MATH 2 Lecture Notes

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1 Chapter 1

1.1 Terminology

Definition A differential equation is an equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables.
· An Ordinary Differential Equation (ODE) involves only ordinary derivatives
· A Partial Differential Equation (PDE) involves partial derivatives.
Definition The order of a DE is the order of the highest-order derivative that appears in the DE
Notation $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2})$
Definition A linear DE is any DE that can be written in form:
$a_0(x)y + a_1(x)y' + a_2(x)y'' \cdots + a_n(x)y^{(n)} = b(x)$
For a DE to be linear:

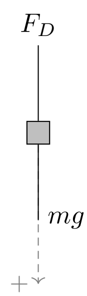
1. Y and all of its derivatives much be of the 1st degree

2. Any term that does not include y or any of its derivatives must be a function of x

1.2 Some Mathematical Models

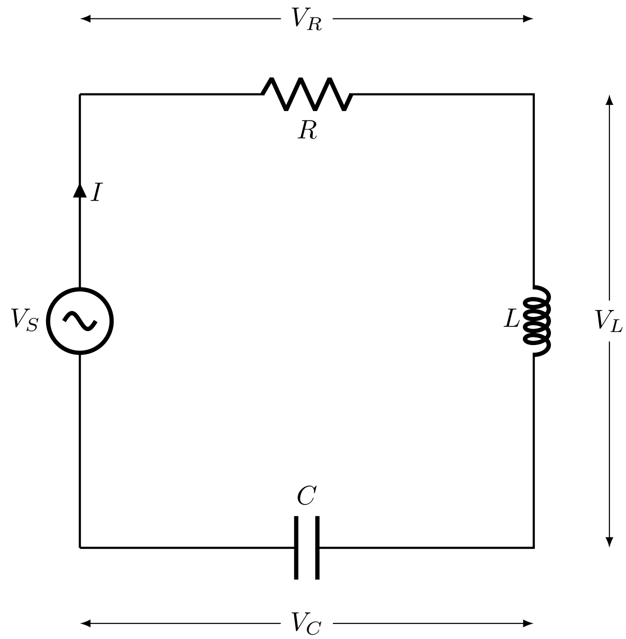
I. Free-falling body

Goal: Find s(t).



Set up a differential equation in S, model it, then solve
$ma = mg$
$\frac{d^2s}{dt^2} = g$
$v = \frac{ds}{dt}, g = \frac{dv}{dt}$
What if there is air resistance. Assume force scales linear with velocity
$\frac{dv}{dt} = g - \frac{kv}{m} \rightarrow \frac{dv}{dt} = g - \frac{k}{m} \cdot \frac{ds}{dt}$

II: Series Circuit



Voltage drops:

$V = L\dfrac{dI}{dt}, V = L\dfrac{d^2q}{dt^2}$

$V = IR, V = R\dfrac{dq}{dt}$

$V = \dfrac{q}{C}$

$E(t) = L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C}$

III: Population Growth

$P = P(t)$ = population at time t — use exponential model

$\dfrac{dp}{dt} \propto P \rightarrow \dfrac{dp}{dt} = kP \rightarrow= Ce^{kt}$ where C is the initial population

IV: Population Growth with Finite Capacity

"Carrying Capacity" = N — uses the logistic growth model

$\dfrac{dp}{dt} \propto$ both P and amount to carrying capacity (N-P)

$\dfrac{dp}{dt} = kP(N - P)$

V: Chemical Reaction

$A + B \rightarrow C$ Concentrations of A and B decreases by amount of C formed

Can we write DE governing the concentration of C x(t)?

The rate at which the reaaction takes place $\propto$ Product of the remaining concentrations of A and B

α initial concentration of A

β initial concentration of B

$\dfrac{dx}{dt} = k(\alpha - x)(\beta - x)$

2 First-Order Differential Equations

2.1 Preliminary Theory

Example DE: $y' = 3y \Rightarrow \boxed{y = Ce^{3x}}$ the general solution where C is an arbitrary constant

Add initial condition $y(0) = 5$ plug in x=0 to $5 = Ce^{3*0}, 5 = C * 1, C = 5 \Leftarrow$ Initial Value Problem
$y = 5e^{3x}$ is the general solution for the Initial Value Problem

Theorem $f(x) = \begin{cases} \dfrac{dy}{dx} = f(x, y) & \text{Differential Equation} \\ y(x_0) = y_0 & \text{Initial Condition} \end{cases}$

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b, c \leq y \leq d$, that contains the point (x_0, y_0) in its interior.

If f(x,y) and $\partial f \partial y$ are continuous on R, then there exists an interval I centered at x_o, and on this interval I there exists a unique solution $y(x)$ for this IVP
Key Questions:
Does every IVP have at least one solution?
If an IVP has a solution is it the only solution?

Meaning of a solution existing "on an Interval" The initial value problem
$\begin{cases} \dfrac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$ has a unique solution. In fact, we can easily verify that $y = \tan x$ satisfies this IVP
However note that there are some inervals on which $y = \tan x$ cannot be a solution for this IVP, such as (-2,2), where the function is discontinuous at $\pm\dfrac{\pi}{2}$ but can be used for (-1,1) since it is continuous at all points within the interval

2.2 Separable Variables (Separable Equations)

Definition: A differential equation that can be written in the form $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$ is said to be separable (or have separable variables).

Example: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$
$h(y)dy = g(x)dx$
$\displaystyle\int h(y)dy = \int g(x)dx$

Example: $dx + e^{3x}dy = 0$
$e^{3x}dy = -dx$
$dy = -\dfrac{dx}{e^{3x}} \rightarrow dy = -e^{-3x}dx \rightarrow \displaystyle\int dy = \int -e^{-3x}dx \rightarrow y = \dfrac{1}{3}e^{-3x} + C$ where C is an arbitrary constant

Substitution: $\dfrac{dy}{dx} = F(ax + bc + c)$ where $b \neq 0$ use the substitution: $u = ax + by + c \Rightarrow \dfrac{du}{dx} = a + b\dfrac{dy}{dx} = \dfrac{1}{b}\left[\dfrac{du}{dx} - a\right]$

Example: $\dfrac{dy}{dx} = \tan^2(x + y)$ let $u = x + y \rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} - 1 \rightarrow \dfrac{du}{dx} - 1 = \tan^2 u \rightarrow \dfrac{du}{dx} = \sec^2 u$
$\displaystyle\int \cos^2 u \, du = \int dx$
$2(x + y) + \sin 2(x + y) = 4x + C \rightarrow 2y - 2x + \sin 2(x + y)$

2.3 Homogenous Equations

2.4 Exact Equations

2.5 Preliminary Theory

3 Example Problems with Solutions

3.1

$\begin{cases} \dfrac{dy}{dx} = 2xy^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$ $\quad y = 0$ and $y = \dfrac{x^6}{27}$ are solutions

$\dfrac{dy}{dx}\dfrac{x^6}{27} = 2x \cdot \dfrac{x^4}{9} = y^{\frac{2}{3}}$

$\begin{cases} \dfrac{dy}{dx} = 2yx^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$ $\quad$ and $y = 0$ is the only solution. This IVP satisfies a certain condition and that makes it have a unique solution

$\begin{cases} \dfrac{dy}{dx} = xy^{\frac{1}{2}} \\ y(0) = 0 \end{cases}$

Does the IVP have a unique solution? When on $\mathbb{R}^2$ is $\dfrac{\partial f}{\partial y}$ continuous? $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}xy^{-\frac{1}{2}} = \dfrac{x}{2\sqrt{y}}$

$\begin{cases} \dfrac{dy}{dx} = 3y \\ y(0) = 5 \end{cases}$ $\quad$ Yes there is a unique solution, $\dfrac{\partial f}{\partial y} = 3$

Determine the region R for which the DE would have a unique solution through a point (x_0, y_0) in the region $\dfrac{dy}{dx} = \sqrt{xy}$

Where on $\mathbb{R}^2$ is $\dfrac{\partial f}{\partial y}$ continuous? $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}(xy)^{-1/2} * \dfrac{\partial}{\partial y}(xy) = \dfrac{x}{2\sqrt{xy}}$

DIY

$\dfrac{dy}{dx} - y = x$

3.2

Solve: $\quad ydx = (2+3x)dy$

$\dfrac{dy}{y} = \dfrac{dx}{2+3x}$

$\displaystyle\int \frac{dy}{y} = \int \frac{dx}{2+3x}$

$\ln|y| = \dfrac{\ln|2+3x|}{3} + C$

$e^{\ln|y|} = e^{\frac{\ln|2+3x|}{3} + C_1}$

$= e^{\frac{\ln|2+3x|}{3}} \cdot e^{C_1}$

$|y| = e^{C_1} \cdot \ln|2+3x|^{\frac{1}{3}}$

$|y| = \left|e^{C_1}\right| \cdot \left|(2+3x)^{\frac{1}{3}}\right|$

$|y| = \left|e^{C_1} \cdot |(2+3x)^{\frac{1}{3}}\right|$

$y = \pm C(2+3x)^{\frac{1}{3}} \qquad x \neq -\dfrac{2}{3}$

3.3

$\dfrac{dy}{dx} = e^x e^{5y}$

$e^{-5y} dy = \dfrac{e^x}{dx}$

$\displaystyle\int e^{-5y} dy = \int e^x dx$

$-\dfrac{1}{5} e^{-5y} = e^x + C_1$

$e^{-5y} = -5e^x - 5C_1$

$-5y = \ln(-5e^x - 5C_1)$

$y = -\dfrac{1}{5}\ln(C - 5e^x)$

3.4

$y' = 2y - y^2$

$\dfrac{dy}{dx} = y(2-y) \rightarrow \displaystyle\int \frac{dy}{y(2-y)} = \int dx \rightarrow \int \left(\frac{0.5}{y} + \frac{0.5}{2-y}\right) dy = \int dx$

$\dfrac{1}{2}\ln|y| - \dfrac{1}{2}\ln|2-y| = x + C_1$

$\ln|y| - \ln|2-y| = 2x + 2C_1$

$\ln\left|\dfrac{y}{2-y}\right| = 2x + 2C_1 \Rightarrow \left|\dfrac{y}{2-y}\right| = e^{2x}e^{2C_1} \rightarrow \dfrac{y}{2-y} = Ce^{2x} \rightarrow y = CE^{2x}(2-y)$

$y = 2Ce^{2x} - Ce^{2x}y \rightarrow (1 + Ce^{2x})y = 2Ce^{2x}$

$\boxed{y = \dfrac{2Ce^{2x}}{1 + Ce^{2x}}} \rightarrow \dfrac{2C}{e^{-2x} + C}$