that the approximation is not exact, but evaluating its accuracy is beyond the scope of this short note. However, it could serve as a great initial guess, obtained at a very small computational price.

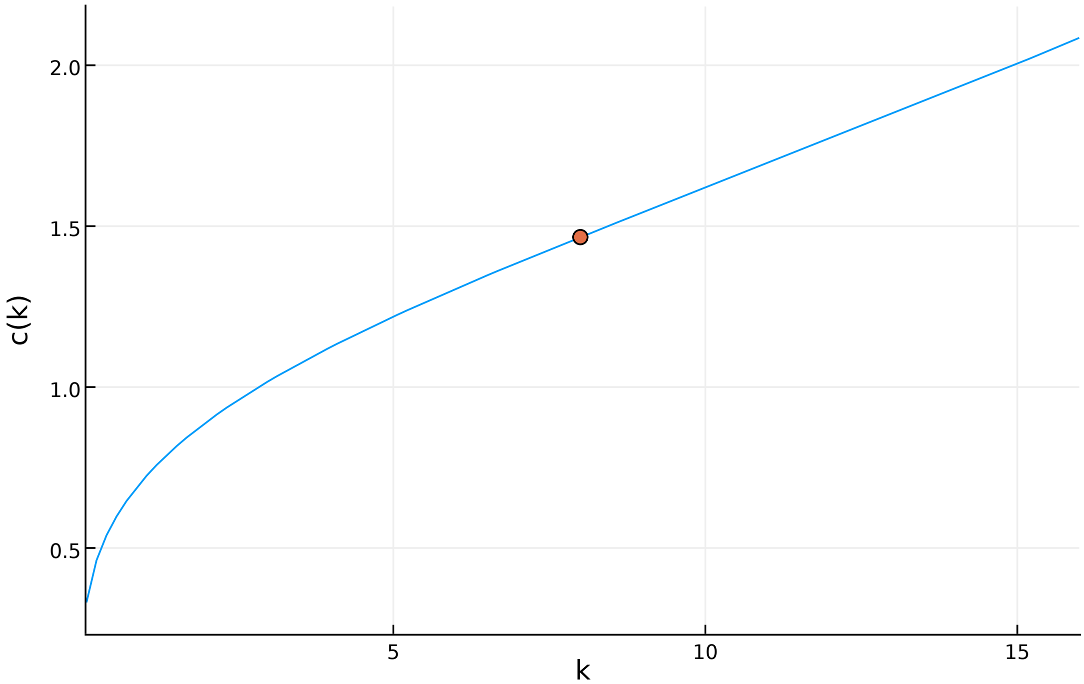

Figure 1: Policy function (steady state as a dot).

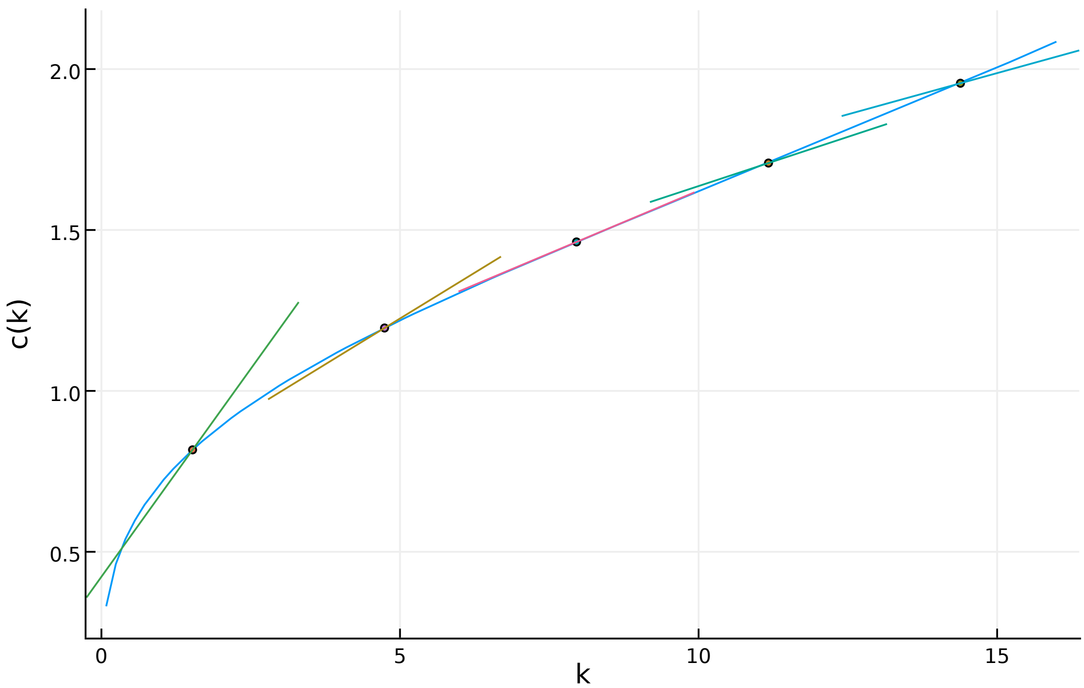

Figure 2: Policy function, with tangents drawn according to $c_1(k)$.

References

Acemoglu, Daron (2008). *Introduction to modern economic growth.* Princeton University Press.

Bezanson, Jeff et al. (2017). "Julia: A Fresh Approach to Numerical Computing". In: *SIAM Review* 59.1, pp. 65–98. DOI: 10.1137/141000671. eprint: http://dx.doi.org/10.1137/141000671. URL: http://dx.doi.org/10.1137/141000671.

Den Haan, Wouter J, Michal L Kobielarz, and Pontus Rendahl (2015). *Exact present solution with consistent future approximation: A gridless algorithm to solve stochastic dynamic models.* Tech. rep.