MATH 5 Lecture Notes

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1 Chapter 1

1.1 Systems of Equations

$\left[\begin{array}{cc|c} a & b & c \\ c & d & e \end{array}\right]$ Linear Equation Example:
$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$
a_{ij} = are coefficients
x_i = variables = unknowns
b_i = constants
The set $S_1, ...S_n$ is a solution if $a_i S_1 + ... + a_{in}S_n = b_i$ is True = consistent for all i
3 possible outcomes
One Solution
No Solutions
Infinitely many Solutions
Row Operations
Interchange Rows
Row Multiplication
Row Addition: Add a constant multiple of one row to another row
Goal: $\left[\begin{array}{cc|c} 1 & 0 & S_1 \\ 0 & 1 & S_2 \end{array}\right]$

1.2 Row Reductions and Echelon Form

Reduced Row Echelon Form

1. Nonzero rows above zero rows
2. Each leading entry in a row is to the right of the pivot in row above
3. All pivots have zero above, below, & left of the pivot
4. All pivots must be 1

Theorem 1: The RREF[A] form is unique, does not imply there is a unique solution

Ex. 1 Not RREF, just Row Echelon Form

$\left[\begin{array}{cccc|c} 3 & 2 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ Zeroes below and left of each pivot. A pivot is the first nonzero entry in each row
Basic variable corresponds to a column with a pivot. In this case x_1, x_3, x_4 are the basic variables
Free variable means there is no pivot in that column. In this case $x_2 = t$ to give it a parameter
$3x_1 + 2x_2 + 3x_3 = 6, x_2 = t$
$x_3 + x_4 = 7, x_3 = 2$
$x_4 = 5, x_4 = 5$
$3x_1 + 2t + 3 \cdot 2 = 6$
$3x_1 + 2t = 0$
$3x_1 = -2t$
$x_1 = -\frac{2}{3}t$ making the parametric solution to the system: $\boxed{(-\frac{2}{3}t, t, 2, 5)}$ or $\boxed{(-\frac{2}{3}x_2, x_2, 2, 5)}$

Ex. 2 Not RREF, just Row Echelon Form

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 2 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ No Solution — system is not consistent in row 3}$$

Ex. 3 Convert to RREF

$$\left[\begin{array}{cccc|c} 3 & 2 & 3 & 0 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \to R_1 \mathrel{+}= -3R_2 \to \left[\begin{array}{cccc|c} 3 & 2 & 0 & -3 & -15 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & -3 & -15 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \to R_1 \mathrel{+}= 3R_3 \to \left[\begin{array}{cccc|c} 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right] \to R_1 \mathrel{/}= 3 \to \left[\begin{array}{cccc|c} 1 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Translate back to equations where $x_2 = t$

$x_4 = 5, x_3 = 2, x_1 + \frac{2}{3}x_2 = 0$

$x_1 = -\frac{2}{3}t$

$x_2 = t$

$x_3 = 2$

$x_4 = 5$ $\boxed{\text{Solution:}(-\frac{2}{3}t, t, 2, 5)}$

If a solution exists and there is a free variable then there are infinitely many solutions

1.3 Vector Equations

1.4 The Matrix Equations Ax=b

1.5 Solution Sets for Linear Systems

1.6 Linear Independence

1.7 Linear Transformations

1.8 The Matrix of a Linear Transformation

2 Chapter 2

2.1 Matrix Operations

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7 Example Problems with Solutions

1: Find the general solution to the homogenous system below

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -11 & 0 \end{array}\right]$$

Interchange 1 and 3 then $R_2 = R_1 - 2R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

Row 2 += Row 3 and Row 3 *= -1

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Row 2 /= 2

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$ Back into equations $x_1 = 0, x_2 + x_4 = 0, x_3 - x_4 = 0$

x_4 is a free variable since there is no pivot and the solution is $\boxed{0, -t, t, t}$

2: Solve for h to make consistent & find all solutions

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ -2 & 1 & 6 & 15 & -21 \end{array}\right]$$

Add first row to 3rd row

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ 0 & 0 & 7 & 14 & -21 \end{array}\right]$$

Scale row 3 by 1/7

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & h \\ 0 & 0 & 1 & 2 & -3 \end{array}\right]$$

A: Consistent if $h = -3$, inconsistent if $h \neq -3$

B: Find Solution when $h = -3$ $\left[\begin{array}{cccc|c} 1 & -1/2 & 0 & -3/2 & 3/2 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solution: $\begin{bmatrix} \frac{3}{2} - \frac{3}{2}t + \frac{1}{2}s \\ s \\ -3 - 2t \\ t \end{bmatrix}$