University of Waterloo
MATH 213, Spring 2015
Assignment 8 Solutions

Question 1

Find the Fourier series of the following function, given over one period, using two methods: real and complex form.

$$f(x) = x^2 \text{ on } (-\pi, \pi)$$

Real

We require coefficients a_0, a_n, b_n.

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x^2 dx && \text{(even integrand)} \\
&= \frac{1}{\pi}\left(\frac{x^3}{3}\right)\Bigg|_0^{\pi} \\
&= \frac{\pi^2}{3} && \text{(1 mark)} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx) dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(nx) dx && \text{(even integrand)} \\
&= \frac{2}{\pi}\left(\frac{(n^2x^2-2)\sin(nx)+2nx\cos(nx)}{n^3}\right)\Bigg|_0^{\pi} \\
&= \frac{2}{\pi}\left(\frac{(n^2\pi^2-2)\sin(n\pi)+2n\pi\cos(n\pi)}{n^3}\right) && \text{(1 mark)}
\end{aligned}$$

Here we note that for multiples of π, sine is always 0 and cosine alternates between -1 and 1. Thus we can rewrite as

$$\begin{aligned}
a_n &= \frac{2}{\pi}\left((-1)^n\frac{2\pi}{n^2}\right) \\
&= (-1)^n\frac{4}{n^2} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx) dx && \text{(odd integrand)} \\
&= 0 && \text{(1 mark)}
\end{aligned}$$

Note that the integrand is composed of two functions, x^2 which is even and $\sin(nx)$ which is odd. We know that the product of an even function and an odd function is an odd function so the entire integrand is odd. Since our interval of integration is symmetric, we know the integral will evaluate to 0. Substituting the coefficients in the series expansion,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx) \qquad \text{(1 mark)}$$

Complex

We require coefficient c_n.

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx} dx \\ &= \frac{1}{2\pi} \left(\frac{e^{-inx}(in^2x^2 + 2nx - 2i)}{n^3} \right) \Bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left(\frac{(n^2\pi^2 - 2)\sin(n\pi) + 2n\pi\cos(n\pi)}{n^3} \right) \\ &= (-1)^n \frac{2}{n^2} \end{aligned} \qquad \text{(1 mark)}$$

Substituting in the series expression,

$$f(x) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{2}{n^2} e^{inx}$$

We note that this is discontinuous at $n = 0$. So we solve for c_0 manually.

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{\pi^2}{3} \end{aligned} \qquad \text{(1 mark)}$$

Therefore,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2} (e^{inx} + e^{-inx}) \qquad \text{(1 mark)}$$

Question 2

Derive the Fourier integral representation of the following function.

$$f(x) = \begin{cases} e^{2x} & 0 \le x < L \\ 0 & x < 0, x \ge L \end{cases}$$

$$
\begin{aligned}
A(\omega) &= \frac{1}{\pi}\int_0^L e^{2x}\cos(\omega x)dx \\
&= \frac{1}{\pi}\left(\frac{e^{2x}(\omega\sin(\omega x)+2\cos(\omega x))}{\omega^2+4}\right)\Bigg|_0^L \\
&= \frac{1}{\pi}\left(\frac{e^{2x}(\omega\sin(\omega x)+2\cos(\omega x))-2}{\omega^2+4}\right)\Bigg|_0^L \\
&= \frac{1}{\pi}\frac{e^{2L}(\omega\sin(L\omega)+2\cos(L\omega))-2}{\omega^2+4} && \text{(1 mark)} \\
B(\omega) &= \frac{1}{\pi}\int_0^L e^{2x}\sin(\omega x)dx \\
&= \frac{1}{\pi}\left(\frac{e^{2x}(2\sin(\omega x)-\omega\cos(\omega x))}{\omega^2+4}\right)\Bigg|_0^L \\
&= \frac{1}{\pi}\left(\frac{e^{2x}(2\sin(\omega x)-\omega\cos(\omega x))+w}{\omega^2+4}\right)\Bigg|_0^L \\
&= \frac{1}{\pi}\frac{e^{2L}(2\sin(L\omega)-\omega\cos(L\omega))+\omega}{\omega^2+4} && \text{(1 mark)}
\end{aligned}
$$

Substituting,

$$
f(x) = \int_0^\infty [\frac{1}{\pi}\frac{e^{2L}(\omega\sin(L\omega)+2\cos(L\omega))-2}{\omega^2+4}\cos(\omega x) + \frac{1}{\pi}\frac{e^{2L}(2\sin(L\omega)-\omega\cos(L\omega))+\omega}{\omega^2+4}\sin(\omega x)]d\omega
$$

(1 mark)

Alternatively,

$$
\begin{aligned}
C(\omega) &= \frac{1}{2\pi}\int_0^L e^{2x}e^{-i\omega x}dx \\
&= \frac{1}{2\pi}\int_0^L e^{(2-i\omega)x}dx \\
&= \frac{1}{2\pi}\frac{i(e^{L(2-i\omega)}-1)}{w+2i} && \text{(2 marks)}
\end{aligned}
$$

Substituting,

$$
f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{i(e^{L(2-i\omega)}-1)}{\omega+2i}\right]e^{i\omega x}d\omega \qquad \text{(1 mark)}
$$