

EXAM 4

MATH 1B, Bach

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1 Identifying Quadric Surfaces

I

Horizontal Traces: Ellipse
Vertical Traces: Parabola
Surface: Elliptic Paraboloid

II

Horizontal Traces: Ellipse
Vertical Traces: Hyperbola
Surface: Hyperboloid of One Sheet

III

Horizontal Traces: Hyperbola
Vertical Traces: Ellipse
Surface: Elliptic Cone

IV

Horizontal Traces: Hyperbola
Vertical Traces: Ellipse
Surface: Hyperboloid of Two Sheets

V

Horizontal Traces: Hyperbola
Vertical Traces: Parabola
Surface: Hyperbolic Paraboloid

VI

Horizontal Traces: Ellipse
Vertical Traces: Ellipse
Surface: Ellipsoid

1.1 For equation (a): $\frac{x^2}{36} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

Let $k = z$ (to test the x/y plane shape, the horizontal tracing)

$$\frac{x^2}{36} + \frac{y^2}{9} = 1 - \frac{z^2}{16} \text{ is an equation for an } \boxed{\text{ellipse}}$$

$$1 - \frac{z^2}{16} \neq 0$$

$$1 \neq \frac{z^2}{16}$$

$$16 \neq z^2$$

$$k \in \mathbb{R} \quad z \neq \pm 4$$

Now, let $k = y$ (to test the x/z plane shape, the vertical tracing)

$$\frac{x^2}{36} + \frac{k^2}{9} + \frac{z^2}{16} = 1$$

$$\frac{x^2}{36} + \frac{z^2}{16} = 1 - \frac{k^2}{9} \text{ is an equation for an } \boxed{\text{ellipse}}$$

$$1 - \frac{k^2}{9} \neq 0$$

$$1 - \frac{k^2}{9} \neq 0$$

$$1 \neq \frac{k^2}{9}$$

$$9 \neq k^2$$

$$k \neq \pm 3$$

Now, let $k = x$ (to test the y/z plane trace, the vertical tracing)

$$\frac{k^2}{36} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

$$\frac{y^2}{9} + \frac{z^2}{16} = 1 - \frac{k^2}{36} \text{ is an equation for an } \boxed{\text{ellipse}}$$

$$1 - \frac{k^2}{36} \neq 0$$

$$1 \neq \frac{k^2}{36}$$

$$36 \neq k^2$$

$$k \in \mathbb{R} \quad k \neq \pm 6$$

As the traces on all 3 planes are ellipses, the shape is an ellipsoid, figure VI (6)

For equation (b): $\frac{x^2}{4} + \frac{y^2}{16} - z = 0$

Let $k = z$ (to test the x/y plane shape, the horizontal tracing)

$$\frac{x^2}{4} + \frac{y^2}{16} - k = 0$$

$$\frac{x^2}{4} + \frac{y^2}{16} = k \text{ This is an equation for an } \boxed{\text{Ellipse}}$$

$$k \in \mathbb{R} \quad k \neq 0$$

Now, let $k = x$ (to test the y/z plane shape, the vertical tracing)

$$\frac{k^2}{4} + \frac{y^2}{16} - z = 0$$

$$\frac{y^2}{16} - z = -\frac{k^2}{4} \text{ This is an equation for a } \boxed{\text{Parabola}}$$

$$-\frac{k^2}{4} \neq 0$$

$$k^2 \neq 0$$

$$k \in \mathbb{R} \quad k \neq 0$$

Now, let $k = y$ (to test the x/z plane shape, the vertical tracing)

$$\frac{x^2}{4} + \frac{k^2}{16} - z = 0$$

$$\frac{x^2}{4} - z = -\frac{k^2}{16} \text{ This is an equation for a } \boxed{\text{Parabola}}$$

$$0 \neq -\frac{k^2}{16}$$

$$0 \neq k^2$$

$$k \in \mathbb{R} \quad k \neq 0$$

Since the equation $\frac{x^2}{4} + \frac{y^2}{16} - z = 0$ can be rewritten as $\frac{x^2}{4} + \frac{y^2}{16} = \frac{z}{1}$ the equation represents an

Elliptic Paraboloid where $c = \pm 1$, $a = \pm 2$, and $b = \pm 4$ and matches with **Figure I**

For equation (c): $\frac{x^2}{4} + \frac{y^2}{16} - \frac{z^2}{4} = 1$

Let $k = z$ (to test the x/y plane shape, the horizontal tracing)

$$\frac{x^2}{4} + \frac{y^2}{16} - \frac{k^2}{4} = 1$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1 + \frac{k^2}{4} \text{ is an equation for an } \boxed{\text{Ellipse}}$$

$$1 + \frac{k^2}{4} \neq 0$$

$\frac{k^2}{4} \neq -1$ has no solutions in the real number spectrum

Meaning $k \in \mathbb{R}$

Now, let $k = x$ (to test the y/z plane shape, the vertical tracing)

$$\frac{k^2}{4} + \frac{y^2}{16} - \frac{z^2}{4} = 1$$

$$\frac{y^2}{16} - \frac{z^2}{4} = 1 - \frac{k^2}{4} \text{ is an equation for a } \boxed{\text{Hyperbola}}$$

$$1 - \frac{k^2}{4} \neq 0$$

$$1 \neq \frac{k^2}{4}$$

$$4 \neq k^2$$

$$k \in \mathbb{R} \quad k \neq \pm 2$$

Now, let $k = y$ (to test the x/z plane shape, the vertical tracing)

$$\frac{x^2}{4} + \frac{k^2}{16} - \frac{z^2}{4} = 1$$

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 - \frac{k^2}{16} \text{ is an equation for a } \boxed{\text{Hyperbola}}$$

$$1 - \frac{k^2}{16} \neq 0$$

$$1 \neq \frac{k^2}{16}$$

$$16 \neq k^2$$

$$k \in \mathbb{R} \quad k \neq \pm 4$$

The equation $\frac{x^2}{4} + \frac{y^2}{16} - \frac{z^2}{4} = 1$ is an equation for a $\boxed{\text{Hyperboloid of One Sheet}}$ where

$a = c = \sqrt{4} = 2$ and $b = \sqrt{16} = 4$, matching

$\boxed{\text{Figure II}}$

For equation (d): $x^2 - 4y^2 - z^2 = 4$

Let $k = z$ (to test the x/y plane shape, the horizontal tracing)

$$x^2 - 4y^2 - k^2 = 4$$

$$x^2 - 4y^2 = 4 + k^2 \text{ is an equation of a } \boxed{\text{Hyperbola}}$$

$$4 + k^2 \neq 0$$

$$k^2 \neq -4 \text{ is true for all real numbers}$$

$$k \in \mathbb{R}$$

Now, let $k = x$ (to test the y/z plane shape, the vertical tracing)

$$k^2 - 4y^2 - z^2 = 4$$

$$-4y^2 - z^2 = 4 - k^2$$

$$4y^2 + z^2 = k^2 - 4 \text{ is an equation for an } \boxed{\text{Ellipse}}$$

$$k^2 - 4 \neq 0$$

$$k^2 \neq 4$$

$$k \in \mathbb{R} \quad k \neq \pm 2$$

Now, let $k = y$ (to test the x/z plane shape, the vertical tracing)

$$x^2 - 4k^2 - z^2 = 4$$

$$x^2 - z^2 = 4 + 4k^2 \text{ is an equation of a } \boxed{\text{Hyperbola}}$$

$$4 + 4k^2 \neq 0$$

$$4k^2 \neq -4$$

$$k^2 \neq -1 \text{ is true for all real numbers}$$

$$k \in \mathbb{R}$$

The equation $x^2 - 4y^2 - z^2 = 4$ can be rewritten as $\frac{x^2}{4} - y^2 - \frac{z^2}{4} = 1$ making the quadric a

$\boxed{\text{Hyperboloid of Two Sheets.}}$ The x and z variables are both negative, making the graph center go through the x axis, corresponding to $\boxed{\text{Figure IV}}$

For equation (e): $9y^2 - 4x^2 - 9z = 0$

Let $k = z$ (to test the x/y plane shape, the horizontal tracing)

$$9y^2 - 4x^2 - 9k = 0$$

$9y^2 - 4x^2 = 9k$ is an equation of a Hyperbola

$$9k \neq 0$$

$$k \in \mathbb{R} \quad k \neq 0$$

Now, let $k = x$ (to test the y/z plane shape, the vertical tracing)

$$9y^2 - 4k^2 - 9z = 0$$

$9y^2 - 9z = 4k^2$ is an equation of a Parabola

$$4k^2 \neq 0$$

$$k \in \mathbb{R} \quad k \neq 0$$

Now, let $k = y$ (to test the x/z plane shape, the vertical tracing)

$$9k^2 - 4x^2 - 9z = 0$$

$-4x^2 - 9z = -9k^2$ is an equation of a Parabola

$$-9k^2 \neq 0$$

$$k^2 \neq 0$$

$$k \in \mathbb{R} \quad k \neq 0$$

The equation $9y^2 - 4x^2 - 9z = 0$ carries the format of a Hyperbolic Paraboloid, represented in Figure V, where the z axis value is the non-exponential and the y variable is the positive squared variable

$$2 \quad \sum_{n=1}^{\infty} (-1)^n \frac{(x+5)^n}{n \cdot 3^n}$$

a) Finding the radius of convergence

Method: Ratio Test $R = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ where the series converges if $|R| < 1$

and where a_n denotes the n_{th} term of the series and r is the radius of convergence

$$a_n = (-1)^n \frac{(x+5)^n}{n \cdot 3^n}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} \frac{(x+5)^{n+1}}{(n+1) \cdot 3^{n+1}}}{(-1)^n \frac{(x+5)^n}{n \cdot 3^n}} \right|$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{(x+5)^{n+1}}{(n+1) \cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x+5)^n} \right|$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{n \cdot (x+5)}{3 \cdot (n+1)} \right|$$

$$R = |x+5| \lim_{n \rightarrow \infty} \left| \frac{n}{3 \cdot (n+1)} \right|$$

$$R = |x+5| \cdot \left| \frac{1}{3} \right|$$

$$\left| \frac{x+5}{3} \right| = 1$$

$$|x+5| = 3$$

$$\boxed{r = 3}$$

The radius of convergence is 3

and the interval of convergence is $-8 < x \leq -2$

b) Finding the Interval of Convergence

The series converges if $\left| \frac{x+5}{3} \right| < 1$

$$-1 < \frac{x+5}{3} < 1$$

$$-3 < x+5 < 3$$

$$-8 < x < -2$$

Testing the endpoints:

Lower: $\sum_{n=1}^{\infty} (-1)^n \frac{(-8+5)^n}{n \cdot 3^n}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(-3)^n}{n \cdot 3^n}$$

The $\frac{(-3)^n}{3^n}$ can be reduced to $(-1)^n$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n}$$

The $(-1)^n (-1)^n$ can be reduced to $(-1)^{2n}$, which can be reduced to 1^n , which is equal to 1 for all real numbers

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the Harmonic Series Test, meaning -8 can not be included in the interval of convergence

Upper: $\sum_{n=1}^{\infty} (-1)^n \frac{(-2+5)^n}{n \cdot 3^n}$

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n \cdot 3^n}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ is } \boxed{\text{convergent by the Alternating Series Test}},$$

meaning that -2 is part of the interval of convergence

After testing endpoints, we can conclude the interval of convergence is $\boxed{-8 < x \leq -2}$

$$\mathbf{3} \quad f(x) = \frac{x^2}{3x+4}$$

Conversion to a power series

To be easily converted to a power series $\sum_{n=0}^{\infty} ar^n$, the equation will need the form of $\frac{a}{1-r}$

We can first divide by x^2 and get $f(x) = x^2 \frac{1}{3x+4}$

Separating out a 4 from the denominator get $f(x) = x^2 \cdot \frac{1}{4} \cdot \frac{1}{\frac{3x}{4} + 1}$

Which can be rearranged as $f(x) = \frac{x^2}{4} \cdot \frac{1}{1 + \frac{3x}{4}}$

When $a = 1$, $r = -\frac{3x}{4}$, the power series can be represented as $f(x) = \frac{x^2}{4} \sum_{n=0}^{\infty} \left(-\frac{3x}{4}\right)^n$

$$f(x) = \frac{x^2}{4} \sum_{n=0}^{\infty} \left(-\frac{3x}{4}\right)^n$$

Distributing the power $f(x) = \frac{x^2}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n \cdot x^n}{4^n}$

Since $x^2 \cdot x^n = x^{n+2}$ and $4 \cdot 4^n = 4^{n+1}$, substituting them in we get $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n \cdot x^{n+2}}{4^{n+1}}$

Similar to section 11.9 example 3, we can shift the indices $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}{4^{n-1}}$

Plugging this back in,
$$\frac{x^2}{3x+4} = \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}{4^{n-1}}$$

Finding radius of convergence

Method: Ratio Test $R = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ where the series converges if $|R| < 1$

and where a_n denotes the n_{th} term of the series and r is the radius of convergence

$$a_n = \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}{4^{n-1}}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n-1} \cdot 3^{n-1} \cdot x^{n+1}}{4^n}}{\frac{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}{4^{n-1}}} \right|$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n-1} \cdot 3^{n-1} \cdot x^{n+1}}{4^n} \cdot \frac{4^{n-1}}{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n} \right|$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{-3x}{4} \right|$$

$$R = \left| \frac{-3x}{4} \right|$$

$$\left| \frac{-3x}{4} \right| = 1$$

$$|-3x| = 4$$

$$3x = 4$$

$$\boxed{r = \frac{4}{3}}$$

Finding interval of convergence

The series converges if $\left| \frac{-3x}{4} \right| < 1$

$$-1 < \frac{3x}{4} < 1$$

$$-4 < 3x < 4$$

$$-\frac{4}{3} < x < \frac{4}{3}$$

Testing the endpoints:

Upper Endpoint: $x = \frac{4}{3}$

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot \frac{4^n}{3}}{4^{n-1}} \\ &= \sum_{n=2}^{\infty} \frac{(-3)^{n-2} \cdot \frac{4^n}{3}}{4^{n-1}} \\ &= \sum_{n=2}^{\infty} \frac{(-3)^{n-2} \cdot \frac{4^{n-2}}{3} \cdot \frac{4^2}{3}}{4^{n-2} \cdot 4} \\ &= \sum_{n=2}^{\infty} \frac{(-4)^{n-2} \cdot \frac{4^2}{3}}{4^{n-2} \cdot 4} \\ &= \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 16}{4 \cdot 9} \\ &= \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 4}{9} \end{aligned}$$

This series is divergent by the Ratio Test, where the ratio is 1, meaning the upper endpoint can not be included in the interval of convergence

Lower Endpoint: $x = -\frac{4}{3}$

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot -\frac{4^n}{3}}{4^{n-1}} \\ &= \sum_{n=2}^{\infty} \frac{(-3)^{n-2} \cdot -\frac{4^n}{3}}{4^{n-1}} \\ &= \sum_{n=2}^{\infty} \frac{(-3)^{n-2} \cdot -\frac{4^{n-2}}{3} \cdot -\frac{4^2}{3}}{4^{n-2} \cdot 4} \\ &= \sum_{n=2}^{\infty} \frac{4^{n-2} \cdot -\frac{4^2}{3}}{4^{n-2} \cdot 4} \\ &= \sum_{n=2}^{\infty} \frac{\left(-\frac{4}{3}\right)^2}{4} \\ &= \sum_{n=2}^{\infty} \frac{16}{4 \cdot 9} \\ &= \sum_{n=2}^{\infty} \frac{4}{9} \end{aligned}$$

This series is divergent by the Ratio Test, where the ratio is 1, meaning the lower endpoint can not be included in the interval of convergence

$$\boxed{\text{Radius of Convergence: } \frac{4}{3}}$$

$$\boxed{\text{Interval of Convergence: } -\frac{4}{3} < x < \frac{4}{3}}$$