MATH 1C Test 2 Notecard

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15.1 Double Integrals over Rectangles

The single variable definite integral: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

Volume as a Double Riemann Sum: $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$

Volume as a Double Integral: $\int \int_R f(x,y)dA = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$

then it is known $\iint\limits_R f(x,y)dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$

Special Case

In the special case that $f(x,y)$ can be factored as a product of a function of x only and a function of y only then it is known the following is true:

$$\iint\limits_R g(x)\, h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

Average Value

The average value of a function f of two variables defined on a rectangle R is:

$$f_{avg} = \frac{1}{A(R)} \iint\limits_R f(x,y)dA$$

where A(R) is the area of R.

15.2 Double Integrals over General Regions

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ s in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

$$\iint\limits_D f(x,y)\, dA = \iint\limits_D F(x,y)\, dA$$

If f is continuous on a Type I region D such that the fixed bounds are vertical, $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, then

$$\iint\limits_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

If f is continuous on a Type II region D such that the fixed bounds are horizontal, $D = \{(x,y) \mid c \le x \le d,\ h_1(x) \le y \le h_2(x)\}$, then

$$\iint\limits_D f(x,y)\, dA = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\, dx\, dy$$

15.3 Double Integrals in Polar Coordinates

$$\iint\limits_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Put all half angle and double angle identities on notecard If f is continuous on a polar region of the form

$$D = \{(r,\theta) | \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

Identities: $x^2 + y^2 = r^2 \quad x = r\cos\theta \quad y = r\sin\theta \quad \tan\theta = \dfrac{y}{x}$

15.4 Applications of Double Integrals

Electric Charge

$$Q = \iint\limits_D \sigma(x,y)\, dA$$

Center of Mass of a Lamina

$\bar{x} = \dfrac{M_y}{m} = \dfrac{1}{m} \iint\limits_D x\rho(x,y)\, dA$ and $\bar{y} = \dfrac{M_x}{m} = \dfrac{1}{m} \iint\limits_D y\rho(x,y)\, dA$ where

$$m = \iint\limits_D \rho(x,y)\, dA$$

15.5 Surface Area

The surface area with equation $z = f(x, y), (x, y)\epsilon D$ where f_x and f_y are continuous is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\; dA$$

$$A(S) = \iint_D \sqrt{\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 + 1}\; dA$$

For the noteacrd:

$$A(S) = \iint_D \sqrt{[f_x]^2 + [f_y]^2 + 1}\; dA$$

15.6 Triple Integrals

$$\iiint_B f(x,y,z)\; dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\; \Delta V \quad \textbf{Wont need on notecard}$$

$$\iiint_B f(x,y,z)\; dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\; dx\; dy\; dz$$

Solutions for general regions:

For a shadow on the x, y plane: $\displaystyle\iiint_B f(x,y,z)\; dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\; dz\right] dA$

For a shadow on the y, z plane: $\displaystyle\iiint_B f(x,y,z)\; dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\; dx\right] dA$

For a shadow on the x, z plane: $\displaystyle\iiint_B f(x,y,z)\; dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\; dy\right] dA$

15.7 Triple Integrals in Cylndrical Coordinates

Coordinate conversion: $P(r, \theta, z) = P(x, y, z)$
Conversions between Rectangular and Cylindrical
Identities: $x^2 + y^2 = r^2 \quad x = r\cos\theta \quad y = r\sin\theta \quad \tan\theta = \dfrac{y}{x} \quad z = z$
Finding a rectangular point given polar coordinates:
$P(r, \theta, z) = (4, -\dfrac{\pi}{4}, -3)$ Since $x = r\cos\theta \quad y = r\sin\theta$

$$x = 4\cos\left(-\frac{\pi}{4}\right),\quad y = 4\sin\left(-\frac{\pi}{4}\right), z = -3$$

$$x = 2\sqrt{2}, y = 2\sqrt{2}, z = -3 \therefore P = (2\sqrt{2}, 2\sqrt{2}, -3)$$

Evaluating Integrals in Cylindrical
For regions that are continuous and $E = \{(x, y, z) | (x, y) \epsilon D, u_1(x, y) \leq z \leq u_2(x, y)\}$

$$\iiint\limits_E f(x, y, z)\, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\; r\; dz\; dr\; d\theta$$

15.8 Triple Integrals in Spherical Coordinates

15.9 Change of Variables in Multiple Integrals