# University of Waterloo
# MATH 213, Spring 2015
# Assignment 5

## Question 1

Find the inverse of the given transform.

a)

$$\frac{3s^2 - 12s^2 - 2s - 56}{(s-7)s(s^2+4)}$$

Using partial fraction expansion,

$$\frac{3s^2 - 12s^2 - 2s - 56}{(s-7)s(s^2+4)} = \frac{A}{s-7} + \frac{B}{s} + \frac{2C}{s^2+4} + \frac{Ds}{s^2+4}$$

Expanding,

$$-9s^2 - 2s - 56 = As(s^2+4) + B(s-7)(s^2+4) + 2Cs(s-7) + Ds(s(s-7))$$
$$-9s^2 - 2s - 56 = As^3 + 4As + Bs^3 + -7Bs^2 + 4Bs - 28B + 2Cs^2 - 14Cs + Ds^3 - 7Ds^2$$

Comparing coefficients, we get the following systems of equations.

$$\begin{aligned} 0 &= A + B + D \\ -9 &= -7B + 2C - 7D \\ -2 &= 4A + 4B - 14C \\ -56 &= -28B \end{aligned}$$

Which gives us $A = \frac{-73}{53}, B = 2, C = \frac{17}{53}, D = \frac{-33}{53}$. Thus our original expression becomes,

$$\frac{3s^2 - 12s^2 - 2s - 56}{(s-7)s(s^2+4)} = \frac{-73}{53}\frac{1}{s-7} + \frac{2}{s} + \frac{17}{53}\frac{2}{s^2+4} + \frac{-33}{53}\frac{s}{s^2+4}$$

(2 marks)

Taking the inverse of the transform, we get

$$L(t) = \frac{-73}{53}e^{7t} + 2 + \frac{17}{53}\sin 2t + \frac{-33}{53}cos2t$$

(1 mark)

b)

$$\frac{s^4+3s^3+2s^2+27s+18}{s^3(s^2+9)}$$

Using partial fraction expansion,

$$\frac{s^4+3s^3+2s^2+27s+18}{s^3(s^2+9)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{3D}{s^2+9}+\frac{Es}{s^2+9}$$

Expanding,

$$s^4+3s^3+2s^2+27s+18 = As^2(s^2+9)+Bs(s^2+9)+C(s^2+9)+3Ds^3+Es(s^3)$$

Comparing coefficients, we get the following system of equations,

$$\begin{aligned} 1 &= A+E \\ 3 &= B+3D \\ 2 &= 9A+C \\ 27 &= 9B \\ 18 &= 9C \end{aligned}$$

Which gives us $A=0, B=3, C=2, D=0, E=1$. Substituting back into the original expression,

$$\frac{s^4+3s^3+2s^2+27s+18}{s^3(s^2+9)} = \frac{3}{s^2}+\frac{2}{s^3}+\frac{s}{s^2+9}$$

(1 mark)

Taking the inverse, we get

$$L(t) = 3t+t^2+\cos 3t$$

(1 mark)

# Question 2

Use the Laplace transform to find the particular solution.

$$y''' - 2y'' + 4y' - 8y = 0$$

$$y(0)=0, y'(0)=0, y''(0)=4$$

First, take the Laplace transform of the equation:

$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 2(s^2Y(s) - sy(0) - y'(0)) + 4(sY(s) - y(0)) - 8Y(s) = 0$
$s^3Y(s) - 4 - 2s^2Y(s) + 4sY(s) - 8Y(s) = 0$
$(s^3 - 2s^2 + 4s - 8)Y(s) = 4$
$Y(s) = \dfrac{4}{s^3 - 2s^2 + 4s - 8}$
$Y(s) = \dfrac{4}{(s-2)(x^2+4)}$ (1 mark)

Using partial fraction decomposition:

$$\frac{4}{(s-2)(x^2+4)} = \frac{A}{s-2} + \frac{Bs}{s^2+4} + \frac{2C}{s^2+4}$$
$$4 = As^2 + 4A + Bs^2 - 2Bs + 2Cs - 4C$$
$$A + B = 0$$
$$-2B + 2C = 0$$
$$4A - 4C = 4 \qquad \text{(1 mark)}$$

Solving for these equations gives $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = -\frac{1}{2}$. (1 mark)
Therefore, (1 mark)

$$Y(s) = \frac{1}{2} * \frac{1}{s-2} - \frac{1}{2} * \frac{s}{s^2+4} - \frac{1}{2} * \frac{2C}{s^2+4}$$

Taking the inverse Laplace: (1 mark)

$$y(t) = \frac{1}{2}e^{2t} - \frac{1}{2}\cos(2t) - \frac{1}{2}\sin(2t)$$