University of Waterloo
MATH 213, Spring 2015
Assignment 3

Question 1

Find the general solution for the following differential equations. Hint: Some roots are complex. You may want to use factor theorem.

a) $y'''' - 8y'' + 72y' - 65y = 0$

This gives the characteristic equation:

$$\lambda^4 - 8\lambda^2 + 72\lambda - 65 = 0$$

(1 mark)

Using factor theorem, we can find roots 1 and -5. The remaining quadratic, $x^2 - 4x + 13$, gives roots $2 \pm 3i$. Using these roots, we get general solution:

$$y = c_1 e^x + c_2 e^{-5x} + c_3 e^{2x} \cos(3x) + c_4 e^{2x} \sin(3x)$$

(1 mark)

b) $y^{(6)} - 4y^{(5)} + 6y^{(4)} - 8y^{(3)} + 9y'' - 4y' + 4y = 0$

This gives the characteristic equation:

$$\lambda^6 - 4\lambda^5 + 6\lambda^4 - 8\lambda^3 + 9\lambda^2 - 4\lambda + 4 = 0$$

(1 mark)

We can factor this to $(x-2)^2(x \pm i)^2$ which gives roots 2 and $\pm i$. We get a general solution:

$$y = (c_1 + c_2 x)e^{2x} + e^x(c_3 + c_4 x)(c_5 \sin(x) + c_6 \cos(x))$$

(2 marks)

Question 2

Find the general solution of the following differential equation using the method of undetermined coefficients.

$$y'' - 8y' + 15y = x + \cos 2x$$

First, we solve the homongenous equation using the characteristic equation:

$$\lambda^2 - 8\lambda + 15 = 0$$

(1 mark)

We get roots 3 and 5 thus,

$$y_h = c_1 e^{3x} + c_2 e^{5x}$$

(1 mark)

Using the method of undetermined coefficients, we get the following assumed form for $x + \cos(2x)$.

$$\begin{aligned}
y_p &= k_1 + k_2 x + k_3 \cos(2x) + k_4 \sin(2x) \\
y_p' &= k_2 - 2k_3 \sin(2x) + 2k_4 \cos(2x) \\
y_p'' &= -4k_3 \cos(2x) - 4k_4 \sin(2x)
\end{aligned}$$

(1 mark)

Substituting into the the original differential equation,

$$\begin{aligned}
&-4k_3 \cos(2x) - 4k_4 \sin(2x) - 8(k_2 - 2k_3 \sin(2x) + 2k_4 \cos(2x)) + 15(k_1 + k_2 x + k_3 \cos(2x) + k_4 \sin(2x)) \\
&= (-8k_2 + 15k_1) + x(15k_2) + \cos(2x)(-4k_3 - 16k_4 + 15k_3) + \sin(2x)(-4k_4 + 16k_3 + 15k_4)
\end{aligned}$$

Setting this expression equal to right hand side $(x + \cos(2x))$ and comparing coefficients, we get a system of equations which resolves to:

$$\begin{aligned}
k_1 &= \frac{8}{225} \\
k_2 &= \frac{1}{15} \\
k_3 &= \frac{11}{377} \\
k_4 &= -\frac{16}{377}
\end{aligned}$$

(1 mark)

Thus the general solution is,

$$y = c_1 e^{3x} + c_2 e^{5x} + \frac{8}{225} + \frac{1}{15}x + \frac{11}{377}\cos(2x) + -\frac{16}{377}\sin(2x)$$

(1 mark)