

Homework 3

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4:

2

If x is an odd integer, then x^3 is odd.

Lets define an odd integer as an integer that can be represented as $2k + 1$ where $k \in \mathbb{Z}$

We have to prove $(2k + 1)^3$ is odd. Doing the binomial expansion we get $8k^3 + 12k^2 + 6k + 1$

$8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$ where $m = 4k^3 + 6k^2 + 3k \in \mathbb{Z}$, making $x^3 = 2m + 1 \checkmark$

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Suppose $x, y \in \mathbb{Z}$. If x and y are odd then xy is odd.

Let x be an integer defined as $2k + 1$ and y be an integer defined as $2l + 1$

$(2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$. Let $m = 2kl + k + l$ making $xy = 2m + 1$

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Suppose x and y are positive real numbers. If $x < y$, then $x^2 < y^2$

$x < y$ multiply both sides by x to get $x^2 < xy$

$x < y$ multiply both sides by y to get $xy < y^2$

Combine them to get $x^2 < xy < y^2$ by transitive property $x^2 < y^2$

5:

6

Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$

The contrapositive states if $x \leq -1$ then $x^3 - x \leq 0$

$x = k : k \in \mathbb{R}, k \leq -1$

$x^3 - x = k^3 - k = k(k^2 - 1)$ Since $k \leq -1$, $k^2 - 1$ is also a nonnegative number.

$k \leq -1 \rightarrow k < 0$ meaning k is negative

For $k = -1$, $k^3 - k = 0$ and for $k < -1$, $k^3 - k < 0$

Meaning $k^3 - k \leq 0$ proving the contrapositive

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Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.

Contrapositive: If a is not odd, then a^2 is divisible by 4

Proof: Define integer $a = 2k$ as an even integer where $k \in \mathbb{Z}$. $(2k)^2 = 4k^2$

$m = k^2$ so $(2k)^2 = 4m$, $m \in \mathbb{Z}$, making all non-odd integers evenly divisible by 4

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Suppose $x, y \in \mathbb{Z}$. If $x + y$ is even, then x and y have the same parity. Parity is defined as the status of being even/odd

Contrapositive: If x and y have opposite parities, then $x + y$ will be odd

Define $2k$ as the integer with the even parity, and $2l + 1$ as the integer with the odd parity. $x, y \in \mathbb{Z}$

Adding up $2k + 2l + 1$, we get $2(k + l) + 1$, an odd number, where the remainder mod 2 is 1, proving the contrapositive

6:

2

Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

Assume the opposite: n^2 is odd and n is even.

Disprove: Assume $n = 2k$. $(2k)^2 = 4k^2$, which will always be an even number by virtue of even coefficient of 4

So there exists a contradiction between the premise and conclusion where n will always be even if n^2 is even

4

Prove that $\sqrt{6}$ is irrational.

Assume $\sqrt{6} \in \mathbb{Q}$

For this to be true: $\exists p, q \in \mathbb{Z}; \gcd(p, q) = 1$

Then $\frac{p}{q} = \sqrt{6} \Rightarrow \frac{p^2}{q^2} = 6 \Rightarrow p^2 = 6q^2$

Since $6q^2$ is even, assume p^2 is also even, meaning $p = 2r \Rightarrow 4r^2 = 6q^2 \Rightarrow 2r^2 = 3q^2$

Since $2r^2$ is even, $3q^2$ is even too, meaning q^2 is even, and q is even, meaning $\gcd(p, q) \geq 2$, making this a contradiction

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There exist no integers a and b for which $21a + 30b = 1$

Assume the opposite: There exist integers a and b for which $21a + 30b = 1$

Write $21a + 30b = 1$ as $3(7a + 10b) = 1$.

Use $n = 7a + 10b$. For the summation to be true, $3n = 1, n \in \mathbb{Z}$ would need to be true, which it is not for any integer, making this a contradiction

11.1

6

Congruence modulo 5 is a relation on the set $A = \mathbb{Z}$. In this relation xRy means $x \equiv y \pmod{5}$. Write out the set R in set-builder notation.

$$R = \{(x, y) \in \mathbb{Z}^2 : 5|(x - y)\}$$

10

Consider the subset $R = (\mathbb{R} \times \mathbb{R}) - \{(x, x) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. What familiar relation on \mathbb{R} is this? Explain. This is $y \neq x$, where its just a line where both coordinates are not the same value.

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In the following exercises, subsets R of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ or $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ are indicated by gray shading. In each case, R is a familiar relation on \mathbb{R} or \mathbb{Z} . State it for the figure labeled “Problem 14.”

It's $R = \{(x, y) : x, y \in \mathbb{R}, y > x\}$

11.2

6

Consider the relation $R = \{(x, x) : x \in \mathbb{Z}\}$ on \mathbb{Z} . Is R reflexive? symmetric? transitive? If a property does not hold, say why. What familiar relation is this?

Well this is the points for the line $y = x$ along the integers. It holds reflexivity and symmetry, and hold transitivity too. If $(a, b) \in R$ and $(b, c) \in R$ then $a = b = c$ making $(a, c) \in R$

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Define a relation on \mathbb{Z} by xRy if $|x - y| < 1$. Is R reflexive? symmetric? transitive? If a property does not hold, say why. What familiar relation is this?

$R = \{(x, y) : x, y \in \mathbb{Z} \times \mathbb{Z}, |x - y| < 1\}$

For absolute value of a difference to be strictly below 1 on the bounds of integers, the difference must equal 0, making this another $y = x, y \in \mathbb{Z}$. It holds reflexivity, symmetry, and transitivity too

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Suppose R is a symmetric and transitive relation on a set A , and there is an element $a \in A$ for which aRx for every $x \in A$. Prove that R is reflexive.

Assume any values a and b . By symmetry, both $(a, b) \in R$ and $(b, a) \in R$

Since aRx for all x , and R is symmetric, by transitivity xRx and is reflexive

11.3

2

Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A . Suppose R has two equivalence classes. Also aRd , bRc and eRd . Write out R as a set.

$\{\{a, d, e\}, \{b, c\}\}$

Relations: $\{(a, a), (a, d), (a, e), (d, e), (d, a), (d, d), (e, e), (e, a), (e, d), (b, b), (b, c), (c, b), (c, c)\}$

6

There are five different equivalence relations on the set $A = \{a, b, c\}$. Describe them all. (Diagrams will suffice.)

Partition $\{\{a\}, \{b\}, \{c\}\}$: Relations $\{(a, a), (b, b), (c, c)\}$

Partition $\{\{a, b\}, \{c\}\}$: Relations: $\{(a, a), (b, b), (c, c), (a, b), (b, a)\}$

Partition $\{\{a, c\}, \{b\}\}$: Relations: $\{(a, a), (b, b), (c, c), (a, c), (c, a)\}$

Partition $\{\{b, c\}, \{a\}\}$: Relations: $\{(a, a), (b, b), (c, c), (c, b), (b, c)\}$

Partition $\{a, b, c\}$ Relations: $\{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c), (b, a), (c, a), (c, b)\}$

16

Define a relation R on \mathbb{Z} by declaring that xRy iff $x^2 \equiv y^2 \pmod{4}$. Prove that R is reflexive, symmetric, and transitive.

$R = \{(x, y) \in \mathbb{Z} : x^2 \equiv y^2 \pmod{4}\}$ which is logically equivalent to $R = \{(x, y) \in \mathbb{Z} : x \equiv y \pmod{2}\}$

Proof of reflexivity:

$$x^2 - x^2 = 0 \text{ since } 4|0 \quad x^2 = x^2$$

Symmetry: If $x^2 - y^2 = 4k$, then multiplying the equation by -1, we get $y^2 - x^2 = -4k$. Since $4|4k$ and $4|-4k$, the relation is symmetric.

Transitivity: If $x^2 - y^2 = 4k$ and $y^2 - z^2 = 4l$, then $y^2 = z^2 + 4l$. Combining them, $x^2 - z^2 - 4l = 4k \rightarrow x^2 - z^2 = 4(k+l)$. Since $4|4(k+l)$, the relation is transitive.

11.4

2

List all the partitions of the set $A = \{a, b, c\}$. Compare your answer to the answer to Exercise 6 of Section 11.3.

Partition $\{\{a\}, \{b\}, \{c\}\}$

Partition $\{\{a, b\}, \{c\}\}$

Partition $\{\{a, c\}, \{b\}\}$

Partition $\{\{b, c\}, \{a\}\}$

Partition $\{a, b, c\}$

6

Consider the partition $P = \{\{0\}, \{\pm 1\}, \{\pm 2\}, \{\pm 3\}, \{\pm 4\}, \dots\}$ of \mathbb{Z} . Describe the equivalence relation whose equivalence classes are the elements of P .

$$R = \{(-x, x) : x \in \mathbb{Z}^\neq\}, |x| = |y| + \{0\}$$

Equivalent to $y = -x, y \in \mathbb{N}$