

# MATH 5 Lecture Notes

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# 1 Chapter 1

## 1.1 Systems of Equations

$\left[ \begin{array}{cc|c} a & b & c \\ c & d & e \end{array} \right]$  Linear Equation Example:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$a_{ij}$  = are coefficients

$x_i$  = variables = unknowns

$b_i$  = constants

The set  $S_1, \dots, S_n$  is a solution if  $a_i S_1 + \dots + a_{in} S_n = b_i$  is True = consistent for all  $i$

### 3 possible outcomes

One Solution

No Solutions

Infinitely many Solutions

### Row Operations

Interchange Rows

Row Multiplication

Row Addition: Add a constant multiple of one row to another row

Goal:  $\left[ \begin{array}{cc|c} 1 & 0 & S_1 \\ 0 & 1 & S_2 \end{array} \right]$

## 1.2 Row Reductions and Echelon Form

### Reduced Row Echelon Form

1. Nonzero rows above zero rows
2. Each leading entry in a row is to the right of the pivot in row above
3. All pivots have zero above, below, & left of the pivot
4. All pivots must be 1

**Theorem 1:** The RREF[A] form is unique, does not imply there is a unique solution

### Ex. 1 Not RREF, just Row Echelon Form

$\left[ \begin{array}{cccc|c} 3 & 2 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$  Zeroes below and left of each pivot. A pivot is the first nonzero entry in each row

Basic variable corresponds to a column with a pivot. In this case  $x_1, x_3, x_4$  are the basic variables

Free variable means there is no pivot in that column. In this case  $x_2 = t$  to give it a parameter

$$3x_1 + 2x_2 + 3x_3 = 6, x_2 = t$$

$$x_3 + x_4 = 7, x_3 = 2$$

$$x_4 = 5, x_4 = 5$$

$$3x_1 + 2t + 3 \cdot 2 = 6$$

$$3x_1 + 2t = 0$$

$$3x_1 = -2t$$

$$x_1 = -\frac{2}{3}t \text{ making the parametric solution to the system: } \boxed{\left(-\frac{2}{3}t, t, 2, 5\right)} \text{ or } \boxed{\left(-\frac{2}{3}x_2, x_2, 2, 5\right)}$$

**Ex. 2 Not RREF, just Row Echelon Form**

$$\left[ \begin{array}{cccc|c} 1 & 2 & 0 & 4 & 2 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \text{ No Solution — system is not consistent in row 3}$$

**Ex. 3 Convert to RREF**

$$\left[ \begin{array}{cccc|c} 3 & 2 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right] \rightarrow R_1 += -3R_2 \rightarrow \left[ \begin{array}{cccc|c} 3 & 2 & 0 & -3 & -15 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 3 & 2 & 0 & -3 & -15 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right] \rightarrow R_1 += 3R_3 \rightarrow \left[ \begin{array}{cccc|c} 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right] \rightarrow R_1 /= 3 \rightarrow \left[ \begin{array}{cccc|c} 1 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right]$$

Translate back to equations where  $x_2 = t$

$$x_4 = 5, x_3 = 2, x_1 + \frac{2}{3}x_2 = 0$$

$$x_1 = -\frac{2}{3}t$$

$$x_2 = t$$

$$x_3 = 2$$

$$x_4 = 5 \quad \boxed{\text{Solution: } (-\frac{2}{3}t, t, 2, 5)}$$

If a solution exists and there is a free variable then there are infinitely many solutions



- 1.3 Vector Equations
- 1.4 The Matrix Equations  $Ax=b$
- 1.5 Solution Sets for Linear Systems
- 1.6 Linear Independence
- 1.7 Linear Transformations
- 1.8 The Matrix of a Linear Transformation
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- 6 Chapter 6
- 6.1 Inner Product Spaces
- 6.2 Orthogonal Sets
- 6.3 Orthogonal Projections
- 6.4 The Gram Schmidt Process

## 7 Example Problems with Solutions

**1: Find the general solution to the homogenous system below**

$$\left[ \begin{array}{cccc|c} 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -11 & 0 \end{array} \right]$$

Interchange 1 and 3 then  $R_2 = R_1 - 2R_3$

$$\left[ \begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array} \right]$$

Row 2 += Row 3 and Row 3 \*= -1

$$\left[ \begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

Row 2 /= 2

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right] \text{ Back into equations } x_1 = 0, x_2 + x_4 = 0, x_3 - x_4 = 0$$

$x_4$  is a free variable since there is no pivot and the solution is  $\boxed{0, -t, t, t}$

**2: Solve for h to make consistent & find all solutions**

$$\left[ \begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ -2 & 1 & 6 & 15 & -21 \end{array} \right]$$

Add first row to 3rd row

$$\left[ \begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ 0 & 0 & 7 & 14 & -21 \end{array} \right]$$

Scale row 3 by 1/7

$$\left[ \begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ 0 & 0 & 1 & 2 & -3 \end{array} \right]$$

A: Consistent if  $h = -3$ , inconsistent if  $h \neq -3$

B: Find Solution when  $h = -3$   $\left[ \begin{array}{cccc|c} 1 & -1/2 & 0 & -3/2 & 3/2 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$

Solution:  $\left[ \begin{array}{c} \frac{3}{2} - \frac{3}{2}t + \frac{1}{2}qs \\ s \\ -3 - 2t \\ t \end{array} \right]$