Homework 3

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4:

2

If x is an odd integer, then x^3 is odd.
Lets define an odd integer as an integer that acn be represented as $2k+1$ where $k \in \mathbb{Z}$
We have to prove $(2k+1)^3$ is odd. Doing the binomial expansion we get $8k^3+12k^2+6k+1$
$8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$ where $m = 4k^3+6k^2+3k \in \mathbb{Z}$, making $x^3 = 2m+1$✓

4

Suppose $x, y \in \mathbb{Z}$. If x and y are odd then xy is odd.
Let x be an integer defined as $2k+1$ and y be an integer defined as $2l+1$
$(2k+1)(2l+1) = 4kl+2k+2l+1 = 2(2kl+k+l)+1$. Let $m = 2kl+k+l$ making $xy = 2m+1$

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Suppose x and y are positive real numbers. If $x < y$, then $x^2 < y^2$
$x < y$ multiply both sides by x to get $x^2 < xy$
$x < y$ multiply both sides by y to get $xy < y^2$
Combine them to get $x^2 < xy < y^2$ by transitive property $x^2 < y^2$

5:

6

Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$
The contrapositive states if $x \leq -1$ then $x^3 - x \leq 0$
$x = k : k \in \mathbb{R}, k \leq -1$
$x^3 - x = k^3 - k = k(k^2-1)$ Since $k \leq -1$, $k^2 - 1$ is also a nonnegative number.
$k \leq -1 \to k < 0$ meaning k is negative
For $k = -1, k^3 - k = 0$ and for $k < -1, k^3 - k < 0$
Meaning $k^3 - k \leq 0$ proving the contrapositive

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Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.
Contrapositive: If a is not odd, then a^2 is divisible by 4
Proof: Define integer $a = 2k$ as an even integer where $k \in \mathbb{Z}$. $(2k)^2 = 4k^2$
$m = k^2$ so $(2k)^2 = 4m, m \in \mathbb{Z}$, making all non-odd integers evenly divisible by 4

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Suppose $x, y \in \mathbb{Z}$. If $x+y$ is even, then x and y have the same parity. Parity is defined as the status of being even/odd

Contrapositive: If x and y have opposite parities, then $x+y$ will be odd

Define $2k$ as the integer with the even parity, and $2l+1$ as the integer with the odd parity. $x, y \in \mathbb{Z}$

Adding up $2k+2l+1$, we get $2(k+l)+1$, an odd number, where the remainder mod 2 is 1, proving the contrapositive

6:

2

Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

Assume the opposite: If n^2 is odd, then n is even.

Disprove: Assume $n = 2k$. $(2k)^2 = 4k^2$, which will always be an even number by virtue of even coefficient of 4

So there exists a contradiction between the premise and conclusion where n will always be even if n^2 is even

4

Prove that $\sqrt{6}$ is irrational.

Assume $\sqrt{6} \in \mathbb{Q}$

For this to be true: $\exists p, q \in \mathbb{Z}; \gcd(p,q)=1$

Then $\dfrac{p}{q} = \sqrt{6} \Rightarrow \dfrac{p^2}{q^2} = 6 \Rightarrow p^2 = 6q^2$

Since $6q^2$ is even, assume p^2 is also even, meaning $p = 2r \Rightarrow 4r^2 = 6q^2 \Rightarrow 2r^2 = 3q^2$

Since $2r^2$ is even, $3q^2$ is even too, meaning q^2 is even, and q is even, meaning $\gcd(p,q) \geq 2$, making this a contradiction

10

There exist no integers a and b for which $21a + 30b = 1$

Assume the opposite: There exist integers a and b for which $21a + 30b = 1$

Write $21a + 30b = 1$ as $3(7a + 8b) = 1$.

Use $n = 7a + 8b$. For the summation to be true, $3n = 1, n \in \mathbb{Z}$ would need to be true, which it is not for any integer, making this a contradiction

11.1

6

Congruence modulo 5 is a relation on the set $A = \mathbb{Z}$. In this relation xRy means $x \equiv y \pmod 5$. Write out the set R in set-builder notation.

$R = \left\{(x,y) \in \mathbb{Z}^2 : 5|(x-y)\right\}$

10

Consider the subset $R = (\mathbb{R} \times \mathbb{R}) - \{(x,x) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. What familiar relation on $\mathbb{R}$ is this? Explain.

This is $y \neq x$, where its just a line where both coordinates are not the same value.

14

In the following exercises, subsets R of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ or $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ are indicated by gray shading. In each case, R is a familiar relation on $\mathbb{R}$ or $\mathbb{Z}$. State it for the figure labeled "Problem 14."
It's $R = \{(x, y) : x, y \in \mathbb{R}\ , y > x\}$

11.2

6

Consider the relation $R = \{(x, x) : x \in \mathbb{Z}\}$ on $\mathbb{Z}$. Is R reflexive? symmetric? transitive? If a property does not hold, say why. What familiar relation is this?
Well this is the points for the line $y = x$ along the integers. It holds reflexivity and symmetry, and hold transitivity too. If $(a, b) \in R$ and $(b, c) \in R$ then $a = b = c$ making $(a, c) \in R$

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Define a relation on $\mathbb{Z}$ by xRy if $|x - y| < 1$. Is R reflexive? symmetric? transitive? If a property does not hold, say why. What familiar relation is this?
$R = \{(x, y) : x, y \in \mathbb{Z} \times \mathbb{Z}, |x - y| < 1\}$
For absolute value of a difference to be strictly below 1 on the bounds of integers, the difference must equal 0, making this another $y = x, y \in \mathbb{Z}$. It holds reflexivity, symmetry, and transitivity too

14

Suppose R is a symmetric and transitive relation on a set A, and there is an element $a \in A$ for which aRx for every $x \in A$. Prove that R is reflexive.
Assume any values a and b. By symmetry, both $(a, b) \in R$ and $(b, a) \in R$
Since aRx for all x, and R is symmetric, by transitivity xRx and is reflexive

11.3

2

Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A. Suppose R has two equivalence classes. Also aRd, bRc and eRd. Write out R as a set.
$\{\{a, d, e\}, \{b, c\}\}$
Relations: $\{(a, a), (a, d), (a, e), (d, e), (d, a), (d, d), (e, e), (e, a), (e, d), (b, b), (b, c), (c, b), (c, c)\}$

6

There are five different equivalence relations on the set $A = \{a, b, c\}$. Describe them all. (Diagrams will suffice.)
Partition $\{\{a\}, \{b\}, \{c\}\}$: Relations $\{(a, a), (b, b), (c, c)\}$
Partition $\{\{a, b\}, \{c\}\}$: Relations: $\{(a, a), (b, b), (c, c), (a, b), (b, a)\}$
Partition $\{\{a, c\}, \{b\}\}$: Relations: $\{(a, a), (b, b), (c, c), (a, c), (c, a)\}$
Partition $\{\{b, c\}, \{a\}\}$: Relations: $\{(a, a), (b, b), (c, c), (c, b), (b, c)\}$
Partition $\{a, b, c\}$ Relations: $\{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c), (b, a), (c, a), (c, b)\}$

16

Define a relation R on $\mathbb{Z}$ by declaring that xRy iff $x^2 \equiv y^2 \pmod 4$. Prove that R is reflexive, symmetric, and transitive.
$R = \left\{(x, y) \in Z : x^2 \equiv y^2 (\text{mod } 4)\right\}$ which is logically equivalent to $R = \{(x, y) \in Z : x \equiv y (\text{mod } 2)\}$

Proof of reflexivity:
$x^2 - x^2 = 0$ since $4|0$ $x^2 = x^2$
Symmetry: If $x^2 - y^2 = 4k$, then multiplying the equation by -1, we get $y^2 - x^2 = -4k$ Since $4|4k$ and $4|-4k$, the relation is symmetric
Transitivity: If $x^2 - y^2 = 4k$ and $y^2 - z^2 = 4l$, then $y^2 = z^2 + 4l$. Combining them, $x^2 - z^2 - 4l = 4k \rightarrow x^2 - z^2 = 4(k + l)$. Since $4|4(k + l)$, the relation is transitive.

11.4

2

List all the partitions of the set $A = \{a, b, c\}$. Compare your answer to the answer to Exercise 6 of Section 11.3.
Partition $\{\{a\}, \{b\}, \{c\}\}$
Partition $\{\{a, b\}, \{c\}\}$
Partition $\{\{a, c\}, \{b\}\}$
Partition $\{\{b, c\}, \{a\}\}$
Partition $\{a, b, c\}$

6

Consider the partition $P = \big\{\{0\}, \{\pm 1\}, \{\pm 2\}, \{\pm 3\}, \{\pm 4\}, \dots\big\}$ of $\mathbb{Z}$. Describe the equivalence relation whose equivalence classes are the elements of P.
$R = \big\{(-x, x) : x \in \mathbb{Z}^{\nvDash}\big\}, |x| = |y| + \{0\}$
Equivalent to $y = -x, y \in \mathbb{N}$