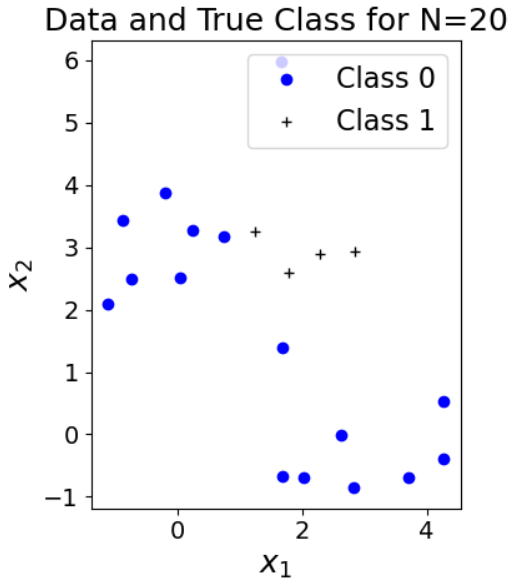
Tyler Cerilli
EECE 5644

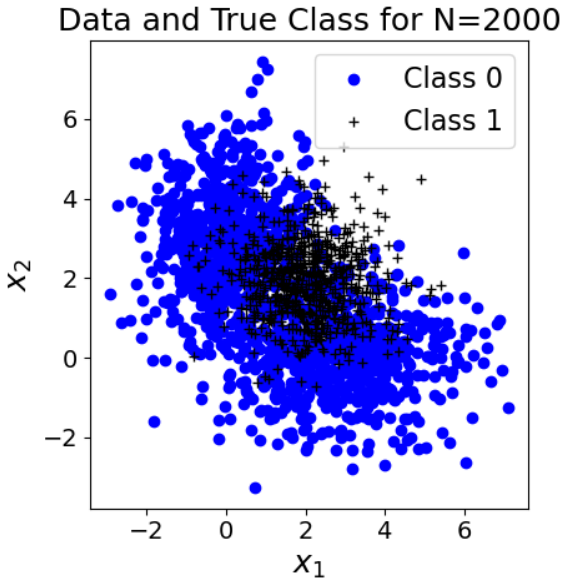
Homework 2

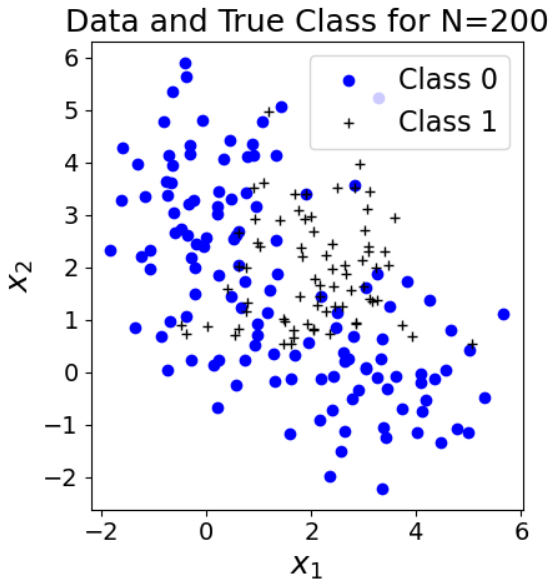
Problem 1

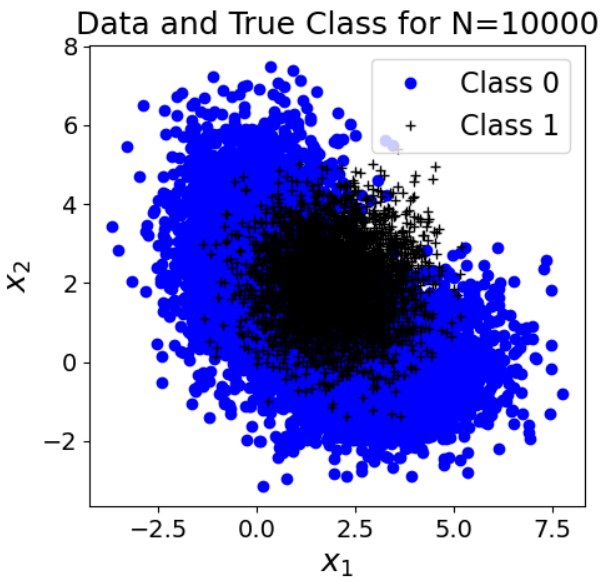
np.random.seed(1)

Number of samples from Class 1: 16, Class 2: 4
Number of samples from Class 1: 127, Class 2: 73
Number of samples from Class 1: 1309, Class 2: 691
Number of samples from Class 1: 6391, Class 2: 3609





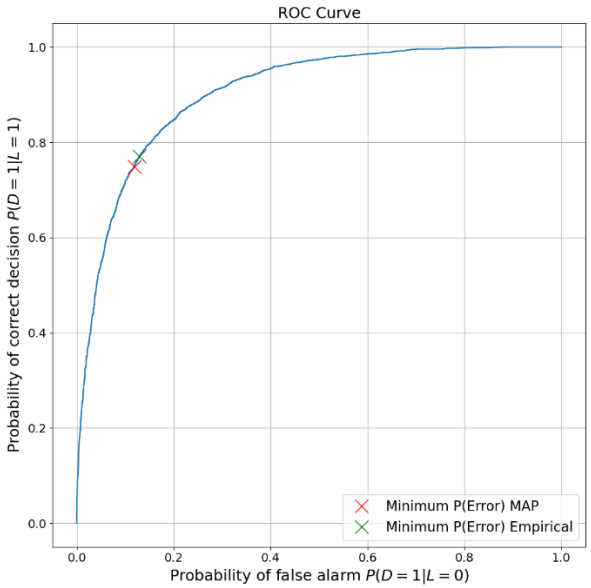


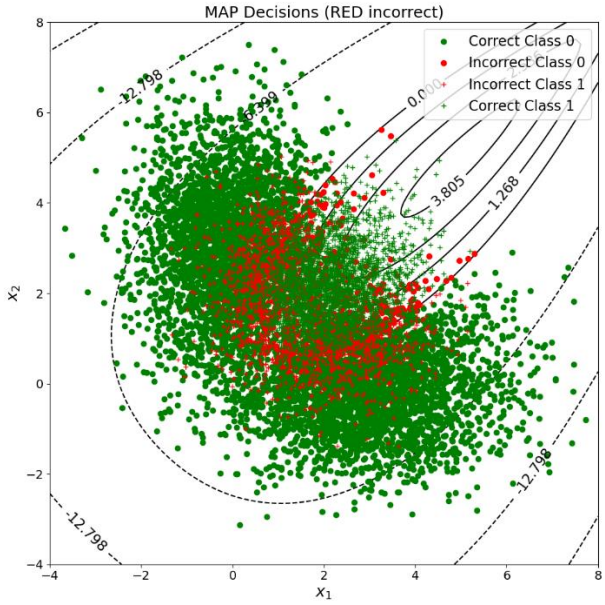


Part 1

Threshold value: 1.8571428571428574
Confusion Matrix MAP (rows: Predicted class, columns: True class):
[[5634 908]
[757 2701]]
Total Number of Misclassified Samples (MAP): 1665

Gamma MAP (Theoretical): 1.8571428571428574
Probability of Error(MAP): 0.16649999999999998
Best Gamma (ERM): 1.649473324917088
Probability of Error(Empirical): 0.1649





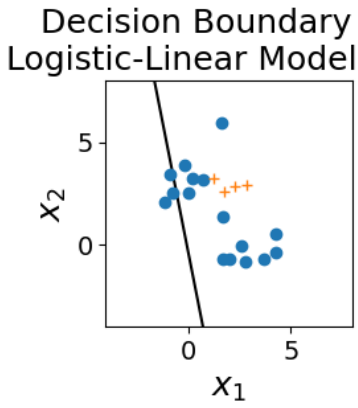
Part 2

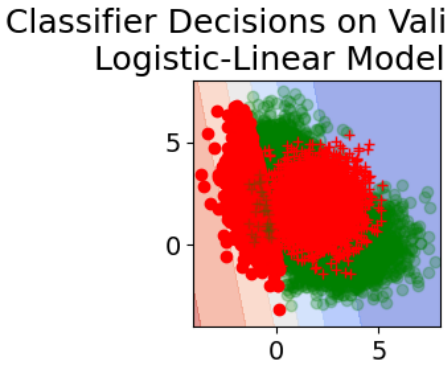
<u>LINEAR</u>
2 batches of size 10:
Logistic-Linear N=20 GD Theta:
[-0.02831339 -0.50197184 -0.09912327]
Logistic-Linear N=20 NLL: 1.2686217324687203
The total error achieved with this classifier is 0.442

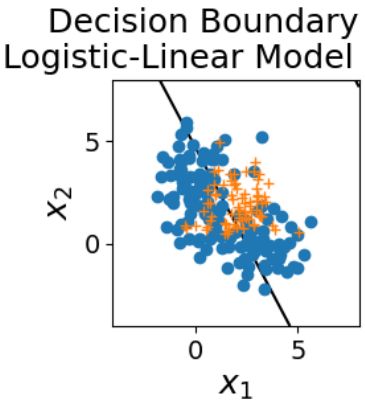
20 batches of size 10:
Logistic-Linear N=200 GD Theta:
[-0.87731068 0.18788255 0.09950977]
Logistic-Linear N=200 NLL: 12.60622593092242
The total error achieved with this classifier is 0.367

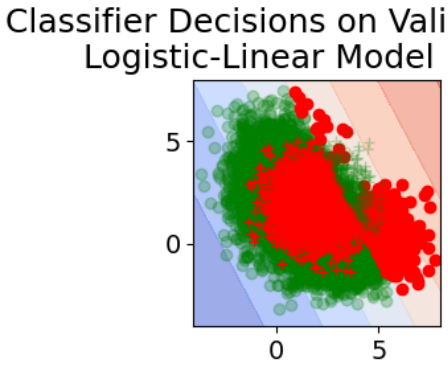
200 batches of size 10:

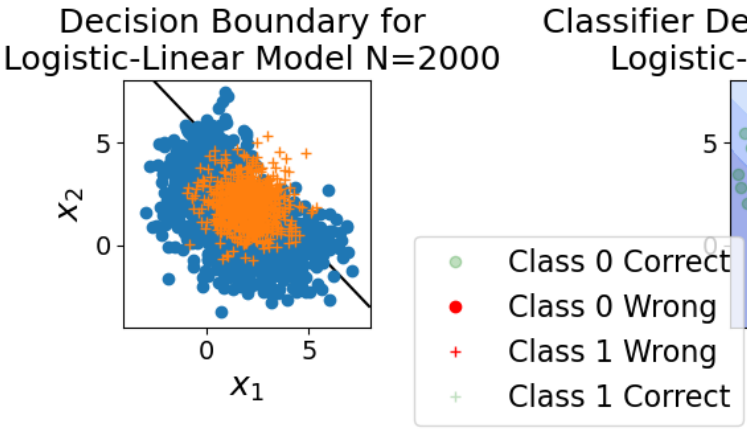
Logistic-Linear N=2000 GD Theta:
[-1.90320918 0.37300914 0.3563989]
Logistic-Linear N=2000 NLL: 121.38572234930577
The total error achieved with this classifier is 0.346







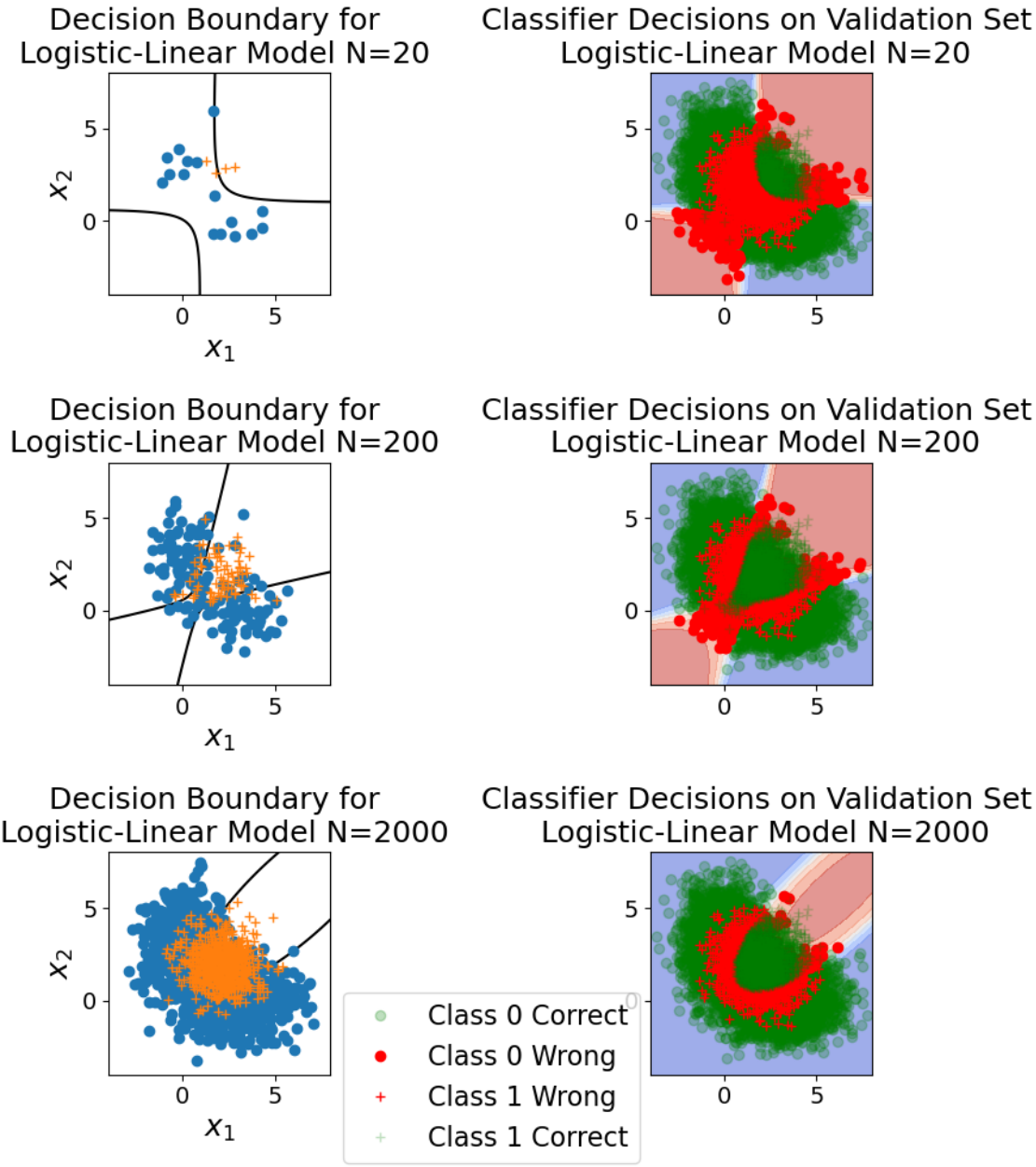




QUADRATIC
2 batches of size 10:
Logistic-Linear N=20 GD Theta:
[0.08970478 -0.81070515 -1.2917546 -0.01122395 1.0294441 -0.0422405]
Logistic-Linear N=20 NLL: 0.4579893514019052
The total error achieved with this classifier is 0.276

20 batches of size 10:
Logistic-Linear N=200 GD Theta:
[0.35070649 -0.18782312 -0.47789395 -0.19579952 0.92950008 -0.20815267]
Logistic-Linear N=200 NLL: 8.668329342392768
The total error achieved with this classifier is 0.196

200 batches of size 10:
Logistic-Linear N=2000 GD Theta:
 [-1.27830878 0.72130575 0.8245824 -0.32236347 0.55910594 -0.36879039]
Logistic-Linear N=2000 NLL: 81.01347096977517
The total error achieved with this classifier is 0.164



Let us look at just the quadratic results, as they are the best, and the same concepts apply to both the quadratic and linear case…
The number of training samples increase the performance of the classifiers drastically. From 20 to 200 to 2000 training samples validated on 10000 samples, the probability of error (performance) decreases from 0.276 (N_{train} =20) to 0.196 (N_{train} =200) to 0.164 (N_{train}=2000). Since we want to mitigate error, a decrease in error is proportional to an increase in performance. The same exact pattern/trend exists in the linear function form as well.

The function form also has a drastic effect and is an especially important component to consider. For both the linear and quadratic case, the best performance is achieved using the highest training set (N_{train}=2000). The linear form achieves a probability of error of 0.346, while the quadratic achieves a probability of error of 0.164. Clearly the quadratic case outperforms the linear case. This is the same across all sizes of training samples tested.

In comparison to the theoretically optimal classifier from Part 1 (p(error) = 0.16649999999999998), the linear form performs much worse, whereas the quadratic form performs better (for this seed). The fits and classifies the data just as well as the theoretically optimal classifier when we have sample set of ~2000 samples.

Problem 2

np.random.seed(7)

10 batches of size 10:

theta start:

[0.20139884 0.80473259 -0.30712783 -0.19619016 -0.69316614 1.56372702
  0.3546326   -0.36475778 -0.32149488 0.10854485]

theta MLE:

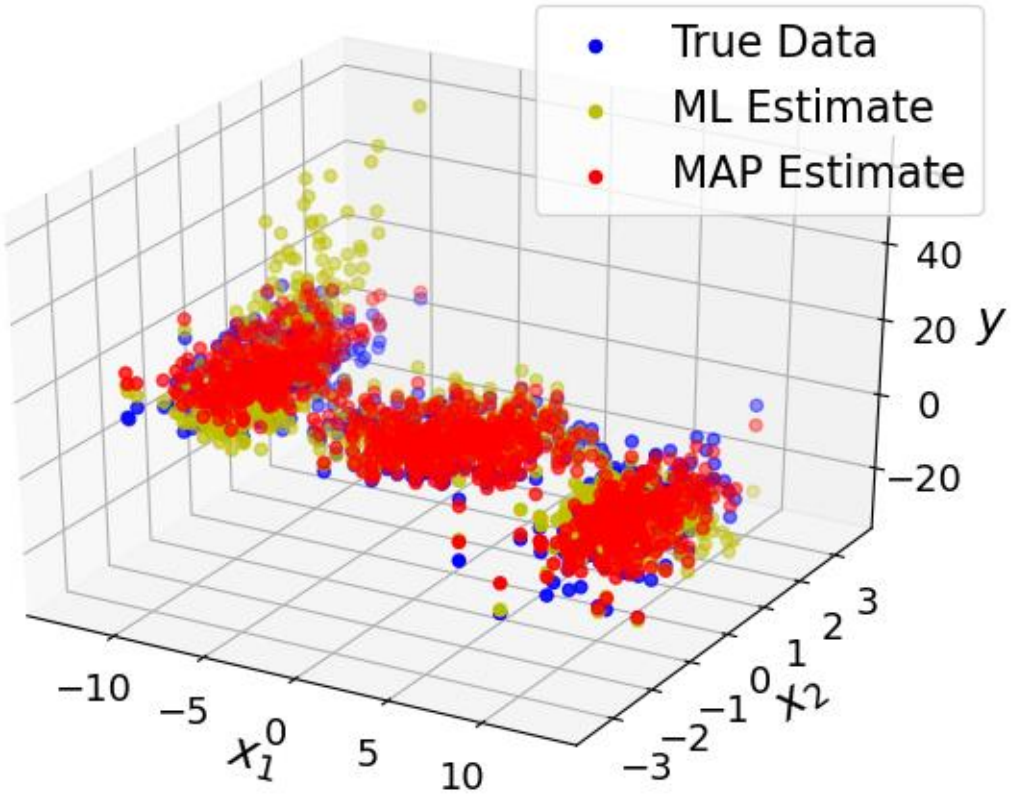
[0.20269365 0.79902827 -0.30291402 -0.00566083 -0.65097462 1.56869389
 -0.01630005 0.02123632 -0.27712615 0.12454508]

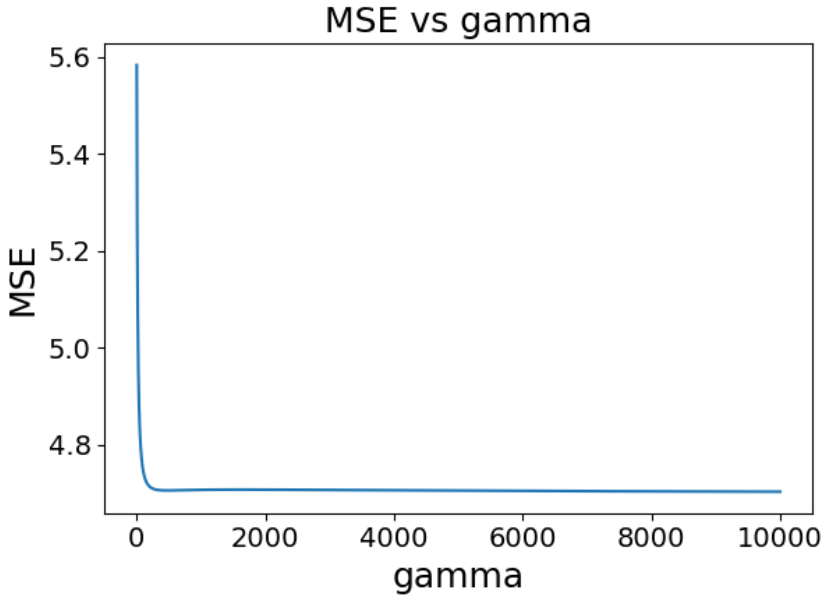
theta MAP:

[-0.3993506   0.03506325 0.2527343 -0.00449423 -0.01386988 0.60370334
 -0.01026672 -0.00096463 -0.0428286 -0.1144727]

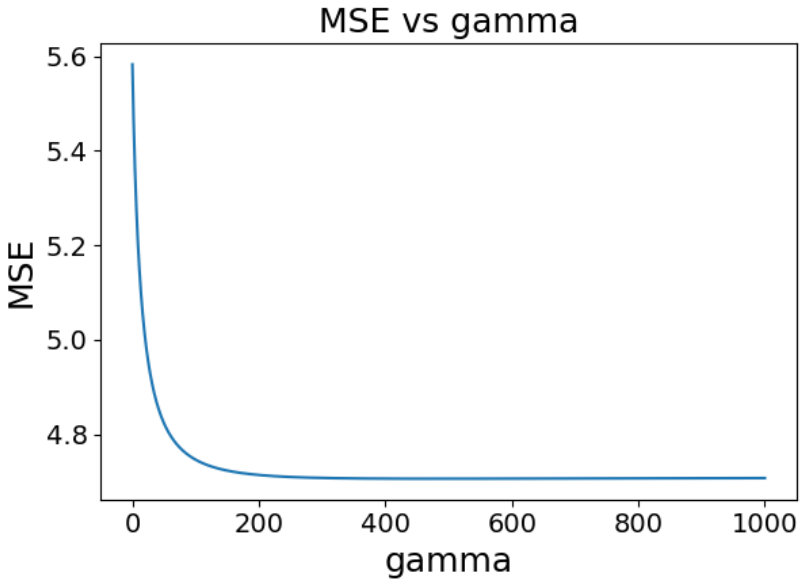
MSE ML (GD): 40.81441651726255

MSE MAP: 5.583247585813262





As gamma in the MAP model increases, the MSE decreases rapidly until about ~ gamma = 50-100 where it flattens out. The curve seems to follow a somewhat exponential decay, especially when we take a closer look at it.
The MAP estimate is related to the ML estimate such that the MSE of the ML estimate is theoretically converging to the MSE point for MAP.



APPENDIX:
PROBLEM 1

```
from email.errors import MultipartInvariantViolationDefect
import matplotlib.pyplot as plt # For general plotting

import numpy as np
from scipy import rand

from scipy.stats import multivariate_normal # MVN not univariate
from sklearn.metrics import confusion_matrix

from modules import prob_utils

from math import ceil, floor


np.set_printoptions(suppress=True)

np.random.seed(1)      # seed 7 is really bad for quadratic

plt.rc('font', size=18)          # controls default text sizes
plt.rc('axes', titlesize=18)     # fontsize of the axes title
plt.rc('axes', labelsize=18)     # fontsize of the x and y labels
plt.rc('xtick', labelsize=14)    # fontsize of the tick labels
plt.rc('ytick', labelsize=14)    # fontsize of the tick labels
plt.rc('legend', fontsize=16)    # legend fontsize
plt.rc('figure', titlesize=18)   # fontsize of the figure title




def gen_data(N):
    mu0 = np.array([[3, 0],
                    [0, 3]])  # Gaussian distributions means
    sigma0 = np.array([[[2, 0],
                        [0, 1]],

                       [[1, 0],
                        [0, 2]]])  # Gaussian distributions covariance matrices

    mu1 = np.array([[2,2]])
    sigma1 = np.array([[[1,0],
                        [0,1]]])
    n = 2 #mu0.shape[0]

    # Class priors
```

```
    priors = np.array([0.65, 0.35])  # Likelihood of each distribution to be
selected
    weights = np.array([.5,.5])
    # Determine number of mixture components
    Cw = len(weights)
    C = len(priors)


    # Output samples and labels
    X = np.zeros([N, n])
    labels = np.zeros(N) # KEEP TRACK OF THIS

    # Plot for original data and their true labels
    labels = np.random.rand(N) >= priors[0]
    L = np.array(range(C))
    Nl = np.array([sum(labels == l) for l in L])
    print("Number of samples from Class 1: {:d}, Class 2: {:d}".format(Nl[0],
Nl[1]))

    gmm_params = prob_utils.GaussianMixturePDFParameters(weights, Cw, mu0,
np.transpose(sigma0))
    gmm_X,_ = prob_utils.generate_mixture_samples(Nl[0], n, gmm_params, True)
    X = np.zeros((N, n))
    X[labels == 0, :] =  gmm_X.T
    X[labels == 1, :] =  multivariate_normal.rvs(mu1[0], sigma1[0], Nl[1])
    return X, labels, Nl




from sys import float_info # Threshold smallest positive floating value

# Generate ROC curve samples
def estimate_roc(discriminant_score, label,N):
    Nlabels = np.array((sum(label == 0), sum(label == 1)))

    sorted_score = sorted(discriminant_score)

    # Use tau values that will account for every possible classification split
    taus = ([sorted_score[0] - float_info.epsilon] +
            sorted_score +
            [sorted_score[-1] + float_info.epsilon])

    # Calculate the decision label for each observation for each gamma
    decisions = [discriminant_score >= t for t in taus]
```

```
    ind10 = [np.argwhere((d==1) & (label==0)) for d in decisions]
    p10 = [len(inds)/Nlabels[0] for inds in ind10]
    ind11 = [np.argwhere((d==1) & (label==1)) for d in decisions]
    p11 = [len(inds)/Nlabels[1] for inds in ind11]

    ind01 = [np.argwhere((d==0) & (label==1)) for d in decisions]
    p01 = [len(inds)/Nlabels[1] for inds in ind01]

    # To find the best value for gamma from the dataset (not theoretical)
    # Here, we find value with lowest probability or error, and convert
    # taus back from log using exp
    prob_error_erm = np.zeros(len(p01))
    for i in range(len(p10)):
        prob_error_erm[i] = np.array((p10[i], p01[i])).dot(Nlabels.T / N)

    best_gamma = np.exp(taus[np.argmin(prob_error_erm)])
    p_error_erm = min(prob_error_erm)
    data_point = np.array([p10[np.argmin(prob_error_erm)],
p11[np.argmin(prob_error_erm)]])#.dot(Nlabels.T / N)

    # ROC has FPR on the x-axis and TPR on the y-axis
    roc = np.array((p10, p11))
    return roc, taus, best_gamma, p_error_erm, data_point




# Define the logistic/sigmoid function
def sigmoid(z):
    return 1.0 / (1 + np.exp(-z))

# Define the prediction function y = 1 / (1 + np.exp(-X*theta))
# X.dot(theta) inputs to the sigmoid referred to as logits
def predict_prob(X, theta):
    logits = X.dot(theta)
    return sigmoid(logits)



# NOTE: This implementation may encounter numerical stability issues...
# Read into the log-sum-exp trick OR use a method like: sklearn.linear_model
import LogisticRegression
def log_reg_loss(theta, X, y):
    # Size of batch
```

```
    B = X.shape[0]

    # Logistic regression model g(X * theta)
    predictions = predict_prob(X, theta)

    # NLL loss, 1/N sum [y*log(g(X*theta)) + (1-y)*log(1-g(X*theta))]
    error = predictions - y
    nll = -np.mean(y*np.log(predictions) + (1 - y)*np.log(1 - predictions))

    # Partial derivative for GD
    g = (1 / B) * X.T.dot(error)

    # Logistic regression loss, NLL (binary cross entropy is another
interpretation)
    return nll, g




# Breaks the matrix X and vector y into batches
def batchify(X, y, batch_size, N):
    X_batch = []
    y_batch = []

    # Iterate over N in batch_size steps, last batch may be < batch_size
    for i in range(0, N, batch_size):
        nxt = min(i + batch_size, N + 1)
        X_batch.append(X[i:nxt, :])
        y_batch.append(y[i:nxt])

    return X_batch, y_batch


def gradient_descent(loss_func, theta0, X, y, N, *args, **kwargs):
    # Mini-batch GD. Stochastic GD if batch_size=1.

    # Break up data into batches and work out gradient for each batch
    # Move parameters theta in that direction, scaled by the step size.

    # Options for total sweeps over data (max_epochs),
    # and parameters, like learning rate and threshold.

    # Default options
    max_epoch = kwargs['max_epoch'] if 'max_epoch' in kwargs else 200
    alpha = kwargs['alpha'] if 'alpha' in kwargs else 0.1
```

```
    epsilon = kwargs['tolerance'] if 'tolerance' in kwargs else 1e-6

    batch_size = kwargs['batch_size'] if 'batch_size' in kwargs else 10

    # Turn the data into batches
    X_batch, y_batch = batchify(X, y, batch_size, N)
    num_batches = len(y_batch)
    print("%d batches of size %d:" % (num_batches, batch_size))

    theta = theta0
    m_t = np.zeros(theta.shape)

    trace = {}
    trace['loss'] = []
    trace['theta'] = []

    # Main loop:
    for epoch in range(1, max_epoch + 1):
        # print("epoch %d\n" % epoch)

        loss_epoch = 0
        for b in range(num_batches):
            X_b = X_batch[b]
            y_b = y_batch[b]
            # print("epoch %d batch %d\n" % (epoch, b))

            # Compute NLL loss and gradient of NLL function
            loss, gradient = loss_func(theta, X_b, y_b, *args)
            loss_epoch += loss

            # Steepest descent update
            theta = theta - alpha * gradient

            # Terminating Condition is based on how close we are to minimum
(gradient = 0)
            if np.linalg.norm(gradient) < epsilon:
                print("Gradient Descent has converged after {}
epochs".format(epoch))
                break

        # Storing the history of the parameters and loss values per epoch
        trace['loss'].append(np.mean(loss_epoch))
        trace['theta'].append(theta)

        # Also break epochs loop
```

```
        if np.linalg.norm(gradient) < epsilon:
            break

    return theta, trace


def create_prediction_score_grid(theta, poly_type):
    # Create coordinate matrices determined by the sample space; can add finer
intervals than 100 if desired
    xx, yy = np.meshgrid(np.linspace(bounds_X[0], bounds_X[1], 200),
np.linspace(bounds_Y[0], bounds_Y[1], 200))
    # Augment grid space with bias ones vector and basis expansion if necessary
    grid = np.c_[xx.ravel(), yy.ravel()]
    grid_aug = np.column_stack((np.ones(200*200), grid))
    if poly_type == 'Q':
        grid_aug = quadratic_transformation(grid_aug)
    # Z matrix are the predictions resulting from sigmoid on the provided model
parameters
    Z = predict_prob(grid_aug, theta).reshape(xx.shape)

    return xx, yy, Z


def plot_prediction_contours(X, theta, ax, poly_type):
    xx, yy, Z = create_prediction_score_grid(theta, poly_type)
    # Once reshaped as a grid, plot contour of probabilities per input feature
(ignoring bias)
    cs = ax.contourf(xx, yy, Z, cmap=plt.cm.coolwarm, alpha=0.55)
    ax.set_xlim([bounds_X[0], bounds_X[1]])
    ax.set_ylim([bounds_Y[0], bounds_Y[1]])


def plot_decision_boundaries(X, labels, theta, ax, poly_type):
    # Plots original class labels and decision boundaries
    ax.plot(X[labels==0, 1], X[labels==0, 2], 'o', label="Class 0")
    ax.plot(X[labels==1, 1], X[labels==1, 2], '+', label="Class 1")

    xx, yy, Z = create_prediction_score_grid(theta, poly_type)
    # Once reshaped as a grid, plot contour of probabilities per input feature
(ignoring bias)
    cs = ax.contour(xx, yy, Z, levels=1, colors='k')

    ax.set_xlabel(r"$x_1$")
    ax.set_ylabel(r"$x_2$")
    ax.set_aspect('equal')
```

```
def report_logistic_classifier_results(X, theta, labels, N_labels, ax,
poly_type):
    """
    Report the probability of error and plot the classified data, plus predicted
    decision contours of the logistic classifier applied to the data given.
    """

    predictions = predict_prob(X, theta)
    # Predicted decisions based on the default 0.5 threshold (higher probability
mass on one side or the other)
    decisions = np.array(predictions >= 0.5)

    # True Negative Probability Rate
    ind_00 = np.argwhere((decisions == 0) & (labels == 0))
    tnr = len(ind_00) / N_labels[0]
    # False Positive Probability Rate
    ind_10 = np.argwhere((decisions == 1) & (labels == 0))
    fpr = len(ind_10) / N_labels[0]
    # False Negative Probability Rate
    ind_01 = np.argwhere((decisions == 0) & (labels == 1))
    fnr = len(ind_01) / N_labels[1]
    # True Positive Probability Rate
    ind_11 = np.argwhere((decisions == 1) & (labels == 1))
    tpr = len(ind_11) / N_labels[1]

    prob_error = fpr*priors[0] + fnr*priors[1]

    print("The total error achieved with this classifier is
{:.3f}\n".format(prob_error))

    # Plot all decisions (green = correct, red = incorrect)
    ax.plot(X[ind_00, 1], X[ind_00, 2], 'og', label="Class 0 Correct", alpha=.25)
    ax.plot(X[ind_10, 1], X[ind_10, 2], 'or', label="Class 0 Wrong")
    ax.plot(X[ind_01, 1], X[ind_01, 2], '+r', label="Class 1 Wrong")
    ax.plot(X[ind_11, 1], X[ind_11, 2], '+g', label="Class 1 Correct", alpha=.25)

    # Draw the decision boundary based on whether its linear (L) or quadratic (Q)
    plot_prediction_contours(X, theta, ax, poly_type)
    ax.set_aspect('equal')
```

```
# Can also use: https://scikit-
learn.org/stable/modules/generated/sklearn.preprocessing.PolynomialFeatures.html
def quadratic_transformation(X):
    n = X.shape[1]
    phi_X = X

    # Take all monic polynomials for a quadratic
    phi_X = np.column_stack((phi_X, X[:, 1] * X[:, 1], X[:, 1] * X[:, 2], X[:, 2]
* X[:, 2]))

    return phi_X


# Options for mini-batch gradient descent
opts = {}
opts['max_epoch'] = 1000
opts['alpha'] = 1e-3
opts['tolerance'] = 1e-3

opts['batch_size'] = 10
# def main():



########################################
###              MAIN              ###
########################################

mu0 = np.array([[3, 0],
                [0, 3]])  # Gaussian distributions means
sigma0 = np.array([[[2, 0],
                    [0, 1]],

                   [[1, 0],
                    [0, 2]]])  # Gaussian distributions covariance matrices

mu1 = np.array([[2,2]])
sigma1 = np.array([[[1,0],
                    [0,1]]])

# Ntrain = np.array([20,200,2000])
# Ntest = 10000
N = np.array([20,200,2000,10000])
# Class priors
priors = np.array([0.65, 0.35])  # Likelihood of each distribution to be selected
```

```
weights = np.array([.5,.5])
# Determine number of mixture components
C = len(priors)

data = []
labels = []
Nl = []
for i in range(len(N)):
    temp1,temp2,Nlabels = gen_data(N[i])
    temp1 = np.column_stack((np.ones(N[i]), temp1))
    data.append(temp1)
    labels.append(temp2)
    Nl.append(np.array((sum(labels[i] == 0), sum(labels[i] == 1))))
# Ny_valid = np.array((sum(labels[3] == 0), sum(labels[3] == 1)))

# Use the validation set's sample space to bound the grid of inputs
# Work out bounds that span the input feature space (x_1 and x_2)
bounds_X = np.array((floor(np.min(data[3][:,1])), ceil(np.max(data[3][:,1]))))
bounds_Y = np.array((floor(np.min(data[3][:,2])), ceil(np.max(data[3][:,2]))))


# Plot the original data and their true labels
# Choose which data set to use [20,200,2000,10000]

# dataset = 3
fig_plot, ax_plot = plt.subplots(2, 2, figsize=(10, 10));
for dataset in range(len(N)):
    if dataset < 2:
        j =0
    else:
        j=1
    ax_plot[dataset%2, j].plot(data[dataset][labels[dataset]==0, 1],
data[dataset][labels[dataset]==0, 2], 'bo', label="Class 0")
    ax_plot[dataset%2, j].plot(data[dataset][labels[dataset]==1, 1],
data[dataset][labels[dataset]==1, 2], 'k+', label="Class 1")
    # plt.plot(Dtrain20[labels2000==0, 0], Dtrain20[labels2000==0, 1], 'bo',
label="Class 0")
    # plt.plot(Dtrain20[labels2000==1, 0], Dtrain20[labels2000==1, 1], 'k+',
label="Class 1")
    ax_plot[dataset%2, j].legend()
    ax_plot[dataset%2, j].set_xlabel(r"$x_1$")
    ax_plot[dataset%2, j].set_ylabel(r"$x_2$")
    ax_plot[dataset%2, j].set_aspect('equal')
    ax_plot[dataset%2, j].set_title("Data and True Class for
N={}".format(data[dataset].shape[0]))
```

```
    # plt.tight_layout()

#    fig = plt.figure(figsize=(9, 9))
    # plt.plot(data[dataset][labels[dataset]==0, 1],
data[dataset][labels[dataset]==0, 2], 'bo', label="Class 0")
    # plt.plot(data[dataset][labels[dataset]==1, 1],
data[dataset][labels[dataset]==1, 2], 'k+', label="Class 1")
    # # plt.plot(Dtrain20[labels2000==0, 0], Dtrain20[labels2000==0, 1], 'bo',
label="Class 0")
    # # plt.plot(Dtrain20[labels2000==1, 0], Dtrain20[labels2000==1, 1], 'k+',
label="Class 1")
    # plt.legend()
    # plt.xlabel(r"$x_1$")
    # plt.ylabel(r"$x_2$")
    # plt.title("Data and True Class Labels")
    # plt.tight_layout()
plt.show()
n = 2 #mu0.shape[0]



# Caculate threshold rule
Lambda = np.ones((C, C)) - np.identity(C)
gamma_map = priors[0] / priors[1]
print(f'Threshold value: {gamma_map}')

# u = np.random.rand(N[3])

L = np.array(range(C))
Nl = np.array([sum(labels[3] == l) for l in L])
# class_conditional_likelihoods = np.array([multivariate_normal.pdf(data[3],
mu[l], Sigma[l]) for l in L])
class_conditional_likelihoods = np.array([0.5*multivariate_normal.pdf(data[3][:,
1:], mu0[0], sigma0[0])+0.5*multivariate_normal.pdf(data[3][:, 1:], mu0[1],
sigma0[1]),\
    multivariate_normal.pdf(data[3][:, 1:], mu1[0], sigma1[0])])
discriminant_score_erm = np.log(class_conditional_likelihoods[1]) -
np.log(class_conditional_likelihoods[0])


decisions_map = discriminant_score_erm >= np.log(gamma_map)
# Get indices and probability estimates of the four decision scenarios:
# (true negative, false positive, false negative, true positive)
```

```
# True Negative Probability
ind_00_map = np.argwhere((decisions_map==0) & (labels[3]==0))
p_00_map = len(ind_00_map) / Nl[0]
# False Positive Probability
ind_10_map = np.argwhere((decisions_map==1) & (labels[3]==0))
p_10_map = len(ind_10_map) / Nl[0]
# False Negative Probability
ind_01_map = np.argwhere((decisions_map==0) & (labels[3]==1))
p_01_map = len(ind_01_map) / Nl[1]
# True Positive Probability
ind_11_map = np.argwhere((decisions_map==1) & (labels[3]==1))
p_11_map = len(ind_11_map) / Nl[1]
# Probability of error for MAP classifier, empirically estimated
prob_error_erm = np.array((p_10_map, p_01_map)).dot(Nl.T / N[3])

print("Confusion Matrix MAP (rows: Predicted class, columns: True class):")
conf_mat = confusion_matrix(decisions_map, labels[3])
print(conf_mat)
correct_class_samples = np.sum(np.diag(conf_mat))
print("Total Number of Misclassified Samples (MAP): {:d}".format(N[3] -
correct_class_samples))

# Construct the ROC for ERM by changing log(gamma)
roc_erm, _, bestGamma, p_error_erm, bestEmpGamma =
estimate_roc(discriminant_score_erm, labels[3],N[3])
roc_map = np.array((p_10_map, p_11_map))


fig_roc, ax_roc = plt.subplots(figsize=(10, 10))
ax_roc.plot(roc_erm[0], roc_erm[1])
ax_roc.plot(roc_map[0], roc_map[1], 'rx', label="Minimum P(Error) MAP",
markersize=16)
ax_roc.plot(bestEmpGamma[0], bestEmpGamma[1], 'gx', mfc='none', label="Minimum
P(Error) Empirical", markersize=16)

ax_roc.legend()
ax_roc.set_xlabel(r"Probability of false alarm $P(D=1|L=0)$")
ax_roc.set_ylabel(r"Probability of correct decision $P(D=1|L=1)$")
plt.grid(True)
plt.title('ROC Curve')
plt.show()
fig_roc;
print('Gamma MAP (Theoretical): ', gamma_map)
print('Probability of Error(MAP): ', prob_error_erm)
print('Best Gamma (ERM): ', bestGamma)
```

```
print('Probability of Error(Empirical): ', p_error_erm)
print()




X = data[3][:, 1:]

fig_disc_grid, ax_disc = plt.subplots(figsize=(10, 10));
# plt.ion() # Re-activate "interactive" mode
horizontal_grid = np.linspace(np.floor(np.min(data[3][:,1])),
np.ceil(np.max(data[3][:,1])), 100)
vertical_grid = np.linspace(np.floor(np.min(data[3][:,2])),
np.ceil(np.max(data[3][:,2])), 100)

ax_disc.plot(X[ind_00_map, 0], X[ind_00_map, 1], 'og', label="Correct Class 0");
ax_disc.plot(X[ind_10_map, 0], X[ind_10_map, 1], 'or', label="Incorrect Class
0");
ax_disc.plot(X[ind_01_map, 0], X[ind_01_map, 1], '+r', label="Incorrect Class
1");
ax_disc.plot(X[ind_11_map, 0], X[ind_11_map, 1], '+g', label="Correct Class 1");

ax_disc.legend();
ax_disc.set_xlabel(r"$x_1$");
ax_disc.set_ylabel(r"$x_2$");
ax_disc.set_title("MAP Decisions (RED incorrect)");
fig_disc_grid.tight_layout();



# Generate a grid of scores that spans the full range of data
[h, v] = np.meshgrid(horizontal_grid, vertical_grid)
# Flattening to feed vectorized matrix in pdf evaluation
gridxy = np.array([h.reshape(-1), v.reshape(-1)])
likelihood_grid_vals = np.array([0.5*multivariate_normal.pdf(gridxy.T, mu0[0],
sigma0[0]) \
    + 0.5*multivariate_normal.pdf(gridxy.T, mu0[1], sigma0[1]), \
    multivariate_normal.pdf(gridxy.T, mu1[0], sigma1[0])])
# Where a score of 0 indicates decision boundary level
# print(likelihood_grid_vals.shape)
discriminant_score_grid_vals = np.log(likelihood_grid_vals[1]) -
np.log(likelihood_grid_vals[0]) - np.log(gamma_map)

# Contour plot of decision boundaries
```

```
discriminant_score_grid_vals =
np.array(discriminant_score_grid_vals).reshape(100, 100)
equal_levels = np.array((0.3, 0.6, 0.9))
min_DSGV = np.min(discriminant_score_grid_vals) * equal_levels[::-1]
max_DSGV = np.max(discriminant_score_grid_vals) * equal_levels
contour_levels = min_DSGV.tolist() + [0] + max_DSGV.tolist()
cs = ax_disc.contour(horizontal_grid, vertical_grid,
discriminant_score_grid_vals.tolist(), contour_levels, colors='k')
ax_disc.clabel(cs, fontsize=16, inline=1)

plt.show()
fig_disc_grid;
# display(fig_disc_grid)









########################################
###             Part 2             ###
########################################

# Use ML parameter estimation to train from the 3 samples.
# Specify as min of negative log likeliehood (NLL)
# Use favorite optimization approach (GD, optimize.minimize in scipy)
# Determine how to use class-label-approx to classify sample to approx min P-
error
# Apply to test data, estimate prob(Error) (use counts of decisions on validation
set)
# (OPTIONAL: Generate plots of decision boundaries)
# PART B
# Repeat for logistic-quadratic-funtion-based approx of class label posterior
functions given a sample
# Compare performancesof classifiers from part B to A.


# Starting point from to search for optimal parameters
theta0_linear = np.random.randn(n+1)
theta0_quadratic = np.random.randn(n+3+1)

fig_decision, ax_decision = plt.subplots(3, 2, figsize=(15, 15));
```

```
print("Training the logistic-linear model with GD per data subset...")

poly_type = 'L'
for i in range(len(N)-1):
    shuffled_indices = np.random.permutation(N[i])

    # Shuffle row-wise X (i.e. across training examples) and labels using same
permuted order
    Xshuf = data[i][shuffled_indices]
    yshuf = labels[i][shuffled_indices]

    X_quad = quadratic_transformation(Xshuf)
    if poly_type=='Q':
        theta_gd, trace = gradient_descent(log_reg_loss, theta0_quadratic,
X_quad, yshuf, N[i], **opts)
    else:
        theta_gd, trace = gradient_descent(log_reg_loss, theta0_linear, Xshuf,
yshuf, N[i], **opts)

    print("Logistic-Linear N={} GD Theta: {}".format(N[i], theta_gd))
    print("Logistic-Linear N={} NLL: {}".format(N[i], trace['loss'][-1]))

    # Convert our trace of parameter and loss function values into NumPy
"history" arrays:
    theta_hist = np.asarray(trace['theta'])
    nll_hist = np.array(trace['loss'])

    if poly_type=='Q':
        plot_decision_boundaries(X_quad, yshuf, theta_gd, ax_decision[i, 0],
poly_type)
    else:
        plot_decision_boundaries(Xshuf, yshuf, theta_gd, ax_decision[i, 0],
poly_type)
    ax_decision[i, 0].set_title("Decision Boundary for \n Logistic-Linear Model
N={}".format(Xshuf.shape[0]))

    # Linear: use validation data (10k samples) and make decisions in report
results routine
    X_valid_quad = quadratic_transformation(data[3])
    if poly_type=='Q':
        report_logistic_classifier_results(X_valid_quad, theta_gd, labels[3], Nl,
ax_decision[i, 1], poly_type)
    else:
```

```
        report_logistic_classifier_results(data[3], theta_gd, labels[3], Nl,
ax_decision[i, 1], poly_type)

    ax_decision[i, 1].set_title("Classifier Decisions on Validation Set \n
Logistic-Linear Model N={}".format(N[i]))


x1_valid_lim = (floor(np.min(data[3][:,1])), ceil(np.max(data[3][:,1])))
x2_valid_lim = (floor(np.min(data[3][:,2])), ceil(np.max(data[3][:,2])))
# Again use the most sampled subset (validation) to define x-y limits
plt.setp(ax_decision, xlim=x1_valid_lim, ylim=x2_valid_lim)

# Adjust subplot positions
plt.subplots_adjust(left=0.05,
                    bottom=0.05,
                    right=0.6,
                    top=0.95,
                    wspace=0.1,
                    hspace=0.3)

# Super plot the legends
handles, labels = ax_decision[0, 1].get_legend_handles_labels()
fig_decision.legend(handles, labels, loc='lower center')

plt.show()
```

PROBLEM2

```
import matplotlib.pyplot as plt # For general plotting

import numpy as np

from scipy.stats import multivariate_normal # MVN not univariate
from sklearn.metrics import confusion_matrix
```

```
from modules import prob_utils
from homework2 import hw2q2
from math import ceil, floor
from sklearn.preprocessing import PolynomialFeatures

np.set_printoptions(suppress=True)

np.random.seed(7)      # seed 7 is really bad for quadratic

plt.rc('font', size=18)          # controls default text sizes
plt.rc('axes', titlesize=18)     # fontsize of the axes title
plt.rc('axes', labelsize=18)     # fontsize of the x and y labels
plt.rc('xtick', labelsize=14)    # fontsize of the tick labels
plt.rc('ytick', labelsize=14)    # fontsize of the tick labels
plt.rc('legend', fontsize=16)    # legend fontsize
plt.rc('figure', titlesize=18)   # fontsize of the figure title



# Breaks the matrix X and vector y into batches
def batchify(X, y, batch_size, N):
    X_batch = []
    y_batch = []

    # Iterate over N in batch_size steps, last batch may be < batch_size
    for i in range(0, N, batch_size):
        nxt = min(i + batch_size, N + 1)
        X_batch.append(X[i:nxt, :])
        y_batch.append(y[i:nxt])

    return X_batch, y_batch


def gradient_descent(loss_func, theta0, X, y, N, *args, **kwargs):
    # Mini-batch GD. Stochastic GD if batch_size=1.

    # Break up data into batches and work out gradient for each batch
    # Move parameters theta in that direction, scaled by the step size.

    # Options for total sweeps over data (max_epochs),
    # and parameters, like learning rate and threshold.

    # Default options
    max_epoch = kwargs['max_epoch'] if 'max_epoch' in kwargs else 200
```

```
    alpha = kwargs['alpha'] if 'alpha' in kwargs else 0.1
    epsilon = kwargs['tolerance'] if 'tolerance' in kwargs else 1e-6

    batch_size = kwargs['batch_size'] if 'batch_size' in kwargs else 10

    # Turn the data into batches
    X_batch, y_batch = batchify(X, y, batch_size, N)
    num_batches = len(y_batch)
    print("%d batches of size %d:" % (num_batches, batch_size))

    theta = theta0
    m_t = np.zeros(theta.shape)

    trace = {}
    trace['loss'] = []
    trace['theta'] = []

    # Main loop:
    for epoch in range(1, max_epoch + 1):
        # print("epoch %d\n" % epoch)

        loss_epoch = 0
        for b in range(num_batches):
            X_b = X_batch[b]
            y_b = y_batch[b]
            # print("epoch %d batch %d\n" % (epoch, b))

            # Compute NLL loss and gradient of NLL function
            loss, gradient = loss_func(theta, X_b, y_b, *args)
            loss_epoch += loss

            # Steepest descent update
            theta = theta - alpha * gradient

            # Terminating Condition is based on how close we are to minimum
(gradient = 0)
            if np.linalg.norm(gradient) < epsilon:
                print("Gradient Descent has converged after {}
epochs".format(epoch))
                break
        # Storing the history of the parameters and loss values per epoch
        trace['loss'].append(np.mean(loss_epoch))
        trace['theta'].append(theta)
        # print(trace['loss'])
        # Also break epochs loop
```

```
        if np.linalg.norm(gradient) < epsilon:
            break

    return theta, trace


def cubic_transformation(X):
    n = X.shape[1]
    phi_X = X

    # Take all monic polynomials for a quadratic
    phi_X = np.column_stack((phi_X, X[:, 1] * X[:, 1],              X[:, 1] *
X[:, 2],              X[:, 2] * X[:, 2],              \
                                    X[:, 1] * X[:, 1] * X[:, 1],    X[:, 1] *
X[:, 1] * X[:, 2],    X[:, 1] * X[:, 2] * X[:, 2],    \
                                    X[:, 2] * X[:, 2] * X[:, 2]
                                    ))

    return phi_X


def plot3(a, b, c, name="Training", mark="o", col="y"):
    # Adjusts the aspect ratio and enlarges the figure (text does not enlarge)
    fig = plt.figure()

    ax = fig.add_subplot(111, projection='3d')
    ax.scatter(a, b, c, marker=mark, color=col)
    ax.set_xlabel(r"$x_1$")
    ax.set_ylabel(r"$x_2$")
    ax.set_zlabel(r"$y$")
    plt.title("{} Dataset".format(name))
    # To set the axes equal for a 3D plot
    # ax.set_prop_cycle(color=['red', 'green', 'blue'])
    # ax.set_box_aspect((np.ptp(a), np.ptp(b), np.ptp(c)))
    # plt.show()

def lin_reg_loss(theta, X, y):
    # Size of batch
    B = X.shape[0]
    # Linear regression model X * theta
    predictions = X.dot(theta)
    # Residual error (X * theta) - y
    error = predictions - y
    # Loss function is MSE
    # print(error)
```

```
    # loss_f = np.mean(error ** 2)

    # loss_f = 0.5*error.T.dot(error)
    loss_f = (X.dot(theta)-y).T.dot(X.dot(theta)-y)
    # print(loss_f)
    # Partial derivative for GD, X^T * ((X * theta) - y)
    g = (1 / B) * X.T.dot(error)
    # g = (X.T.dot(error) - X.T.dot(y))

    return loss_f, g


def MAP_gamma(X,y,gamma):
    theta =
np.linalg.inv(X.T.dot(X)+gamma*np.identity(X.shape[1])).dot(X.T.dot(y))
    return theta

def mean_square_err(X,y,theta):
    y_predict = X.dot(theta) #+ noiseV
    ### MSE
    mse = np.mean((y - y_predict)**2)
    return mse

"""
cubic polynomial y = c(x,theta) + v
           where v = Gauss(0,sigma**2)
x = [1,x1,x2,x1x1,x1x2,x2x2,x1x1x1,x1x1x2,x1x2x2,x2x2x2]
10 terms (including bias)
"""

# Options for mini-batch gradient descent
opts = {}
opts['max_epoch'] = 100
opts['alpha'] = 1e-6
opts['tolerance'] = 1e-3

opts['batch_size'] = 10



def main():
    # mu = np.array([[0,0,0,0,0,0,0,0,0,0]])
    mu = np.zeros(10)
    sigma2 = 1
    sigma = np.identity(10)*sigma2
```

```
    mu = 0
    sigma = 1


    Ntrain = 100
    Nvalidate = 1000
    # xTrain= hw2q2.generateData(Ntrain)
    # xVal = hw2q2.generateData(Nval)
    xTrain, yTrain, xValidate, yValidate = hw2q2.hw2q2()
    noiseT = multivariate_normal.rvs(mu,sigma,Ntrain)
    noiseV = multivariate_normal.rvs(mu,sigma,Nvalidate)
    # shuffled_indices = np.random.permutation(N[i])
    # # Shuffle row-wise X (i.e. across training examples) and labels using same
permuted order
    # Xshuf = data[i][shuffled_indices]
    # yshuf = labels[i][shuffled_indices]

    xAugT = np.column_stack((np.ones(Ntrain), xTrain))
    yAug = np.column_stack((np.ones(Ntrain), yTrain))
    X3train = cubic_transformation(xAugT) #+ noiseT

    xAugV = np.column_stack((np.ones(Nvalidate), xValidate))
    X3validate = cubic_transformation(xAugV) #+ noiseT

    # poly = PolynomialFeatures(3)
    # X3 = poly.fit_transform(xTrain)
    # # print(xTrain[0])
    # print(X3[0])

    nCubic = X3train.shape[1]
    theta0 = np.random.randn(nCubic)
    # theta0 = np.zeros(nCubic)

    theta_gd, trace = gradient_descent(lin_reg_loss, theta0, X3train, yTrain,
Ntrain, **opts)
    theta_MAP = MAP_gamma(X3train,yTrain,0)

    #Results
    print('theta start:')
    print(theta0)
    print('theta MLE:')
    print(theta_gd)
    print('theta MAP:')
    print(theta_MAP)
    print()
```

```
    # print("Mini-batch GD Theta: ", theta_gd)
    # print("MSE: ", trace['loss'][-1])

    ### Now compare to test data and calc accuracy using Mean Square Error ###

    mse_gd = mean_square_err(X3validate,yValidate,theta_gd)
    mse_MAP = mean_square_err(X3validate,yValidate,theta_MAP)
    print('MSE GD:', mse_gd)
    print('MSE MAP:', mse_MAP)

    # y = c + noise

    # X_gd = multivariate_normal.rvs(theta_gd,sigma,Nvalidate) + noiseV
    # X_MAP = multivariate_normal.rvs(theta_MAP,sigma,Nvalidate) + noiseV
    # print()
    y_MAP = X3validate.dot(theta_MAP) + noiseV
    y_gd = X3validate.dot(theta_gd) + noiseV




    fig = plt.figure()

    ax = fig.add_subplot(111, projection='3d')
    ax.scatter(xValidate[:, 0], xValidate[:, 1], yValidate, marker='o',
color='b', label='True Data')
    ax.scatter(X3validate[:, 1], X3validate[:, 2], y_gd, marker='o', color='y',
label='ML Estimate')
    ax.scatter(X3validate[:, 1], X3validate[:, 2], y_MAP, marker='o', color='r',
label='MAP Estimate')
    ax.set_xlabel(r"$x_1$")
    ax.set_ylabel(r"$x_2$")
    ax.set_zlabel(r"$y$")
    ax.legend()
    plt.show()

    ######################################
    ###         Varying gamma          ###
    ######################################

    trials = 10001
    gamma = np.linspace(0.0001,1000,trials)
    # print(gamma)
    mse_range = []
```

```
    for i in range(trials):
        theta_temp = MAP_gamma(X3train,yTrain,gamma[i])
        mse_range.append(mean_square_err(X3validate,yValidate,theta_temp))
    plt.plot(gamma,mse_range)
    plt.title("MSE vs gamma")
    plt.xlabel('gamma')
    plt.ylabel('MSE')
    # plt.xscale('log')
    # plt.yscale('log')
    plt.show()

    return

if __name__ == '__main__':
    main()
```