

**University of Waterloo**  
**MATH 213, Spring 2015**  
**Assignment 7**

**Question 1 (7 marks)**

Find the inverse of the given transform in two different ways: using partial fractions and using the convolution theorem.

$$F(s) = \frac{7}{(s-3)s^3}$$

**Partial Fraction Expansion**

$$\begin{aligned}\frac{7}{(s-3)s^3} &= \frac{A}{s-3} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} \\ 7 &= As^3 + B(s-3) + Cs(s-3) + Ds^2(s-3) \\ 7 &= (A+D)s^3 + (C-3D)s^2 + (B-3C)s - 3B\end{aligned}\tag{1 mark}$$

Comparing coefficients, we get a system of equations:

$$\begin{aligned}A + D &= 0 \\ C - 3D &= 0 \\ B - 3C &= 0 \\ -3B &= 7\end{aligned}$$

We get  $A = \frac{7}{27}$ ,  $B = \frac{-7}{3}$ ,  $C = \frac{-7}{9}$ ,  $D = \frac{-7}{27}$ . So the original expression becomes,

$$\frac{7}{(s-3)s^3} = \frac{7}{27} \left( \frac{1}{s-3} \right) - \frac{7}{3} \frac{1}{s^3} - \frac{7}{9} \frac{1}{s^2} - \frac{7}{27} \frac{1}{s}\tag{1 mark}$$

Taking the inverse Laplace transform,

$$f(t) = \frac{7}{27}e^{3t} - \frac{7}{6}t^2 - \frac{7}{9}t - \frac{7}{27}\tag{1 mark}$$

**Convolution Theorem**

We separate  $F(s)$  into the following equation:(1 mark)

$$7 * \frac{1}{s^3} * \frac{1}{s-3}$$

Now we do the convolution:

$$\begin{aligned}
 f(t) &= 7 \int_0^t \frac{1}{2} \tau^2 e^{3(t-\tau)} d\tau & (1 \text{ mark}) \\
 &= \frac{7e^{3t}}{2} \int_0^t \tau^2 e^{-3\tau} d\tau \\
 &= \frac{7e^{3t}}{2} \left( \frac{\tau^2 e^{-3\tau}}{-3} \Big|_0^t + \frac{2}{3} \int_0^t \tau e^{-3\tau} d\tau \right) \\
 &= \frac{7e^{3t}}{2} \left( -\frac{t^2 e^{-3t}}{3} + \frac{2}{3} \left( -\frac{\tau e^{-3\tau}}{3} \Big|_0^t + \frac{1}{3} \int_0^t e^{-3\tau} d\tau \right) \right) \\
 &= \frac{7e^{3t}}{2} \left( -\frac{t^2 e^{-3t}}{3} + \frac{2}{3} \left( -\frac{t e^{-3t}}{3} - \frac{1}{9} e^{-3\tau} \Big|_0^t \right) \right) \\
 &= \frac{7e^{3t}}{2} \left( -\frac{t^2 e^{-3t}}{3} + \frac{2}{3} \left( -\frac{t e^{-3t}}{3} - \frac{1}{9} e^{-3t} + \frac{1}{9} \right) \right) \\
 &= -\frac{7t^2}{6} + \frac{7e^{3t}}{3} \left( -\frac{t e^{-3t}}{3} - \frac{1}{9} e^{-3t} + \frac{1}{9} \right) & (1 \text{ mark for solving integration}) \\
 &= -\frac{7t^2}{6} - \frac{7t}{9} - \frac{7}{27} + \frac{7e^{3t}}{27} & (1 \text{ mark})
 \end{aligned}$$

## Question 2 (3 marks)

Solve for  $x(t)$  on  $0 \leq t < \infty$ ,

$$x'' - 7x' = \delta(t - 11)$$

Taking the Laplace transform of the equation,

$$\begin{aligned}
 s^2 x(s) - 7s x(s) &= e^{-11s} \\
 x(s) &= \frac{e^{-11s}}{s(s-7)} & (1 \text{ mark})
 \end{aligned}$$

Using partial fraction expansion,

$$\begin{aligned}
 \frac{1}{s(s-7)} &= \frac{A}{s} + \frac{B}{s-7} \\
 A(s-7) + Bs &= 1
 \end{aligned}$$

Comparing coefficients,

$$A = \frac{-1}{7}, B = \frac{1}{7}$$

(1 mark)

Substituting back into the original expression,

$$\frac{e^{-11s}}{s(s-7)} = e^{-11s} \left( -\frac{1}{7s} + \frac{1}{7(s-7)} \right)$$

Taking the inverse Laplace transform,

$$x(t) = \frac{-1}{7}u(t-11) + \frac{1}{7}e^{7(t-11)}u(t-11)$$

(1 mark)