University of Waterloo
MATH 213, Spring 2015
Assignment 7

Question 1 (7 marks)

Find the inverse of the given transform in two different ways: using partial fractions and using the convolution theorem.

$$F(s) = \frac{7}{(s-3)s^3}$$

Partial Fraction Expansion

$$\begin{aligned}
\frac{7}{(s-3)s^3} &= \frac{A}{s-3} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} \\
7 &= As^3 + B(s-3) + Cs(s-3) + Ds^2(s-3) \\
7 &= (A+D)s^3 + (C-3D)s^2 + (B-3C)s - 3B && \text{(1 mark)}
\end{aligned}$$

Comparing coefficients, we get a system of equations:

$$\begin{aligned}
A + D &= 0 \\
C - 3D &= 0 \\
B - 3C &= 0 \\
-3B &= 7
\end{aligned}$$

We get $A = \frac{7}{27}, B = \frac{-7}{3}, C = \frac{-7}{9}, D = \frac{-7}{27}$. So the original expression becomes,

$$\frac{7}{(s-3)s^3} = \frac{7}{27}\left(\frac{1}{s-3}\right) - \frac{7}{3}\frac{1}{s^3} - \frac{7}{9}\frac{1}{s^2} - \frac{7}{27}\frac{1}{s}$$

(1 mark)

Taking the inverse Laplace transform,

$$f(t) = \frac{7}{27}e^{3t} - \frac{7}{6}t^2 - \frac{7}{9}t - \frac{7}{27}$$

(1 mark)

Convolution Theorem

We separate $F(s)$ into the following equation:

$$7 * \frac{1}{s^3} * \frac{1}{s-3}$$

Now we do the convolution:

$$\begin{aligned}
f(t) &= 7\int_0^t \frac{1}{2}\tau^2 e^{3(t-\tau)}d\tau \\
&= \frac{7e^{3t}}{2}\int_0^t \tau^2 e^{-3\tau}d\tau \\
&= \frac{7e^{3t}}{2}\left(\frac{\tau^2 e^{-3\tau}}{-3}\Bigg|_0^t + \frac{2}{3}\int_0^t \tau e^{-3\tau}d\tau\right) \\
&= \frac{7e^{3t}}{2}\left(-\frac{t^2e^{-3t}}{3} + \frac{2}{3}\left(-\frac{\tau e^{-3\tau}}{3}\Bigg|_0^t + \frac{1}{3}\int_0^t e^{-3\tau}d\tau\right)\right) \\
&= \frac{7e^{3t}}{2}\left(-\frac{t^2e^{-3t}}{3} + \frac{2}{3}\left(-\frac{te^{-3t}}{3} - \frac{1}{9}e^{-3\tau}\Bigg|_0^t\right)\right) \\
&= \frac{7e^{3t}}{2}\left(-\frac{t^2e^{-3t}}{3} + \frac{2}{3}\left(-\frac{te^{-3t}}{3} - \frac{1}{9}e^{-3t} + \frac{1}{9}\right)\right) \\
&= -\frac{7t^2}{6} + \frac{7e^{3t}}{3}\left(-\frac{te^{-3t}}{3} - \frac{1}{9}e^{-3t} + \frac{1}{9}\right) \\
&= -\frac{7t^2}{6} - \frac{7t}{9} - \frac{7}{27} + \frac{7e^{3t}}{27}
\end{aligned}$$

Question 2 (3 marks)

Solve for $x(t)$ on $0 \leq t < \infty$,

$$x'' - 7x' = \delta(t-11)$$

Taking the Laplace transform of the equation,

$$\begin{aligned}
s^2x(s) - 7sx(s) &= e^{-11s} \\
x(s) &= \frac{e^{-11s}}{s(s-7)} \qquad \text{(1 mark)}
\end{aligned}$$

Using partial fraction expansion,

$$\begin{aligned}
\frac{1}{s(s-7)} &= \frac{A}{s} + \frac{B}{s-7} \\
A(s-7) + Bs &= 1
\end{aligned}$$

Comparing coefficients,

$$A = \frac{-1}{7}, B = \frac{11}{7}$$

(1 mark)

Substituting back into the original expression,

$$\frac{e^{-11s}}{s(s-7)} = e^{-11s}\left(\frac{1}{7s} - \frac{1}{7(s-7)}\right)$$

Taking the inverse Laplace transform,

$$x(t) = \frac{-1}{7}u(t-11) + \frac{1}{7}e^{7(t-11)}u(t-11)$$

(1 mark)