MATH 2 Lecture Notes

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1 Chapter 1

1.1 Terminology

Definition A differential equation is an equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables.
· An Ordinary Differential Equation (ODE) involves only ordinary derivatives
· A Partial Differential Equation (PDE) involves partial derivatives.
Definition The order of a DE is the order of the highest-order derivative that appears in the DE
Notation $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2})$
Definition A linear DE is any DE that can be written in form:
$a_0(x)y + a_1(x)y' + a_2(x)y'' \cdots + a_n(x)y^{(n)} = b(x)$
For a DE to be linear:

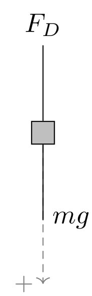
1. Y and all of its derivatives much be of the 1st degree

2. Any term that does not include y or any of its derivatives must be a function of x

1.2 Some Mathematical Models

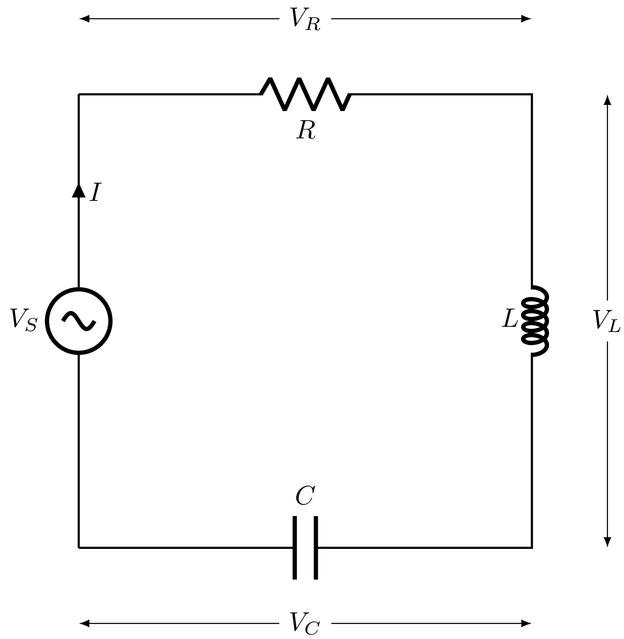
I. Free-falling body

Goal: Find s(t).



Set up a differential equation in S, model it, then solve
$ma = mg$
$\frac{d^2s}{dt^2} = g$
$v = \frac{ds}{dt}, g = \frac{dv}{dt}$
What if there is air resistance. Assume force scales linear with velocity
$\frac{dv}{dt} = g - \frac{kv}{m} \to \frac{dv}{dt} = g - \frac{k}{m} \cdot \frac{ds}{dt}$

II: Series Circuit



Voltage drops:

$V = L\dfrac{dI}{dt}, V = L\dfrac{d^2q}{dt^2}$

$V = IR, V = R\dfrac{dq}{dt}$

$V = \dfrac{q}{C}$

$E(t) = L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C}$

III: Population Growth

$P = P(t) =$ population at time t — use exponential model

$\dfrac{dp}{dt} \propto P \rightarrow \dfrac{dp}{dt} = kP \rightarrow= Ce^{kt}$ where C is the initial population

IV: Population Growth with Finite Capacity

"Carrying Capacity" = N — uses the logistic growth model

$\dfrac{dp}{dt} \propto$ both P and amount to carrying capacity (N-P)

$\dfrac{dp}{dt} = kP(N-P)$

V: Chemical Reaction

$A + B \rightarrow C$ Concentrations of A and B decreases by amount of C formed

Can we write DE governing the concentration of C x(t)?

The rate at which the reaaction takes place $\propto$ Product of the remaining concentrations of A and B

α initial concentration of A

β initial concentration of B

$\dfrac{dx}{dt} = k(\alpha - x)(\beta - x)$

2 First-Order Differential Equations

2.1 Preliminary Theory

Example DE: $y' = 3y \Rightarrow \boxed{y = Ce^{3x}}$ the general solution where C is an arbitrary constant

Add initial condition $y(0) = 5$ plug in x=0 to $5 = Ce^{3*0}, 5 = C * 1, C = 5 \Leftarrow$ Initial Value Problem
$y = 5e^{3x}$ is the general solution for the Initial Value Problem

2.1.1 Theorem

$$f(x) = \begin{cases} \dfrac{dy}{dx} = f(x, y) & \text{Differential Equation} \\ y(x_0) = y_0 & \text{Initial Condition} \end{cases}$$

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b, c \leq y \leq d$, that contains the point (x_0, y_0) in its interior.

If f(x,y) and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists an interval I centered at x_o, and on this interval I there exists a unique solution $y(x)$ for this IVP

2.1.2 Key Questions:

Does every IVP have at least one solution?
If an IVP has a solution is it the only solution?

Meaning of a solution existing "on an Interval" The initial value problem
$\begin{cases} \dfrac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$ has a unique solution. In fact, we can easily verify that $y = \tan x$ satisfies this IVP
However note that there are some inervals on which $y = \tan x$ cannot be a solution for this IVP, such as (-2,2), where the function is discontinuous at $\pm\dfrac{\pi}{2}$ but can be used for (-1,1) since it is continuous at all points within the interval

2.2 Separable Variables (Separable Equations)

2.2.1 Definition:

A differential equation that can be written in the form $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$ is said to be separable (or have separable variables).

Example: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$

$h(y)dy = g(x)dx$

$\displaystyle\int h(y)dy = \int g(x)dx$

Example: $dx + e^{3x}dy = 0$

$e^{3x}dy = -dx$

$dy = -\dfrac{dx}{e^{3x}} \rightarrow dy = -e^{-3x}dx \rightarrow \displaystyle\int dy = \int -e^{-3x}dx \rightarrow y = \dfrac{1}{3}e^{-3x} + C$ where C is an arbitrary constant

2.2.2 Substitution

$\dfrac{dy}{dx} = F(ax+bc+c)$ where $b \neq 0$ use the substitution: $u = ax+by+c \Rightarrow \dfrac{du}{dx} = a+b\dfrac{dy}{dx} = \dfrac{1}{b}\left[\dfrac{du}{dx} - a\right]$
Example: $\dfrac{dy}{dx} = \tan^2(x+y)$ let $u = x+y \rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} - 1 \rightarrow \dfrac{du}{dx} - 1 = \tan^2 u \rightarrow \dfrac{du}{dx} = \sec^2 u$
$\displaystyle\int \cos^2 u \, du = \int dx$
$2(x+y) + \sin 2(x+y) = 4x + C \rightarrow 2y - 2x + \sin 2(x+y)$
Solve: $\dfrac{dy}{dx} = (y+3)^2$ By inspection $y = -3$ is a solution. This is the only solution because $f(x,y) = (x+3)^2$ is continuous on $\mathbb{R}^2$ and $\dfrac{\partial f}{\partial x}$ is continuous on$\mathbb{R}$ so it is the only solution Why solving by separation is not possible $\displaystyle\int (y+3)^-2dy = \int dx \rightarrow (y+3)^-2/-1 = x + C_1 \rightarrow \dfrac{1}{y+3} = -x - C_1 \rightarrow$
$y+3 = \dfrac{1}{c-x} \rightarrow y = -3 + \dfrac{1}{c-x}$
$y(0) = -3 \rightarrow 0 = \dfrac{1}{c}$ where there is no real c that solves that equation, making this not possible

2.3 Homogeneous Equations

What do we do if the DE is not separable?

2.3.1 Definition

A function $f(x,y)$ is said to be **homogeneous of degree** n if, for $x, y,$ and$t$where $f(x,y)$ and $f(tx,ty)$ are defined:

$$f(tx,ty) = t^n f(x,y)$$

2.3.2 Example

Determine wheteher each function is homogeneous:
a: $f(x,y) = x^3 - 7x^2y + 4y^3 \rightarrow f(tx,ty) = (tx^3) - 7(tx)^2(ty) + 4(ty)^3$
$t^3x^3 - 7t^3x^2y + 4t^3y^3$
$t^3(x^3 - 7x^2y - 4y^3) = t^3 f(x,y)$
How to tell quickly wheteher $f(x,y)$ is homogeneous:
Each term must have the same combined degree
Example: $x^3 - 7x^2y + 4y^3$ is D3, $x^2 + y^2 - 4x$ is not, $\sqrt{x^5 + 4y^5}$ is with D 2.5, $\dfrac{3y}{x} - 2$ is D0

2.3.3 Differential Equation form

$M(x,y)dx + N(x,y)dy = 0$ is called a homogeneous differential equation if the functions M and N are both homogeneous of the same degree
If $f(x,y)$ is homogeneous of degree n then $f(x,y)$ can be written as:
$f(x,y) = f(x \times 1, x \times \dfrac{y}{x}) = x^n f(1, \dfrac{y}{x})$
or $f(x,y) = y^n f(\dfrac{x}{y}, 1)$

2.3.4 Substitution

to solve a homogeneous DE make the substitution: $y = ux \ \ (u = \dfrac{y}{x})$ or$x = vy \ \ (v = \dfrac{x}{y})$

2.3.5 Example

$(y^2 + xy)dx + x^2dy = 0 \rightarrow y = ux \rightarrow dy = (udx + xdu)$
$(u^2x^2 + ux^2)dx + x^2(udx + xdU) = 0$
$u^2x^2dx + ux^2dx + ux^2dx + x^3du = 0$
$ux^2(u + 2)dx + x^3du = 0$
$\displaystyle\int \frac{1}{u(u+2)}du = -\int \frac{1}{x}dx$
Partial Fraction Decomposition: $\displaystyle\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2} \rightarrow A = \frac{1}{2}, B = -\frac{1}{2}$ Back to solving
$\displaystyle\int \left[\frac{0.5}{u} - \frac{0.5}{u+2}\right] = -\int \frac{1}{x}dx$
$0.5 \ln|u| - 1/2 \ln|u + 2| = -\ln|x| + C_1$
$\displaystyle\ln\left|\frac{u}{u+2}\right| = 2C_1 - 2\ln|x|$
$\displaystyle\left|\frac{u}{u+2}\right| = e^{2C_1} \cdot e^{-2\ln|x|} = e^{2C_1} \cdot |x^{-2}| \Rightarrow \left|\frac{u}{u+2}\right| = \left|e^{2C_1} \cdot x^{-2}\right| \Rightarrow \left|\frac{u}{u+2} = \frac{C}{x^2}\right|$
$ux^2 = X(u + 2) \Rightarrow ux^2 = Cu + 2c \rightarrow ux^2 - Cu = 2C$
$\displaystyle u(x^2 - c) = 2C \Rightarrow u = \frac{2C}{x^2 - C} \Rightarrow \frac{y}{x} = \frac{2Cx}{x^2 - C},\ x \neq 0$

2.4 Exact Equations

2.5 Preliminary Theory

3 Example Problems with Solutions

3.1

$$\begin{cases} \dfrac{dy}{dx} = 2xy^{\frac{2}{3}} \\ y(0) = 0 \end{cases} \quad y = 0 \text{ and } y = \frac{x^6}{27} \text{ are solutions}$$

$$\frac{dy}{dx}\frac{x^6}{27} = 2x \cdot \frac{x^4}{9} = y^{\frac{2}{3}}$$

$\begin{cases} \dfrac{dy}{dx} = 2yx^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$ and $y = 0$ is the only solution. This IVP satisfies a certain condition and that makes it have a unique solution

$$\begin{cases} \dfrac{dy}{dx} = xy^{\frac{1}{2}} \\ y(0) = 0 \end{cases}$$

Does the IVP have a unique solution? When on $\mathbb{R}^2$ is $\dfrac{\partial f}{\partial y}$ continuous? $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}xy^{-\frac{1}{2}} = \dfrac{x}{2\sqrt{y}}$

$\begin{cases} \dfrac{dy}{dx} = 3y \\ y(0) = 5 \end{cases}$ Yes there is a unique solution, $\dfrac{\partial f}{\partial y} = 3$

Determine the region R for which the DE would have a unique solution through a point (x_0, y_0) in the region $\dfrac{dy}{dx} = \sqrt{xy}$

Where on $\mathbb{R}^2$ is $\dfrac{\partial f}{\partial y}$ continuous? $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}(xy)^{-1/2} * \dfrac{\partial}{\partial y}(xy) = \dfrac{x}{2\sqrt{xy}}$

DIY

$$\frac{dy}{dx} - y = x$$

3.2

Solve: $ydx = (2 + 3x)dy$

$$\frac{dy}{y} = \frac{dx}{2 + 3x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{2 + 3x}$$

$$\ln|y| = \frac{\ln|2 + 3x|}{3} + C$$

$$e^{\ln|y|} = e^{\frac{\ln|2+3x|}{3} + C_1}$$

$$= e^{\frac{\ln|2+3x|}{3}} \cdot e^{C_1}$$

$$|y| = e^{C_1} \cdot \ln|2 + 3x|^{\frac{1}{3}}$$

$$|y| = \left|e^{C_1}\right| \cdot \left|(2 + 3x)^{\frac{1}{3}}\right|$$

$$|y| = \left|e^{C_1} \cdot |(2 + 3x)^{\frac{1}{3}}\right|$$

$$y = \pm C(2 + 3x)^{\frac{1}{3}} \qquad x \neq -\frac{2}{3}$$

3.3

$\frac{dy}{dx} = e^x e^{5y}$
$e^{-5y} dy = \frac{e^x}{dx}$
$\int e^{-5y} dy = \int e^x dx$
$-\frac{1}{5} e^{-5y} = e^x + C_1$
$e^{-5y} = -5e^x - 5C_1$
$-5y = \ln(-5e^x - 5C_1)$
$y = -\frac{1}{5} \ln(C - 5e^x)$

3.4

$y' = 2y - y^2$
$\frac{dy}{dx} = y(2-y) \to \int \frac{dy}{y(2-y)} = \int dx \to \int \left(\frac{0.5}{y} + \frac{0.5}{2-y} \right) dy = \int dx$
$\frac{1}{2} \ln|y| - \frac{1}{2} \ln|2-y| = x + C_1$
$\ln|y| - \ln|2-y| = 2x + 2C_1$
$\ln\left|\frac{y}{2-y}\right| = 2x + 2C_1 \Rightarrow \left|\frac{y}{2-y}\right| = e^{2x} e^{2C_1} \to \frac{y}{2-y} = Ce^{2x} \to y = CE^{2x}(2-y)$
$y = 2Ce^{2x} - Ce^{2x} y \to (1 + Ce^{2x})y = 2Ce^{2x}$
$\boxed{y = \frac{2Ce^{2x}}{1 + Ce^{2x}}} \to \frac{2C}{e^{-2x} + C}$

3.5

$(x - y)dx + xdy = 0$
Substitution $y = ux \Rightarrow dy = udx + xdu$
$(x - ux)dx + x(udx + xdu) = 0$
$xdx - uxdx + uxdx + x^2 du = 0$
$xdx + x^2 du = 0$
$\int du = -\int \frac{1}{x} dx \Rightarrow u = -\ln|x| + C$
$u = \frac{y}{x}$
$\frac{y}{x} = C - \ln|x|$
$y = Cx - x\ln|x|$