On the implementation of an optimal detector in a computational auditory model

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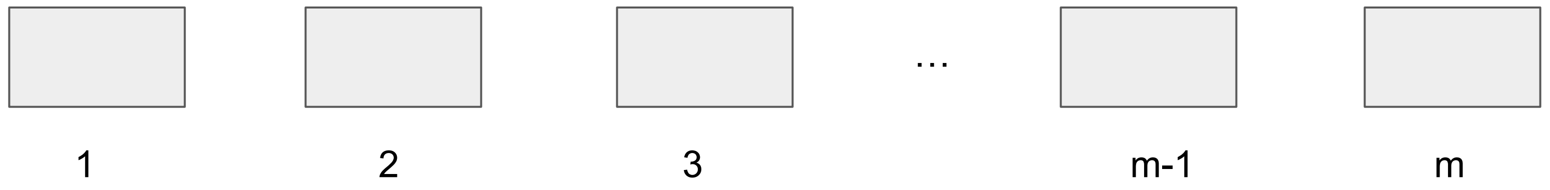
Utility of computational auditory modeling

Why do we need computational modelling in hearing research?

- Powerful tool for understanding the functioning of the human auditory system
- Is intended to simulate human behavior in response to acoustic stimuli

Tone-in-noise detection experiment within *m*-AFC framework

- *m* sound intervals



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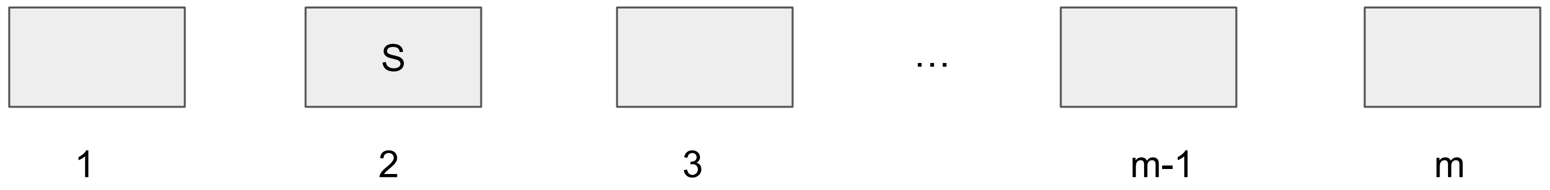
- 1 random interval contains target signal embedded in noise (**S**)



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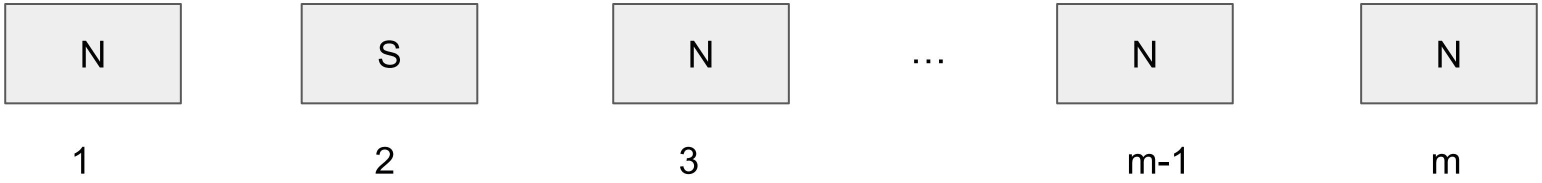

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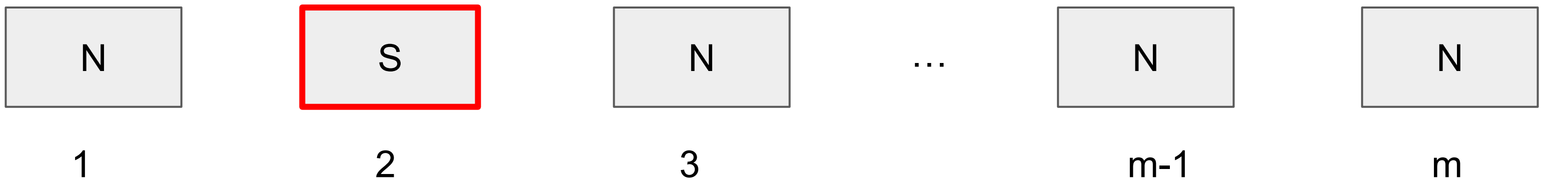

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- *m* sound intervals
- 1 random interval contains target signal embedded in noise (**S**)
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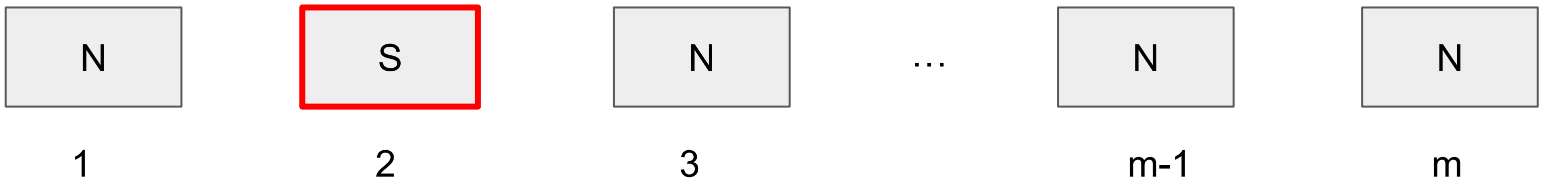
- Tracking rule: e.g. 1-up-2-down (Levitt, 1971) $\rightarrow \quad \widehat{P_m(C)}$



Model Structure

Acoustic Stimuli

N

S

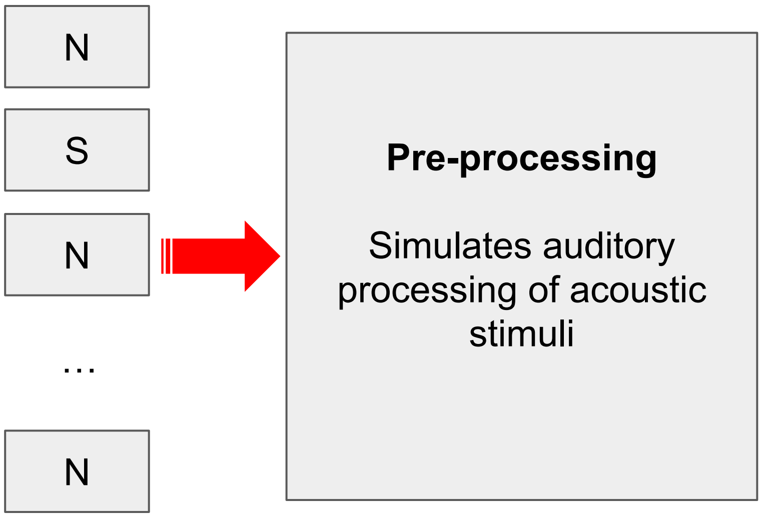
N

…

N

Model Structure

Acoustic Stimuli



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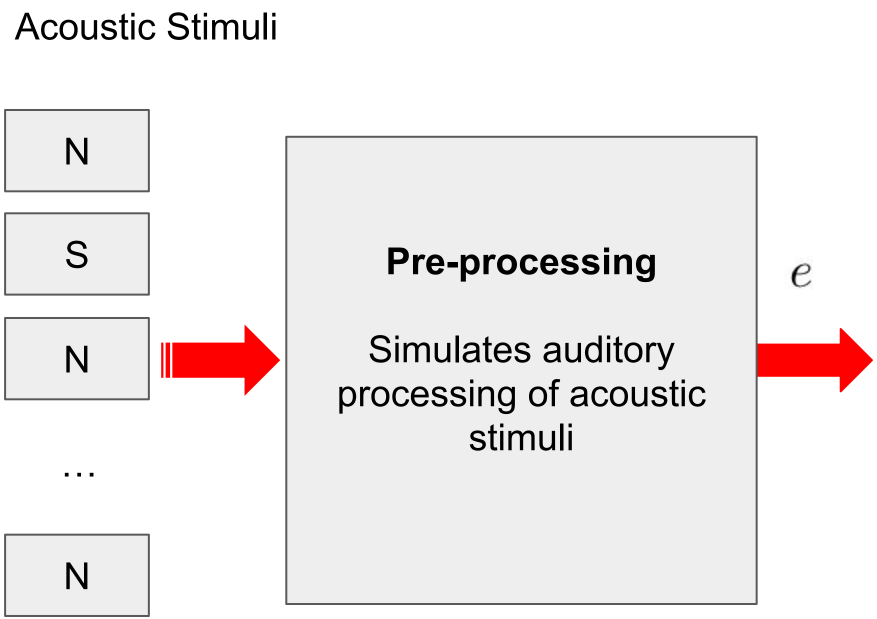

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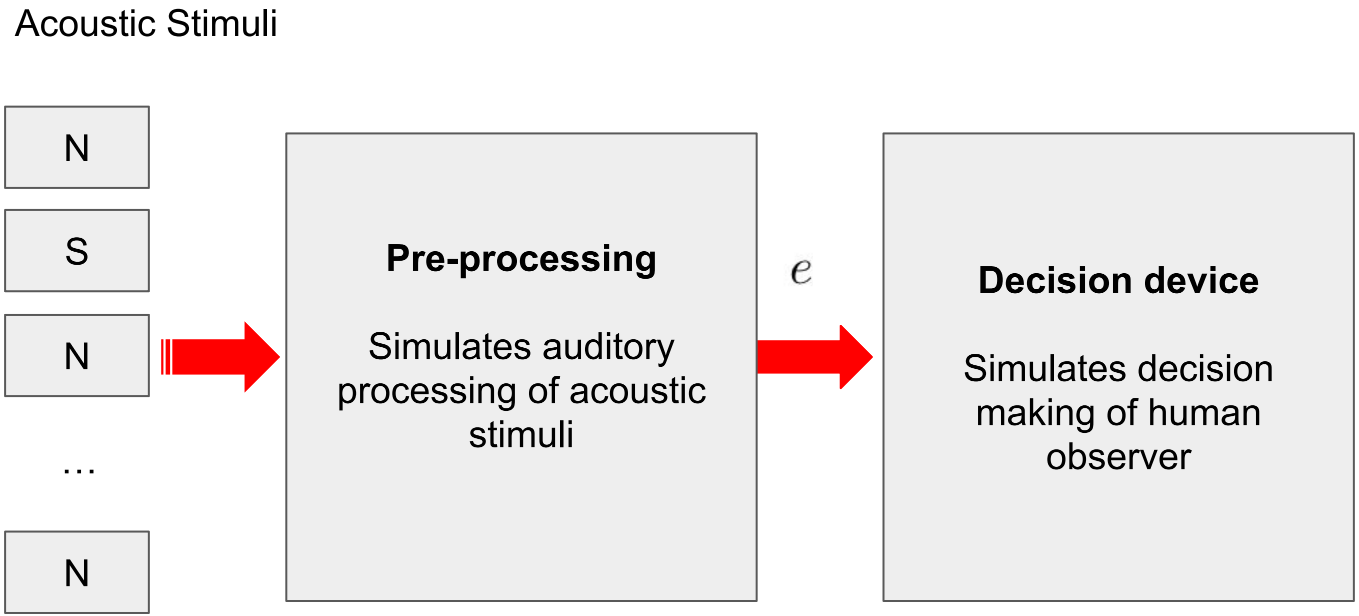

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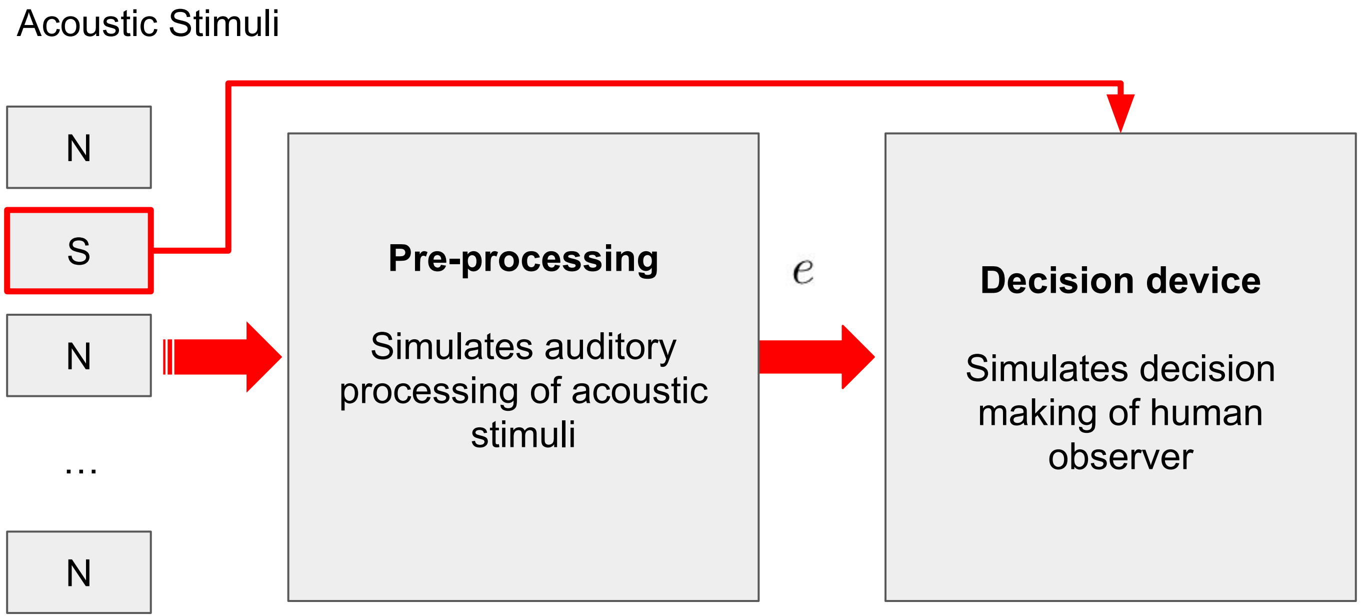

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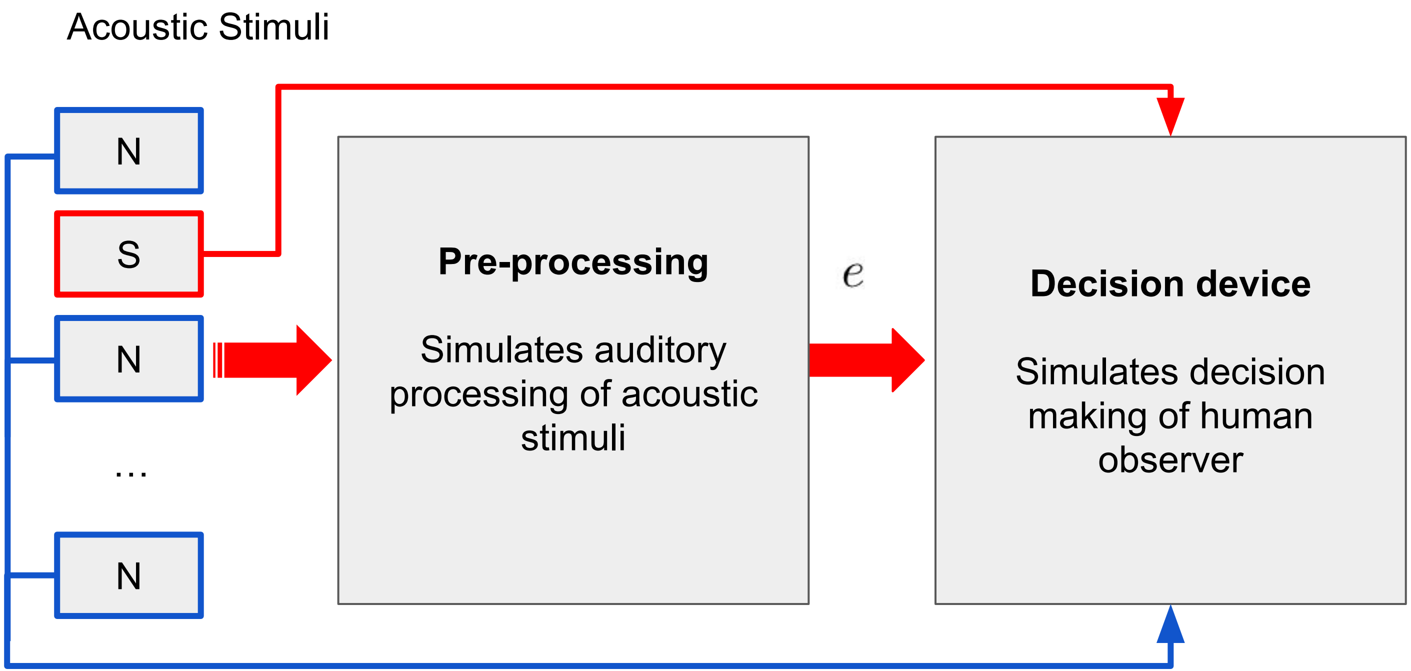

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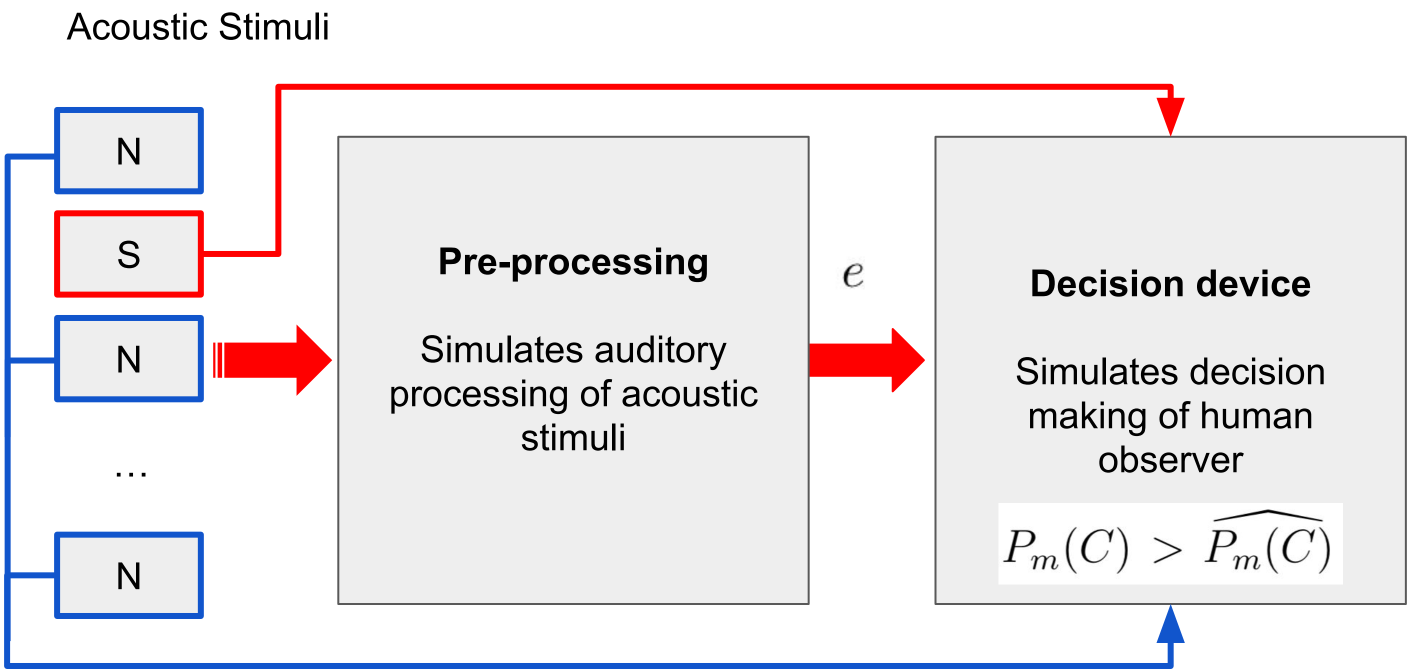

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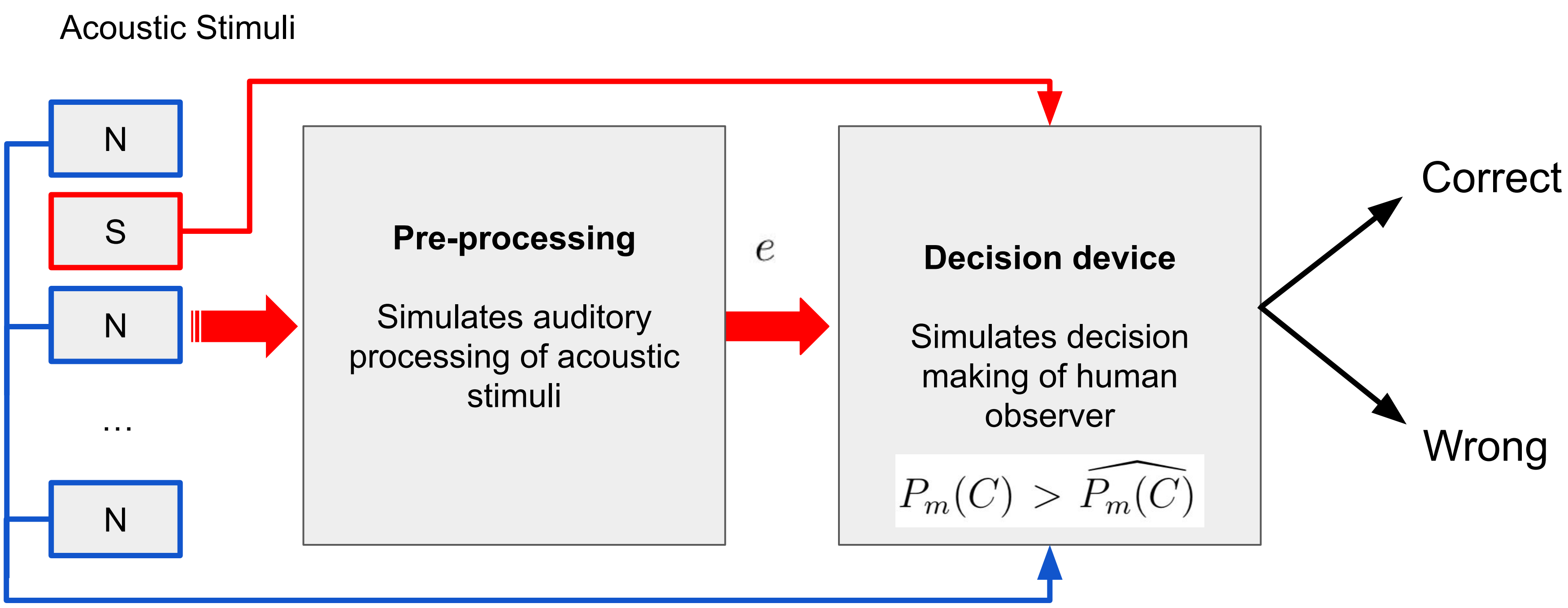
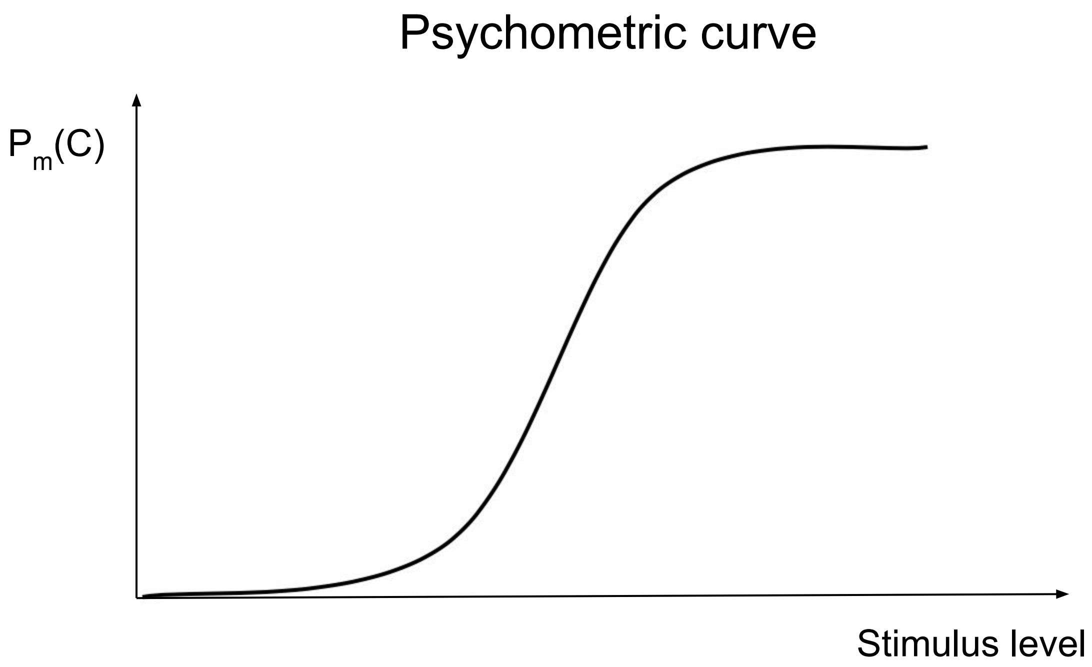
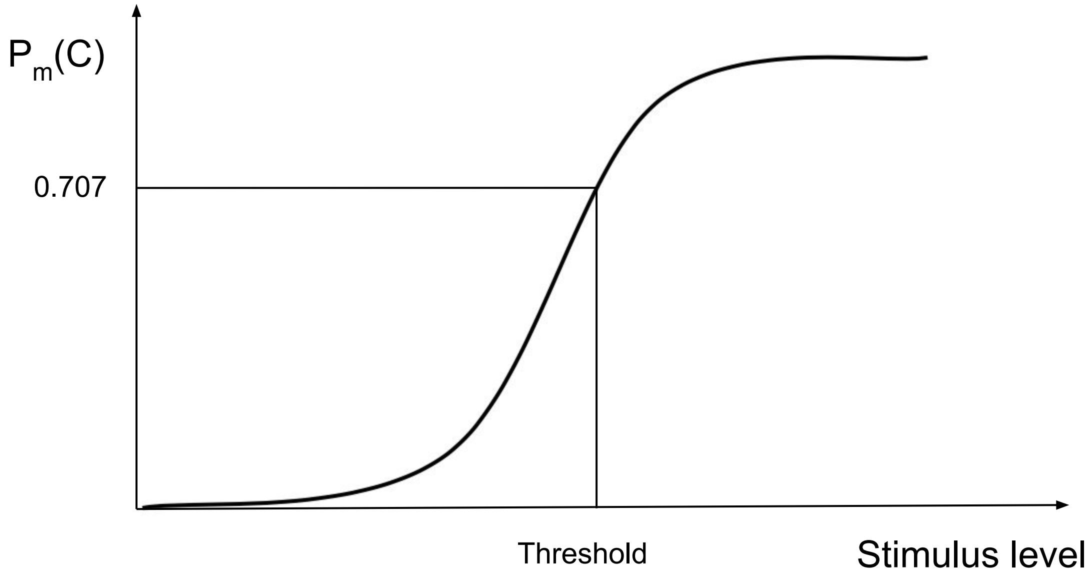


Decision making: 1-up-2-down



Decision making: 1-up-2-down

Psychometric curve



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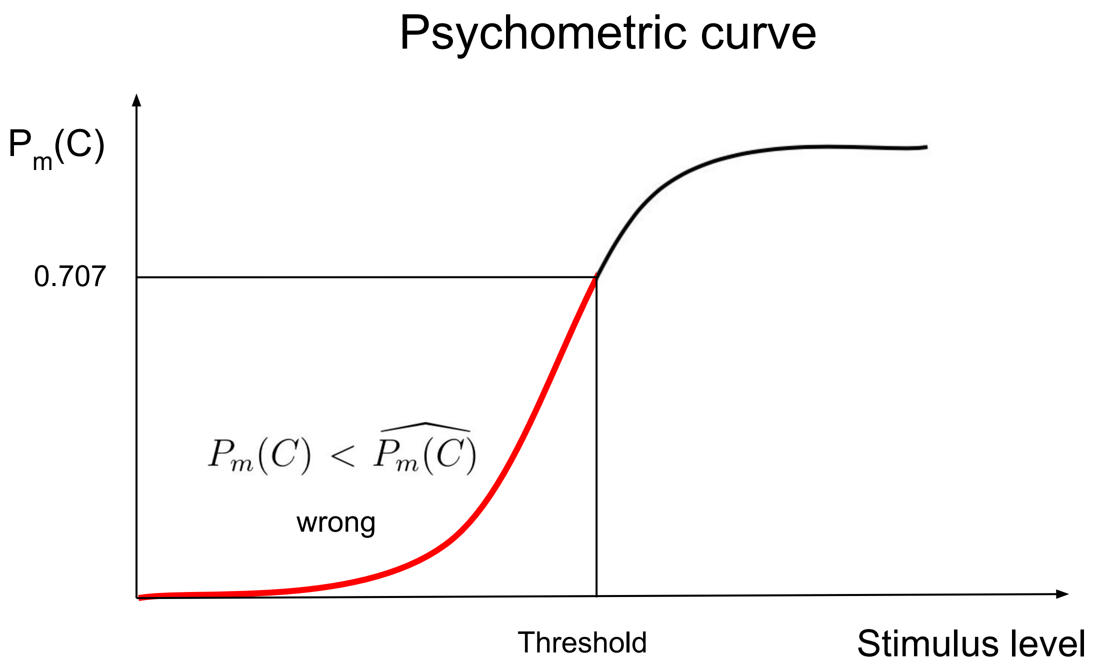

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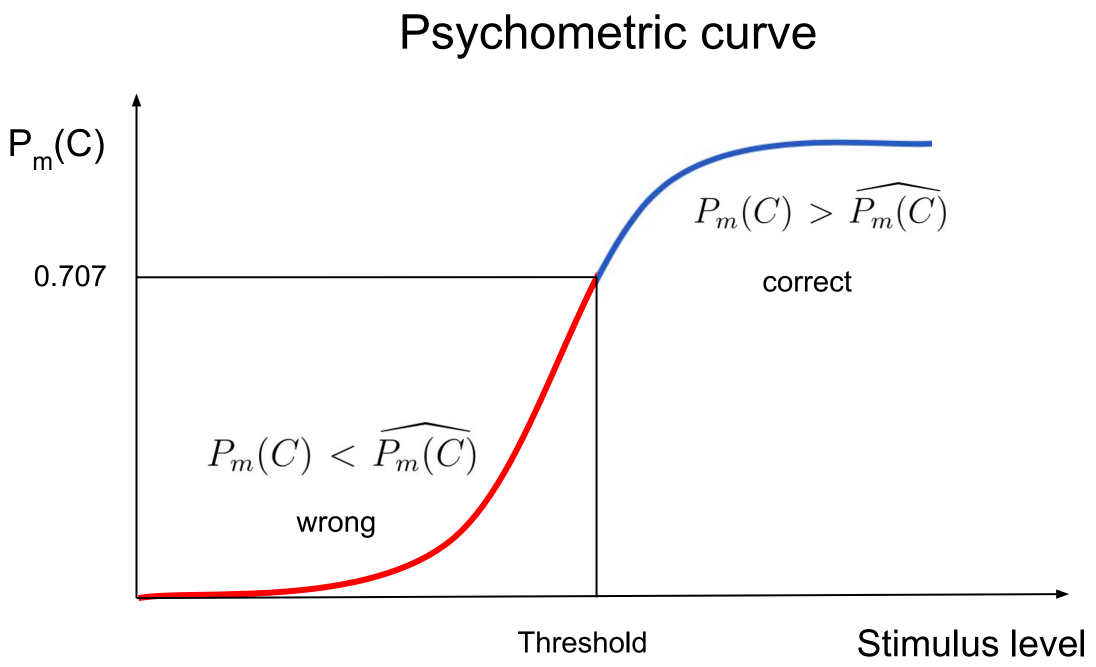

Template of the target stimulus

- Beginning of the experiment: highly detectable tone (high sound level)

e_S

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e_S e_N

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$$e_S - e_N$$

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$$e_S - e_N = s \longrightarrow$$ **template:** auditory image of “what to listen for”

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- Beginning of the experiment: highly detectable tone (high sound level)

$$e_S - e_N = s$$ ➡ **template:** auditory image of “what to listen for”

- All other experimental trials:

$$\delta_i = e_i - e_N \qquad i = 1, ..., m$$ ➡ **difference signal**

Statistical properties of the internal representation

Limited fidelity in the auditory system:
uncertainty about the representation of the stimulus

→ Modeled with noise added to δ_i

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$$f(\delta_i|N)$$

$$f(\delta_i|S)$$

Statistical properties of the internal representation

Limited fidelity in the auditory system:
uncertainty about the representation of the stimulus → Modeled with noise added to □$_i$

$$f(\delta_i|N) \rightarrow \mathcal{N}(0, \sigma)$$

$$f(\delta_i|S) \rightarrow \mathcal{N}(s, \sigma)$$

Assumption:
equal variance Gaussian distributions

Likelihood ratio and Decision rule

- Likelihood ratio: a single number to express the evidence of the signal in each interval

$$l(\delta_i) = \frac{f(\delta_i|S)}{f(\delta_i|N)}$$

→ Decision rules based on it are **OPTIMAL**
(any other decision rule will not give a better performance on average)

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$$\ln l(\delta_i) = \frac{1}{\sigma^2}\left\{ \underbrace{\sum_n \delta_i(n) s(n)}_{\text{correlation }(\rho)} - \underbrace{\frac{1}{2}\sum_n s(n)^2}_{\text{constant}} \right\}$$

Likelihood ratio and Decision rule

Correlation: monotonically related to likelihood ratio

Also Gaussian distributed

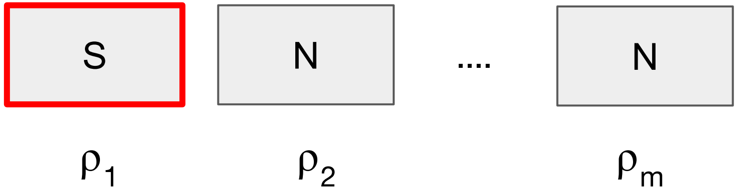
$$\begin{cases} f(\rho_i|N) = \mathcal{N}(0,\sigma) \\ f(\rho_i|S) = \mathcal{N}(d,\sigma) \end{cases}$$

Any decision rule based on correlation is also OPTIMAL

DECISION RULE: select the interval with the highest correlation with the template

The objective: maximize the probability of correct responses

- Equal prior probabilities



- Decision rule: select the interval with the largest correlation with the template

$$\arg\max_{i=1\ldots m} \rho_i$$

$$\boxed{\rho_1 = k}$$

Correct answer if and only if

$$\boxed{\rho_i < k \quad \forall\, i = 2, ..., m}$$

The objective: maximize the probability of correct responses

$$P_m(C) = P(\rho_1 = k, \rho_2 < k, \rho_3 < k, ..., \rho_m < k)$$

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An alternative way to calculate $P_m(C)$

$$\begin{cases} \rho_1 = k = \max\limits_{i=1\ldots m} \rho_i \\ \rho_i < k \quad \forall\, i = 2, \ldots, m \end{cases}$$

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$\rho_i \quad \forall\, i = 2, \dots, m$

- m-1 i.i.d. Gaussian random variables
- their maximum is approximately Gaussian distributed (Cramer, 1946) $\mathcal{N}(d^*, \sigma^*)$

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$$\Rightarrow \max_{i=2\ldots m} \rho_i - \rho_1$$

$$\max_{i=2\ldots m} \rho_i \rightarrow \mathcal{N}(d^*, \sigma^*) \qquad \rho_1 \rightarrow \mathcal{N}(d, \sigma)$$

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$$\Rightarrow \underbrace{\max_{i=2\ldots m} \rho_i}_{\mathcal{N}(d^*, \sigma^*)} - \underbrace{\rho_1}_{\mathcal{N}(d, \sigma)} \Rightarrow \mathcal{N}(d^* - d, \sqrt{\sigma^{*2} + \sigma^2})$$

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$$\gamma = \frac{d^*-d}{\sqrt{\sigma^{*2}+\sigma^2}}$$

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$$\boxed{= \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{\gamma}{\sqrt{2}}\right)\right]} \Rightarrow \quad P_m(C) > \widehat{P_m(C)}$$

Thanks!