

Assignment 6

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1 Problem Set 1

1.1 Question 1

First we find the equation for this graph.

$$f(t) = t - (t-3)H(t-3) - 3H(t-3) + H(t-3) - H(t-4) + (-(t-4)+1)H(t-4) + (t-5)H(t-5) + 2(t-5)H(t-5)$$

Then we take the Laplace transform.

$$\begin{aligned} L\{f(t)\} &= \frac{1}{s^2} - \frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s} + \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{e^{-4s}}{s} + \frac{e^{-5s}}{s^2} + 2\frac{e^{-5s}}{s^2} \\ &= \frac{1 - e^{-3s} - e^{-4s} + 3e^{-5s}}{s^2} - \frac{2e^{-3s}}{s} \end{aligned}$$

1.2 Question 2

We can use the theorem for periodic functions,

$$L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T f(t)e^{-st} dt$$

Substituting the function $f(t) = \sin(t)$ with period $T = 2\pi$, we get

$$L\{\sin(t)\} = \frac{1}{1 - e^{-s2\pi}} \int_0^{2\pi} \sin(t)e^{-st} dt$$

Using integration by parts, we can simplify the integral.

$$\begin{aligned} \int_0^{2\pi} \sin(t)e^{-st} dt &= \frac{e^{-st} \sin(t)}{-s} \Big|_0^{2\pi} + \frac{1}{s} \int_0^{2\pi} e^{-st} \cos(t) dt \\ \int_0^{2\pi} \sin(t)e^{-st} dt &= \frac{e^{-st} \sin(t)}{-s} \Big|_0^{2\pi} + \frac{1}{s} \left(\frac{e^{-st} \cos(t)}{-s} \Big|_0^{2\pi} - \frac{1}{s} \int_0^{2\pi} e^{-st} \sin(t) dt \right) \\ \left(1 + \frac{1}{s^2}\right) \int_0^{2\pi} \sin(t)e^{-st} dt &= \frac{e^{-st} \sin(t)}{-s} \Big|_0^{2\pi} + \frac{1}{s} \frac{e^{-st} \cos(t)}{-s} \Big|_0^{2\pi} \\ \frac{s^2 + 1}{s^2} \int_0^{2\pi} \sin(t)e^{-st} dt &= \frac{1}{s} \left(\frac{e^{-s2\pi}}{-s} - \frac{e^0}{-s} \right) \\ \int_0^{2\pi} \sin(t)e^{-st} dt &= \frac{1 - e^{-s2\pi}}{s^2 + 1} \end{aligned}$$

Substituting into the origin expression,

$$\begin{aligned} L\{\sin(t)\} &= \frac{1}{1 - e^{-s2\pi}} \int_0^{2\pi} \sin(t)e^{-st} dt \\ &= \frac{1}{1 - e^{-s2\pi}} \frac{1 - e^{-s2\pi}}{s^2 + 1} \\ &= \frac{1}{s^2 + 1} \end{aligned}$$

Therefore the Laplace transform of $\sin(t)$ is $\frac{1}{s^2 + 1}$.

2 Problem Set 2

2.1 Question 1

$$\cos(4t), 0 < t < 2\pi$$

We know that

$$\begin{aligned} L\{\cos(4t)\} &= \frac{1}{1 - e^{-sT}} \int_0^T \cos(4t)e^{-st} dt \\ &= \frac{1}{1 - e^{-s2\pi}} \int_0^{2\pi} \cos(4t)e^{-st} dt \end{aligned}$$

Use integration by parts to find the integral

$$\begin{aligned} \int_0^{2\pi} \cos(4t)e^{-st} dt &= \left. \frac{e^{-st} \cos(4t)}{-s} \right|_0^{2\pi} - \frac{4}{s} \int_0^{2\pi} e^{-st} \sin(4t) dt \\ &= \left. \frac{e^{-st} \cos(4t)}{-s} \right|_0^{2\pi} - \frac{4}{s} \left(\left. \frac{e^{-st} \sin(4t)}{-s} \right|_0^{2\pi} + \frac{4}{s} \int_0^{2\pi} e^{-st} \cos(4t) dt \right) \\ \left(1 + \frac{16}{s^2} \right) \int_0^{2\pi} \cos(4t)e^{-st} dt &= \left. \frac{e^{-st} \cos(4t)}{-s} \right|_0^{2\pi} - \frac{4}{s} \left. \frac{e^{-st} \sin(4t)}{-s} \right|_0^{2\pi} \\ \left(\frac{s^2 + 16}{s^2} \right) \int_0^{2\pi} \cos(4t)e^{-st} dt &= \frac{e^{-s2\pi}}{-s} - \frac{1}{-s} - 0 + 0 \\ &= \frac{1 - e^{-s2\pi}}{s} \\ \int_0^{2\pi} \cos(4t)e^{-st} dt &= \frac{s(1 - e^{-s2\pi})}{s^2 + 16} \end{aligned}$$

Substitute back into the original equation

$$\begin{aligned} L\{\cos(4t)\} &= \frac{1}{1 - e^{-s2\pi}} \int_0^{2\pi} \cos(4t)e^{-st} dt \\ &= \frac{1}{1 - e^{-s2\pi}} * \frac{s(1 - e^{-s2\pi})}{s^2 + 16} \\ &= \frac{s}{s^2 + 16} \end{aligned}$$

So the Laplace transform of $\cos(4t)$ is $\frac{s}{s^2+16}$.

2.2 Question 2

Assuming $f(t) = 0$ when $t \leq 0$ and $t \geq 4$. As the question is unclear.

$$x'' - x = f(t), x(0) = 0, x'(0) = 0, f(t) = t \text{ when } 0 < t < 4$$

Turn into a heaviside function

$$x'' - x = t - 4u(t-4) - (t-4)u(t-4)$$

Take the Laplace of both sides

$$\begin{aligned}
x'' - x &= t - 4u(t-4) - (t-4)u(t-4) \\
s^2 X(s) - X(s) &= \frac{1}{s^2} - \frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} \\
(s^2 - 1)X(s) &= \frac{1}{s^2} - \frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} \\
X(s) &= \frac{1}{s^2(s-1)(s+1)} - \frac{4e^{-4s}}{s(s-1)(s+1)} - \frac{e^{-4s}}{s^2(s-1)(s+1)}
\end{aligned}$$

Use partial fraction decomposition.

$$\begin{aligned}
\frac{1}{s^2(s-1)(s+1)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1} \\
1 &= As^3 - As + Bs^2 - B + Cs^3 + Cs^2 +Ds^3 -Ds^2 \\
A + C + D &= 0, B + C - D = 0, -A = 0, -B = 1 \\
A = 0, B &= -1, C = \frac{1}{2}, D = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{s(s-1)(s+1)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} \\
1 &= As^2 - A + Bs^2 + Bs + Cs^2 - Cs \\
A + B + C &= 0, B - C = 0, -A = 1 \\
A = -1, B &= \frac{1}{2}, C = \frac{1}{2}
\end{aligned}$$

Putting it all together

$$\begin{aligned}
X(s) &= \frac{1}{s^2(s-1)(s+1)} - \frac{4e^{-4s}}{s(s-1)(s+1)} - \frac{e^{-4s}}{s^2(s-1)(s+1)} \\
X(s) &= -\frac{1}{s^2} + \frac{1}{2} \frac{1}{s-1} - \frac{1}{2} \frac{1}{s+1} + 4e^{-4s} \left(-\frac{1}{s} + \frac{1}{2} \frac{1}{s-1} + \frac{1}{2} \frac{1}{s+1} \right) + e^{-4s} \left(-\frac{1}{s^2} + \frac{1}{2} \frac{1}{s-1} - \frac{1}{2} \frac{1}{s+1} \right) \\
X(s) &= -\frac{1}{s^2} + \frac{1}{2} \frac{1}{s-1} - \frac{1}{2} \frac{1}{s+1} + e^{-4s} \left(-\frac{4}{s} - \frac{1}{s^2} + \frac{5}{2} \frac{1}{s-1} + \frac{3}{2} \frac{1}{s+1} \right) \\
x(t) &= -t + \frac{e^t}{2} - \frac{e^{-t}}{2} + u(t-4) \left(-4 - (t-4) + \frac{5e^{t-4}}{2} + \frac{3e^{-(t-4)}}{2} \right)
\end{aligned}$$