

University of Waterloo
MATH 213, Spring 2015
Assignment 6

Question 1

Use the Laplace transform to find the particular solution of the following functions

$$y' + y = \sin(t - 3)u(t - 3) + tu(t - 3) - 3u(t - 3), y(0) = 0$$

Where

$$u(t) = \begin{cases} 0 & \text{when } t < 0 \\ 1 & \text{when } t \geq 0 \end{cases}$$

Take the Laplace of both sides:

$$y' + y = \sin(t - 3)u(t - 3) + tu(t - 3) - 3u(t - 3)$$

$$y' + y = \sin(t - 3)u(t - 3) + (t - 3)u(t - 3)$$

$$sY(s) + Y(s) = \frac{e^{-3s}}{s^2 + 1} + \frac{e^{-3s}}{s^2}$$

$$(s + 1)Y(s) = e^{-3s} \left(\frac{1}{s^2 + 1} + \frac{1}{s^2} \right)$$

$$Y(s) = e^{-3s} \left(\frac{1}{(s^2 + 1)(s + 1)} + \frac{1}{s^2(s + 1)} \right) \quad (1 \text{ mark})$$

Use partial fraction decomposition:

$$\frac{1}{(s^2 + 1)(s + 1)} = \frac{As}{s^2 + 1} + \frac{B}{s^2 + 1} + \frac{C}{s + 1}$$

$$1 = As^2 + As + Bs + B + Cs^2 + C$$

$$A + C = 0, A + B = 0, B + C = 1$$

$$\therefore A = -0.5, C = 0.5, B = 0.5 \quad (1 \text{ mark})$$

$$\frac{1}{s^2(s + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 1}$$

$$1 = As^2 + As + Bs + B + Cs^2$$

$$A + C = 0, A + B = 0, B = 1$$

$$\therefore A = -1, B = 1, C = 1 \quad (1 \text{ mark})$$

Take the inverse Laplace:

$$Y(s) = e^{-3s} \left(\frac{1}{(s^2 + 1)(s + 1)} + \frac{1}{s^2(s + 1)} \right)$$

$$Y(s) = e^{-3s} \left(-\frac{1}{2} \frac{s}{s^2 + 1} + \frac{1}{2} \frac{1}{s^2 + 1} + \frac{1}{2} \frac{1}{s + 1} - \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s + 1} \right)$$

$$y(t) = \left(-\frac{1}{2} \cos(t - 3) + \frac{1}{2} \sin(t - 3) + \frac{3e^{-(t-3)}}{2} - 1 + (t - 3) \right) u(t - 3) \quad (2 \text{ marks})$$

Question 2