Interpreting hydrologic response using transfer function models with time-varying parameters: an example from the Virginia Blue Ridge

Jeffrey G. Chanat[1,2] and John P. Norton[1,3]

[1] Integrated Catchment Assessment & Management Centre, Australian National University

[2] Department of Environmental Sciences, University of Virginia, USA; jgc3n@virginia.edu

[3] School of Engineering, University of Birmingham, UK

Abstract: Streamflow hydrographs represent the integrated effects of hydrologic processes operating over a wide range of spatial and temporal scales. Constant-parameter transfer function models have been shown to represent the relationship between effective rainfall and streamflow adequately in many cases, but are limited in ability to reveal detailed behaviour by their aggregation of watershed dynamics in a linear, time-invariant model. One way of examining details of catchment dynamics while retaining a simple model structure is to fit linear but time-varying models. We illustrate this approach applied to seven years' daily rainfall and streamflow data from a 10 km^2 forested watershed in Virginia. Low-order models with an output-offset term (modeled as zero, constant or time-varying) are fitted by extended least squares estimation with optimal smoothing, treating time-varying model parameters as random walks. The extent of time variation is restricted to keep the ratio of mean-squared one-step-prediction error to mean-squared residual close to unity, so that parameter updates track changes in watershed input-output dynamics rather than noise in the record. All models turn out to have one dominant pole, indicating that with the slowest components of the flow record accounted for by an offset, watershed response can be well modeled as a single, linear reservoir with varying gain and time constant. The reservoir time constant varies fairly smoothly between two and thirteen days. Patterns in the evolution of the time constant and steady-state gain correspond to physically interpretable events in the hydrologic record, including snow accumulation and melt and extreme summer storms. Models of the sort presented here have potential applications in baseflow filtering, in revealing subtle changes in hydrologic response, and in identifying anomalies in the records.

Keywords: *Transfer functions; rainfall-runoff; time-varying*

1. INTRODUCTION

Runoff processes in forested watersheds operate over a wide range of temporal scales. Transfer function models are an empirical approach for describing these complex dynamics in a manner consistent with the information content in rainfall-runoff records (Jakeman et al., 1990; Jakeman and Hornberger, 1993). The results frequently suggest that runoff can be related to effective precipitation with low-order, constant-parameter models that are mathematically equivalent to one or two linear reservoirs connected in series or parallel. This representation may lend insight into underlying physical processes (e.g., Jakeman et al., 1993), and permits the dynamic response characteristics of different catchments to be compared (Post and Jakeman, 1996). However, even in cases where catchment dynamics can be summarized with low-order models, there is no physical reason to expect the underlying processes to be stationary. One approach to overcoming this potential limitation of constant-parameter models while retaining a simple model structure is to fit linear but time-varying models. Time-varying models derived by fixed-interval smoothing have been used to characterize nonstationarity in catchment response and suggest refinements to constant-parameter models (Norton, 1975; 1976; Young and Beven, 1991; Young and Beven, 1994; Beven, 2001). In this paper we present a similarly motivated technique, using extended least squares with optimal smoothing, applied to a model structure which includes an output-offset term. This term, which may also be time-varying, allows for behaviour not within the scope of the rest of the low-order model, such as baseflow. The overall aim is to characterize and interpret the response of a small forested catchment, on daily and longer time scales.

2. METHODS

2.1. Study Area

The Staunton River drains a 10.5 km^2 watershed on the eastern flank of the Blue Ridge in central Virginia, in the mid-Atlantic region of the eastern US. The stream gauge latitude and longitude are 38° 26' 42" and 78° 22' 38", and the gauge elevation is 480 m. Mean annual precipitation for water years 1993-1999 was 1770 mm, with precipitation more or less evenly distributed over the year. Snow cover in the watershed is discontinuous during the winter months, with maximum annual snow depths ranging from 20 cm to 120 cm over the study period. The watershed is fairly steep, with a divide-to-outlet gradient of 29%, and completely forested, primarily with second-to-third-growth mixed hardwoods. Mean annual runoff for water years 1993-1999 was 763 mm, with roughly 65 percent occurring between November and April.

2.2. Modeling Approach

A transfer function model with time-varying parameters was fitted to a daily record of rainfall and streamflow from October 1, 1992 to August 16, 1999 (Figure 1a). Because time-varying models cannot be applied in a predictive sense to data sets other than those with which they were formulated, the entire period of record was used as the calibration period; that is, there was no model validation period. The model in output-error form is

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})}u(t) + \frac{C(q^{-1})}{A(q^{-1})}e(t) + d(t),$$

where $y(t)$ represents streamflow at time t, $u(t)$ represents rainfall, $e(t)$ is a zero-mean white noise sequence and $d(t)$ is an offset. $A(q^{-1})$, $B(q^{-1})$, and $C(q^{-1})$ are polynomials in the backshift operator, with $A(q^{-1})$ and $C(q^{-1})$ monic. In initial experiments $A(q^{-1})$ and $B(q^{-1})$ were of degree 3. This allows for some variation in the pure delay between rainfall and streamflow (as can be seen in records of individual storms) and identification of up to three linear storages in series or parallel, ensuring that no significant component is overlooked.

The coefficients of $A(q^{-1})$ and $B(q^{-1})$ are modeled as random walks. The extent of variation of each is tuned via the specified mean-square value of its forcing, minimizing the mean-squared residuals (MSR) while not allowing the ratio ρ of mean-squared one-step prediction errors (MSPE) to MSR to rise far or rapidly from its constant-parameter value of unity. This prevents excessive variation in the model parameter estimates due only to noise, which confers no predictive power, while allowing tracking of systematic variation in dynamics, which does improve predictive power. The coefficients of $C(q^{-1})$ are estimated as constants so as to leave it to $A(q^{-1})$, $B(q^{-1})$ and $d(t)$ to reflect as much as possible of the systematic behaviour. A time-invariant noise model in output-error form would require $C(q^{-1})$ to cancel variation in $A(q^{-1})$ exactly, but that would introduce products of coefficients from $A(q^{-1})$ and the constant noise-model parameters into the equation-error form of the model, destroying the linearity required by extended least squares and optimal smoothing. The offset term was modeled as zero, constant, or time varying. Model performance was measured by the ratio of the standard deviation of the model-output residuals to that of the observed output. Details of the estimation algorithm may be found in Norton and Chanat (2003).

Dominant modal components in the rainfall-runoff transfer function were identified by partial fraction decomposition of $B(q^{-1})/A(q^{-1})$ and examination of each component. In some cases complex conjugate poles with small and rapidly varying residues arose. They have little predictive power and are driven by very-short-term features of the records which cannot be fitted by the remainder of the model.

3. RESULTS

Based on assessment of MSR and ratio ρ, the variances of the zero-mean, white noise sequences driving the variations of each parameter in $A(q^{-1})$ and $B(q^{-1})$ were tuned to 10^{-5} and 10^{-6} respectively. The offset term was driven by a sequence with variance 0.001 when not modeled as constant. Inclusion of a constant offset reduces the small mean of the residuals virtually to zero but its addition reduces statistical efficiency, increasing the standard deviation (s.d.) of the residuals and RMSE by about 1%. Allowing the offset to vary reduces the ratio of RMSE to s.d. of flow from 0.3023 to 0.2948; ρ is between 1.02 and 1.03 in all cases. Only the model with the time-varying offset will be discussed hereafter.

Model steady-state gain (SSG), defined as *(sum of coefficients in* $B(q^{-1})$*)/(sum of coefficients in* $A(q^{-1})$*)*, varied substantially but smoothly between 0.11 and 0.53 mm/mm, with a mean of 0.27 mm/mm (Figure 1b). Most of the model gain was contributed by a single dominant mode, with significant differences only evident for roughly the first 200 days of the record and between days 1200 and 1700 (Figure 1b). This suggests that with the slowest runoff component accounted for by an offset, catchment response is quite well modeled as a single, linear storage with smoothly varying gain

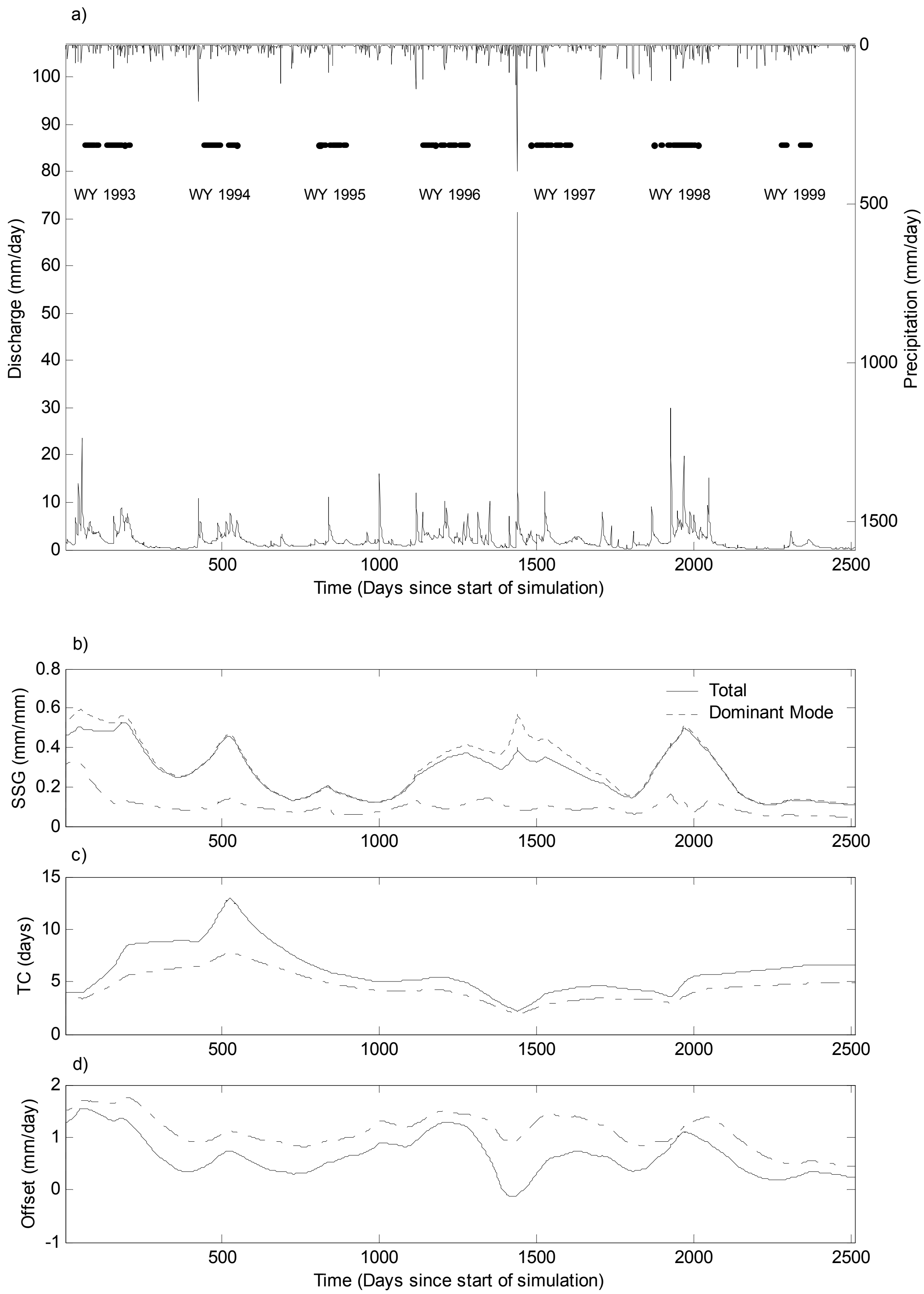

Figure 1. Observed precipitation, flow, and periods with snow cover, modeled steady-state gain, time constant, and offset term, Staunton River, October 1, 1992 – August 16, 1999. a) precipitation, flow, and periods with snow cover (horizontal bars); b) steady-state gain (SSG), SSG of dominant mode, and SSG using effective precipitation as input (dash-dot line); c) time constant (TC). Dash-dot line as in b); d) offset term. Dash-dot line as in b).

and time constant. The variation in the SSGs suggests a seasonal cycle, with the SSG higher in the winter months and lower in summer. The extent of seasonal variation varies from year to year, with smallest differences in water years (WYs) 1995 and 1999. A sharp peak punctuates this pattern at day 1440, coinciding with an exceptional rainfall event (Figure 1b).

The time constant $-1/ln(dominant\ pole)$ associated with the dominant pole varies between 2.2 and 13.0 days, with a mean of 6.0 days (Figure 1c). The time constant varies more smoothly than the SSGs and does not exhibit the seasonal pattern. Prominent features include a distinct peak near day 530 and local minima near days 1440 and 1930. The dominant time constant is notably lower when estimated using effective rainfall (Figure 1c).

Thirty-two percent of the modeled flow volume is accounted for by the time-varying offset (Figure 1d), which has a mean of 0.65 mm/day and a maximum of 1.57 mm/day. The offset is transiently negative near day 1440, but the use of effective rainfall removes this anomaly (Figure 1d).

4. DISCUSSION

In periods of high SSG, a greater proportion of the rainfall ends up as streamflow. This might be expected when the catchment is least able to store water; i.e. is relatively wet. The seasonal pattern in the SSG thus suggests variation in catchment wetness due to evapotranspiration and rainfall. The greatest seasonal variation in SSG might be expected in the years with the wettest winters. This tendency is evident from both inspection of the relative magnitude of the winter flows (Figure 1a) and tabulation of relative winter precipitation (Table 1). Seasonal differences in SSG are least pronounced in the driest winters (WYs 1995 and 1999), and more pronounced in the wetter winters (e.g., WYs 1998 and 1994).

The seasonal variation in SSG suggests an obvious modification of the model structure: inclusion of a nonlinear module to account for catchment wetness. To examine the effect of such a module, the total rainfall was processed using the nonlinear rainfall filter of the IHACRES model (e.g., Jakeman and Hornberger, 1993), with a catchment drying time constant of 10 days and temperature modulation factor of 2.0. The resulting effective rainfall time series was used as input to the time-varying model. The net effect on model performance is a 1% reduction in RMSE, as a larger mean is removed from the residuals. The resulting time series of model SSG shows appreciably less seasonality, although the remaining seasonal pattern suggests that further improvement could be achieved through fine-tuning (Figure 1b). The reduction in dominant time constant on using effective rainfall (Figure 1c), also suggests that the influence of the slow soil-moisture dynamics is at least partly accounted for by the evapotranspiration loss module.

The peak in model SSG near time step 1440 is associated with the heaviest precipitation event of the record: a 4-day period of 544 mm total precipitation in late summer, with nearly 400 mm on the last day. It is likely that an event of this magnitude resulted in complete saturation of the watershed, with nearly all precipitation appearing as streamflow. The resulting peak in SSG is therefore comparable in magnitude to those evident in the wettest winters. The peak in estimated SSG is inevitably spread out by the change-inhibiting effect of the specified values for mean-square changes of the model parameters; Weston and Norton (1997) offer an optimal-smoothing-based technique specifically intended for detection and quantification of such isolated abrupt events.

The dominant time constant of the response shows two prominent minima near days 1440 (the event discussed above) and 1950 (Figure 1c). The second follows the second largest runoff event, in January 1998. These minima are consistent with the expectation that in the heaviest events rain falls onto on a saturated catchment and follows the quickest pathways to the stream.

Maxima in the time constant occur in periods of slowest runoff response. One possible cause is storage of water in snowpack, followed by slow release during snowmelt. The most prominent

Table 1. Winter Precipitation and Snow Cover Index[1] Over Study Period

	Water Year						
	1993	1994	1995	1996	1997	1998	1999
October-March Precipitation (mm)	864	888	663	914	765	1222	740
Snow Cover Index (days-cm)	712	988	300	470	520	1620	235

[1]Snow cover index = (Number of days with snow cover)x(Median depth of snow cover when present)

maximum in the time constant occurs in the winter of WY 1994 (Figure 1c). To test whether this maximum is due to melting of an unusually heavy snowpack, a coarse index of snow cover was computed for each winter season. The number of days with snow cover was multiplied by the median snow depth over those days. The results indicate that WY 1998 had by far the largest snow cover, followed by WYs 1994 and 1993 (Table 1). Water years 1993 and 1994 do have the highest winter time constants, but the effect of heavy snow cover is not so evident in the time constant for WY 1998, although the time constant rises markedly in the few weeks around day 2000. A likely explanation is that the specified mean-square values of the parameter changes inhibited rapid recovery of the model's time constant from its minimum due to heavy rainfall in January 1998. Two other large rainfall events later that season may have also contributed to keeping the time constant low (Figure 1a).

After these results were obtained, we discovered that the gauge on the Staunton River was not operating between days 959 and 1469 because of a major flood. Surrogate data, derived from regression relationships with nearby watersheds, had been inserted for this period. Some features of the model results suggest effects of the substitution. First, the interval during which the offset term is negative, indicating overestimation of flow by the input-output part of the model, coincides with the largest storm in the record, which occurs during the period of surrogate data (Figure 1d). Secondly, the longest and most pronounced period when the SSGs of the entire model and the dominant mode differ substantially, indicating higher-than-first-order dynamics, is between roughly days 1200 and 1700, significantly overlapping the period of surrogate data (Figure 1b). These results suggest that, in addition to being an interpretive tool, the time-varying model may be useful for identifying anomalies in the record.

The question arises whether the modeled offset term can be used to identify "baseflow", loosely defined as the component of the total flow due to the slowest, and usually unmodelled, components of the catchment rainfall-runoff dynamics. More generally, the decomposition of total flow response into slower and quicker components invites comparison with two-compartment constant-parameter models as produced by IHACRES. However, the estimated offset term also responds to any systematic undermodelling or mismodelling in the rest of the model, such as significantly too-large or too-small values of the mean-square changes in the parameters or omitted influences which are not completely accommodated by the random-walk parameter representations, such as that of temperature on evapotranspiration loss.

The flexibility of linear, low-order, time-varying models allows them to track, to a useful extent, complicated dynamics which are in fact both non-linear and, being both distributed and subject to varying pure delay, infinite-order. The price paid for this richness of behaviour is that it makes decomposition into modal and offset components at any given instant non-unique; a change in relative tuning of the time variation of the various parameters and offset can alter the components for a given quality of fit. Separation of effects, e.g. into baseflow and faster rainfall-runoff relations, or into the effects of rainfall, evapotanspiration and snowmelt on runoff, thus depends on tuning by reference to what behaviour is known to be credible, and checking against known catchment. properties. Even then, a range of time-varying models (narrow and informative, if the tuning is done with care) may fit the catchment's short-term behaviour. By contrast, time-invariant, low-order, lumped, models, as used by IHACRES, cannot model complicated dynamics in detail but can successfully model average behaviour over sufficient periods and at suitable time scales for that behaviour to be uniquely defined and consistent.

5. CONCLUSIONS

Linear, time-varying models, obtained by extended least squares estimation and optimal smoothing, can be effective in refining model structure (e.g. revealing dependences of parameters on variables), revealing changes in dynamics and identifying anomalies in records. This ability can usefully be applied to rainfall-runoff records from catchments subject to large seasonal and year-to-year variations and to unmeasured snowmelt.

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