

# Homework 3

Tejas Patel

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**4:**

**2**

If  $x$  is an odd integer, then  $x^3$  is odd.

Lets define an odd integer as an integer that acn be represented as  $2k + 1$  where  $k \in \mathbb{Z}$

We have to prove  $(2k + 1)^3$  is odd. Doing the binomial expansion we get  $8k^3 + 12k^2 + 6k + 1$

$8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$  where  $m = 4k^3 + 6k^2 + 3k \in \mathbb{Z}$ , making  $x^3 = 2m + 1$  ✓

**4**

Suppose  $x, y \in \mathbb{Z}$ . If  $x$  and  $y$  are odd then  $xy$  is odd.

Let  $x$  be an integer defined as  $2k + 1$  and  $y$  be an integer defined as  $2l + 1$

$(2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$ . Let  $m = 2kl + k + l$  making  $xy = 2m + 1$

**18**

Suppose  $x$  and  $y$  are positive real numbers. If  $x < y$ , then  $x^2 < y^2$

$x < y$  multiply both sides by  $x$  to get  $x^2 < xy$

$x < y$  multiply both sides by  $y$  to get  $xy < y^2$

Combine them to get  $x^2 < xy < y^2$  by transitive property  $x^2 < y^2$

**5:**

**6**

Suppose  $x \in \mathbb{R}$ . If  $x^3 - x > 0$  then  $x > -1$

The contrapositive states if  $x \leq -1$  then  $x^3 - x \leq 0$

$x = k : k \in \mathbb{R}, k \leq -1$

$x^3 - x = k^3 - k = k(k^2 - 1)$  Since  $k \leq -1$ ,  $k^2 - 1$  is also a nonnegative number.

$k \leq -1 \rightarrow k < 0$  meaning  $k$  is negative

For  $k = -1, k^3 - k = 0$  and for  $k < -1, k^3 - k < 0$

Meaning  $k^3 - k \leq 0$  proving the contrapositive

**12**

Suppose  $a \in \mathbb{Z}$ . If  $a^2$  is not divisible by 4, then  $a$  is odd.

Contrapositive: If  $a$  is not odd, then  $a^2$  is divisible by 4

Proof: Define integer  $a = 2k$  as an even integer where  $k \in \mathbb{Z}$ .  $(2k)^2 = 4k^2$

$m = k^2$  so  $(2k)^2 = 4m, m \in \mathbb{Z}$ , making all non-odd integers evenly divisible by 4

## 16

Suppose  $x, y \in \mathbb{Z}$ . If  $x + y$  is even, then  $x$  and  $y$  have the same parity. Parity is defined as the status of being even/odd

Contrapositive: If  $x$  and  $y$  have opposite parities, then  $x + y$  will be odd

Define  $2k$  as the integer with the even parity, and  $2l + 1$  as the integer with the odd parity.  $x, y \in \mathbb{Z}$

Adding up  $2k + 2l + 1$ , we get  $2(k + l) + 1$ , an odd number, where the remainder mod 2 is 1, proving the contrapositive

## 6:

### 2

Suppose  $n \in \mathbb{Z}$ . If  $n^2$  is odd, then  $n$  is odd.

Assume the opposite:  $n^2$  is odd and  $n$  is even.

Disprove: Assume  $n = 2k$ .  $(2k)^2 = 4k^2$ , which will always be an even number by virtue of even coefficient of 4

So there exists a contradiction between the premise and conclusion where  $n$  will always be even if  $n^2$  is even

### 4

Prove that  $\sqrt{6}$  is irrational.

Assume  $\sqrt{6} \in \mathbb{Q}$

For this to be true:  $\exists p, q \in \mathbb{Z}; \gcd(p, q) = 1$

Then  $\frac{p}{q} = \sqrt{6} \Rightarrow \frac{p^2}{q^2} = 6 \Rightarrow p^2 = 6q^2$

Since  $6q^2$  is even, assume  $p^2$  is also even, meaning  $p = 2r \Rightarrow 4r^2 = 6q^2 \Rightarrow 2r^2 = 3q^2$

Since  $2r^2$  is even,  $3q^2$  is even too, meaning  $q^2$  is even, and  $q$  is even, meaning  $\gcd(p, q) \geq 2$ , making this a contradiction

## 10

There exist no integers  $a$  and  $b$  for which  $21a + 30b = 1$

Assume the opposite: There exist integers  $a$  and  $b$  for which  $21a + 30b = 1$

Write  $21a + 30b = 1$  as  $3(7a + 10b) = 1$ .

Use  $n = 7a + 10b$ . For the summation to be true,  $3n = 1, n \in \mathbb{Z}$  would need to be true, which it is not for any integer, making this a contradiction

## 11.1

### 6

Congruence modulo 5 is a relation on the set  $A = \mathbb{Z}$ . In this relation  $xRy$  means  $x \equiv y \pmod{5}$ . Write out the set  $R$  in set-builder notation.

$R = \{(x, y) \in \mathbb{Z}^2 : 5 \mid (x - y)\}$

### 10

Consider the subset  $R = (\mathbb{R} \times \mathbb{R}) - \{(x, x) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$ . What familiar relation on  $\mathbb{R}$  is this? Explain. This is  $y \neq x$ , where its just a line where both coordinates are not the same value.

## 14

In the following exercises, subsets  $R$  of  $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$  or  $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$  are indicated by gray shading. In each case,  $R$  is a familiar relation on  $\mathbb{R}$  or  $\mathbb{Z}$ . State it for the figure labeled “Problem 14.”

It's  $R = \{(x, y) : x, y \in \mathbb{R}, y > x\}$

## 11.2

### 6

Consider the relation  $R = \{(x, x) : x \in \mathbb{Z}\}$  on  $\mathbb{Z}$ . Is  $R$  reflexive? symmetric? transitive? If a property does not hold, say why. What familiar relation is this?

Well this is the points for the line  $y = x$  along the integers. It holds reflexivity and symmetry, and hold transitivity too. If  $(a, b) \in R$  and  $(b, c) \in R$  then  $a = b = c$  making  $(a, c) \in R$

### 8

Define a relation on  $\mathbb{Z}$  by  $xRy$  if  $|x - y| < 1$ . Is  $R$  reflexive? symmetric? transitive? If a property does not hold, say why. What familiar relation is this?

$R = \{(x, y) : x, y \in \mathbb{Z} \times \mathbb{Z}, |x - y| < 1\}$

For absolute value of a difference to be strictly below 1 on the bounds of integers, the difference must equal 0, making this another  $y = x, y \in \mathbb{Z}$ . It holds reflexivity, symmetry, and transitivity too

## 14

Suppose  $R$  is a symmetric and transitive relation on a set  $A$ , and there is an element  $a \in A$  for which  $aRx$  for every  $x \in A$ . Prove that  $R$  is reflexive.

Assume any values  $a$  and  $b$ . By symmetry, both  $(a, b) \in R$  and  $(b, a) \in R$

Since  $aRx$  for all  $x$ , and  $R$  is symmetric, by transitivity  $xRx$  and is reflexive

## 11.3

### 2

Let  $A = \{a, b, c, d, e\}$ . Suppose  $R$  is an equivalence relation on  $A$ . Suppose  $R$  has two equivalence classes. Also  $aRd$ ,  $bRc$  and  $eRd$ . Write out  $R$  as a set.

$\{\{a, d, e\}, \{b, c\}\}$

Relations:  $\{(a, a), (a, d), (a, e), (d, e), (d, a), (d, d), (e, e), (e, a), (e, d), (b, b), (b, c), (c, b), (c, c)\}$

### 6

There are five different equivalence relations on the set  $A = \{a, b, c\}$ . Describe them all. (Diagrams will suffice.)

Partition  $\{\{a\}, \{b\}, \{c\}\}$ : Relations  $\{(a, a), (b, b), (c, c)\}$

Partition  $\{\{a, b\}, \{c\}\}$ : Relations:  $\{(a, a), (b, b), (c, c), (a, b), (b, a)\}$

Partition  $\{\{a, c\}, \{b\}\}$ : Relations:  $\{(a, a), (b, b), (c, c), (a, c), (c, a)\}$

Partition  $\{\{b, c\}, \{a\}\}$ : Relations:  $\{(a, a), (b, b), (c, c), (c, b), (b, c)\}$

Partition  $\{a, b, c\}$  Relations:  $\{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c), (b, a), (c, a), (c, b)\}$

## 16

Define a relation  $R$  on  $\mathbb{Z}$  by declaring that  $xRy$  iff  $x^2 \equiv y^2 \pmod{4}$ . Prove that  $R$  is reflexive, symmetric, and transitive.

$R = \{(x, y) \in \mathbb{Z} : x^2 \equiv y^2 \pmod{4}\}$  which is logically equivalent to  $R = \{(x, y) \in \mathbb{Z} : x \equiv y \pmod{2}\}$

Proof of reflexivity:

$$x^2 - x^2 = 0 \text{ since } 4|0 \quad x^2 = x^2$$

Symmetry: If  $x^2 - y^2 = 4k$ , then multiplying the equation by -1, we get  $y^2 - x^2 = -4k$ . Since  $4|4k$  and  $4|-4k$ , the relation is symmetric.

Transitivity: If  $x^2 - y^2 = 4k$  and  $y^2 - z^2 = 4l$ , then  $y^2 = z^2 + 4l$ . Combining them,  $x^2 - z^2 - 4l = 4k \rightarrow x^2 - z^2 = 4(k + l)$ . Since  $4|4(k + l)$ , the relation is transitive.

## 11.4

### 2

List all the partitions of the set  $A = \{a, b, c\}$ . Compare your answer to the answer to Exercise 6 of Section 11.3.

Partition  $\{\{a\}, \{b\}, \{c\}\}$

Partition  $\{\{a, b\}, \{c\}\}$

Partition  $\{\{a, c\}, \{b\}\}$

Partition  $\{\{b, c\}, \{a\}\}$

Partition  $\{a, b, c\}$

### 6

Consider the partition  $P = \{\{0\}, \{\pm 1\}, \{\pm 2\}, \{\pm 3\}, \{\pm 4\}, \dots\}$  of  $\mathbb{Z}$ . Describe the equivalence relation whose equivalence classes are the elements of  $P$ .

$$R = \{(-x, x) : x \in \mathbb{Z}^\times\}, |x| = |y| + \{0\}$$

Equivalent to  $y = -x, y \in \mathbb{N}$