

MATH 5 Lecture Notes

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1 Chapter 1

1.1 Systems of Equations

$\left[\begin{array}{cc|c} a & b & c \\ c & d & e \end{array} \right]$ Linear Equation Example:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

a_{ij} = are coefficients

x_i = variables = unknowns

b_i = constants

The set S_1, \dots, S_n is a solution if $a_i S_1 + \dots + a_{in} S_n = b_i$ is True = consistent for all i

3 possible outcomes

One Solution

No Solutions

Infinitely many Solutions

Row Operations

Interchange Rows

Row Multiplication

Row Addition: Add a constant multiple of one row to another row

Goal: $\left[\begin{array}{cc|c} 1 & 0 & S_1 \\ 0 & 1 & S_2 \end{array} \right]$

1.2 Row Reductions and Echelon Form

Reduced Row Echelon Form

1. Nonzero rows above zero rows
2. Each leading entry in a row is to the right of the pivot in row above
3. All pivots have zero above, below, & left of the pivot
4. All pivots must be 1

Theorem 1: The RREF[A] form is unique, does not imply there is a unique solution

Ex. 1 Not RREF, just Row Echelon Form

$\left[\begin{array}{cccc|c} 3 & 2 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$ Zeroes below and left of each pivot. A pivot is the first nonzero entry in each row

Basic variable corresponds to a column with a pivot. In this case x_1, x_3, x_4 are the basic variables

Free variable means there is no pivot in that column. In this case $x_2 = t$ to give it a parameter

$$3x_1 + 2x_2 + 3x_3 = 6, x_2 = t$$

$$x_3 + x_4 = 7, x_3 = 2$$

$$x_4 = 5, x_4 = 5$$

$$3x_1 + 2t + 3 \cdot 2 = 6$$

$$3x_1 + 2t = 0$$

$$3x_1 = -2t$$

$$x_1 = -\frac{2}{3}t \text{ making the parametric solution to the system: } \boxed{\left(-\frac{2}{3}t, t, 2, 5\right)} \text{ or } \boxed{\left(-\frac{2}{3}x_2, x_2, 2, 5\right)}$$

Ex. 2 Not RREF, just Row Echelon Form

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 2 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \text{ No Solution — system is not consistent in row 3}$$

Ex. 3 Convert to RREF

$$\left[\begin{array}{cccc|c} 3 & 2 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right] \rightarrow R_1 += -3R_2 \rightarrow \left[\begin{array}{cccc|c} 3 & 2 & 0 & -3 & -15 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & -3 & -15 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right] \rightarrow R_1 += 3R_3 \rightarrow \left[\begin{array}{cccc|c} 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right] \rightarrow R_1 /= 3 \rightarrow \left[\begin{array}{cccc|c} 1 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right]$$

Translate back to equations where $x_2 = t$

$$x_4 = 5, x_3 = 2, x_1 + \frac{2}{3}x_2 = 0$$

$$x_1 = -\frac{2}{3}t$$

$$x_2 = t$$

$$x_3 = 2$$

$$x_4 = 5 \quad \boxed{\text{Solution: } (-\frac{2}{3}t, t, 2, 5)}$$

If a solution exists and there is a free variable then there are infinitely many solutions

1.3 Vector Equations

If $\vec{a}_1 \dots \vec{a}_j$ spans \mathbb{R}^n then any \vec{b} in \mathbb{R}^n can be written as a Linear Combination of $\vec{a}_1 \dots \vec{a}_j$ so there exists $x_1 \dots x_j$ so that $x_1 \vec{a}_1 \dots x_j \vec{a}_j = \vec{b}$

In math: $\vec{a}_1 \dots \vec{a}_j$ spans $\mathbb{R}^n \Leftrightarrow \forall \vec{b} \in \mathbb{R}^n \exists x_1 \dots x_j \in \mathbb{R} \text{ so } x_1 \vec{a}_1 \dots x_j \vec{a}_j = \vec{b}$

Theorem: If $\vec{v}_1, \dots, \vec{v}_p, \vec{b}, \in \mathbb{R}^n$ the following statements are equivalent:

1. \vec{b} is a linear combination of $\vec{v}_1, \dots, \vec{v}_p$
2. $\exists x_1, \dots, x_p \in \mathbb{R} \text{ so } x_1 \vec{v}_1 + \dots x_p \vec{v}_p = \vec{b}$
3. $\vec{b} \in \text{span} \{ \vec{v}_1, \dots, \vec{v}_p \}$
4. The linear system corresponding to augmented matrix with columns $\left[\vec{v}_1 \dots \vec{v}_p | \vec{b} \right]$ is consistent
5. The reduced row echelon matrix of this augmented matrix does not contain a pivot in the last column $[0 \dots 0 | 1]$

1.4 The Matrix Equations $Ax=b$

$$\text{Matrix Multiplication } A \cdot \vec{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \vec{a}_1 + b_2 \vec{a}_2 + b_3 \vec{a}_3$$

Matrix A times a vector \vec{b}

Compute the product or state that it is undefined $\begin{bmatrix} 3 & -7 \\ 8 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \end{bmatrix} = \begin{bmatrix} -47 \\ -52 \\ 21 \end{bmatrix}$ Write the

following system as a vector equation or matrix equation as indicated $x_1 \begin{bmatrix} 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} +$

$$x_3 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

As a matrix Equation: $x_1 \begin{bmatrix} 6 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Solve $\begin{bmatrix} 1 & 3 & -1 \\ 2 & -4 & 1 \\ -4 & 18 & -5 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & -10 & 3 & -2b_1 + b_2 \\ 0 & 30 & -9 & 4b_1 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & -10 & 3 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + 3b_2 + b_3 \end{bmatrix}$$

If $-2b_1 + 3b_2 + b_3 = 0$ the system has infinitely many solutions. If its not on the plane then the system is inconsistent

Theorem: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A , either they are all true or all false.

- a For each \mathbf{b} in \mathbb{R}^m , the equation $Ax = b$ has a solution
- Each B in \mathbb{R}^m is a linear combination of the columns in A
- a The Columns of A span \mathbb{R}^m
- a RREF of A has a pivot position in every row

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7 Example Problems with Solutions

1: Find the general solution to the homogenous system below

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -11 & 0 \end{array} \right]$$

Interchange 1 and 3 then $R_2 = R_1 - 2R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array} \right]$$

Row 2 += Row 3 and Row 3 *= -1

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

Row 2 /= 2

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right] \text{ Back into equations } x_1 = 0, x_2 + x_4 = 0, x_3 - x_4 = 0$$

x_4 is a free variable since there is no pivot and the solution is $\boxed{0, -t, t, t}$

2: Solve for h to make consistent & find all solutions

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ -2 & 1 & 6 & 15 & -21 \end{array} \right]$$

Add first row to 3rd row

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ 0 & 0 & 7 & 14 & -21 \end{array} \right]$$

Scale row 3 by 1/7

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ 0 & 0 & 1 & 2 & -3 \end{array} \right]$$

A: Consistent if $h = -3$, inconsistent if $h \neq -3$

B: Find Solution when $h = -3$ $\left[\begin{array}{cccc|c} 1 & -1/2 & 0 & -3/2 & 3/2 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$

Solution: $\left[\begin{array}{c} \frac{3}{2} - \frac{3}{2}t + \frac{1}{2}qs \\ s \\ -3 - 2t \\ t \end{array} \right]$