MATH 5 Lecture Notes

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Contents

1 Chapter 1

1.1 Systems of Equations

$\left[\begin{array}{cc|c} a & b & c \\ c & d & e \end{array}\right]$ Linear Equation Example:
$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$
a_{ij} = are coefficients
x_i = variables = unknowns
b_i = constants
The set $S_1, ...S_n$ is a solution if $a_i S_1 + ... + a_{in} S_n = b_i$ is True = consistent for all i
3 possible outcomes
One Solution
No Solutions
Infinitely many Solutions
Row Operations
Interchange Rows
Row Multiplication
Row Addition: Add a constant multiple of one row to another row
Goal: $\left[\begin{array}{cc|c} 1 & 0 & S_1 \\ 0 & 1 & S_2 \end{array}\right]$

1.2 Row Reductions and Echelon Form

Reduced Row Echelon Form

1. Nonzero rows above zero rows
2. Each leading entry in a row is to the right of the pivot in row above
3. All pivots have zero above, below, & left of the pivot
4. All pivots must be 1

Theorem 1: The RREF[A] form is unique, does not imply there is a unique solution

Ex. 1 Not RREF, just Row Echelon Form

$\left[\begin{array}{cccc|c} 3 & 2 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ Zeroes below and left of each pivot. A pivot is the first nonzero entry in each row
Basic variable corresponds to a column with a pivot. In this case x_1, x_3, x_4 are the basic variables
Free variable means there is no pivot in that column. In this case $x_2 = t$ to give it a parameter
$3x_1 + 2x_2 + 3x_3 = 6, x_2 = t$
$x_3 + x_4 = 7, x_3 = 2$
$x_4 = 5, x_4 = 5$
$3x_1 + 2t + 3 \cdot 2 = 6$
$3x_1 + 2t = 0$
$3x_1 = -2t$
$x_1 = -\frac{2}{3}t$ making the parametric solution to the system: $\boxed{(-\frac{2}{3}t, t, 2, 5)}$ or $\boxed{(-\frac{2}{3}x_2, x_2, 2, 5)}$

Ex. 2 Not RREF, just Row Echelon Form

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 2 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ No Solution — system is not consistent in row 3}$$

Ex. 3 Convert to RREF

$$\left[\begin{array}{cccc|c} 3 & 2 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \rightarrow R_1 += -3R_2 \rightarrow \left[\begin{array}{cccc|c} 3 & 2 & 0 & -3 & -15 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & -3 & -15 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \rightarrow R_1 += 3R_3 \rightarrow \left[\begin{array}{cccc|c} 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \rightarrow R_1 /= 3 \rightarrow \left[\begin{array}{cccc|c} 1 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Translate back to equations where $x_2 = t$

$x_4 = 5, x_3 = 2, x_1 + \frac{2}{3}x_2 = 0$

$x_1 = -\frac{2}{3}t$

$x_2 = t$

$x_3 = 2$

$x_4 = 5$ $\boxed{\text{Solution:}(-\frac{2}{3}t, t, 2, 5)}$

If a solution exists and there is a free variable then there are infinitely many solutions

1.3 Vector Equations

If $\vec{a}_1 ... \vec{a}_j$ spans $\mathbb{R}^n$ then any $\vec{b}$ in $\mathbb{R}^n$ can be written as a Linear Combination of $\vec{a}_1 ... \vec{a}_j$ so there exists $x_1 ... x_j$ so that $x_1\vec{a}_1 ... x_j\vec{a}_j = \vec{b}$

In math: $\vec{a}_1 ... \vec{a}_j$ spans $\mathbb{R}^n \Leftrightarrow \forall \vec{b} \epsilon \mathbb{R}^n \exists x_1 ... x_j \mathbb{R}$ so $x_1\vec{a}_1 ... x_j\vec{a}_j = \vec{b}$

Theorem: If $\vec{v_1}, ..., \vec{v_p}, \vec{b}, \epsilon\mathbb{R}^n$ the following statements are equivalent:

1. $\vec{b}$ is a linear combination of $\vec{v_1}, ..., \vec{v_p}$
2. $\exists x_1, ..., x_p \epsilon \mathbb{R}$ so $x_1\vec{v_1} + ... x_p\vec{v_p} = b$
3. $\vec{b}\epsilon$ spans $\vec{v_1}, ..., \vec{v_p}$
4. The linear system corresponding to augmented matrix with columns $\left[\vec{v_1} ... \vec{v_p} | \vec{b}\right]$ is consistent
5. The reduced row echelon matrix of this augmented matrix does not contain a pivot in the last column $[0...0|1]$

1.4 The Matrix Equations Ax=b

Matrix Multiplication $A \cdot \vec{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1\vec{a}_1 + b_2\vec{a}_2 + b_3\vec{a}_3$

Matrix A times a vector $\vec{b}$

Compute the product or state that it is undefined $\begin{bmatrix} 3 & -7 \\ 8 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \end{bmatrix} = \begin{bmatrix} -47 \\ -52 \\ 21 \end{bmatrix}$ **Write the following system as a vector equation or matrix equation as indicated** $x_1 \begin{bmatrix} 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

As a matrix Equation: $x_1 \begin{bmatrix} 6 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Solve $\begin{bmatrix} 1 & 3 & -1 \\ 2 & -4 & 1 \\ -4 & 18 & -5 \end{bmatrix}$ and $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & -10 & 3 & -2b_1 + b_2 \\ 0 & 30 & -9 & 4b_1 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & -10 & 3 & -2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + 3b_2 + b_3 \end{bmatrix}$$

If $-2b_1 + 3b_2 + b_3 = 0$ the system has infinitely many solutions. If its not on the plane then the system is inconsistent

Theorem: Let A be an $m \times n$ matrix. Then the following statements arae logically equivalent. That is, for a particular A, either they are all true or all false.

a For each **b** in $\mathbb{R}^m$, the equation $Ax = b$ has a solution

  Each B in $\mathbb{R}^m$ is a linear combination of the columns in A

a The Columns of A span $\mathbb{R}^m$

a RREF of A has a pivot position in every row

7 Example Problems with Solutions

1: Find the general solution to the homogenous system below

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -11 & 0 \end{array}\right]$$

Interchange 1 and 3 then $R_2 = R_1 - 2R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

Row 2 += Row 3 and Row 3 *= -1

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Row 2 /= 2

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$ Back into equations $x_1 = 0, x_2 + x_4 = 0, x_3 - x_4 = 0$

x_4 is a free variable since there is no pivot and the solution is $\boxed{0, -t, t, t}$

2: Solve for h to make consistent & find all solutions

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ -2 & 1 & 6 & 15 & -21 \end{array}\right]$$

Add first row to 3rd row

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ 0 & 0 & 7 & 14 & -21 \end{array}\right]$$

Scale row 3 by 1/7

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ 0 & 0 & 1 & 2 & -3 \end{array}\right]$$

A: Consistent if $h = -3$, inconsistent if $h \neq -3$

B: Find Solution when $h = -3$ $$\left[\begin{array}{cccc|c} 1 & -1/2 & 0 & -3/2 & 3/2 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: $$\begin{bmatrix} \frac{3}{2} - \frac{3}{2}t + \frac{1}{2}qs \\ s \\ -3 - 2t \\ t \end{bmatrix}$$