MATH 2 Lecture Notes

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1 Chapter 1

1.1 Terminology

Definition A differential equation is an equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables.
· An Ordinary Differential Equation (ODE) involves only ordinary derivatives
· A Partial Differential Equation (PDE) involves partial derivatives.
Definition The order of a DE is the order of the highest-order derivative that appears in the DE
Notation $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2})$
Definition A linear DE is any DE that can be written in form:
$a_0(x)y + a_1(x)y' + a_2(x)y'' \cdots + a_n(x)y^{(n)} = b(x)$
For a DE to be linear:

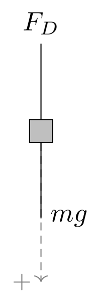
1. Y and all of its derivatives much be of the 1st degree

2. Any term that does not include y or any of its derivatives must be a function of x

1.2 Some Mathematical Models

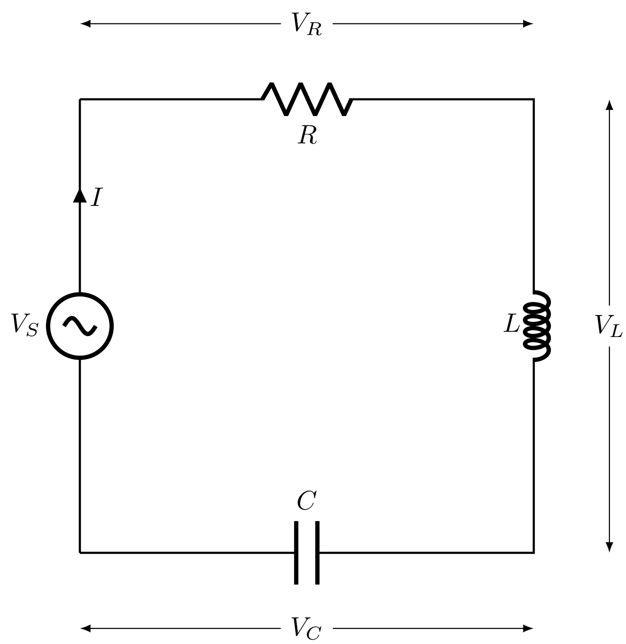
I. Free-falling body

Goal: Find s(t).



Set up a differential equation in S, model it, then solve
$ma = mg$
$\frac{d^2s}{dt^2} = g$
$v = \frac{ds}{dt}, g = \frac{dv}{dt}$
What if there is air resistance. Assume force scales linear with velocity
$\frac{dv}{dt} = g - \frac{kv}{m} \rightarrow \frac{dv}{dt} = g - \frac{k}{m} \cdot \frac{ds}{dt}$

II: Series Circuit



Voltage drops:

$V = L\dfrac{dI}{dt}, V = L\dfrac{d^2q}{dt^2}$

$V = IR, V = R\dfrac{dq}{dt}$

$V = \dfrac{q}{C}$

$E(t) = L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C}$

III: Population Growth

$P = P(t) =$ population at time t — use exponential model

$\dfrac{dp}{dt} \propto P \rightarrow \dfrac{dp}{dt} = kP \rightarrow = Ce^{kt}$ where C is the initial population

IV: Population Growth with Finite Capacity

"Carrying Capacity" = N — uses the logistic growth model

$\dfrac{dp}{dt} \propto$ both P and amount to carrying capacity (N-P)

$\dfrac{dp}{dt} = kP(N - P)$

V: Chemical Reaction

$A + B \rightarrow C$ Concentrations of A and B decreases by amount of C formed
Can we write DE governing the concentration of C x(t)?
The rate at which the reaaction takes place $\propto$ Product of the remaining concentrations of A and B
α initial concentration of A
β initial concentration of B

$\dfrac{dx}{dt} = k(\alpha - x)(\beta - x)$

2 First-Order Differential Equations

2.1 Preliminary Theory

Example DE: $y' = 3y \Rightarrow \boxed{y = Ce^{3x}}$ the general solution where C is an arbitrary constant

Add initial condition $y(0) = 5$ plug in x=0 to $5 = Ce^{3*0}, 5 = C * 1, C = 5 \Leftarrow$ Initial Value Problem
$y = 5e^{3x}$ is the general solution for the Initial Value Problem

Theorem $f(x) = \begin{cases} \dfrac{dy}{dx} = f(x,y) & \text{Differential Equation} \\ y(x_0) = y_0 & \text{Initial Condition} \end{cases}$

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b, c \leq y \leq d$, that contains the point (x_0, y_0) in its interior.

If f(x,y) and $\partial f \partial y$ are continuous on R, then there exists an interval I centered at x_o, and on this interval I there exists a unique solution $y(x)$ for this IVP

Key Questions:
Does every IVP have at least one solution?
If an IVP has a solution is it the only solution?

2.2 Separable Variables

2.3 Homogenous Equations

2.4 Exact Equations

2.5 Preliminary Theory

3 Example Problems with Solutions

3.1

$\begin{cases} \dfrac{dy}{dx} = 2xy^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$ $\quad y = 0$ and $y = \dfrac{x^6}{27}$ are solutions

$\dfrac{dy}{dx}\dfrac{x^6}{27} = 2x \cdot \dfrac{x^4}{9} = y^{\frac{2}{3}}$

$\begin{cases} \dfrac{dy}{dx} = 2yx^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$ $\quad$ and $y = 0$ is the only solution. This IVP satisfies a certain condition and that makes it have a unique solution

$\begin{cases} \dfrac{dy}{dx} = xy^{\frac{1}{2}} \\ y(0) = 0 \end{cases}$

Does the IVP have a unique solution? When on $\mathbb{R}^2$ is $\dfrac{\partial f}{\partial y}$ continuous? $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}xy^{-\frac{1}{2}} = \dfrac{x}{2\sqrt{y}}$

$\begin{cases} \dfrac{dy}{dx} = 3y \\ y(0) = 5 \end{cases}$ $\quad$ Yes there is a unique solution, $\dfrac{\partial f}{\partial y} = 3$

Determine the region R for which the DE would have a unique solution through a point (x_0, y_0) in the region $\dfrac{dy}{dx} = \sqrt{xy}$

Where on $\mathbb{R}^2$ is $\dfrac{\partial f}{\partial y}$ continuous? $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}(xy)^{-1/2} * \dfrac{\partial}{\partial y}(xy) = \dfrac{x}{2\sqrt{xy}}$

DIY

$\dfrac{dy}{dx} - y = x$