MATH 2 Lecture Notes

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Tuesday, 14 January, 2025

Contents

1 Chapter 1

1.1 Terminology

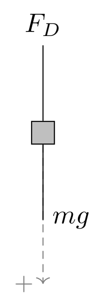
Definition A differential equation is an equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables.
· An Ordinary Differential Equation (ODE) involves only ordinary derivatives
· A Partial Differential Equation (PDE) involves partial derivatives.
Definition The order of a DE is the order of the highest-order derivative that appears in the DE
Notation $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2})$
Definition A linear DE is any DE that can be written in form:
$a_0(x)y + a_1(x)y' + a_2(x)y'' \cdots + a_n(x)y^{(n)} = b(x)$
For a DE to be linear:

1. Y and all of its derivatives much be of the 1st degree
2. Any term that does not include y or any of its derivatives must be a function of x

1.2 Some Mathematical Models

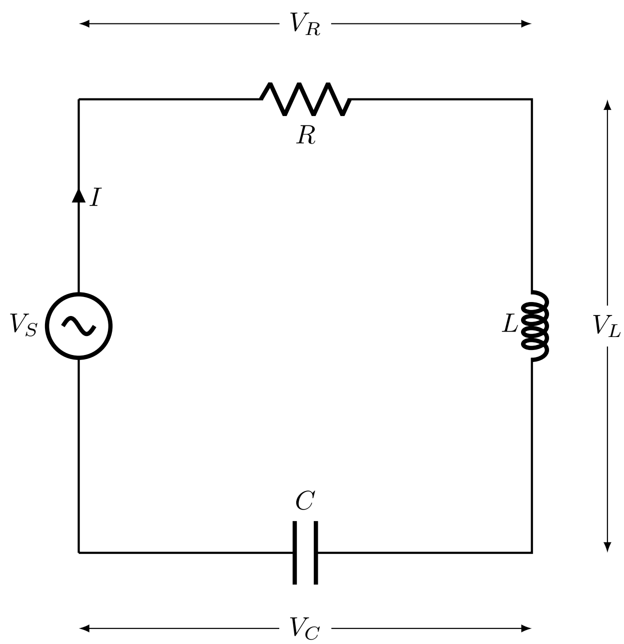
I. Free-falling body

Goal: Find s(t).



Set up a differential equation in S, model it, then solve
$ma = mg$
$\frac{d^2s}{dt^2} = g$
$v = \frac{ds}{dt}, g = \frac{dv}{dt}$
What if there is air resistance. Assume force scales linear with velocity
$\frac{dv}{dt} = g - \frac{kv}{m} \rightarrow \frac{dv}{dt} = g - \frac{k}{m} \cdot \frac{ds}{dt}$

II: Series Circuit



Voltage drops:

$V = L\dfrac{dI}{dt}, V = L\dfrac{d^2q}{dt^2}$

$V = IR, V = R\dfrac{dq}{dt}$

$V = \dfrac{q}{C}$

$E(t) = L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C}$

III: Population Growth

$P = P(t) =$ population at time t — use exponential model

$\dfrac{dp}{dt} \propto P \rightarrow \dfrac{dp}{dt} = kP \rightarrow = Ce^{kt}$ where C is the initial population

IV: Population Growth with Finite Capacity

"Carrying Capacity" = N — uses the logistic growth model

$\dfrac{dp}{dt} \propto$ both P and amount to carrying capacity (N-P)

$\dfrac{dp}{dt} = kP(N - P)$

V: Chemical Reaction

$A + B \rightarrow C$ Concentrations of A and B decreases by amount of C formed

Can we write DE governing the concentration of C x(t)?

The rate at which the reaaction takes place $\propto$ Product of the remaining concentrations of A and B

α initial concentration of A

β initial concentration of B

$\dfrac{dx}{dt} = k(\alpha - x)(\beta - x)$

2 First-Order Differential Equations

2.1 Preliminary Theory

Example DE: $y' = 3y \Rightarrow \boxed{y = Ce^{3x}}$ the general solution where C is an arbitrary constant

Add initial condition $y(0) = 5$ plug in x=0 to $5 = Ce^{3*0}, 5 = C * 1, C = 5 \Leftarrow$ Initial Value Problem
$y = 5e^{3x}$ is the general solution for the Initial Value Problem

2.1.1 Theorem

$$f(x) = \begin{cases} \dfrac{dy}{dx} = f(x, y) & \text{Differential Equation} \\ y(x_0) = y_0 & \text{Initial Condition} \end{cases}$$

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b, c \leq y \leq d$, that contains the point (x_0, y_0) in its interior.

If f(x,y) and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists an interval I centered at x_o, and on this interval I there exists a unique solution $y(x)$ for this IVP

2.1.2 Key Questions:

Does every IVP have at least one solution?
If an IVP has a solution is it the only solution?

Meaning of a solution existing "on an Interval" The initial value problem

$\begin{cases} \dfrac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$ has a unique solution. In fact, we can easily verify that $y = \tan x$ satisfies this IVP

However note that there are some inervals on which $y = \tan x$ cannot be a solution for this IVP, such as (-2,2), where the function is discontinuous at $\pm\dfrac{\pi}{2}$ but can be used for (-1,1) since it is continuous at all points within the interval

2.2 Separable Variables (Separable Equations)

2.2.1 Definition:

A differential equation that can be written in the form $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$ is said to be separable (or have separable variables).

Example: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$

$h(y)dy = g(x)dx$

$\int h(y)dy = \int g(x)dx$

Example: $dx + e^{3x}dy = 0$

$e^{3x}dy = -dx$

$dy = -\dfrac{dx}{e^{3x}} \rightarrow dy = -e^{-3x}dx \rightarrow \int dy = \int -e^{-3x}dx \rightarrow y = \dfrac{1}{3}e^{-3x} + C$ where C is an arbitrary constant

2.2.2 Substitution

$\dfrac{dy}{dx} = F(ax+bc+c)$ where $b \neq 0$ use the substitution: $u = ax+by+c \Rightarrow \dfrac{du}{dx} = a+b\dfrac{dy}{dx} = \dfrac{1}{b}\left[\dfrac{du}{dx} - a\right]$

Example: $\dfrac{dy}{dx} = \tan^2(x+y)$ let $u = x+y \rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} - 1 \rightarrow \dfrac{du}{dx} - 1 = \tan^2 u \rightarrow \dfrac{du}{dx} = \sec^2 u$

$\displaystyle\int \cos^2 u\, du = \int dx$

$2(x+y) + \sin 2(x+y) = 4x + C \rightarrow 2y - 2x + \sin 2(x+y)$

Solve: $\dfrac{dy}{dx} = (y+3)^2$ By inspection $y = -3$ is a solution. This is the only solution because $f(x,y) = (x+3)^2$ is continuous on $\mathbb{R}^2$ and $\dfrac{\partial f}{\partial x}$ is continuous on$\mathbb{R}$ so it is the only solution Why solving by separation is not possible $\displaystyle\int (y+3)^{-}2dy = \int dx \rightarrow (y+3)^{-}2/-1 = x + C_1 \rightarrow \dfrac{1}{y+3} = -x - C_1 \rightarrow$
$y + 3 = \dfrac{1}{c-x} \rightarrow y = -3 + \dfrac{1}{c-x}$

$y(0) = -3 \rightarrow 0 = \dfrac{1}{c}$ where there is no real c that solves that equation, making this not possible

2.3 Homogeneous Equations

What do we do if the DE is not separable?

2.3.1 Definition

A function $f(x,y)$ is said to be **homogeneous of degree** n if, for $x, y,$ and$t$where $f(x,y)$ and $f(tx,ty)$ are defined:

$$f(tx,ty) = t^n f(x,y)$$

2.3.2 Example

Determine wheteher each function is homogeneous:
a: $f(x,y) = x^3 - 7x^2y + 4y^3 \rightarrow f(tx,ty) = (tx^3) - 7(tx)^2(ty) + 4(ty)^3$
$t^3x^3 - 7t^3x^2y + 4t^3y^3$
$t^3(x^3 - 7x^2y - 4y^3) = t^3 f(x,y)$
How to tell quickly wheteher $f(x,y)$ is homogeneous:
Each term must have the same combined degree
Example: $x^3 - 7x^2y + 4y^3$ is D3, $x^2 + y^2 - 4x$ is not, $\sqrt{x^5 + 4y^5}$ is with D 2.5, $\dfrac{3y}{x} - 2$ is D0

2.3.3 Differential Equation form

$M(x,y)dx + N(x,y)dy = 0$ is called a homogeneous differential equation if the functions M and N are both homogeneous of the same degree
If $f(x,y)$ is homogeneous of degree n then $f(x,y)$ can be written as:
$f(x,y) = f\left(x \times 1, x \times \dfrac{y}{x}\right) = x^n f\left(1, \dfrac{y}{x}\right)$
or $f(x,y) = y^n f\left(\dfrac{x}{y}, 1\right)$

2.3.4 Substitution

To solve a homogeneous DE make the substitution: $y = ux \;\; (u = \dfrac{y}{x})$ or$x = vy \;\; (v = \dfrac{x}{y})$

2.3.5 Example

$(y^2 + xy)dx + x^2dy = 0 \to y = ux \to dy = (udx + xdu)$
$(u^2x^2 + ux^2)dx + x^2(udx + xdu) = 0$
$u^2x^2dx + ux^2dx + ux^2dx + x^3du = 0$
$ux^2(u + 2)dx + x^3du = 0$
$\int \frac{1}{u(u+2)}du = -\int \frac{1}{x}dx$
Partial Fraction Decomposition: $\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2} \to A = \frac{1}{2}, B = -\frac{1}{2}$ Back to solving
$\int \left[\frac{0.5}{u} - \frac{0.5}{u+2}\right] = -\int \frac{1}{x}dx$
$0.5\ln|u| - 1/2\ln|u+2| = -\ln|x| + C_1$
$\ln\left|\frac{u}{u+2}\right| = 2C_1 - 2\ln|x|$
$\left|\frac{u}{u+2}\right| = e^{2C_1} \cdot e^{-2\ln|x|} = e^{2C_1} \cdot |x^{-2}| \Rightarrow \left|\frac{u}{u+2}\right| = \left|e^{2C_1} \cdot x^{-2}\right| \Rightarrow \left|\frac{u}{u+2} = \frac{C}{x^2}\right|$
$ux^2 = X(u+2) \Rightarrow ux^2 = Cu + 2c \to ux^2 - Cu = 2C$
$u(x^2 - c) = 2C \Rightarrow u = \frac{2C}{x^2 - C} \Rightarrow \frac{y}{x} = \frac{2Cx}{x^2 - C},\ x \neq 0$

2.4 Exact Equations

Recall from Math 1C: Let $F(x, y) = \langle 3x^2 - 7y, -7x + 2y\rangle$

1. If F a conservative vector field
   i.e., Is there a function $f(x, y)$ such that ∇f? Yes, -7=-7

2. If F is indeed conservative, what is f?
   $x^3 - 7xy + g(y) = f(x, y)$
   $-7x + 2y, g'(y) = 2y$
   $f(x, y) = x^3 - 7xy + y^2 + k$

2.4.1 Definition

A differential equation in the form $M(x, y)dx + N(x, y)dy = 0$ where $M_y = N_x$, is called an exact differential equation.

2.4.2 Solve the DE

$(3x^2 - 7y)dx + (-7x + 2y)dy = 0$
Using 1C techniques it is $f(x, y) = x^3 - 2xy + y^2 + k$
Set this f = c. $f(x, y) = x^3 - 2xy + y^2 = c$ take k=0 in every problem
If the DE is not exact, sometimes we can make it exact by multiplying by magical quantity $\mu(x, y)$

2.4.3 Example:

Solve the DE:
$(x + y)dx + xlnxdy = 0$ using $\mu(x, y) = \frac{1}{x}$
$\left(\frac{x+y}{x}\right)dx + \ln|x|\, dy = 0$ is now exact.
Solution: $f(x, y) = x + y\ln x = c$

2.5 Linear Equations

Recall: First Order Linear DE is a DE in the form $a_1(x)\dfrac{dy}{dx} + a_0(y)y = g(x), \qquad a_1(x) \neq 0$

Divide both sidex by $a_1(x) \Rightarrow \dfrac{dy}{dx} + \dfrac{a_0(x)}{a_1(x)}y = \dfrac{g(x)}{a_1(x)}$ where $P(x) = \dfrac{a_0(x)}{a_1(x)}$ and $f(x) = \dfrac{g(x)}{a_1(x)}$

$\dfrac{dy}{dx} + P(x)y = f(x)$ There is an integrating factor $\mu(x)$ that turns this DE into an exact DE

$dy + P(x)ydx = f(x)dx \rightarrow dy\,[P(x)y - f(x)]\,dx = 0$

$\mu(x)dy + \mu(x)\,[P(x)y - f(x)]\,dx = 0 \rightarrow \mu'(x) = \mu(x)P(x)$

$\dfrac{d\mu}{dx} = \mu P \rightarrow \displaystyle\int \frac{d\mu}{mu} = \int P(x) \rightarrow \ln\mu = \int P(x)\ dx$

$\mu(x) = e^{\int P(x)\ dx} \Rightarrow e^{\int P(x)\ dx}\dfrac{dy}{dx} + e^{\int P(x)\ dx}P(x)y = e^{\int P(x)\ dx}f(x)$

$\dfrac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x) \rightarrow e^{\int P(x)dx}y = \displaystyle\int e^{\int P(x)dx}f(x)dx$ $\boxed{y = e^{-\int P(x)dx}\displaystyle\int e^{\int P(x)dx}f(x)dx}$

2.5.1 Procedure to follow for every Linear DE

1. Rewrite the linear DE in the form $\dfrac{dy}{dx} + P(x)y = f(x)$
2. Find the integrating factor $\mu(x) = e^{\int P(x)dx}$
3. Multiply each side of the DE by $\mu(x)$
4. Rewrite the left side as $\dfrac{d}{dx}\left[\mu(x)\cdot y\right]$
5. Integrate both sides with respect to x and retreive an implicitly expressed solution
6. Solve for y

2.6 What method to use to solve?

First ask is it exact? ($M_y = M_x$)
Yes: Use the method in §2.4
No: Is it linear? (in y or x)
Yes: Use the method in §2.5
No: Is it separable?
Yes: §2.2
No: Homogeneous?
Yes: Use a substitution §2.3
No: Good luck. or use inspection

3 Applications of First-Order Differential Equation

3.1 Orthogonal Trajectories

· Consider the family of curves $y = cx^3$ Question: Which DE should be solved to get this family as its solutions?
Steps:

1. Find $\frac{dy}{dx} = 3cx^2$
2. Eliminate c"

$y = cx^3 c = \frac{y}{x^3}$
$\frac{dy}{dx} = 3\frac{y}{x^3}x^2 \rightarrow \frac{3y}{x}$
 · The two curves are orthogonal if their tangent lines are orthogonal at the point of intersection i.e. The derivatives are the negative reciprocals of each other

3.1.1 Example

Show that $y = x^3$ and $x^2 + 3y^2 = 4$ are orthogonal at their points of intersection, (1,1) and (-1,-1)
$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \rightarrow = 3$ at $x = 1$ and 3 at $x = -1$
$2x + 6y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{x}{3y} = \frac{-1}{3}$ at both $x = 1$ and $x = -1$ meaning it is orthogonal

3.1.2 Definition

When all the curves of one family of curves intersect orthogonally all the curves of another family, then the families are said to be orthogonal trajectories of each other

3.2 Applications of Linear Equations

$\frac{dN}{dt} = kN \rightarrow N = Ce^{kt}$ for bacterial growth rate. Nothing else here, just an applications section

3.3 Applications of Nonlinear Equations

Logistic Model of Population Growth

1: End Behaviour (Steady State Solution) as $t \rightarrow \infty P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}} \rightarrow P(t) = \frac{aP_0}{bP_0} = \frac{a}{b}$

2: Concavity Analysis (Point of Inflection) $\frac{dP}{dt} = P(a - bP)$
$\frac{d^2P}{dt^2} = \frac{dP}{dt}(a - 2bP) = P(a - bP)(a - 2bP) = 0$
For inflection point $a - 2bP = 0 \rightarrow a = 2bP \rightarrow P = \frac{a}{2b} \rightarrow P = \frac{N}{2}$ 3 cases of initial conditions

$$\begin{cases} 0 < P_0 < \frac{a}{2b} & \text{Hits inflection point while rising to CC} \\ \frac{a}{2b} < P_0 < \frac{a}{b} & \text{Population grows at a decreasing rate to CC} \\ P_0 > \frac{a}{b} & \text{Population falls to the carying capacity} \end{cases}$$

4 Linear DE of Higher Order

4.1 Preliminary Theory

Initial Value Problem $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = g(x)$
Initial Conditions $y(x_0) = y_0 ... y^{(n-1)}(x_0) = y_0^{(n-1)}$
Theorem Let easch $a_j(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every... CONTINUE LATER
Boundary-Value Problem for 2nd order Linear DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$
$y(a) = y_0 \qquad y(b) = y_1$
Example: $y'' + 16y = 0 \quad y(0) = 0 \quad y(\pi/2) = 0$
$y = \sin 4x$ and $y = \cos 4x$ are solutions so $y(x) = c_1 \cos 4x + c_2 \sin 4x$
$y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 0$
$y(\pi/2) = c_1 \cos(2\pi) + c_2 \sin(2\pi) = c_1 = 0$ so $y(x) = c_2 \sin 4x$ is a solution

4.2 Constructing a Second Solution from a Known Solution

General Formula

Given $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ and $a_2(x) \neq 0$ and $y_1(x) \neq 0$ is a solution of this DE, find $y_2(x)$

Divide by $a_2(x)$: $y'' + \overset{P(x)}{\cancel{\frac{a_1(x)}{a_2(x)}}}y' + \overset{Q(x)}{\cancel{\frac{a_0(x)}{a_2(x)}}}y = 0$
$y'' + Py' + Qy = 0, \quad y_2 = uy_1 \rightarrow y_2' = uy'1 + u'y_1 \rightarrow y_2'' = uy_1'' + 2u'y_1' + u''y_1$
Plug it all in: $u''y_1 + 2u'y_1' + uy_1'' + Py'y_1 + Puy_1' + Quy_1 = 0$

$u''y_1 + u'(2y_1' + Py_1) + u\overset{0}{\cancel{(y_1'' + Py_1' + Qy_1)}} = 0$
$u''y_1 + u'(2y_1' + Py_1) = 0$ Let w=u' and w'=u''
$y_1w' + (2y_1' + Py_1)w = 0$
$w' + \frac{2y_1' + Py_1}{y_1}w = 0 \quad \mu = e^{\int \frac{2y_1'}{y_1} + Pdx}$
$w = c_1 y_1^{-2} e^{-\int Pdx} = u' = e^{-\int Pdx} y_1^2$

4.2.1 General Reduction of Order Formula

$y = C_1y_1 + C_2y_2$ where $\boxed{y_2 = y_1 \int \frac{e^{-\int Pdx}}{y_1^2}dx}$

4.3 Homogeneous Linear Equations w/ Constant Coefficients

In the DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, take
$a_2(x) = a, \qquad a_1(x) = b, \qquad a_0(x) = c$
so we have a 2ns-order Homogeneous Linear DE with *constant coefficients*

$$ay'' + by' + cy$$

What does a typical solution look like?
$\begin{cases} y = e^{mx} \\ y' = me^{mx} \\ y'' = m^2e^{mx} \end{cases}$ so $am^2e^{mx} + bme^{mx} + ce^{mx} = 0 = e^{mx}(am^2 + bc + c)$

4.3.1 Auxilliary Equation for a DE

$am^2 + bm + c = 0$

4.3.2 Three Scenarios for the Auxilliary Equation

$$\begin{cases} \text{If } \; b^2 - 4ac > 0 & \text{two real roots } \; y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \\ \text{If } \; b^2 - 4ac = 0 & \text{one real root } \; y = c_1 e^{mx} + c_2 x e^{mx} \\ \text{If } \; b^2 - 4ac < 0 & \text{No real roots, 2 distinct complex roots}, y = e^{\alpha x}\left(C_1 \cos\beta t + C_2 \sin\beta t\right) \; m = \alpha \pm i\beta \end{cases}$$

4.4 Undetermined Coefficients - Superposition Approach

• Nonhomogeneous Linear DE with constant coefficients:

$$ay'' + by' + cy = g(x)$$

Recall from Section 4.1: The general solution is:

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x)$ is the general solution $ay'' + by' + cy = 0$
$y_p(x)$ is one particular colution of $ay'' + by' + cy = g(x)$
The big question: How do we find $y_p(x)$?

4.4.1 Trial Particular Solutions

g(x)	Form of y_p
constant	A
$2x - 7$	$Ax + B$
$-x^2 + 3$	$Ax^2 + Bx + C$
$\sin kx$ or $\cos kx$	$A\cos kx + B\sin kx$
e^{kx}	Ae^{kx}
$(2x-7)e^{kx}$	$(Ax + B)e^{kx}$
$x^2 e^{kx}$	$(Ax^2 + Bx + C)e^{kx}$
$e^{kx}\cos lx \; or e^{kx}\sin lx$	$e^{kx}(A\cos lx + B\sin lx)$
$5x^2 \sin kx$	$(Ax^2 + Bx + C)\cos kx + (Dx^2 + Ex + F)\sin kx$
$xe^{kx}\cos lx$	$(Ax + B)e^{kx}\cos lx + (Cx + D)e^{kx}\sin lx$

4.5 Variation of Parameters

Given: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$
$\Rightarrow y'' + P(x)y' + Q(x)y = f(x)$
Goal: Find y_p

4.5.1 Idea:

Let $y_1, \; y_2$ be the two linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

Then we will look for y_p of the form:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

4.5.2 Derivation:

$y_p' = u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2'$ by product rule

Assume $u_1'y_1 + u_2'y_2 = 0 \to y_p' = u_1y_1' + u_2y_2'$

$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2' + y_2'$ then substitute into $y'' + Py' + Qy = f$

$$u_1(\cancelto{0}{y_1'' + Py_1' + Qy_1}) + u_2(\cancelto{0}{y_2'' + Py_2' + Qy_2}) + u_1'y_1' + u_2' + y_2' = f$$

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2' + y_2' = f \end{cases} \quad \to y_p = u_1y_1 + u_2y_2$$

4.5.3 Example

$y'' + 9y = \cos 3x$

$m^2 + 9 = 0 \to m = \pm 3i \to y_c = C_1 \cos 3x + C_2 \sin 3x$

$$y_p = u_1y_1 + u_2y_2 \text{ where } u_1' = \frac{\begin{bmatrix} 0 & \sin 3x \\ \cos 3x & 3\cos 3x \end{bmatrix}}{\begin{bmatrix} \cos 3x & \sin 3x \\ 3\sin 3x & 3\cos 3x \end{bmatrix}} = -\frac{1}{3}\sin 3x \cos 3x = -\frac{1}{6}\sin 6x$$

$$u_1 = \int -\frac{1}{6}\sin 6x dx = \frac{\cos 6x}{36}$$

$$u_2' = \frac{\begin{bmatrix} \cos 3x & 0 \\ -3\sin 3x & \cos 3x \end{bmatrix}}{\begin{bmatrix} \cos 3x & \sin 3x \\ 3\sin 3x & 3\cos 3x \end{bmatrix}} = \frac{\cos^2 3x}{3} = \int \frac{\cos^2 3x}{3} dx = \int \frac{1 + \cos 6x}{6} dx = \frac{x}{6} + \frac{\sin 6x}{36}$$

$$y_p = \frac{\cos 6x}{36}\cos 3x + \frac{x}{6} + \frac{\sin 6x}{36}\sin 3x = \frac{1}{36}\cos 3x + \frac{x}{6}\sin 3x$$

$$y_p = C_1\cos 3x + C_2 \sin 3x + \frac{1}{36}\cos 3x + \frac{x}{6}\sin 3x = \boxed{C_1 \cos 3x + C_2 \sin 3x + \frac{1}{6}x \sin 3x}$$

Initial Value Problem $y(0) = 1, \quad y'(0) = 0$

$$C_1\cos 3x + \cancelto{0}{C_2 \sin 3x + \frac{1}{6}x\sin 3x} \to C_1 \cos(0) = 1 \to C_1 = 1$$

$$y'(0) \to \cancelto{0}{-3\sin 3x} + 3C_2 \cos 3x + \frac{1}{6}x + 3\cos 3x + \cancelto{0}{\frac{1}{6}\sin 3x}$$

$$3C_2 \cos 3x + \frac{1}{6}x + 3\cos 3x = 0 \to 3C_2 = 0 \text{ so } C_2 = 0$$

$$\boxed{y(x) = \cos 3x + \frac{1}{6}x \sin 3x}$$

5 Applications of 2nd-Order DE

5.1 Simple Harmonic Motion

$\dfrac{d^2x}{dt^2} + \omega^2 s = 0$ where $\omega = \sqrt{\dfrac{k}{m}}$
Auxilliary Equation: $m^2\omega^2 = 0 \quad m = 0 \pm \beta i$
$C_1 cos\omega t + C_3 \sin \omega t = A \sin(\omega t + \phi)$ where $A = \sqrt{C_1^2 + C_2^2}$, $\sin\phi = \dfrac{C_2}{A}$, $\cos\phi = \dfrac{C_2}{A}$, $\tan\phi = \dfrac{C_1}{C_2}$

5.1.1 Example

A 16lb weight is attached to a spring, stretching it 1.28 feet. This weight is released 2 feet above the equilibrium position, with initial downward velocity of 1.5 ft/s

DE: $m\dfrac{d^2x}{dt^2} + kx = 0 \begin{cases} \text{Weight} & w = mg = \dfrac{1}{2}\text{Slug} \\ k & F = k \cdot s = 16 = k * 1.28, \quad 12.5\dfrac{lb}{ft} \end{cases}$

$\dfrac{d^2x}{dt^2} + 25x = 0, \quad x(0) = -2, \quad x'(0) = -1.5$
Solve the equation: $x(t) = C_1 \cos 5t + C_2 \sin 5t$
$x(t) = -2\cos 5t + 0.3 \sin 5t$
Period is $\dfrac{2\pi}{5}$ seconds or $\dfrac{5}{2\pi}$ vps

5.2 Damped Motion

Copy Notes from Canvas

5.3 Forced Motion

With the presence of an external force, the equation of motion has another extra term:
$x\dfrac{d^2x}{dt^2} = -kx - \beta\dfrac{dx}{dt} + f(t)$ where $k > 0$

5.3.1 Example

8lb weight, 32/13 ft stretch when hung freely, starts 3 feet below equilibrium, find equation of motion if $f(t) = 18.75 \sin 2t$ is applied and damping force is 1.5x instantaneous velocity
$m\dfrac{d^2x}{dt^2} + \beta\dfrac{dx}{dt} + kx = f(t)$
$m = \dfrac{1}{4}$, $\beta = 1.5$ $k : 8 = k \cdot \dfrac{32}{13} = \dfrac{13}{4}$
$\dfrac{1}{4}x'' + \dfrac{6}{4}x' + \dfrac{13}{4}x = 18.75 \sin 2t$
$x'' + 6x' + 13x = 75 \sin 2t$ $x(0) = 3$ $x'(0) = 0$
$x_p = A\cos 2t + B \sin 2t$ $A = -4$, $B = 3$
$x(t) = c_1 e^{-3t} \cos 2t + c_2 e^{-3t} \sin 2t - 4\cos 2t + 3 \sin 2t$
with initial conditions $c_1 = 7$ $c_2 = \dfrac{15}{2}$
$x(t) = 7e^{-3t}\cos 2t + \dfrac{15}{2}e^{-3t}\sin 2t - 4\cos 2t + 3\sin 2t$
$-4\cos 2t + 3\sin 2t$ is the steady state solution and
$7e^{-3t}\cos 2t + \dfrac{15}{2}e^{-3t}\sin 2t$ is the transient solution

5.4 Electric Circuits & Other Analogous Systems

LRC Circuit: For charge $q = q(t)$ and currtent $i = i(t) = \frac{dq}{dt}$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E(t)$$

5.4.1 Example

$L = 2H,\ R = 8\Omega,\ C = \frac{1}{18}F,\ E(t) = 10\sin(t)\ V,\ q(0) = 0,\ i(0) = 0\ 2\frac{d^2q}{dt^2} + 8\frac{dq}{dt} + 18\frac{q}{c} = 10\sin t$
$= \frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 9\frac{q}{c} = 5\sin t$
$\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 9\frac{q}{c} = 5\sin t \Rightarrow m^2 + 4m + 9 = 0 \to m = -2 \pm \sqrt{5}i$
$q_c(t) = c_1e^{-2t}\cos\sqrt{5}t + c_2e^{-2t}\sin\sqrt{5}t$
$y_p = -\frac{1}{4}\cos t + \frac{1}{2}\sin t$ time for initial conditions
$c_1 = \frac{1}{4}\ \ c_2 = 0 \Rightarrow \boxed{q(t) = c_1e^{-2t}\cos\sqrt{5}t - \frac{1}{4}\cos t + \frac{1}{2}\sin t}$
Steady state charge: $q = -\frac{1}{4}\cos t + \frac{1}{2}\sin t$
Steady state current: $i = \frac{1}{4}\sin t + \frac{1}{2}\cos t$

6 DE with Variable Coefficients

6.1 Cauchy-Euler Equations

Differential Equations of the form:

$$a_nx^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + ... + a_1xy' + a_0y = g(x)$$

These are considered equidimensional, where the degree of each monomial coefficient function equals the order of the derivative of y in each term

6.1.1 General Formula for the Auxilliary Equation

$am(m-1) + bm + c = 0$ is equivalent to $am^2 + (b-a)m + c = 0$

6.1.2 Forms of Solutions

$$\begin{cases} \text{Two Real Roots} & y = c_1x^{m_1} + c_2x^{m_2} \\ \text{One Repeated Real Root} & y = c_1x^{m_1} + c_2x^{m_1}\ln x \\ \text{Complex Conjugate Roots where } m = \alpha \pm i\beta & y = x^\alpha\left[c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)\right] \end{cases}$$

6.1.3 Example Case 1

Solve $x^2y'' + 4xy' - 4y = 0$
Assume $y = x^m \Rightarrow -4y = -4x^m$
$y' = mx^{m-1} \Rightarrow 4xy' = 4mx^m$
$y'' = m(m-1)x^{m-2} \Rightarrow x^m$
$x^2y'' + 4xy' - 4y = m(m-1)x^m + 4mx^m - 4x^m \Rightarrow [m(m-1) + 4m - 4]x^m = 0$

$m^2 - m + 4m - 4 = 0$ is the auxilliary equation $m = 1,\ \ -4 \Rightarrow y_1 = x',\ \ y_2 = x^{-4}$
$y = C_1 x + C_2 x^{-4}$

6.1.4 Example Case 2

$$9x^2y'' + 3xy' + y = 0 \Rightarrow a = 9,\ b = 3,\ c = 1 \Rightarrow (3m-1)^2 \Rightarrow m = \frac{1}{3}$$

$y_1 = x^{\frac{1}{3}},\ y_2 = c_2 x^{\frac{1}{3}} \ln x$
$y = x^{\frac{1}{3}} + c_2 x^{\frac{1}{3}} \ln x$

6.1.5 Example Case 3

Solve the IVP: $x^2y'' + 3x'y + 3y = 0 \qquad y(1) = 1 \qquad y'(1) = -5$
$m^2 + 2m + 3 = 0 \Rightarrow m = -1 \pm \sqrt{2}iy = x^{\frac{1}{3}} + c_2 x^{\frac{1}{3}} \ln x$
$y = x^{-1}\left[c_1 \cos(\sqrt{2}\ln x) + c_2 \sin(\sqrt{2}\ln x)\right]$
Now time to plug in initial conditions
$1 = 1\left[c_1 \cancelto{1}{\cos 0} + c_2 \cancelto{0}{\sin 0}\right] \Rightarrow c_1 = 1$
$y'(x) = x^{-1}\left[-c_1 \sin(\sqrt{2}\ln x)\frac{\sqrt{2}}{x} + c_1 \frac{\sqrt{2}}{x}\cos(\sqrt{2}\ln x)\right] + x^{-2}\left[c_1 \cos(\sqrt{2}\ln x) + c_2 \sin(\sqrt{2}\ln x)\right]$
$-5 = y'(1) = c_2 \cdot \sqrt{2} - 1 \Rightarrow c_2 = -2\sqrt{2}$
$y(x) = x^{-1}\left[\cos\left(\sqrt{2}\ln x\right) - 2\sqrt{2}\sin\left(\sqrt{2}\ln x\right)\right]$

6.2 Review of Power Series; Power-Series Solutions

7 Example Problems with Solutions

7.1

$\begin{cases} \frac{dy}{dx} = 2xy^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$ $\quad y = 0$ and $y = \frac{x^6}{27}$ are solutions

$\frac{dy}{dx}\frac{x^6}{27} = 2x \cdot \frac{x^4}{9} = y^{\frac{2}{3}}$

$\begin{cases} \frac{dy}{dx} = 2yx^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$ $\quad$ and $y = 0$ is the only solution. This IVP satisfies a certain condition and that makes it have a unique solution

$\begin{cases} \frac{dy}{dx} = xy^{\frac{1}{2}} \\ y(0) = 0 \end{cases}$

Does the IVP have a unique solution? When on $\mathbb{R}^2$ is $\frac{\partial f}{\partial y}$ continuous? $\frac{\partial f}{\partial y} = \frac{1}{2}xy^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$

$\begin{cases} \frac{dy}{dx} = 3y \\ y(0) = 5 \end{cases}$ $\quad$ Yes there is a unique solution, $\frac{\partial f}{\partial y} = 3$

Determine the region R for which the DE would have a unique solution through a point (x_0, y_0) in the region $\frac{dy}{dx} = \sqrt{xy}$

Where on $\mathbb{R}^2$ is $\frac{\partial f}{\partial y}$ continuous? $\frac{\partial f}{\partial y} = \frac{1}{2}(xy)^{-1/2} * \frac{\partial}{\partial y}(xy) = \frac{x}{2\sqrt{xy}}$

DIY

$\frac{dy}{dx} - y = x$

7.2 $\quad ydx = (2 + 3x)dy$

Solve: $\quad ydx = (2 + 3x)dy$

$\frac{dy}{y} = \frac{dx}{2 + 3x}$

$\int \frac{dy}{y} = \int \frac{dx}{2 + 3x}$

$\ln|y| = \frac{\ln|2 + 3x|}{3} + C$

$e^{\ln|y|} = e^{\frac{\ln|2+3x|}{3} + C_1}$

$= e^{\frac{\ln|2+3x|}{3}} \cdot e^{C_1}$

$|y| = e^{C_1} \cdot \ln|2 + 3x|^{\frac{1}{3}}$

$|y| = \left|e^{C_1}\right| \cdot \left|(2 + 3x)^{\frac{1}{3}}\right|$

$|y| = \left|e^{C_1} \cdot |(2 + 3x)^{\frac{1}{3}}\right|$

$y = \pm C(2 + 3x)^{\frac{1}{3}} \qquad x \neq -\frac{2}{3}$

7.3 $\frac{dy}{dx} = e^x e^{5y}$

$\frac{dy}{dx} = e^x e^{5y}$
$e^{-5y} dy = \frac{e^x}{dx}$
$\int e^{-5y} dy = \int e^x dx$
$-\frac{1}{5} e^{-5y} = e^x + C_1$
$e^{-5y} = -5e^x - 5C_1$
$-5y = \ln(-5e^x - 5C_1)$
$y = -\frac{1}{5} \ln(C - 5e^x)$

7.4 $y' = 2y - y^2$

$y' = 2y - y^2$
$\frac{dy}{dx} = y(2-y) \to \int \frac{dy}{y(2-y)} = \int dx \to \int \left(\frac{0.5}{y} + \frac{0.5}{2-y}\right) dy = \int dx$
$\frac{1}{2} \ln |y| - \frac{1}{2} \ln |2-y| = x + C_1$
$\ln |y| - \ln |2-y| = 2x + 2C_1$
$\ln \left|\frac{y}{2-y}\right| = 2x + 2C_1 \Rightarrow \left|\frac{y}{2-y}\right| = e^{2x} e^{2C_1} \to \frac{y}{2-y} = Ce^{2x} \to y = CE^{2x}(2-y)$
$y = 2Ce^{2x} - Ce^{2x} y \to (1 + Ce^{2x})y = 2Ce^{2x}$
$\boxed{y = \frac{2Ce^{2x}}{1 + Ce^{2x}}} \to \frac{2C}{e^{-2x} + C}$

7.5 $(x-y)dx + xdy = 0$

$(x-y)dx + xdy = 0$
Substitution $y = ux \Rightarrow dy = udx + xdu$
$(x - ux)dx + x(udx + xdu) = 0$
$xdx - uxdx + uxdx + x^2 du = 0$
$xdx + x^2 du = 0$
$\int du = -\int \frac{1}{x} dx \Rightarrow u = -\ln |x| + C$
$u = \frac{y}{x}$
$\frac{y}{x} = C - \ln |x|$
$y = Cx - x \ln |x|$

7.6 $(x^3 + y^2)dx = 3xy^2 dy = 0$

$(x^3 + y^2)dx = 3xy^2 dy = 0$ is conservative, so find $f(x,y)$ that satisfies $M = f_x, N = f_y$
$\int (x^3 + y^3)dx = \frac{x^4}{4} + xy^3 + g(y)$ and $\int xy^2 dy = xy^3 + g'(y)$
$f(x,y) = \frac{x^4}{4} + xy^3 = C$

7.7 Logistic Growth Rumor

Big Mouth John brings a juicy rumor to a town of 5000. Assume logistic growth. After 5 days 200 people have heard it. How many people will have heard it after 7 days? $\frac{dP}{dt} = kP(5000-P) = P(5000k - kP) \to a = 5000k$

$P(t) = \dfrac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$

$P(t) = \dfrac{5000k \cdot 1}{k \cdot 1 + (5000k - k \cdot 1)e^{-5000kt}} = \dfrac{5000k}{k + 4999ke^{-5000kt}} = \dfrac{5000}{1 + 4999e^{-5000kt}}$

From here use $P(5) = 200$ to determine k

$P(t) = \dfrac{5000}{1 + 4999e^{-5000kt}} \Rightarrow 200 = \dfrac{5000}{1 + 4999e^{-25000k}}$

$1 + 4999e^{-25000k} = 25$

$e^{-25000k} = \dfrac{24}{4999}$

$k = -\dfrac{1}{25000}\ln\left(\dfrac{25}{4999}\right) = 2.13557\text{E} - 4$

Now plug in for P(7)=1303.3603 people

7.8 Chemical Reaction

Compound C is formed as a reaction of A and B $A + B \to C$. The resulting reaction is such that

1. For each gram of B, 3 grams of A are used
2. Initially 40g of A 25g of B
3. 10 mins after start, 20g of C is formed
4. Reaction rate is proportional to amounts of A and B

(a) Determine the amount of C at time t
(b) How much C is formed in 15 minutes
(c) How much C formes at $t = \infty$

$\frac{dx}{dt} = k_1(40 - 0.75x)(25 - 0.25x) = \frac{k_1}{160}(160 - 3x)(100 - x) = k(160 - 3x)(100 - x)$

$\frac{dx}{dt} = k(160 - 3x)(100 - x) = \int \frac{1}{(160 - 3x)(100 - x)}dx = \int kdt$

$= \int \frac{3/140}{160 - 3x} - \frac{1/140}{100 - x}dx = kt + C_1$

$= \frac{1}{140}\ln\left|\frac{100 - x}{160 - 3x}\right| = kt + C_1 = \frac{100 - x}{160 - 3x} = c_2e^{140kt} \Rightarrow c_2 = \frac{5}{8}$ by $x(0) = 0$

$= \frac{100 - x}{160 - 3x} = \frac{5}{8}e^{140kt} \quad x(10) = 20 \to k = \frac{1}{1400}\ln\frac{32}{25}$

$x(t) = \dfrac{100(e^{140kt} - 1)}{\frac{15}{8}e^{140kt} - 1}$ where $k = \dfrac{1}{1400}\ln\dfrac{32}{25}$

$x(15) = 26.12705$ grams of C

$\lim\limits_{t\to\infty} \dfrac{100(e^{140kt} - 1)}{\frac{15}{8}e^{140kt} - 1} = \dfrac{160}{3}$ grams of C

7.9 Find the auxilliary equation for the DE

$3y'' + 5y' - 2y = 0$
$3m^2 + 5m - 2 = 0$

7.10 $3m^2 + 5m - 2 = 0$

$3m^2 + 5m - 2 = 0$
Solutions to quadratic formula $m_1 = \frac{1}{3}$ $m_2 = -2$
Solution to DE $y = c_1 e^{\frac{x}{3}} + c_2 e^{-2x}$

7.11 $m^2 + 4m + 4 = 0$

$m^2 + 4m + 4 = 0$
$(m+2)^2 = 0, \quad m = -2$
$y = c_1 e^{-2x} + c_2 x e^{-2x}$

7.12 $y'' - 2y' - 3y = -6x^2 + x - 2$

Find y_c
$y'' - 2y' - 3y = 0$ gets an auxilliary equation
$m^2 - 2m - 3 = 0 \Rightarrow (m+1)(m-3) = 0 \Rightarrow m_2 = -1, \;\; m_2 = -3$
$y_2 = c_1 e^{-x} + c_2 e^{3x}$
Find y_p
$y_p(x) = Ax^2 + Bx + C$
$y_p'(x) = 2Ax + B$
$y_p''(x) = 2A$
Plug it in
$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = -6x^2 + x - 2$
$\Rightarrow -3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = -6x^2 + x - 2$
$-3A = -6 \quad -4A - 3B = 1 \quad 2A - 2B - 3C = -2$
$A = 2, \quad B = -3, \quad C = 4$ **making the equation** $y_p(x) = 2x^2 - 3x + 4$
Solution to the nonhomogeneous equation $y = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} + 2x^2 - 3x + 4$

7.13 $y'' + 2y' + y = 3 \sin 2x$

$m^2 + 2m + 1 = 0 \Rightarrow m- = 1, \quad m = -1$
$y_c = c_1 e^{-x} + c_2 x e^{-x}$
$y_p = \cos 2x + B \sin 2x$
$y_p'(x) = -2A \sin 2x + 2x \cos 2x$
$y_p'' = -4A \cos 2x - 4B \sin 2x$
$y_p'' + 2y_p' + y_p = (-3A + 4B)\cos 2x + (-4A - 3B)\sin 2x = 3 \sin 2x \Rightarrow A = -\frac{12}{25} \quad B = -\frac{9}{25}$
$\Rightarrow y_p = -\frac{12}{25}\cos 2x - \frac{9}{25}\sin 2x$
$y = c_1 e^{-x} + c_2 x e^{-x} - \frac{12}{25}\cos 2x - \frac{9}{25}\sin 2x$