Homework 4

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Chapter 10

2

Base case of 1: $1^2 = 1$ and $\dfrac{n(n+1)(2n+1)}{6} \Rightarrow \dfrac{1*(1+1)(2+1)}{6} = 1$
Inductive Step: let $n = k$
$\dfrac{k(k+1)(2k+1)}{6}$ add $(k+1)^2 = k^2 + 2k + 1 = \dfrac{6k^2 + 12k + 6}{6}$
$\dfrac{k(k+1)(2k+1) + 6k^2 + 12k + 6}{6} = \dfrac{6 + 13k + 9k^2 + 2k^3}{6} = \dfrac{(2k+3)(k+1)(k+2)}{6}$
Sub in $k = n$ and simplify: $\dfrac{(2n+3)(n+1)(n+2)}{6} = \dfrac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

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Sums of $n(n+1) = \dfrac{n(n+1)(n+2)}{3}$
Base case $n = 1:\ 1*2 = 2$ and $\dfrac{1*2*3}{3} = 2$
Inductive step: Let $n = k \Rightarrow \dfrac{k(k+1)(k+2)}{3}$
Add $(k+1)(k+2) = k^2 + 3k + 2 = \dfrac{3k^2 + 9k + 6}{3} \Rightarrow \dfrac{(k(k+1)(k+2)) + 3k^2 + 9k + 6}{3} = \dfrac{6 + 11k + 6k^2 + k^3}{3}$
$= \dfrac{(k+1)(k+2)(k+3)}{3}$ switch k back to n and get $n+1$ out
$= \dfrac{(n+1)((n+1)+1)((n+1)+2)}{3}$ checks out

8

Base case of 1: $\dfrac{1}{(1+1)!} = \dfrac{1}{2}$
Inductive step: Let $n = k$
$1 - \dfrac{1}{(k+1)!} + \dfrac{k+1}{(k+2)!}$ which Wolfram evaluated to $1 - \dfrac{1}{\Gamma(k+3)}$ which is equivalent to $1 - \dfrac{1}{k+2}$
Reverting it back to n and factoring out $n+1$ we get $1 - \dfrac{1}{(n+1)+1}$

13.7

4

$\lim_{n\to\infty} 1 - \dfrac{1}{2^n} = 1$ by definition
Breaking it up termwise, $1 \to 1$ and $-\dfrac{1}{n^2} \to 0$. $1 + 0 = 1$ by the Sum law for limits and definition of series convergence

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$\lim_{n\to\infty} \frac{5n^2+n+1}{4n^2+2} = \frac{5}{4}$
Since both the numerator and denominator are of the same degree and both are polynomials, we can use the difference in the degrees to determine the outcome. Numerator > Denominator diverges to infinity, Numerator < Denominator converges on 0, and Numerator = Denominator converges on the quotient of their coefficients

8

Since there is no indexing variable n in the series, the series does not change value, hence it continues on as c as $n \to \infty$

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By Definition 13.5, $\lim_{n\to\infty} \{a_n\} = L$ and $\lim_{n\to\infty} \{b_n\} = M$
By the Sum Law of Limits, since for both limits we are taking $n \to \infty$, $\lim_{n\to\infty} \{a_n\} + \lim_{n\to\infty} \{b_n\} = L + M$ is equivalent to $\lim_{n\to\infty} \{a_n + b_n\} = L + M$

Excercises from Notes

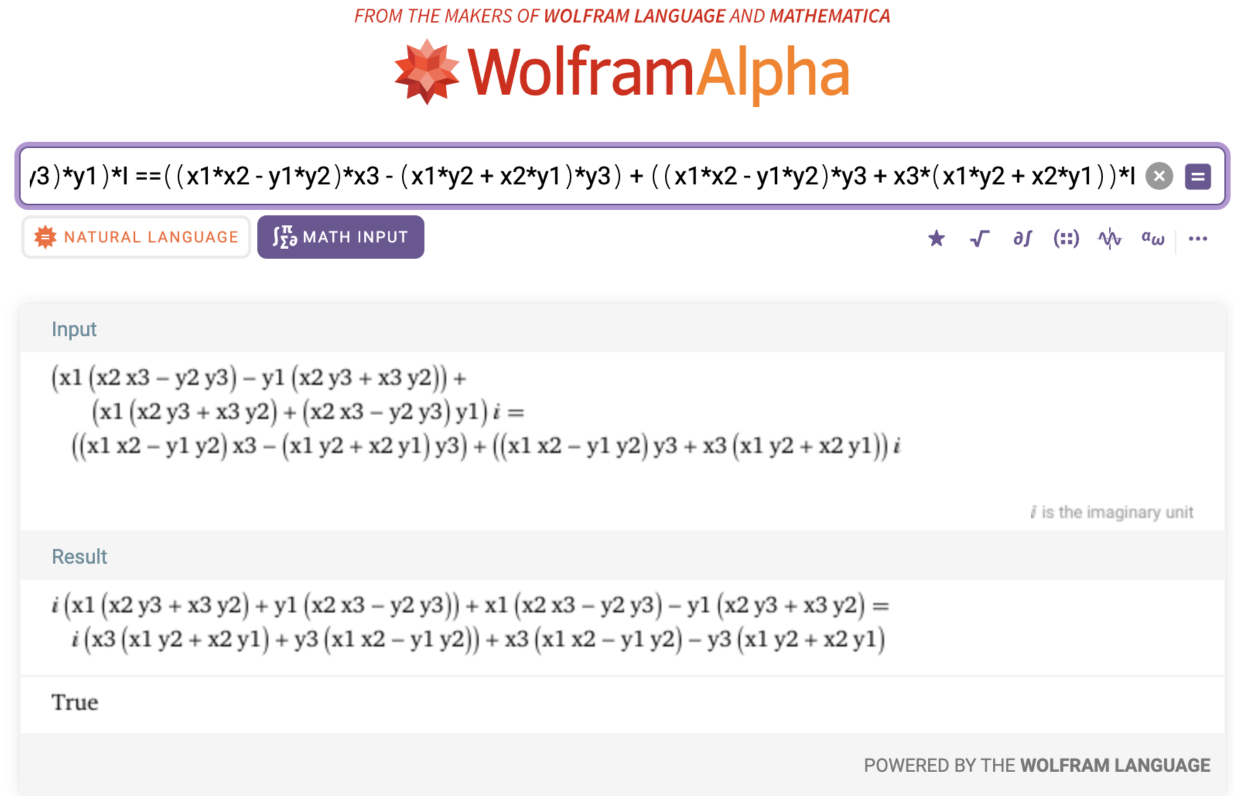
4.1

$z_1 = x_1 + y_1 i,\ z_2 = x_2 + y_2 i,\ z_3 = x_3 + y_3 i$

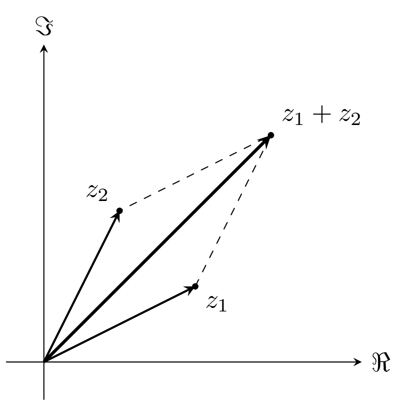
$z_1(z_2 z_3) = (z_1 z_2) z_3$
$z_1 \cdot ((x_2x_3 - y_2y_3) + (x_2y_3 + x_3y_2)i) = ((x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i) \cdot z_2$

$(x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + x_3y_2)) + (x_1(x_2y_3 + x_3y_2) + (x_2x_3 - y_2y_3)y_1)i =$
$((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + x_2y_1)y_3) + ((x_1x_2 - y_1y_2)y_3 + x_3(x_1y_2 + x_2y_1))i$

Bless Mathematica for this evaluation



4.2



4.3

$$
\begin{array}{c|l}
n & w \\
0 & \cos\left(\frac{0\pi}{2}\right) + i\sin\left(\frac{0\pi}{2}\right) = 1 \\
1 & \cos\left(\frac{1\pi}{2}\right) + i\sin\left(\frac{1\pi}{2}\right) = i \\
2 & \cos\left(\frac{2\pi}{2}\right) + i\sin\left(\frac{2\pi}{2}\right) = -1 \\
3 & \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i
\end{array}
$$

4.4

$$-2\sqrt{3}+2i = 4\,e^{i\frac{5\pi}{6}} \quad\Longrightarrow\quad z=\sqrt{2}\,e^{i\left(\frac{5\pi}{24}+\frac{k\pi}{2}\right)}=\sqrt{2}\Big(\cos\big(\tfrac{5\pi}{24}+\tfrac{k\pi}{2}\big)+i\sin\big(\tfrac{5\pi}{24}+\tfrac{k\pi}{2}\big)\Big),\ k=0,1,2,3.$$

4.5

$$\begin{aligned}|z_1+z_2|^2 &= (z_1+z_2)(\overline{z_1}+\overline{z_2}) = |z_1|^2+|z_2|^2+2\ \mathrm{Re}(z_1\overline{z_2})\\ &\le |z_1|^2+|z_2|^2+2|z_1||z_2| = (|z_1|+|z_2|)^2.\end{aligned}$$

4.6

$0=z^5-1=(z-1)(z^4+z^3+z^2+z+1)$
So $z^4+z^3+z^2+z+1=0$

4.7

$$1-z=1-e^{2\pi ik/n}=-2i\,e^{\pi ik/n}\,\sin\left(\frac{\pi k}{n}\right),$$
so
$$(1-z)^{2n}=\big(-2i\big)^{2n}e^{2\pi ik}\,\sin^{2n}\left(\frac{\pi k}{n}\right)=(-4)^n\,\sin^{2n}\left(\frac{\pi k}{n}\right)\in\mathbb{R}.$$
Thus $(1-z)^{2n}$ is real.