MATH 2 Lecture Notes

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1 Chapter 1

1.1 Terminology

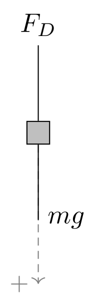
Definition A differential equation is an equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables.
· An Ordinary Differential Equation (ODE) involves only ordinary derivatives
· A Partial Differential Equation (PDE) involves partial derivatives.
Definition The order of a DE is the order of the highest-order derivative that appears in the DE
Notation $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2})$
Definition A linear DE is any DE that can be written in form:
$a_0(x)y + a_1(x)y' + a_2(x)y'' \cdots + a_n(x)y^{(n)} = b(x)$
For a DE to be linear:

1. Y and all of its derivatives much be of the 1st degree
2. Any term that does not include y or any of its derivatives must be a function of x

1.2 Some Mathematical Models

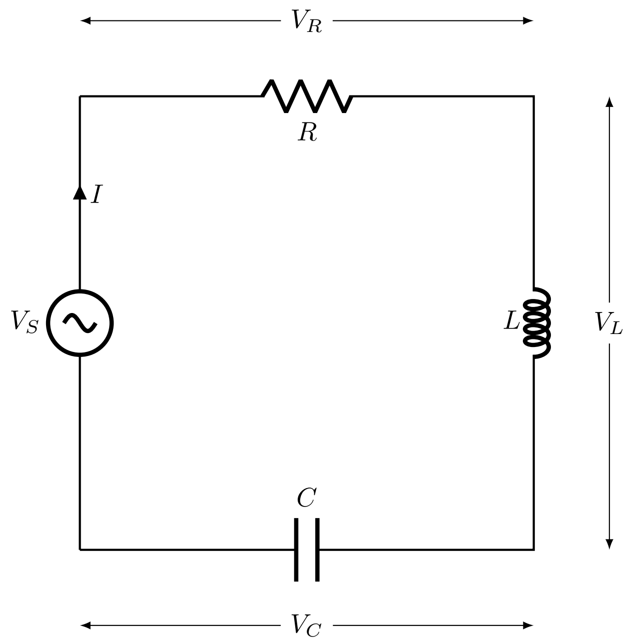
I. Free-falling body

Goal: Find s(t).



Set up a differential equation in S, model it, then solve
$ma = mg$
$\frac{d^2s}{dt^2} = g$
$v = \frac{ds}{dt}, g = \frac{dv}{dt}$
What if there is air resistance. Assume force scales linear with velocity
$\frac{dv}{dt} = g - \frac{kv}{m} \rightarrow \frac{dv}{dt} = g - \frac{k}{m} \cdot \frac{ds}{dt}$

II: Series Circuit



Voltage drops:

$V = L\dfrac{dI}{dt}, V = L\dfrac{d^2q}{dt^2}$

$V = IR, V = R\dfrac{dq}{dt}$

$V = \dfrac{q}{C}$

$E(t) = L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C}$

III: Population Growth

$P = P(t)$ = population at time t — use exponential model

$\dfrac{dp}{dt} \propto P \to \dfrac{dp}{dt} = kP \to= Ce^{kt}$ where C is the initial population

IV: Population Growth with Finite Capacity

"Carrying Capacity" = N — uses the logistic growth model

$\dfrac{dp}{dt} \propto$ both P and amount to carrying capacity (N-P)

$\dfrac{dp}{dt} = kP(N - P)$

V: Chemical Reaction

$A + B \to C$ Concentrations of A and B decreases by amount of C formed
Can we write DE governing the concentration of C x(t)?
The rate at which the reaaction takes place $\propto$ Product of the remaining concentrations of A and B
α initial concentration of A
β initial concentration of B

$\dfrac{dx}{dt} = k(\alpha - x)(\beta - x)$

2 First-Order Differential Equations

2.1 Preliminary Theory

Example DE: $y' = 3y \Rightarrow \boxed{y = Ce^{3x}}$ the general solution where C is an arbitrary constant

Add initial condition $y(0) = 5$ plug in x=0 to $5 = Ce^{3*0}, 5 = C * 1, C = 5 \Leftarrow$ Initial Value Problem
$y = 5e^{3x}$ is the general solution for the Initial Value Problem

2.1.1 Theorem

$$f(x) = \begin{cases} \dfrac{dy}{dx} = f(x, y) & \text{Differential Equation} \\ y(x_0) = y_0 & \text{Initial Condition} \end{cases}$$

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b, c \leq y \leq d$, that contains the point (x_0, y_0) in its interior.

If f(x,y) and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists an interval I centered at x_o, and on this interval I there exists a unique solution $y(x)$ for this IVP

2.1.2 Key Questions:

Does every IVP have at least one solution?
If an IVP has a solution is it the only solution?

Meaning of a solution existing "on an Interval" The initial value problem

$$\begin{cases} \dfrac{dy}{dx} = 1 + y^2 \\ y(0) = 0 \end{cases}$$

has a unique solution. In fact, we can easily verify that $y = \tan x$ satisfies this IVP
However note that there are some inervals on which $y = \tan x$ cannot be a solution for this IVP, such as (-2,2), where the function is discontinuous at $\pm\dfrac{\pi}{2}$ but can be used for (-1,1) since it is continuous at all points within the interval

2.2 Separable Variables (Separable Equations)

2.2.1 Definition:

A differential equation that can be written in the form $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$ is said to be separable (or have separable variables).

Example: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$

$h(y)dy = g(x)dx$

$\displaystyle\int h(y)dy = \int g(x)dx$

Example: $dx + e^{3x}dy = 0$

$e^{3x}dy = -dx$

$dy = -\dfrac{dx}{e^{3x}} \rightarrow dy = -e^{-3x}dx \rightarrow \displaystyle\int dy = \int -e^{-3x}dx \rightarrow y = \dfrac{1}{3}e^{-3x} + C$ where C is an arbitrary constant

2.2.2 Substitution

$\frac{dy}{dx} = F(ax+bc+c)$ where $b \neq 0$ use the substitution: $u = ax+by+c \Rightarrow \frac{du}{dx} = a+b\frac{dy}{dx} = \frac{1}{b}\left[\frac{du}{dx} - a\right]$

Example: $\frac{dy}{dx} = \tan^2(x+y)$ let $u = x+y \rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1 \rightarrow \frac{du}{dx} - 1 = \tan^2 u \rightarrow \frac{du}{dx} = \sec^2 u$

$\int \cos^2 u \, du = \int dx$

$2(x+y) + \sin 2(x+y) = 4x + C \rightarrow 2y - 2x + \sin 2(x+y)$

Solve: $\frac{dy}{dx} = (y+3)^2$ By inspection $y = -3$ is a solution. This is the only solution because $f(x,y) = (x+3)^2$ is continuous on $\mathbb{R}^2$ and $\frac{\partial f}{\partial x}$ is continuous on$\mathbb{R}$ so it is the only solution Why solving by separation is not possible $\int (y+3)^-2dy = \int dx \rightarrow (y+3)^-2/-1 = x + C_1 \rightarrow \frac{1}{y+3} = -x - C_1 \rightarrow$
$y+3 = \frac{1}{c-x} \rightarrow y = -3 + \frac{1}{c-x}$

$y(0) = -3 \rightarrow 0 = \frac{1}{c}$ where there is no real c that solves that equation, making this not possible

2.3 Homogeneous Equations

What do we do if the DE is not separable?

2.3.1 Definition

A function $f(x,y)$ is said to be **homogeneous of degree** n if, for $x, y,$ and$t$where $f(x,y)$ and $f(tx,ty)$ are defined:

$$f(tx,ty) = t^n f(x,y)$$

2.3.2 Example

Determine wheteher each function is homogeneous:
a: $f(x,y) = x^3 - 7x^2y + 4y^3 \rightarrow f(tx,ty) = (tx^3) - 7(tx)^2(ty) + 4(ty)^3$
$t^3x^3 - 7t^3x^2y + 4t^3y^3$
$t^3(x^3 - 7x^2y - 4y^3) = t^3 f(x,y)$
How to tell quickly wheteher $f(x,y)$ is homogeneous:
Each term must have the same combined degree
Example: $x^3 - 7x^2y + 4y^3$ is D3, $x^2 + y^2 - 4x$ is not, $\sqrt{x^5 + 4y^5}$ is with D 2.5, $\frac{3y}{x} - 2$ is D0

2.3.3 Differential Equation form

$M(x,y)dx + N(x,y)dy = 0$ is called a homogeneous differential equation if the functions M and N are both homogeneous of the same degree
If $f(x,y)$ is homogeneous of degree n then $f(x,y)$ can be written as:
$f(x,y) = f\left(x \times 1, x \times \frac{y}{x}\right) = x^n f\left(1, \frac{y}{x}\right)$
or $f(x,y) = y^n f\left(\frac{x}{y}, 1\right)$

2.3.4 Substitution

To solve a homogeneous DE make the substitution: $y = ux \;\; (u = \frac{y}{x})$ or$x = vy \;\; (v = \frac{x}{y})$

2.3.5 Example

$(y^2 + xy)dx + x^2dy = 0 \to y = ux \to dy = (udx + xdu)$
$(u^2x^2 + ux^2)dx + x^2(udx + xdu) = 0$
$u^2x^2dx + ux^2dx + ux^2dx + x^3du = 0$
$ux^2(u + 2)dx + x^3du = 0$
$\int \frac{1}{u(u+2)}du = -\int \frac{1}{x}dx$
Partial Fraction Decomposition: $\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2} \to A = \frac{1}{2}, B = -\frac{1}{2}$ Back to solving
$\int \left[\frac{0.5}{u} - \frac{0.5}{u+2}\right] = -\int \frac{1}{x}dx$
$0.5 \ln|u| - 1/2 \ln|u + 2| = -\ln|x| + C_1$
$\ln\left|\frac{u}{u+2}\right| = 2C_1 - 2\ln|x|$
$\left|\frac{u}{u+2}\right| = e^{2C_1} \cdot e^{-2\ln|x|} = e^{2C_1} \cdot |x^{-2}| \Rightarrow \left|\frac{u}{u+2}\right| = \left|e^{2C_1} \cdot x^{-2}\right| \Rightarrow \left|\frac{u}{u+2} = \frac{C}{x^2}\right|$
$ux^2 = X(u + 2) \Rightarrow ux^2 = Cu + 2c \to ux^2 - Cu = 2C$
$u(x^2 - c) = 2C \Rightarrow u = \frac{2C}{x^2 - C} \Rightarrow \frac{y}{x} = \frac{2Cx}{x^2 - C},\ x \neq 0$

2.4 Exact Equations

Recall from Math 1C: Let $F(x, y) = \langle 3x^2 - 7y, -7x + 2y\rangle$

1. If F a conservative vector field
   i.e., Is there a function $f(x, y)$ such that ∇f? Yes, -7=-7

2. If F is indeed conservative, what is f?
   $x^3 - 7xy + g(y) = f(x, y)$
   $-7x + 2y, g'(y) = 2y$
   $f(x, y) = x^3 - 7xy + y^2 + k$

2.4.1 Definition

A differential equation in the form $M(x, y)dx + N(x, y)dy = 0$ where $M_y = N_x$, is called an exact differential equation.

2.4.2 Solve the DE

$(3x^2 - 7y)dx + (-7x + 2y)dy = 0$
Using 1C techniques it is $f(x, y) = x^3 - 2xy + y^2 + k$
Set this f = c. $f(x, y) = x^3 - 2xy + y^2 = c$ take k=0 in every problem
If the DE is not exact, sometimes we can make it exact by multiplying by magical quantity $\mu(x, y)$

2.4.3 Example:

Solve the DE:
$(x + y)dx + xlnxdy = 0$ using $\mu(x, y) = \frac{1}{x}$
$\left(\frac{x+y}{x}\right) dx + \ln|x|\ dy = 0$ is now exact.
Solution: $f(x, y) = x + y\ln x = c$

2.5 Linear Equations

Recall: First Order Linear DE is a DE in the form $a_1(x)\dfrac{dy}{dx} + a_0(y)y = g(x), \qquad a_1(x) \neq 0$

Divide both sidex by $a_1(x) \Rightarrow \dfrac{dy}{dx} + \dfrac{a_0(x)}{a_1(x)}y = \dfrac{g(x)}{a_1(x)}$ where $P(x) = \dfrac{a_0(x)}{a_1(x)}$ and $f(x) = \dfrac{g(x)}{a_1(x)}$

$\dfrac{dy}{dx} + P(x)y = f(x)$ There is an integrating factor $\mu(x)$ that turns this DE into an exact DE

$dy + P(x)ydx = f(x)dx \to dy\,[P(x)y - f(x)]\,dx = 0$

$\mu(x)dy + \mu(x)\,[P(x)y - f(x)]\,dx = 0 \to \mu'(x) = \mu(x)P(x)$

$\dfrac{d\mu}{dx} = \mu P \to \displaystyle\int \frac{d\mu}{mu} = \int P(x) \to \ln\mu = \int P(x)\;dx$

$\mu(x) = e^{\int P(x)\;dx} \Rightarrow e^{\int P(x)\;dx}\dfrac{dy}{dx} + e^{\int P(x)\;dx}P(x)y = e^{\int P(x)\;dx}f(x)$

$\dfrac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x) \to e^{\int P(x)dx}y = \displaystyle\int e^{\int P(x)dx}f(x)dx \; \boxed{y = e^{-\int P(x)dx}\displaystyle\int e^{\int P(x)dx}f(x)dx}$

2.5.1 Procedure to follow for every Linear DE

1. Rewrite the linear DE in the form $\dfrac{dy}{dx} + P(x)y = f(x)$
2. Find the integrating factor $\mu(x) = e^{\int P(x)dx}$
3. Multiply each side of the DE by $\mu(x)$
4. Rewrite the left side as $\dfrac{d}{dx}\,[\mu(x)\cdot y]$
5. Integrate both sides with respect to x and retreive an implicitly expressed solution
6. Solve for y

2.6 What method to use to solve?

First ask is it exact? ($M_y = M_x$)
Yes: Use the method in §2.4
No: Is it linear? (in y or x)
Yes: Use the method in §2.5
No: Is it separable?
Yes: §2.2
No: Homogeneous?
Yes: Use a substitution §2.3
No: Good luck. or use inspection

3 Applications of First-Order Differential Equation

3.1 Orthogonal Trajectories

· Consider the family of curves $y = cx^3$ Question: Which DE should be solved to get this family as its solutions?
Steps:

1. Find $\dfrac{dy}{dx} = 3cx^2$
2. Eliminate c"

$y = cx^3 c = \dfrac{y}{x^3}$
$\dfrac{dy}{dx} = 3\dfrac{y}{x^3}x^2 \rightarrow \dfrac{3y}{x}$
· The two curves are orthogonal if their tangent lines are orthogonal at the point of intersection i.e. The derivatives are the negative reciprocals of each other

3.1.1 Example

Show that $y = x^3$ and $x^2 + 3y^2 = 4$ are orthogonal at their points of intersection, (1,1) and (-1,-1)
$y = x^3 \Rightarrow \dfrac{dy}{dx} = 3x^2 \rightarrow = 3$ at $x = 1$ and 3 at $x = -1$
$2x + 6y\dfrac{dy}{dx} = 0 \rightarrow \dfrac{dy}{dx} = -\dfrac{x}{3y} = \dfrac{-1}{3}$ at both $x = 1$ and $x = -1$ meaning it is orthogonal

3.1.2 Definition

When all the curves of one family of curves intersect orthogonally all the curves of another family, then the families are said to be orthogonal trajectories of each other

4 Example Problems with Solutions

4.1

$$\begin{cases} \dfrac{dy}{dx} = 2xy^{\frac{2}{3}} \\ y(0) = 0 \end{cases} \quad y = 0 \text{ and } y = \frac{x^6}{27} \text{ are solutions}$$

$$\frac{dy}{dx}\frac{x^6}{27} = 2x \cdot \frac{x^4}{9} = y^{\frac{2}{3}}$$

$$\begin{cases} \dfrac{dy}{dx} = 2yx^{\frac{2}{3}} \\ y(0) = 0 \end{cases}$$ and $y = 0$ is the only solution. This IVP satisfies a certain condition and that makes it have a unique solution

$$\begin{cases} \dfrac{dy}{dx} = xy^{\frac{1}{2}} \\ y(0) = 0 \end{cases}$$

Does the IVP have a unique solution? When on $\mathbb{R}^2$ is $\dfrac{\partial f}{\partial y}$ continuous? $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}xy^{-\frac{1}{2}} = \dfrac{x}{2\sqrt{y}}$

$$\begin{cases} \dfrac{dy}{dx} = 3y \\ y(0) = 5 \end{cases} \quad \text{Yes there is a unique solution, } \frac{\partial f}{\partial y} = 3$$

Determine the region R for which the DE would have a unique solution through a point (x_0, y_0) in the region $\dfrac{dy}{dx} = \sqrt{xy}$

Where on $\mathbb{R}^2$ is $\dfrac{\partial f}{\partial y}$ continuous? $\dfrac{\partial f}{\partial y} = \dfrac{1}{2}(xy)^{-1/2} * \dfrac{\partial}{\partial y}(xy) = \dfrac{x}{2\sqrt{xy}}$

DIY

$$\frac{dy}{dx} - y = x$$

4.2

Solve: $ydx = (2 + 3x)dy$

$$\frac{dy}{y} = \frac{dx}{2 + 3x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{2 + 3x}$$

$$\ln|y| = \frac{\ln|2 + 3x|}{3} + C$$

$$e^{\ln|y|} = e^{\frac{\ln|2+3x|}{3} + C_1}$$

$$= e^{\frac{\ln|2+3x|}{3}} \cdot e^{C_1}$$

$$|y| = e^{C_1} \cdot \ln|2 + 3x|^{\frac{1}{3}}$$

$$|y| = \left|e^{C_1}\right| \cdot \left|(2 + 3x)^{\frac{1}{3}}\right|$$

$$|y| = \left|e^{C_1} \cdot |(2 + 3x)^{\frac{1}{3}}\right|$$

$$y = \pm C(2 + 3x)^{\frac{1}{3}} \qquad x \neq -\frac{2}{3}$$

4.3

$$\frac{dy}{dx} = e^x e^{5y}$$
$$e^{-5y}dy = \frac{e^x}{dx}$$
$$\int e^{-5y}dy = \int e^x dx$$
$$-\frac{1}{5}e^{-5y} = e^x + C_1$$
$$e^{-5y} = -5e^x - 5C_1$$
$$-5y = \ln(-5e^x - 5C_1)$$
$$y = -\frac{1}{5}\ln(C - 5e^x)$$

4.4

$$y' = 2y - y^2$$
$$\frac{dy}{dx} = y(2-y) \to \int \frac{dy}{y(2-y)} = \int dx \to \int \left(\frac{0.5}{y} + \frac{0.5}{2-y}\right) dy = \int dx$$
$$\frac{1}{2}\ln|y| - \frac{1}{2}\ln|2-y| = x + C_1$$
$$\ln|y| - \ln|2-y| = 2x + 2C_1$$
$$\ln\left|\frac{y}{2-y}\right| = 2x + 2C_1 \Rightarrow \left|\frac{y}{2-y}\right| = e^{2x}e^{2C_1} \to \frac{y}{2-y} = Ce^{2x} \to y = CE^{2x}(2-y)$$
$$y = 2Ce^{2x} - Ce^{2x}y \to (1 + Ce^{2x})y = 2Ce^{2x}$$
$$\boxed{y = \frac{2Ce^{2x}}{1 + Ce^{2x}}} \to \frac{2C}{e^{-2x} + C}$$

4.5

$$(x-y)dx + xdy = 0$$
Substitution $y = ux \Rightarrow dy = udx + xdu$
$$(x - ux)dx + x(udx + xdu) = 0$$
$$xdx - uxdx + uxdx + x^2du = 0$$
$$xdx + x^2du = 0$$
$$\int du = -\int \frac{1}{x}dx \Rightarrow u = -\ln|x| + C$$
$$u = \frac{y}{x}$$
$$\frac{y}{x} = C - \ln|x|$$
$$y = Cx - x\ln|x|$$

4.6

$(x^3 + y^2)dx = 3xy^2dy = 0$ is conservative, so find $f(x, y)$ that satisfies $M = f_x, N = f_y$

$\int (x^3 + y^3)dx = \frac{x^4}{4} + xy^3 + g(y)$ and $\int xy^2dy = xy^3 + g'(y)$

$$f(x, y) = \frac{x^4}{4} + xy^3 = C$$