Assignment 5

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1 Problem Set 1

1.1 Problem 1

1) $f(t) = e^{2^{t^2}}$
This function is not exponential order. For some K, c, T,

$$\frac{e^{2^{t^2}}}{Ke^{ct}} = \frac{1}{K}e^{2^{t^2}-ct}$$

As t approaches infinity, the expression in the exponent approaches infinity regardless of our selection of c.

2) $f(t) = 2t^3$

$$\lim_{t\to\infty} \frac{2t^3}{Ke^{ct}} = \lim_{t\to\infty} \frac{6t^2}{cKe^{ct}} = \lim_{t\to\infty} \frac{12t}{c^2Ke^{ct}} = \lim_{t\to\infty} \frac{12}{c^3Ke^{ct}}$$

Evaluating this limit, we see that as t approaches infinity, the ratio approaches zero. Therefore $2t^3$ is of exponential order. Pick $K = 100, c = 1, T = 1$.

1.2 Problem 2

$$F(s) = \frac{1}{s^2 + 6s + 8}$$

Using partial fraction decomposition,

$$\begin{aligned} \frac{1}{s^2+6s+8} &= \frac{A}{s+2} + \frac{B}{s+4} \\ 1 &= As + 4A + Bs + 2B \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} 0 &= A + B \\ 1 &= 4A + 2B \end{aligned}$$

We get $A = 1/2$ and $B = -1/2$. Substituting,

$$F(s) = \frac{1}{s^2+6s+8} = \frac{1}{2(s+2)} + \frac{-1}{2(s+4)}$$

Taking the inverse Laplace transform,

$$f(t) = \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}$$

2 Problem Set 2

2.1 Problem 1

$$y'' + 5y' + 4y = 0, y(0) = 1, y'(0) = 2$$

First we take the Laplace of both sides.

$$\begin{aligned}
&s^2Y(s) - sy(0) - y'(0) + 5(sY(s) - y(0)) + 4Y(s) = 0 \\
&s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 4Y(s) = 0 \\
&s^2Y(s) - s - 2 + 5sY(s) - 5 + 4Y(s) = 0 \\
&s^2Y(s) + 5sY(s) + 4Y(s) = s + 7 \\
&(s^2 + 5s + 4)Y(s) = s + 7 \\
&Y(s) = \frac{s+7}{s^2+5s+4} \\
&Y(s) = \frac{s+7}{(s+1)(s+4)}
\end{aligned}$$

Use partial fraction decomposition to split $Y(s)$.

$$\begin{aligned}
&\frac{s+7}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4} \\
&s + 7 = As + 4A + Bs + B \\
&\therefore A + B = 1, 4A + B = 7 \\
&\therefore A = 2, B = -1
\end{aligned}$$

Take the inverse Laplace to find $y(t)$.

$$\begin{aligned}
&Y(s) = \frac{s+7}{(s+1)(s+4)} \\
&Y(s) = \frac{2}{s+1} - \frac{1}{s+4} \\
&y(t) = 2e^{-t} + e^{-4t}
\end{aligned}$$

Therefore, the particular solution is $y(t) = 2e^{-t} + e^{-4t}$.

2.2 Problem 2

$$x'' + 7x' + 12x = 2 + e^{-t}, x(0) = 0, x'(0) = 1$$

First we take the Laplace of both sides.

$$\begin{aligned}
&s^2X(s) - sx(0) - x'(0) + 7sX(s) - 7x(0) + 12X(s) = \frac{2}{s} + \frac{1}{s+1} \\
&(s^2 + 7s + 12)X(s) = \frac{2}{s} + \frac{1}{s+1} + 1 \\
&(s+4)(s+3)X(s) = \frac{2}{s} + \frac{1}{s+1} + 1 \\
&X(s) = \frac{2}{s(s+4)(s+3)} + \frac{1}{(s+1)(s+4)(s+3)} + \frac{1}{(s+4)(s+3)}
\end{aligned}$$

Use partial fraction decomposition to split $X(s)$.

$$\begin{aligned}
&\frac{2}{s(s+4)(s+3)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+3} \\
&2 = As^2 + 7As + 12A + Bs^2 + 3Bs + Cs^2 + 4Cs \\
&A + B + C = 0, 7A + 3B + 4C = 0, 12A = 2 \\
&A = \frac{1}{6}, B = \frac{1}{2}, C = -\frac{2}{3}
\end{aligned}$$

$$
\begin{aligned}
&\frac{1}{(s+1)(s+4)(s+3)} = \frac{A}{s+1} + \frac{B}{s+4} + \frac{C}{s+3} \\
&1 = As^2 + 7As + 12A + Bs^2 + 4Bs + 3B + Cs^2 + 5Cs + 4C \\
&A + B + C = 0, 7A + 4B + 5C = 0, 12A + 3B + 4C = 1 \\
&A = \frac{1}{6}, B = \frac{1}{3}, C = -\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3} \\
&1 = As + 3A + Bs + 4B \\
&A + B = 0, 3A + 4B = 1 \\
&A = -1, B = 1
\end{aligned}
$$

Take the inverse Laplace to find $x(t)$.

$$
\begin{aligned}
&X(s) = \frac{1}{6}\frac{1}{s} + \frac{1}{2}\frac{1}{s+4} - \frac{2}{3}\frac{1}{s+3} + \frac{1}{6}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4} - \frac{1}{2}\frac{1}{s+3} - \frac{1}{s+4} + \frac{1}{s+3} \\
&X(s) = \frac{1}{6}\frac{1}{s} - \frac{1}{6}\frac{1}{s+4} - \frac{1}{6}\frac{1}{s+3} + \frac{1}{6}\frac{1}{s+1} \\
&x(t) = \frac{1}{6} - \frac{1}{6}e^{-4t} - \frac{1}{6}e^{-3t} + \frac{1}{6}e^{-t}
\end{aligned}
$$

Therefore, the particular solution is $x(t) = \frac{1}{6} - \frac{1}{6}e^{-4t} - \frac{1}{6}e^{-3t} + \frac{1}{6}e^{-t}$.