

# MATH 2 Lecture Notes

Tejas Patel

Tuesday, 14 January, 2025

## Contents

<b>1</b>	<b>Chapter 1</b>	<b>4</b>
1.1	Terminology . . . . .	4
1.2	Some Mathematical Models . . . . .	4
<b>2</b>	<b>First-Order Differential Equations</b>	<b>6</b>
2.1	Preliminary Theory . . . . .	6
2.1.1	<b>Theorem</b> . . . . .	6
2.1.2	<b>Key Questions:</b> . . . . .	6
2.2	Separable Variables (Separable Equations) . . . . .	6
2.2.1	<b>Definition:</b> . . . . .	6
2.2.2	Substitution . . . . .	7
2.3	Homogeneous Equations . . . . .	7
2.3.1	Definition . . . . .	7
2.3.2	Example . . . . .	7
2.3.3	Differential Equation form . . . . .	7
2.3.4	Substitution . . . . .	7
2.3.5	Example . . . . .	8
2.4	Exact Equations . . . . .	8
2.4.1	Definition . . . . .	8
2.4.2	Solve the DE . . . . .	8
2.4.3	Example: . . . . .	8
2.5	Linear Equations . . . . .	9
2.5.1	Procedure to follow for every Linear DE . . . . .	9
2.6	What method to use to solve? . . . . .	9
<b>3</b>	<b>Applications of First-Order Differential Equation</b>	<b>10</b>
3.1	Orthogonal Trajectories . . . . .	10
3.1.1	Example . . . . .	10
3.1.2	Definition . . . . .	10
3.2	Applications of Linear Equations . . . . .	10
3.3	Applications of Nonlinear Equations . . . . .	10
<b>4</b>	<b>Linear DE of Higher Order</b>	<b>11</b>
4.1	Preliminary Theory . . . . .	11
4.2	Constructing a Second Solution from a Known Solution . . . . .	11
4.2.1	General Reduction of Order Formula . . . . .	11
4.3	Homogeneous Linear Equations w/ Constant Coefficients . . . . .	11
4.3.1	Auxilliary Equation for a DE . . . . .	12
4.3.2	Three Scenarios for the Auxilliary Equation . . . . .	12

4.4	Undetermined Coefficients - Superposition Approach . . . . .	12
4.4.1	Trial Particular Solutions . . . . .	12
4.5	Variation of Parameters . . . . .	12
4.5.1	Idea: . . . . .	12
4.5.2	Derivation: . . . . .	13
4.5.3	Example . . . . .	13
<b>5</b>	<b>Applications of 2nd-Order DE</b>	<b>14</b>
5.1	Simple Harmonic Motion . . . . .	14
5.1.1	Example . . . . .	14
5.2	Damped Motion . . . . .	14
5.2.1	Differential Equation form . . . . .	14
5.3	Forced Motion . . . . .	14
5.3.1	Example . . . . .	14
5.4	Electric Circuits & Other Analogous Systems . . . . .	15
5.4.1	Example . . . . .	15
<b>6</b>	<b>DE with Variable Coefficients</b>	<b>15</b>
6.1	Cauchy-Euler Equations . . . . .	15
6.1.1	General Formula for the Auxilliary Equation . . . . .	15
6.1.2	Steps to solving for an auxilliary equation . . . . .	15
6.1.3	Forms of Solutions . . . . .	16
6.1.4	Example Case 1 . . . . .	16
6.1.5	Example Case 2 . . . . .	16
6.1.6	Example Case 3 . . . . .	16
6.2	Review of Power Series; Power-Series Solutions . . . . .	16
6.2.1	Example . . . . .	16
6.2.2	Write $\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$ as one series . . . . .	17
6.2.3	Use Power Series to solve $y' - 2xy = 0$ . . . . .	17
6.2.4	$4y'' + y = 0$ . . . . .	17
<b>7</b>	<b>Laplace Transformations</b>	<b>18</b>
7.1	Laplace Transform: . . . . .	18
7.1.1	Definition . . . . .	18
7.1.2	Example $\mathcal{L}$ . . . . .	18
7.1.3	Theorem . . . . .	18
7.1.4	Theorem . . . . .	18
7.1.5	Forms of Laplace Transformations . . . . .	18
7.2	Inverse Laplace Transform: . . . . .	19
7.2.1	Example . . . . .	19
7.2.2	Theorem . . . . .	19
<b>8</b>	<b>Example Problems with Solutions</b>	<b>20</b>
8.1	. . . . .	20
8.2	$ydx = (2 + 3x)dy$ . . . . .	20
8.3	$\frac{dy}{dx} = e^x e^{5y}$ . . . . .	21
8.4	$y' = 2y - y^2$ . . . . .	21
8.5	$(x - y)dx + xdy = 0$ . . . . .	21
8.6	$(x^3 + y^2)dx = 3xy^2dy = 0$ . . . . .	21
8.7	Logistic Growth Rumor . . . . .	22

8.8	Chemical Reaction . . . . .	22
8.9	Find the auxilliary equation for the DE . . . . .	22
8.10	$3m^2 + 5m - 2 = 0$ . . . . .	23
8.11	$m^2 + 4m + 4 = 0$ . . . . .	23
8.12	$y'' - 2y' - 3y = -6x^2 + x - 2$ . . . . .	23
8.13	$y'' + 2y' + y = 3 \sin 2x$ . . . . .	23

# 1 Chapter 1

## 1.1 Terminology

**Definition** A differential equation is an equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables.

· An Ordinary Differential Equation (ODE) involves only ordinary derivatives

· A Partial Differential Equation (PDE) involves partial derivatives.

**Definition** The order of a DE is the order of the highest-order derivative that appears in the DE

**Notation**  $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2})$

**Definition** A linear DE is any DE that can be written in form:

$$a_0(x)y + a_1(x)y' + a_2(x)y'' \cdots + a_n(x)y^{(n)} = b(x)$$

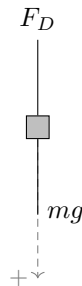
For a DE to be linear:

1. Y and all of its derivatives must be of the 1st degree
2. Any term that does not include y or any of its derivatives must be a function of x

## 1.2 Some Mathematical Models

### I. Free-falling body

Goal: Find  $s(t)$ .



Set up a differential equation in S, model it, then solve

$$ma = mg$$

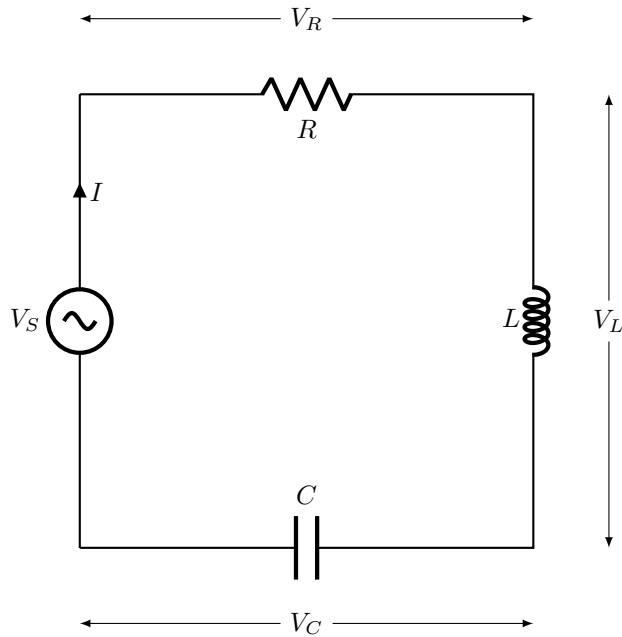
$$\frac{d^2s}{dt^2} = g$$

$$v = \frac{ds}{dt}, g = \frac{dv}{dt}$$

What if there is air resistance. Assume force scales linear with velocity

$$\frac{dv}{dt} = g - \frac{kv}{m} \rightarrow \frac{dv}{dt} = g - \frac{k}{m} \cdot \frac{ds}{dt}$$

## II: Series Circuit



Voltage drops:

$$V = L \frac{dI}{dt}, V = L \frac{d^2 q}{dt^2}$$

$$V = IR, V = R \frac{dq}{dt}$$

$$V = \frac{q}{C}$$

$$E(t) = L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C}$$

## III: Population Growth

$P = P(t)$  = population at time  $t$  — use exponential model

$$\frac{dp}{dt} \propto P \rightarrow \frac{dp}{dt} = kP \rightarrow C e^{kt} \text{ where } C \text{ is the initial population}$$

## IV: Population Growth with Finite Capacity

"Carrying Capacity" =  $N$  — uses the logistic growth model

$$\frac{dp}{dt} \propto \text{both } P \text{ and amount to carrying capacity } (N-P)$$

$$\frac{dp}{dt} = kP(N - P)$$

## V: Chemical Reaction

$A + B \rightarrow C$  Concentrations of  $A$  and  $B$  decreases by amount of  $C$  formed

Can we write DE governing the concentration of  $C$   $x(t)$ ?

The rate at which the reaction takes place  $\propto$  Product of the remaining concentrations of  $A$  and  $B$

$\alpha$  initial concentration of  $A$

$\beta$  initial concentration of  $B$

$$\frac{dx}{dt} = k(\alpha - x)(\beta - x)$$

## 2 First-Order Differential Equations

### 2.1 Preliminary Theory

Example DE:  $y' = 3y \Rightarrow \boxed{y = Ce^{3x}}$  the general solution where C is an arbitrary constant

Add initial condition  $y(0) = 5$  plug in  $x=0$  to  $5 = Ce^{3*0}$ ,  $5 = C * 1$ ,  $C = 5 \Leftarrow$  Initial Value Problem  
 $y = 5e^{3x}$  is the general solution for the Initial Value Problem

#### 2.1.1 Theorem

$$f(x) = \begin{cases} \frac{dy}{dx} = f(x, y) & \text{Differential Equation} \\ y(x_0) = y_0 & \text{Initial Condition} \end{cases}$$

Let R be a rectangular region in the xy-plane defined by  $a \leq x \leq b, c \leq y \leq d$ , that contains the point  $(x_0, y_0)$  in its interior.

If  $f(x, y)$  and  $\frac{\partial f}{\partial y}$  are continuous on R, then there exists an interval I centered at  $x_0$ , and on this interval I there exists a unique solution  $y(x)$  for this IVP

#### 2.1.2 Key Questions:

Does every IVP have at least one solution?

If an IVP has a solution is it the only solution?

**Meaning of a solution existing "on an Interval"** The initial value problem

$$\begin{cases} \frac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases} \text{ has a unique solution. In fact, we can easily verify that } y = \tan x \text{ satisfies this IVP}$$

However note that there are some intervals on which  $y = \tan x$  cannot be a solution for this IVP, such as  $(-2, 2)$ , where the function is discontinuous at  $\pm \frac{\pi}{2}$  but can be used for  $(-1, 1)$  since it is continuous at all points within the interval

## 2.2 Separable Variables (Separable Equations)

### 2.2.1 Definition:

A differential equation that can be written in the form  $\frac{dy}{dx} = \frac{g(x)}{h(y)}$  is said to be separable (or have separable variables).

**Example:**  $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

$$h(y)dy = g(x)dx$$

$$\int h(y)dy = \int g(x)dx$$

**Example:**  $dx + e^{3x}dy = 0$

$$e^{3x}dy = -dx$$

$$dy = -\frac{dx}{e^{3x}} \rightarrow dy = -e^{-3x}dx \rightarrow \int dy = \int -e^{-3x}dx \rightarrow y = \frac{1}{3}e^{-3x} + C \text{ where C is an arbitrary constant}$$

### 2.2.2 Substitution

$\frac{dy}{dx} = F(ax + by + c)$  where  $b \neq 0$  use the substitution:  $u = ax + by + c \Rightarrow \frac{du}{dx} = a + b\frac{dy}{dx} = \frac{1}{b} \left[ \frac{du}{dx} - a \right]$

Example:  $\frac{dy}{dx} = \tan^2(x + y)$  let  $u = x + y \rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1 \rightarrow \frac{du}{dx} - 1 = \tan^2 u \rightarrow \frac{du}{dx} = \sec^2 u$

$$\int \cos^2 u \, du = \int dx$$

$$2(x + y) + \sin 2(x + y) = 4x + C \rightarrow 2y - 2x + \sin 2(x + y)$$

**Solve:**  $\frac{dy}{dx} = (y + 3)^2$  By inspection  $y = -3$  is a solution. This is the only solution because  $f(x, y) = (x + 3)^2$  is continuous on  $\mathbb{R}^2$  and  $\frac{\partial f}{\partial x}$  is continuous on  $\mathbb{R}$  so it is the only solution Why solving by

separation is not possible  $\int (y + 3)^{-2} dy = \int dx \rightarrow (y + 3)^{-2} / -1 = x + C_1 \rightarrow \frac{1}{y + 3} = -x - C_1 \rightarrow$

$$y + 3 = \frac{1}{-x - C_1} \rightarrow y = -3 + \frac{1}{-x - C_1}$$

$y(0) = -3 \rightarrow 0 = \frac{1}{c}$  where there is no real  $c$  that solves that equation, making this not possible

## 2.3 Homogeneous Equations

What do we do if the DE is not separable?

### 2.3.1 Definition

A function  $f(x, y)$  is said to be **homogeneous of degree  $n$**  if, for  $x, y$ , and  $t$  where  $f(x, y)$  and  $f(tx, ty)$  are defined:

$$f(tx, ty) = t^n f(x, y)$$

### 2.3.2 Example

Determine whether each function is homogeneous:

$$a: f(x, y) = x^3 - 7x^2y + 4y^3 \rightarrow f(tx, ty) = (tx)^3 - 7(tx)^2(ty) + 4(ty)^3$$

$$t^3x^3 - 7t^3x^2y + 4t^3y^3$$

$$t^3(x^3 - 7x^2y + 4y^3) = t^3 f(x, y)$$

How to tell quickly whether  $f(x, y)$  is homogeneous:

Each term must have the same combined degree

Example:  $x^3 - 7x^2y + 4y^3$  is D3,  $x^2 + y^2 - 4x$  is not,  $\sqrt{x^5 + 4y^5}$  is with D 2.5,  $\frac{3y}{x} - 2$  is D0

### 2.3.3 Differential Equation form

$M(x, y)dx + N(x, y)dy = 0$  is called a homogeneous differential equation if the functions  $M$  and  $N$  are both homogeneous of the same degree

If  $f(x, y)$  is homogeneous of degree  $n$  then  $f(x, y)$  can be written as:

$$f(x, y) = f\left(x \times 1, x \times \frac{y}{x}\right) = x^n f\left(1, \frac{y}{x}\right)$$

$$\text{or } f(x, y) = y^n f\left(\frac{x}{y}, 1\right)$$

### 2.3.4 Substitution

To solve a homogeneous DE make the substitution:  $y = ux$  ( $u = \frac{y}{x}$ ) or  $x = vy$  ( $v = \frac{x}{y}$ )

### 2.3.5 Example

$$(y^2 + xy)dx + x^2dy = 0 \rightarrow y = ux \rightarrow dy = (udx + xdu)$$

$$(u^2x^2 + ux^2)dx + x^2(udx + xdu) = 0$$

$$u^2x^2dx + ux^2dx + ux^2dx + x^3du = 0$$

$$ux^2(u+2)dx + x^3du = 0$$

$$\int \frac{1}{u(u+2)} du = - \int \frac{1}{x} dx$$

Partial Fraction Decomposition:  $\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2} \rightarrow A = \frac{1}{2}, B = -\frac{1}{2}$  Back to solving

$$\int \left[ \frac{0.5}{u} - \frac{0.5}{u+2} \right] = - \int \frac{1}{x} dx$$

$$0.5 \ln |u| - 1/2 \ln |u+2| = -\ln |x| + C_1$$

$$\ln \left| \frac{u}{u+2} \right| = 2C_1 - 2 \ln |x|$$

$$\left| \frac{u}{u+2} \right| = e^{2C_1} \cdot e^{-2 \ln |x|} = e^{2C_1} \cdot |x|^{-2} \Rightarrow \left| \frac{u}{u+2} \right| = |e^{2C_1} \cdot x^{-2}| \Rightarrow \left| \frac{u}{u+2} \right| = \frac{C}{x^2}$$

$$ux^2 = X(u+2) \Rightarrow ux^2 = Cu + 2c \rightarrow ux^2 - Cu = 2C$$

$$u(x^2 - c) = 2C \Rightarrow u = \frac{2C}{x^2 - C} \Rightarrow \frac{y}{x} = \frac{2Cx}{x^2 - C}, x \neq 0$$

## 2.4 Exact Equations

Recall from Math 1C: Let  $F(x, y) = \langle 3x^2 - 7y, -7x + 2y \rangle$

1. If F a conservative vector field  
i.e., Is there a function  $f(x, y)$  such that  $\nabla f$ ? Yes,  $-7=-7$
2. If F is indeed conservative, what is f?  

$$x^3 - 7xy + g(y) = f(x, y)$$

$$-7x + 2y, g'(y) = 2y$$

$$f(x, y) = x^3 - 7xy + y^2 + k$$

### 2.4.1 Definition

A differential equation in the form  $M(x, y)dx + N(x, y)dy = 0$  where  $M_y = N_x$ , is called an exact differential equation.

### 2.4.2 Solve the DE

$$(3x^2 - 7y)dx + (-7x + 2y)dy = 0$$

Using 1C techniques it is  $f(x, y) = x^3 - 2xy + y^2 + k$

Set this f = c.  $f(x, y) = x^3 - 2xy + y^2 = c$  take k=0 in every problem

If the DE is not exact, sometimes we can make it exact by multiplying by magical quantity  $\mu(x, y)$

### 2.4.3 Example:

Solve the DE:

$$(x + y)dx + x \ln x dy = 0 \text{ using } \mu(x, y) = \frac{1}{x}$$

$$\left( \frac{x+y}{x} \right) dx + \ln |x| dy = 0 \text{ is now exact.}$$

$$\text{Solution: } f(x, y) = x + y \ln x = c$$



## 2.5 Linear Equations

Recall: First Order Linear DE is a DE in the form  $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ ,  $a_1(x) \neq 0$

Divide both sides by  $a_1(x) \Rightarrow \frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$  where  $P(x) = \frac{a_0(x)}{a_1(x)}$  and  $f(x) = \frac{g(x)}{a_1(x)}$

$\frac{dy}{dx} + P(x)y = f(x)$  There is an integrating factor  $\mu(x)$  that turns this DE into an exact DE

$$dy + P(x)ydx = f(x)dx \rightarrow dy [P(x)y - f(x)] dx = 0$$

$$\mu(x)dy + \mu(x) [P(x)y - f(x)] dx = 0 \rightarrow \mu'(x) = \mu(x)P(x)$$

$$\frac{d\mu}{dx} = \mu P \rightarrow \int \frac{d\mu}{\mu} = \int P(x) \rightarrow \ln \mu = \int P(x) dx$$

$$\mu(x) = e^{\int P(x) dx} \Rightarrow e^{\int P(x) dx} \frac{dy}{dx} + e^{\int P(x) dx} P(x)y = e^{\int P(x) dx} f(x)$$

$$\frac{d}{dx} [e^{\int P(x) dx} y] = e^{\int P(x) dx} f(x) \rightarrow e^{\int P(x) dx} y = \int e^{\int P(x) dx} f(x) dx \quad \boxed{y = e^{-\int P(x) dx} \int e^{\int P(x) dx} f(x) dx}$$

### 2.5.1 Procedure to follow for every Linear DE

1. Rewrite the linear DE in the form  $\frac{dy}{dx} + P(x)y = f(x)$
2. Find the integrating factor  $\mu(x) = e^{\int P(x) dx}$
3. Multiply each side of the DE by  $\mu(x)$
4. Rewrite the left side as  $\frac{d}{dx} [\mu(x) \cdot y]$
5. Integrate both sides with respect to x and retrieve an implicitly expressed solution
6. Solve for y

## 2.6 What method to use to solve?

First ask is it exact? ( $M_y = M_x$ )

Yes: Use the method in §2.4

No: Is it linear? (in y or x)

Yes: Use the method in §2.5

No: Is it separable?

Yes: §2.2

No: Homogeneous?

Yes: Use a substitution §2.3

No: Good luck. or use inspection

## 3 Applications of First-Order Differential Equation

### 3.1 Orthogonal Trajectories

· Consider the family of curves  $y = cx^3$  Question: Which DE should be solved to get this family as its solutions?

Steps:

1. Find  $\frac{dy}{dx} = 3cx^2$

2. Eliminate  $c$

$$y = cx^3 \Rightarrow c = \frac{y}{x^3}$$

$$\frac{dy}{dx} = 3 \frac{y}{x^3} x^2 \rightarrow \frac{3y}{x}$$

· The two curves are orthogonal if their tangent lines are orthogonal at the point of intersection  
i.e. The derivatives are the negative reciprocals of each other

#### 3.1.1 Example

Show that  $y = x^3$  and  $x^2 + 3y^2 = 4$  are orthogonal at their points of intersection, (1,1) and (-1,-1)

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \rightarrow 3 \text{ at } x = 1 \text{ and } 3 \text{ at } x = -1$$

$$2x + 6y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{x}{3y} = -\frac{1}{3} \text{ at both } x = 1 \text{ and } x = -1 \text{ meaning it is orthogonal}$$

#### 3.1.2 Definition

When all the curves of one family of curves intersect orthogonally all the curves of another family, then the families are said to be orthogonal trajectories of each other

### 3.2 Applications of Linear Equations

$$\frac{dN}{dt} = kN \rightarrow N = Ce^{kt} \text{ for bacterial growth rate. Nothing else here, just an applications section}$$

### 3.3 Applications of Nonlinear Equations

#### Logistic Model of Population Growth

- 1: End Behaviour (Steady State Solution) as  $t \rightarrow \infty P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}} \rightarrow P(t) = \frac{aP_0}{bP_0} = \frac{a}{b}$

- 2: Concavity Analysis (Point of Inflection)  $\frac{dP}{dt} = P(a - bP)$

$$\frac{d^2P}{dt^2} = \frac{dP}{dt}(a - 2bP) = P(a - bP)(a - 2bP) = 0$$

For inflection point  $a - 2bP = 0 \rightarrow a = 2bP \rightarrow P = \frac{a}{2b} \rightarrow P = \frac{N}{2}$  3 cases of initial conditions

$$\begin{cases} 0 < P_0 < \frac{a}{2b} & \text{Hits inflection point while rising to CC} \\ \frac{a}{2b} < P_0 < \frac{a}{b} & \text{Population grows at a decreasing rate to CC} \\ P_0 > \frac{a}{b} & \text{Population falls to the carrying capacity} \end{cases}$$

## 4 Linear DE of Higher Order

### 4.1 Preliminary Theory

Initial Value Problem  $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$

Initial Conditions  $y(x_0) = y_0 \dots y^{(n-1)}(x_0) = y_0^{(n-1)}$

**Theorem** Let each  $a_j(x)$  be continuous on an interval I and let  $a_n(x) \neq 0$  for every... CONTINUE LATER

Boundary-Value Problem for 2nd order Linear DE  $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$y(a) = y_0 \quad y(b) = y_1$

Example:  $y'' + 16y = 0 \quad y(0) = 0 \quad y(\pi/2) = 0$

$y = \sin 4x$  and  $y = \cos 4x$  are solutions so  $y(x) = c_1 \cos 4x + c_2 \sin 4x$

$y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 0$

$y(\pi/2) = c_1 \cos(2\pi) + c_2 \sin(2\pi) = c_1 = 0$  so  $y(x) = c_2 \sin 4x$  is a solution

### 4.2 Constructing a Second Solution from a Known Solution

#### General Formula

Given  $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$  and  $a_2(x) \neq 0$  and  $y_1(x) \neq 0$  is a solution of this DE, find  $y_2(x)$

Divide by  $a_2(x)$ :  $y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$

$y'' + Py' + Qy = 0, \quad y_2 = uy_1 \rightarrow y_2' = uy_1' + u'y_1 \rightarrow y_2'' = uy_1'' + 2u'y_1' + u''y_1$

Plug it all in:  $u''y_1 + 2u'y_1' + uy_1'' + Py_1'y_1' + Qy_1uy_1 = 0$

$u''y_1 + u'(2y_1' + Py_1) + u(y_1'' + Py_1'y_1' + Qy_1) = 0$

$u''y_1 + u'(2y_1' + Py_1) = 0$  Let  $w = u'$  and  $w' = u''$

$y_1w' + (2y_1' + Py_1)w = 0$

$w' + \frac{2y_1' + Py_1}{y_1}w = 0 \quad \mu = e^{\int \frac{2y_1' + Py_1}{y_1} dx}$

$w = c_1 y_1^{-2} e^{-\int P dx} = u' = e^{-\int P dx} y_1^2$

#### 4.2.1 General Reduction of Order Formula

$y = C_1 y_1 + C_2 y_2$  where  $y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx$

### 4.3 Homogeneous Linear Equations w/ Constant Coefficients

In the DE  $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ , take

$a_2(x) = a, \quad a_1(x) = b, \quad a_0(x) = c$

so we have a 2ns-order Homogeneous Linear DE with *constant coefficients*

$$ay'' + by' + cy$$

What does a typical solution look like?

$$\begin{cases} y = e^{mx} \\ y' = me^{mx} \\ y'' = m^2 e^{mx} \end{cases} \quad \text{so } am^2 e^{mx} + bme^{mx} + ce^{mx} = 0 = e^{mx}(am^2 + bm + c)$$

#### 4.3.1 Auxilliary Equation for a DE

$$am^2 + bm + c = 0$$

#### 4.3.2 Three Scenarios for the Auxilliary Equation

$$\begin{cases} \text{If } b^2 - 4ac > 0 & \text{two real roots } y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \\ \text{If } b^2 - 4ac = 0 & \text{one real root } y = c_1 e^{mx} + c_2 x e^{mx} \\ \text{If } b^2 - 4ac < 0 & \text{No real roots, 2 distinct complex roots, } y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) \quad m = \alpha \pm i\beta \end{cases}$$

### 4.4 Undetermined Coefficients - Superposition Approach

- Nonhomogeneous Linear DE with constant coefficients:

$$ay'' + by' + cy = g(x)$$

**Recall from Section 4.1:** The general solution is:

$$y(x) = y_c(x) + y_p(x)$$

where  $y_c(x)$  is the general solution  $ay'' + by' + cy = 0$

$y_p(x)$  is one particular solution of  $ay'' + by' + cy = g(x)$

**The big question:** How do we find  $y_p(x)$ ?

#### 4.4.1 Trial Particular Solutions

$g(x)$	Form of $y_p$
constant	$A$
$2x - 7$	$Ax + B$
$-x^2 + 3$	$Ax^2 + Bx + C$
$\sin kx$ or $\cos kx$	$A \cos kx + B \sin kx$
$e^{kx}$	$Ae^{kx}$
$(2x - 7)e^{kx}$	$(Ax + B)e^{kx}$
$x^2 e^{kx}$	$(Ax^2 + Bx + C)e^{kx}$
$e^{kx} \cos lx$ or $e^{kx} \sin lx$	$e^{kx} (A \cos lx + B \sin lx)$
$5x^2 \sin kx$	$(Ax^2 + Bx + C) \cos kx + (Dx^2 + Ex + F) \sin kx$
$x e^{kx} \cos lx$	$(Ax + B)e^{kx} \cos lx + (Cx + D)e^{kx} \sin lx$

### 4.5 Variation of Parameters

Given:  $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$\Rightarrow y'' + P(x)y' + Q(x)y = f(x)$

Goal: Find  $y_p$

#### 4.5.1 Idea:

Let  $y_1, y_2$  be the two linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

Then we will look for  $y_p$  of the form:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

#### 4.5.2 Derivation:

$y'_p = u'_1 y_1 + u'_2 y_2 + u_1 y'_1 + u_2 y'_2$  by product rule

Assume  $u'_1 y_1 + u'_2 y_2 = 0 \rightarrow y'_p = u_1 y'_1 + u_2 y'_2$

$y''_p = u_1 y''_1 + u_2 y''_2 + u'_1 y'_1 + u'_2 y'_2$  then substitute into  $y'' + Py' + Qy = f$

$$\begin{aligned} & \cancel{u_1(y''_1 + Py'_1 + Qy_1) + u_2(y''_2 + Py'_2 + Qy_2)} + u'_1 y'_1 + u'_2 y'_2 = f \\ & \begin{cases} u'_1 y_1 + u'_2 y_2 = 0 \\ u'_1 y'_1 + u'_2 y'_2 = f \end{cases} \rightarrow y_p = u_1 y_1 + u_2 y_2 \end{aligned}$$

#### 4.5.3 Example

$$y'' + 9y = \cos 3x$$

$$m^2 + 9 = 0 \rightarrow m = \pm 3i \rightarrow y_c = C_1 \cos 3x + C_2 \sin 3x$$

$$y_p = u_1 y_1 + u_2 y_2 \text{ where } u'_1 = \frac{\begin{bmatrix} 0 & \sin 3x \\ \cos 3x & 3 \cos 3x \end{bmatrix}}{\begin{bmatrix} \cos 3x & \sin 3x \\ 3 \sin 3x & 3 \cos 3x \end{bmatrix}} = -\frac{1}{3} \sin 3x \cos 3x = -\frac{1}{6} \sin 6x$$

$$u_1 = \int -\frac{1}{6} \sin 6x dx = \frac{\cos 6x}{36}$$

$$u'_2 = \frac{\begin{bmatrix} \cos 3x & 0 \\ -3 \sin 3x & \cos 3x \end{bmatrix}}{\begin{bmatrix} \cos 3x & \sin 3x \\ 3 \sin 3x & 3 \cos 3x \end{bmatrix}} = \frac{\cos^2 3x}{3} = \int \frac{\cos^2 3x}{3} dx = \int \frac{1 + \cos 6x}{6} dx = \frac{x}{6} + \frac{\sin 6x}{36}$$

$$y_p = \frac{\cos 6x}{36} \cos 3x + \frac{x}{6} + \frac{\sin 6x}{36} \sin 3x = \frac{1}{36} \cos 3x + \frac{x}{6} \sin 3x$$

$$y_p = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{36} \cos 3x + \frac{x}{6} \sin 3x = \boxed{C_1 \cos 3x + C_2 \sin 3x + \frac{1}{6} x \sin 3x}$$

**Initial Value Problem**  $y(0) = 1, \quad y'(0) = 0$

$$C_1 \cos 3x + C_2 \sin 3x + \frac{1}{6} x \sin 3x \rightarrow C_1 \cos(0) = 1 \rightarrow C_1 = 1$$

$$y'(0) \rightarrow \cancel{-3 \sin 3x} + 3C_2 \cos 3x + \frac{1}{6} x + 3 \cos 3x + \cancel{\frac{1}{6} \sin 3x}$$

$$3C_2 \cos 3x + \frac{1}{6} x + 3 \cos 3x = 0 \rightarrow 3C_2 = 0 \text{ so } C_2 = 0$$

$$\boxed{y(x) = \cos 3x + \frac{1}{6} x \sin 3x}$$

## 5 Applications of 2nd-Order DE

### 5.1 Simple Harmonic Motion

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \text{ where } \omega = \sqrt{\frac{k}{m}}$$

$$\text{Auxilliary Equation: } m^2\omega^2 = 0 \quad m = 0 \pm \beta i$$

$$C_1 \cos \omega t + C_2 \sin \omega t = A \sin(\omega t + \phi) \text{ where } A = \sqrt{C_1^2 + C_2^2}, \quad \sin \phi = \frac{C_2}{A}, \quad \cos \phi = \frac{C_1}{A}, \quad \tan \phi = \frac{C_1}{C_2}$$

#### 5.1.1 Example

A 16lb weight is attached to a spring, stretching it 1.28 feet. This weight is released 2 feet above the equilibrium position, with initial downward velocity of 1.5 ft/s

$$\text{DE: } m \frac{d^2x}{dt^2} + kx = 0 \quad \begin{cases} \text{Weight} & w = mg = \frac{1}{2} \text{ Slug} \\ k & F = k \cdot s = 16 = k \cdot 1.28, \quad 12.5 \frac{lb}{ft} \end{cases}$$

$$\frac{d^2x}{dt^2} + 25x = 0, \quad x(0) = -2, \quad x'(0) = -1.5$$

$$\text{Solve the equation: } x(t) = C_1 \cos 5t + C_2 \sin 5t$$

$$x(t) = -2 \cos 5t + 0.3 \sin 5t$$

$$\text{Period is } \frac{2\pi}{5} \text{ seconds or } \frac{5}{2\pi} \text{ vps}$$

### 5.2 Damped Motion

#### 5.2.1 Differential Equation form

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0 \text{ Forms of solutions}$$

### 5.3 Forced Motion

With the presence of an external force, the equation of motion has another extra term:

$$x \frac{d^2x}{dt^2} = -kx - \beta \frac{dx}{dt} + f(t) \text{ where } k > 0$$

#### 5.3.1 Example

8lb weight, 32/13 ft stretch when hung freely, starts 3 feet below equilibrium, find equation of motion if  $f(t) = 18.75 \sin 2t$  is applied and damping force is 1.5x instantaneous velocity

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = f(t)$$

$$m = \frac{1}{4}, \quad \beta = 1.5 \quad k : 8 = k \cdot \frac{32}{13} = \frac{13}{4}$$

$$\frac{1}{4} x'' + \frac{6}{4} x' + \frac{13}{4} x = 18.75 \sin 2t$$

$$x'' + 6x' + 13x = 75 \sin 2t \quad x(0) = 3 \quad x'(0) = 0$$

$$x_p = A \cos 2t + B \sin 2t \quad A = -4, \quad B = 3$$

$$x(t) = c_1 e^{-3t} \cos 2t + c_2 e^{-3t} \sin 2t - 4 \cos 2t + 3 \sin 2t$$

$$\text{with initial conditions } c_1 = 7 \quad c_2 = \frac{15}{2}$$

$$x(t) = 7e^{-3t} \cos 2t + \frac{15}{2} e^{-3t} \sin 2t - 4 \cos 2t + 3 \sin 2t$$

$-4 \cos 2t + 3 \sin 2t$  is the steady state solution and  
 $7e^{-3t} \cos 2t + \frac{15}{2}e^{-3t} \sin 2t$  is the transient solution

## 5.4 Electric Circuits & Other Analogous Systems

LRC Circuit: For charge  $q = q(t)$  and current  $i = i(t) = \frac{dq}{dt}$

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{c} = E(t)$$

### 5.4.1 Example

$$L = 2H, R = 8\Omega, C = \frac{1}{18}F, E(t) = 10 \sin(t) V, q(0) = 0, i(0) = 0 \quad 2 \frac{d^2 q}{dt^2} + 8 \frac{dq}{dt} + 18 \frac{q}{c} = 10 \sin t$$

$$= \frac{d^2 q}{dt^2} + 4 \frac{dq}{dt} + 9 \frac{q}{c} = 5 \sin t$$

$$\frac{d^2 q}{dt^2} + 4 \frac{dq}{dt} + 9 \frac{q}{c} = 5 \sin t \Rightarrow m^2 + 4m + 9 = 0 \rightarrow m = -2 \pm \sqrt{5}i$$

$$q_c(t) = c_1 e^{-2t} \cos \sqrt{5}t + c_2 e^{-2t} \sin \sqrt{5}t$$

$$y_p = -\frac{1}{4} \cos t + \frac{1}{2} \sin t \quad \text{time for initial conditions}$$

$$c_1 = \frac{1}{4} \quad c_2 = 0 \Rightarrow \boxed{q(t) = c_1 e^{-2t} \cos \sqrt{5}t - \frac{1}{4} \cos t + \frac{1}{2} \sin t}$$

$$\text{Steady state charge: } q = -\frac{1}{4} \cos t + \frac{1}{2} \sin t$$

$$\text{Steady state current: } i = \frac{1}{4} \sin t + \frac{1}{2} \cos t$$

## 6 DE with Variable Coefficients

### 6.1 Cauchy-Euler Equations

Differential Equations of the form:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = g(x)$$

These are considered equidimensional, where the degree of each monomial coefficient function equals the order of the derivative of  $y$  in each term

#### 6.1.1 General Formula for the Auxilliary Equation

$$am(m-1) + bm + c = 0 \text{ is equivalent to } am^2 + (b-a)m + c = 0$$

#### 6.1.2 Steps to solving for an auxilliary equation

1. Consider  $y = x^r$ ,  $y' = r x^{r-1}$ ,  $y'' = r(r-1)x^{r-2}$  etc.
2. Plug that in for each  $y$  and its derivatives, simplify the  $x^r$  terms
3. Factor out  $x^r$  and thats your equation

### 6.1.3 Forms of Solutions

$$\left\{ \begin{array}{ll} \text{Two Real Roots} & y = c_1 x^{m_1} + c_2 x^{m_2} \\ \text{One Repeated Real Root} & y = c_1 x^{m_1} + c_2 x^{m_1} \ln x \\ \text{Complex Conjugate Roots where } m = \alpha \pm i\beta & y = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)] \end{array} \right.$$

### 6.1.4 Example Case 1

Solve  $x^2 y'' + 4xy' - 4y = 0$

Assume  $y = x^m \Rightarrow -4y = -4x^m$

$$y' = mx^{m-1} \Rightarrow 4xy' = 4mx^m$$

$$y'' = m(m-1)x^{m-2} \Rightarrow x^m$$

$$x^2 y'' + 4xy' - 4y = m(m-1)x^m + 4mx^m - 4x^m \Rightarrow [m(m-1) + 4m - 4]x^m = 0$$

$$m^2 - m + 4m - 4 = 0 \text{ is the auxilliary equation } m = 1, -4 \Rightarrow y_1 = x', y_2 = x^{-4}$$

$$y = C_1 x + C_2 x^{-4}$$

### 6.1.5 Example Case 2

$$9x^2 y'' + 3xy' + y = 0 \Rightarrow a = 9, b = 3, c = 1 \Rightarrow (3m-1)^2 \Rightarrow m = \frac{1}{3}$$

$$y_1 = x^{\frac{1}{3}}, y_2 = c_2 x^{\frac{1}{3}} \ln x$$

$$y = x^{\frac{1}{3}} + c_2 x^{\frac{1}{3}} \ln x$$

### 6.1.6 Example Case 3

Solve the IVP:  $x^2 y'' + 3x' y + 3y = 0 \quad y(1) = 1 \quad y'(1) = -5$

$$m^2 + 2m + 3 = 0 \Rightarrow m = -1 \pm \sqrt{2}iy = x^{\frac{1}{3}} + c_2 x^{\frac{1}{3}} \ln x$$

$$y = x^{-1} [c_1 \cos(\sqrt{2} \ln x) + c_2 \sin(\sqrt{2} \ln x)]$$

Now time to plug in initial conditions

$$1 = 1 \left[ c_1 \cos 0 + c_2 \sin 0 \right] \Rightarrow c_1 = 1$$

$$y'(x) = x^{-1} \left[ -c_1 \sin(\sqrt{2} \ln x) \frac{\sqrt{2}}{x} + c_1 \frac{\sqrt{2}}{x} \cos(\sqrt{2} \ln x) \right] + x^{-2} [c_1 \cos(\sqrt{2} \ln x) + c_2 \sin(\sqrt{2} \ln x)]$$

$$-5 = y'(1) = c_2 \cdot \sqrt{2} - 1 \Rightarrow c_2 = -2\sqrt{2}$$

$$y(x) = x^{-1} \left[ \cos(\sqrt{2} \ln x) - 2\sqrt{2} \sin(\sqrt{2} \ln x) \right]$$

## 6.2 Review of Power Series; Power-Series Solutions

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

### 6.2.1 Example

Where does  $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$  converge?

$$L = \lim_{k \rightarrow \infty} \left| \frac{x^{k+1}}{(k+1)^2} \right| = \lim_{k \rightarrow \infty} \left| \frac{x^{k+1}}{x^k} \right| \cdot \left| \frac{k^2}{(k+1)^2} \right| = |x| \cdot \lim_{k \rightarrow \infty} \frac{k^2}{(k+1)^2} \text{ for } x=1 \text{ and } x=-1 \text{ both } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ and}$$

$$\sum_{n=1}^{\infty} \frac{-1^n}{n^2} \text{ converges by the p series test and alternating series test}$$



**6.2.2 Write  $\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$  as one series**

Write it out manually

$$\text{First: } 2c_1 + 4c_2x + 6c_3x^2 + 8c_4x^3 + \dots = 2c_1 + \sum_{n=2}^{\infty} 2nc_n x^{n-1} = \sum_{k=n-1}^{\infty} 2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k$$

$$\text{2nd: } 6c_0x + 6c_1x^2 + 6c_2x^3 = \sum_{n=0}^{\infty} 6c_n x^{n+1} = \sum_{k=n-1}^{\infty} 6c_{k-1}x^k$$

$$\text{Combine them: } 2c_1 + \sum_{k=1}^{\infty} (2(k+1)c_{k+1} + 6c_k - 1)x^k$$

**6.2.3 Use Power Series to solve  $y' - 2xy = 0$**

$$y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \quad 2xy = 2x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} 2c_n x^{n+1}$$

$$y' - 2xy = \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} 2c_n x^{n+1} \quad \text{Let } k = n - 1, \quad n = k + 1$$

$$y' = c_1 + \sum_{k=1}^{\infty} (k+1)c_{k+1}x^k \quad -2xy = -\sum_{k=1}^{\infty} 2c_{k-1}x^k$$

$$y' - 2xy = c_1 + \sum_{k=1}^{\infty} (k+1)c_{k+1}x^k - \sum_{k=1}^{\infty} 2c_{k-1}x^k = c_1 + \sum_{k=1}^{\infty} [(k+1)c_{k+1} - 2c_{k-1}]x^k = 0$$

$$c_1 = 0, \quad (k+1)c_{k+1} = 2c_{k-1} \Rightarrow c_{k+1} = \frac{2c_{k-1}}{k+1}$$

$$c_0 \text{ is an arbitrary constant } c_1 = 0 \quad c_{k+1} = \frac{2c_{k-1}}{k+1}$$

$$c_2 = c_0 \quad c_3 = 0 \quad c_4 = \frac{1}{2}c_0 \quad c_6 = \frac{c_0}{6} \quad c_8 = \frac{c_0}{24}$$

$$\begin{cases} 0 & \text{If } n \text{ is odd} \\ \frac{c_0}{\left(\frac{n}{2}\right)!} & \text{If } n \text{ is even} \end{cases} \quad \text{If } m = \frac{n}{2} \Rightarrow \frac{c_0}{m!}$$

$$\sum_{m=0}^{\infty} \frac{c_0}{m!} x^{2m} = c_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{m!} \text{ which converts to } c_0 e^{x^2}$$

**6.2.4  $4y'' + y = 0$**

$$y = \sum_{n=0}^{\infty} c_n x^n \quad y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$4y'' = \sum_{n=2}^{\infty} 4n(n-1)c_n x^{n-2} \quad y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow \sum_{k=0}^{\infty} (4(k+2)(k+1)c_{k+2} + c_k)x^k = 0$$

$$\text{Solve for } c_{k+2} = -\frac{c_k}{4(k+1)(k+2)} \quad k \in \mathbb{Z}$$

$$c_2 = -\frac{c_0}{8} \quad c_3 = \frac{c_1}{24} \quad c_4 = \frac{c_0}{4^2 \cdot 4!} \quad c_5 = \frac{c_1}{4^2 \cdot 5!}$$

$$\text{General Pattern: } \begin{cases} c_n = (-1)^{\frac{n}{2}} \frac{c_0}{4^{\frac{n}{2}} \cdot n!} & \text{When } n \text{ is even} \\ c_n = c_{2m+1} = (-1)^m \cdot \frac{c_1}{4^m \cdot (2m+1)!} & \text{When } n \text{ is odd} \end{cases}$$

$$c_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\frac{x}{2}\right)^{2m} + c_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left(\frac{x}{2}\right)^{2m+1}$$

Converts to  $y = c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2}$

## 7 Laplace Transformations

### 7.1 Laplace Transform:

- Method used to solve certain DEs in an easier way
- Converts DE/IVP into simpler equations

#### 7.1.1 Definition

Let  $f(t)$  be a function, where  $t \geq 0$  then

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt, \quad s > 0$$

#### 7.1.2 Example $\mathcal{L}$

$$\int_0^{\infty} e^{-st} * 1 dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} dt = \lim_{b \rightarrow \infty} \left[ \frac{e^{-st}}{-s} \right]_{t=0}^{t=b} \text{ results in } \frac{1}{s}$$

#### 7.1.3 Theorem

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

#### 7.1.4 Theorem

If  $f$  is piecewise continuous on  $[0, \infty)$ , and  $f$  is of **exponential order**, then  $\mathcal{L}\{f(t)\}$  exists for  $s > c$

#### 7.1.5 Forms of Laplace Transformations

$$t^n \Rightarrow \frac{n!}{s^{n+1}}$$

$$e^{at} \Rightarrow \frac{1}{s-a}$$

$$\cos(kt) = \frac{s}{k^2 + s^2}$$

$$\sin(kt) = \frac{k}{k^2 + s^2}$$

$$\cosh(kt) = \frac{s}{s^2 - k^2}$$

$$\sinh(kt) = \frac{k}{s^2 - k^2}$$

$$te^{at} = \frac{1}{(s-a)^2}$$

$$t^n e^{at} = \frac{n!}{(s-a)^{n+1}}$$

$$\cos^2 t = \frac{1}{2} \mathcal{L}\{1 + \cos 2t\} = \frac{1}{2} \left( \frac{1}{s} + \frac{s}{s^2 + 4} \right)$$

$$e^{at} \cos kt = \frac{s-a}{(s-a)^2 + k^2}$$

$$e^{at} \sin kt = \frac{k}{(s-a)^2 + k^2}$$

## 7.2 Inverse Laplace Transform:

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

### 7.2.1 Example

Inverse Laplace Transform  $\frac{1}{s^5} = \mathcal{L}^{-1}\left\{\frac{1}{4!} \frac{4!}{s^5}\right\} = \frac{1}{4!} \mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{t^4}{4!} = \frac{t^4}{24}$

2: ILT:  $\mathcal{L}^{-1}\left\{\frac{5s-4}{s^2-3}\right\} = 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+3}\right\} - 4\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2+3}\right\} = 5\cos\sqrt{3}t - \frac{4}{\sqrt{3}}\sin\sqrt{3}t$

3: ILT:  $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/12}{s+1} + \frac{-1/3}{s-2} + \frac{1/4}{s-3}\right\}$  distribute the transform  
NEEDS COMPLETION

4: ILT:  $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s+2)^3}\right\} \Rightarrow \text{PFD} \Rightarrow \frac{-1}{16} + \frac{1}{8}t + \frac{1}{16}e^{-2t} - \frac{1}{8}t^2e^{-2t}$

### 7.2.2 Theorem

Every Laplace transform  $\rightarrow 0$  as  $s \rightarrow \infty$

## 8 Example Problems with Solutions

### 8.1

$$\begin{cases} \frac{dy}{dx} = 2xy^{\frac{2}{3}} \\ y(0) = 0 \end{cases} \quad y = 0 \text{ and } y = \frac{x^6}{27} \text{ are solutions}$$

$$\frac{dy}{dx} \frac{x^6}{27} = 2x \cdot \frac{x^4}{9} = y^{\frac{2}{3}}$$

$$\begin{cases} \frac{dy}{dx} = 2yx^{\frac{2}{3}} \\ y(0) = 0 \end{cases} \quad \text{and } y = 0 \text{ is the only solution. This IVP satisfies a certain condition and that makes}$$

it have a unique solution

$$\begin{cases} \frac{dy}{dx} = xy^{\frac{1}{2}} \\ y(0) = 0 \end{cases}$$

Does the IVP have a unique solution? When on  $\mathbb{R}^2$  is  $\frac{\partial f}{\partial y}$  continuous?  $\frac{\partial f}{\partial y} = \frac{1}{2}xy^{-\frac{1}{2}} = \frac{x}{2\sqrt{y}}$

$$\begin{cases} \frac{dy}{dx} = 3y \\ y(0) = 5 \end{cases} \quad \text{Yes there is a unique solution, } \frac{\partial f}{\partial y} = 3$$

Determine the region R for which the DE would have a unique solution through a point  $(x_0, y_0)$  in the region  $\frac{dy}{dx} = \sqrt{xy}$

Where on  $\mathbb{R}^2$  is  $\frac{\partial f}{\partial y}$  continuous?  $\frac{\partial f}{\partial y} = \frac{1}{2}(xy)^{-1/2} * \frac{\partial}{\partial y}(xy) = \frac{x}{2\sqrt{xy}}$

**DIY**

$$\frac{dy}{dx} - y = x$$

### 8.2 $ydx = (2 + 3x)dy$

**Solve:**  $ydx = (2 + 3x)dy$

$$\frac{dy}{y} = \frac{dx}{2 + 3x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{2 + 3x}$$

$$\ln |y| = \frac{\ln |2 + 3x|}{3} + C$$

$$e^{\ln |y|} = e^{\frac{\ln |2 + 3x|}{3} + C_1}$$

$$= e^{\frac{\ln |2 + 3x|}{3}} \cdot e^{C_1}$$

$$|y| = e^{C_1} \cdot \ln |2 + 3x|^{\frac{1}{3}}$$

$$|y| = |e^{C_1}| \cdot \left| (2 + 3x)^{\frac{1}{3}} \right|$$

$$|y| = \left| e^{C_1} \cdot (2 + 3x)^{\frac{1}{3}} \right|$$

$$y = \pm C(2 + 3x)^{\frac{1}{3}} \quad x \neq -\frac{2}{3}$$

$$8.3 \quad \frac{dy}{dx} = e^x e^{5y}$$

$$\frac{dy}{dx} = e^x e^{5y}$$

$$e^{-5y} dy = \frac{e^x}{dx}$$

$$\int e^{-5y} dy = \int e^x dx$$

$$-\frac{1}{5} e^{-5y} = e^x + C_1$$

$$e^{-5y} = -5e^x - 5C_1$$

$$-5y = \ln(-5e^x - 5C_1)$$

$$y = -\frac{1}{5} \ln(C - 5e^x)$$

$$8.4 \quad y' = 2y - y^2$$

$$y' = 2y - y^2$$

$$\frac{dy}{dx} = y(2 - y) \rightarrow \int \frac{dy}{y(2 - y)} = \int dx \rightarrow \int \left( \frac{0.5}{y} + \frac{0.5}{2 - y} \right) dy = \int dx$$

$$\frac{1}{2} \ln |y| - \frac{1}{2} \ln |2 - y| = x + C_1$$

$$\ln |y| - \ln |2 - y| = 2x + 2C_1$$

$$\ln \left| \frac{y}{2 - y} \right| = 2x + 2C_1 \Rightarrow \left| \frac{y}{2 - y} \right| = e^{2x} e^{2C_1} \rightarrow \frac{y}{2 - y} = C e^{2x} \rightarrow y = C e^{2x} (2 - y)$$

$$y = 2C e^{2x} - C e^{2x} y \rightarrow (1 + C e^{2x}) y = 2C e^{2x}$$

$$\boxed{y = \frac{2C e^{2x}}{1 + C e^{2x}}} \rightarrow \frac{2C}{e^{-2x} + C}$$

$$8.5 \quad (x - y)dx + xdy = 0$$

$$(x - y)dx + xdy = 0$$

$$\text{Substitution } y = ux \Rightarrow dy = udx + xdu$$

$$(x - ux)dx + x(udx + xdu) = 0$$

$$xdx - uxdx + uxdx + x^2 du = 0$$

$$xdx + x^2 du = 0$$

$$\int du = - \int \frac{1}{x} dx \Rightarrow u = -\ln |x| + C$$

$$u = \frac{y}{x}$$

$$\frac{y}{x} = C - \ln |x|$$

$$y = Cx - x \ln |x|$$

$$8.6 \quad (x^3 + y^2)dx = 3xy^2 dy = 0$$

$$(x^3 + y^2)dx = 3xy^2 dy = 0 \text{ is conservative, so find } f(x, y) \text{ that satisfies } M = f_x, N = f_y$$

$$\int (x^3 + y^3)dx = \frac{x^4}{4} + xy^3 + g(y) \text{ and } \int xy^2 dy = xy^3 + g'(y)$$

$$f(x, y) = \frac{x^4}{4} + xy^3 = C$$

## 8.7 Logistic Growth Rumor

Big Mouth John brings a juicy rumor to a town of 5000. Assume logistic growth. After 5 days 200 people have heard it. How many people will have heard it after 7 days?  $\frac{dP}{dt} = kP(5000 - P) =$

$$P(5000k - kP) \rightarrow a = 5000k$$

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

$$P(t) = \frac{5000k \cdot 1}{k \cdot 1 + (5000k - k \cdot 1)e^{-5000kt}} = \frac{5000k}{k + 4999ke^{-5000kt}} = \frac{5000}{1 + 4999e^{-5000kt}}$$

From here use  $P(5) = 200$  to determine  $k$

$$P(t) = \frac{5000}{1 + 4999e^{-5000kt}} \Rightarrow 200 = \frac{5000}{1 + 4999e^{-25000k}}$$

$$1 + 4999e^{-25000k} = 25$$

$$e^{-25000k} = \frac{24}{4999}$$

$$k = -\frac{1}{25000} \ln\left(\frac{24}{4999}\right) = 2.13557E - 4$$

Now plug in for  $P(7) = 1303.3603$  people

## 8.8 Chemical Reaction

Compound C is formed as a reaction of A and B  $A + B \rightarrow C$ . The resulting reaction is such that

1. For each gram of B, 3 grams of A are used
2. Initially 40g of A 25g of B
3. 10 mins after start, 20g of C is formed
4. Reaction rate is proportional to amounts of A and B

(a) Determine the amount of C at time  $t$

(b) How much C is formed in 15 minutes

(c) How much C forms at  $t = \infty$

$$\frac{dx}{dt} = k_1(40 - 0.75x)(25 - 0.25x) = \frac{k_1}{160}(160 - 3x)(100 - x) = k(160 - 3x)(100 - x)$$

$$\frac{dx}{dt} = k(160 - 3x)(100 - x) = \int \frac{1}{(160 - 3x)(100 - x)} dx = \int k dt$$

$$= \int \frac{3/140}{160 - 3x} - \frac{1/140}{100 - x} dx = kt + C_1$$

$$= \frac{1}{140} \ln \left| \frac{100 - x}{160 - 3x} \right| = kt + C_1 = \frac{100 - x}{160 - 3x} = c_2 e^{140kt} \Rightarrow c_2 = \frac{5}{8} \text{ by } x(0) = 0$$

$$= \frac{100 - x}{160 - 3x} = \frac{5}{8} e^{140kt} \quad x(10) = 20 \rightarrow k = \frac{1}{1400} \ln \frac{32}{25}$$

$$x(t) = \frac{100(e^{140kt} - 1)}{\frac{15}{8}e^{140kt} - 1} \text{ where } k = \frac{1}{1400} \ln \frac{32}{25}$$

$$x(15) = 26.12705 \text{ grams of } C$$

$$\lim_{t \rightarrow \infty} \frac{100(e^{140kt} - 1)}{\frac{15}{8}e^{140kt} - 1} = \frac{160}{3} \text{ grams of } C$$

## 8.9 Find the auxilliary equation for the DE

$$3y'' + 5y' - 2y = 0$$

$$3m^2 + 5m - 2 = 0$$

$$8.10 \quad 3m^2 + 5m - 2 = 0$$

$$3m^2 + 5m - 2 = 0$$

Solutions to quadratic formula  $m_1 = \frac{1}{3}$   $m_2 = -2$

Solution to DE  $y = c_1 e^{\frac{x}{3}} + c_2 e^{-2x}$

$$8.11 \quad m^2 + 4m + 4 = 0$$

$$m^2 + 4m + 4 = 0$$

$$(m + 2)^2 = 0, \quad m = -2$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$8.12 \quad y'' - 2y' - 3y = -6x^2 + x - 2$$

**Find**  $y_c$

$y'' - 2y' - 3y = 0$  gets an auxilliary equation

$$m^2 - 2m - 3 = 0 \Rightarrow (m + 1)(m - 3) = 0 \Rightarrow m_1 = -1, \quad m_2 = 3$$

$$y_c = c_1 e^{-x} + c_2 e^{3x}$$

**Find**  $y_p$

$$y_p(x) = Ax^2 + Bx + C$$

$$y_p'(x) = 2Ax + B$$

$$y_p''(x) = 2A$$

Plug it in

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = -6x^2 + x - 2$$

$$\Rightarrow -3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = -6x^2 + x - 2$$

$$-3A = -6 \quad -4A - 3B = 1 \quad 2A - 2B - 3C = -2$$

$$A = 2, \quad B = -3, \quad C = 4 \text{ making the equation } y_p(x) = 2x^2 - 3x + 4$$

Solution to the nonhomogeneous equation  $y = y_c + y_p = c_1 e^{-x} + c_2 e^{3x} + 2x^2 - 3x + 4$

$$8.13 \quad y'' + 2y' + y = 3 \sin 2x$$

$$m^2 + 2m + 1 = 0 \Rightarrow m = -1, \quad m = -1$$

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

$$y_p = \cos 2x + B \sin 2x$$

$$y_p'(x) = -2A \sin 2x + 2x \cos 2x$$

$$y_p''(x) = -4A \cos 2x - 4B \sin 2x$$

$$y_p'' + 2y_p' + y_p = (-3A + 4B) \cos 2x + (-4A - 3B) \sin 2x = 3 \sin 2x \Rightarrow A = -\frac{12}{25} \quad B = -\frac{9}{25}$$

$$\Rightarrow y_p = -\frac{12}{25} \cos 2x - \frac{9}{25} \sin 2x$$

$$y = c_1 e^{-x} + c_2 x e^{-x} - \frac{12}{25} \cos 2x - \frac{9}{25} \sin 2x$$