EXAM 4

MATH 1B, Bach

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1 Identifying Quadric Surfaces

I

Horizontal Traces: Ellipse
Vertical Traces: Parabola
Surface: Elliptic Paraboloid

II

Horizontal Traces: Ellipse
Vertical Traces: Hyperbola
Surface: Hyperboloid of One Sheet

III

Horizontal Traces: Hyperbola
Vertical Traces: Ellipse
Surface: Elliptic Cone

IV

Horizontal Traces: Hyperbola
Vertical Traces: Ellipse
Surface: Hyperboloid of Two Sheets

V

Horizontal Traces: Hyperbola
Vertical Traces: Parabola
Surface: Hyperbolic Paraboloid

VI

Horizontal Traces: Ellipse
Vertical Traces: Ellipse
Surface: Ellipsoid

1.1 For equation (a): $\dfrac{x^2}{36}+\dfrac{y^2}{9}+\dfrac{z^2}{16}=1$

Let $k=z$ (to test the x/y plane shape, the horizontal tracing)

$\dfrac{x^2}{36}+\dfrac{y^2}{9}=1-\dfrac{z^2}{16}$ is an equation for an $\boxed{\text{ellipse}}$

$1-\dfrac{z^2}{16}\neq 0$

$1\neq\dfrac{z^2}{16}$

$16\neq z^2$

$k\epsilon\mathbb{R}\quad z\neq\pm 4$

Now, let $k=y$ (to test the x/z plane shape, the vertical tracing)

$\dfrac{x^2}{36}+\dfrac{k^2}{9}+\dfrac{z^2}{16}=1$

$\dfrac{x^2}{36}+\dfrac{z^2}{16}=1-\dfrac{k^2}{9}$ is an equation for an $\boxed{\text{ellipse}}$

$1-\dfrac{k^2}{9}\neq 0$

$1-\dfrac{k^2}{9}\neq 0$

$1\neq\dfrac{k^2}{9}$

$9\neq k^2$

$k\neq\pm 3$

Now, let $k=x$ (to test the y/z plane trace, the vertical tracing)

$\dfrac{k^2}{36}+\dfrac{y^2}{9}+\dfrac{z^2}{16}=1$

$\dfrac{y^2}{9}+\dfrac{z^2}{16}=1-\dfrac{k^2}{36}$ is an equation for an $\boxed{\text{ellipse}}$

$1-\dfrac{k^2}{36}\neq 0$

$1\neq\dfrac{k^2}{36}$

$36\neq k^2$

$k\epsilon\mathbb{R}\quad k\neq\pm 6$

As the traces on all 3 planes are ellipses, $\boxed{\text{the shape is an ellipsoid, figure VI (6)}}$

For equation (b): $\dfrac{x^2}{4}+\dfrac{y^2}{16}-z=0$

Let $k=z$ (to test the x/y plane shape, the horizontal tracing)

$\dfrac{x^2}{4}+\dfrac{y^2}{16}-k=0$

$\dfrac{x^2}{4}+\dfrac{y^2}{16}=k$ This is an equation for an $\boxed{\text{Ellipse}}$

$k\epsilon\mathbb{R}\quad k\neq 0$

Now, let $k=x$ (to test the y/z plane shape, the vertical tracing)

$\dfrac{k^2}{4}+\dfrac{y^2}{16}-z=0$

$\dfrac{y^2}{16}-z=-\dfrac{k^2}{4}$ This is an equation for a $\boxed{\text{Parabola}}$

$-\dfrac{k^2}{4}\neq 0$

$k^2\neq 0$

$k\epsilon\mathbb{R}\quad k\neq 0$

Now, let $k=y$ (to test the x/z plane shape, the vertical tracing)

$\dfrac{x^2}{4}+\dfrac{k^2}{16}-z=0$

$\dfrac{x^2}{4}-z=-\dfrac{k^2}{16}$ This is an equation for a $\boxed{\text{Parabola}}$

$0\neq-\dfrac{k^2}{16}$

$0\neq k^2$

$k\epsilon\mathbb{R}\quad k\neq 0$

Since the equation $\dfrac{x^2}{4}+\dfrac{y^2}{16}-z=0$ can be rewritten as $\dfrac{x^2}{4}+\dfrac{y^2}{16}=\dfrac{z}{1}$ the equation represents an $\boxed{\textbf{Elliptic Paraboloid}\text{ where } c=\pm 1,\ a=\pm 2, \text{ and } b=\pm 4}$ and matches with $\boxed{\textbf{Figure I}}$

For equation (c): $\dfrac{x^2}{4}+\dfrac{y^2}{16}-\dfrac{z^2}{4}=1$

Let $k=z$ (to test the x/y plane shape, the horizontal tracing)

$\frac{x^2}{4}+\frac{y^2}{16}-\frac{k^2}{4}=1$

$\frac{x^2}{4}+\frac{y^2}{16}=1+\frac{k^2}{4}$ is an equation for an $\boxed{\text{Ellipse}}$

$1+\frac{k^2}{4}\neq 0$

$\frac{k^2}{4}\neq -1$ has no solutions in the real number spectrum

Meaning $k\epsilon\mathbb{R}$

Now, let $k=x$ (to test the y/z plane shape, the vertical tracing)

$\frac{k^2}{4}+\frac{y^2}{16}-\frac{z^2}{4}=1$

$\frac{y^2}{16}-\frac{z^2}{4}=1-\frac{k^2}{4}$ is an equation for a $\boxed{\text{Hyperbola}}$

$1-\frac{k^2}{4}\neq 0$

$1\neq\frac{k^2}{4}$

$4\neq k^2$

$k\epsilon\mathbb{R}\quad k\neq\pm2$

Now, let $k=y$ (to test the x/z plane shape, the vertical tracing)

$\frac{x^2}{4}+\frac{k^2}{16}-\frac{z^2}{4}=1$

$\frac{x^2}{4}-\frac{z^2}{4}=1-\frac{k^2}{16}$ is an equation for a $\boxed{\text{Hyperbola}}$

$1-\frac{k^2}{16}\neq 0$

$1\neq\frac{k^2}{16}$

$16\neq k^2$

$k\epsilon\mathbb{R}\quad k\neq\pm4$

The equation $\frac{x^2}{4}+\frac{y^2}{16}-\frac{z^2}{4}=1$ is an equation for a $\boxed{\textbf{Hyperboloid of One Sheet}}$ where $a=c=\sqrt{4}=2$ and $b=\sqrt{16}=4$, matching $\boxed{\textbf{Figure II}}$

For equation (d): $x^2-4y^2-z^2=4$

Let $k=z$ (to test the x/y plane shape, the horizontal tracing)

$x^2-4y^2-k^2=4$

$x^2-4y^2=4+k^2$ is an equation of a $\boxed{\text{Hyperbola}}$

$4+k^2\neq 0$

$k^2\neq -4$ is true for all real numbers

$k\epsilon\mathbb{R}$

Now, let $k=x$ (to test the y/z plane shape, the vertical tracing)

$k^2-4y^2-z^2=4$

$-4y^2-z^2=4-k^2$

$4y^2+z^2=k^2-4$ is an equation for an $\boxed{\text{Ellipse}}$

$k^2-4\neq 0$

$k^2\neq 4$

$k\epsilon\mathbb{R}\quad k\neq\pm2$

Now, let $k=y$ (to test the x/z plane shape, the vertical tracing)

$x^2-4k^2-z^2=4$

$x^2-z^2=4+4k^2$ is an equation of a $\boxed{\text{Hyperbola}}$

$4+4k^2\neq 0$

$4k^2\neq -4$

$k^2\neq -1$ is true for all real numbers

$k\epsilon\mathbb{R}$

The equation $x^2-4y^2-z^2=4$ can be rewritten as $\frac{x^2}{4}-y^2-\frac{z^2}{4}=1$ making the quadric a $\boxed{\text{Hyperboloid of Two Sheets.}}$ The x and z variables are both negative, making the graph center go through the x axis, corresponding to $\boxed{\text{Figure IV}}$

For equation (e): $9y^2 - 4x^2 - 9z = 0$

Let $k = z$ (to test the x/y plane shape, the horizontal tracing)

$9y^2 - 4x^2 - 9k = 0$

$9y^2 - 4x^2 = 9k$ is an equation of a $\boxed{\text{Hyperbola}}$

$9k \neq 0$

$k \epsilon \mathbb{R} \quad k \neq 0$

Now, let $k = x$ (to test the y/z plane shape, the vertical tracing)

$9y^2 - 4k^2 - 9z = 0$

$9y^2 - 9z = 4k^2$ is an equation of a $\boxed{\text{Parabola}}$

$4k^2 \neq 0$

$k \epsilon \mathbb{R} \quad k \neq 0$

Now, let $k = y$ (to test the x/z plane shape, the vertical tracing)

$9k^2 - 4x^2 - 9z = 0$

$-4x^2 - 9z = -9k^2$ is an equation of a $\boxed{\text{Parabola}}$

$-9k^2 \neq 0$

$k^2 \neq 0$

$k \epsilon \mathbb{R} \quad k \neq 0$

The equation $9y^2 - 4x^2 - 9z = 0$ carries the format of a $\boxed{\text{Hyperbolic Paraboloid, represented in Figure V}}$, where the z axis value is the non-exponential and the y variable is the positive squared variable

2 $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{(x+5)^n}{n \cdot 3^n}$

a) Finding the radius of convergence

Method: Ratio Test $R = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ where the series converges if $|R| < 1$

and where a_n denotes the n_{th} term of the series and r is the radius of convergence

$$a_n = (-1)^n \frac{(x+5)^n}{n \cdot 3^n}$$

$$R = \lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n+1}}\frac{(x+5)^{n+1}}{(n+1)\cdot 3^{n+1}}}{\cancel{(-1)^n}\frac{(x+5)^n}{n\cdot 3^n}}\right|$$

$$R = \lim_{n\to\infty}\left|\frac{(x+5)^{\cancel{n+1}}}{(n+1)\cdot 3^{\cancel{n+1}}}\cdot\frac{n\cdot \cancel{3^n}^{1}}{\cancel{(x+5)^n}^{1}}\right|$$

$$R = \lim_{n\to\infty}\left|\frac{n\cdot(x+5)}{3\cdot(n+1)}\right|$$

$$R = |x+5|\lim_{n\to\infty}\left|\frac{n}{3\cdot(n+1)}\right|$$

$$R = |x+5|\cdot\left|\frac{1}{3}\right|$$

$$\left|\frac{x+5}{3}\right| = 1$$

$$|x+5| = 3$$

$$\boxed{r = 3}$$

The radius of convergence is 3

and the interval of convergence is $-8 < x \leq -2$

b) Finding the Interval of Convergence

The series converges if $\left|\frac{x+5}{3}\right| < 1$

$$-1 < \frac{x+5}{3} < 1$$

$$-3 < x+5 < 3$$

$$-8 < x < -2$$

Testing the endpoints:

Lower: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(-8+5)^n}{n\cdot 3^n}$

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-3)^n}{n\cdot 3^n}$$

The $\frac{(-3)^n}{3^n}$ can be reduced to $(-1)^n$

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n}{n}$$

The $(-1)^n(-1)^n$ can be reduced to $(-1)^{2n}$, which can be reduced to 1^n, which is equal to 1 for all real numbers

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ $\boxed{\text{diverges by the Harmonic Series Test}}$, meaning -8 can not be included in the interval of convergence

Upper: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(-2+5)^n}{n\cdot 3^n}$

$$\sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n\cdot 3^n}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cancel{3^n}^{1}}{n\cdot \cancel{3^n}^{1}}$$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is $\boxed{\text{convergent by the Alternating Series Test}}$, meaning that -2 is part of the interval of convergence

After testing endpoints, we can conclude the interval of convergence is $\boxed{-8 < x \leq -2}$

3 $f(x) = \dfrac{x^2}{3x+4}$

Conversion to a power series

To be easily converted to a power series $\displaystyle\sum_{n=0}^{\infty} ar^n$, the equation will need the form of $\dfrac{a}{1-r}$

We can first divide by x^2 and get $f(x) = x^2 \dfrac{1}{3x+4}$

Separating out a 4 from the denominator get $f(x) = x^2 \cdot \dfrac{1}{4} \cdot \dfrac{1}{\frac{3x}{4}+1}$

Which can be rearranged as $f(x) = \cdot \dfrac{x^2}{4} \cdot \dfrac{1}{1+\frac{3x}{4}}$

When $a = 1$, $r = -\dfrac{3x}{4}$, the power series can be represented as $f(x) = \dfrac{x^2}{4} \displaystyle\sum_{n=0}^{\infty} \left(-\frac{3x}{4}\right)^n$

$$f(x) = \frac{x^2}{4} \sum_{n=0}^{\infty} \left(-\frac{3x}{4}\right)^n$$

Distributing the power $f(x) = \dfrac{x^2}{4} \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n \cdot x^n}{4^n}$

Since $x^2 \cdot x^n = x^{n+2}$ and $4 \cdot 4^n = 4^{n+1}$, substituting them in we get $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n \cdot x^{n+2}}{4^{n+1}}$

Similar to section 11.9 example 3, we can shift the indices $f(x) = \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}{4^{n-1}}$

Plugging this back in, $\boxed{\dfrac{x^2}{3x+4} = \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}{4^{n-1}}}$

Finding radius of convergence

Method: Ratio Test $R = \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ where the series converges if $|R| < 1$

and where a_n denotes the n_{th} term of the series and r is the radius of convergence

$$a_n = \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}{4^{n-1}}$$

$$R = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n-1} \cdot 3^{n-1} \cdot x^{n+1}}{4^n}}{\dfrac{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}{4^{n-1}}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{(-1)^{\cancel{n-1}} \cdot 3^{\cancel{n-1}} \cdot x^{\cancel{n+1}}}{4^{\cancel{n}}} \cdot \frac{4^{\cancel{n-1}}}{\cancelto{1}{(-1)^{n-2} \cdot 3^{n-2} \cdot x^n}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{-3x}{4} \right|$$

$$R = \left|\frac{-3x}{4}\right|$$

$$\left|\frac{-3x}{4}\right| = 1$$

$$|-3x| = 4$$

$$3x = 4$$

$$\boxed{r = \frac{4}{3}}$$

Finding interval of convergence

The series converges if $\left|\frac{-3x}{4}\right| < 1$

$$-1 < \frac{3x}{4} < 1$$

$$-4 < 3x < 4$$

$$-\frac{4}{3} < x < \frac{4}{3}$$

Testing the endpoints:

Upper Endpoint: $x = \frac{4}{3}$

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot \frac{4}{3}^n}{4^{n-1}}$$

$$= \sum_{n=2}^{\infty} \frac{(-3)^{n-2} \cdot \frac{4}{3}^n}{4^{n-1}}$$

$$= \sum_{n=2}^{\infty} \frac{(-3)^{n-2} \cdot \frac{4}{3}^{n-2} \cdot \frac{4}{3}^2}{4^{n-2} \cdot 4}$$

$$= \sum_{n=2}^{\infty} \frac{(-4)^{n-2} \cdot \frac{4}{3}^2}{4^{n-2} \cdot 4}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 16}{4 \cdot 9}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 4}{9}$$

This series is divergent by the Ratio Test, where the ratio is 1, meaning the upper endpoint can not be included in the interval of convergence

Lower Endpoint: $x = -\frac{4}{3}$

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-2} \cdot 3^{n-2} \cdot -\frac{4}{3}^n}{4^{n-1}}$$

$$= \sum_{n=2}^{\infty} \frac{(-3)^{n-2} \cdot -\frac{4}{3}^n}{4^{n-1}}$$

$$= \sum_{n=2}^{\infty} \frac{(-3)^{n-2} \cdot -\frac{4}{3}^{n-2} \cdot -\frac{4}{3}^2}{4^{n-2} \cdot 4}$$

$$= \sum_{n=2}^{\infty} \frac{\cancelto{1}{4^{n-2}} \cdot -\frac{4}{3}^2}{\cancelto{1}{4^{n-2}} \cdot 4}$$

$$= \sum_{n=2}^{\infty} \frac{\left(-\frac{4}{3}\right)^2}{4}$$

$$= \sum_{n=2}^{\infty} \frac{16}{4 \cdot 9}$$

$$= \sum_{n=2}^{\infty} \frac{4}{9}$$

This series is divergent by the Ratio Test, where the ratio is 1, meaning the lower endpoint can not be included in the interval of convergence

$\boxed{\textbf{Radius of Convergence: } \frac{4}{3}}$

$\boxed{\textbf{Interval of Convergence: } -\frac{4}{3} < x < \frac{4}{3}}$

4 $f(x) = 3^{-2x}$

Finding the first 6 coefficients $\dfrac{f^{(n)}(0)}{n!}$

0:$f(x) = 3^{-2x} = 9^{-x} = \dfrac{1}{9^x}$ at $x = 0$

$\dfrac{1}{9^0} = 1$ then dividing by 0! $\dfrac{1}{0!} = \boxed{1}$

1:$f'(x) = \dfrac{d}{dx} 3^{-2x} = \dfrac{d}{dx} 9^{-x} = \dfrac{d}{dx}\dfrac{1}{9^x}$

using the quotient rule $\dfrac{d}{dx}\dfrac{1}{9^x} = \dfrac{0 \cdot 9^x - 9^x(\ln 9)}{9^{2x}}$

$= \dfrac{-9^x(\ln 9)}{9^{2x}}$

$= \dfrac{-\ln 9}{9^x}$ where $x = 0$

$= \dfrac{-\ln 9}{9^0} = -\ln 9$

dividing by 1! we get $\boxed{\dfrac{-\ln 9}{1!}}$

2:$f''(x) = \dfrac{d}{dx}\dfrac{-\ln 9}{9^x}$

Using the quotient rule, again
$\dfrac{d}{dx}\dfrac{-\ln 9}{9^x} = \dfrac{0 \cdot 9^x - (-\ln 9 \cdot 9^x(\ln 9))}{9^{2x}}$

$= \dfrac{\ln 9 \cdot 9^x(\ln 9)}{9^{2x}}$

$= \dfrac{\ln 9(\ln 9)}{9^x}$ where $x = 0$

$= (\ln 9)(\ln 9)$ or $(\ln(9))^2$

Dividing by 2! we get $\boxed{\dfrac{(\ln(9))^2}{2!}}$

3: $f'''(x) = \dfrac{d}{dx}\dfrac{(\ln(9))^2}{9^x}$

Using the quotient rule, again
$\dfrac{d}{dx}\dfrac{(\ln(9))^2}{9^x} = \dfrac{0 * 9^x - (\ln(9))^2 \cdot 9^x \ln 9}{9^{2x}}$

$= \dfrac{-(\ln(9))^2 \cdot 9^x \ln 9}{9^{2x}}$

$= \dfrac{-(\ln(9))^3}{9^x}$ where $x = 0$

$= -(\ln(9))^3$

dividing by 3! we get $\boxed{\dfrac{-(\ln(9))^3}{3!}}$

4: $f^{(4)}(x) = \dfrac{d}{dx}\dfrac{-(\ln(9))^3}{9^x}$

Using the quotient rule, again $\dfrac{d}{dx}\dfrac{-(\ln(9))^3}{9^x} = \dfrac{0 * 9^x - (-(\ln(9))^3 \cdot 9^x \ln 9}{9^{2x}}$

$= \dfrac{(\ln(9))^3 \cdot 9^x \ln 9}{9^{2x}}$

$= \dfrac{(\ln(9))^4}{9^x}$ where $x = 0$, evaluates to $(\ln(9))^4$

Dividing by 4! we get $\boxed{\dfrac{(\ln(9))^4}{4!}}$

5: $f^{(5)}(x) = \dfrac{d}{dx}\dfrac{(\ln(9))^4}{9^x}$

Using the quotient rule, for the last time
$\dfrac{d}{dx}\dfrac{(\ln(9))^4}{9^x} = \dfrac{0 * 9^x - (\ln(9))^4 \cdot 9^x \ln 9}{9^{2x}}$

$= \dfrac{0 * 9^x - (\ln(9))^4 \cdot 9^x \ln 9}{9^{2x}}$

$= \dfrac{-(\ln(9))^4 \cdot 9^x \ln 9}{9^{2x}}$

$= \dfrac{-(\ln(9))^5 \cdot 9^x}{9^{2x}}$

$= \dfrac{-(\ln(9))^5}{9^x}$ where $x = 0$, evaluates to $-(\ln(9))^5$

Dividing by 5! we get $\boxed{\dfrac{-(\ln(9))^5}{5!}}$

First 6 coefficients:

0. 1
1. $\dfrac{-\ln 9}{1!}$
2. $\dfrac{(\ln(9))^2}{2!}$
3. $\dfrac{-(\ln(9))^3}{3!}$
4. $\dfrac{(\ln(9))^4}{4!}$
5. $\dfrac{-(\ln(9))^5}{5!}$

The emergent pattern seems to be $\boxed{\dfrac{f^{(n)}(0)}{n!} = \dfrac{(-\ln(9))^n}{n!}}$

The Maclaurin Series

Using the known coefficient pattern, we can multiply the general term by x^n and make it a summation

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-\ln(9))^n}{n!} x^n}$$

5 Line L passes through $A(4, -3, 5)$ and $B(-2, -1, 8)$

For the purposes of this problem, I will use the vector going in the direction from points A to B

Vector Equation

Parametric Equations

Symmetric Equation