Question 1

Find the general solution to the below equation using the integrating factor method.

$$y' + y = 4x + 5x^2$$

Multiply both sides by the integrating factor.

$$\sigma y' + \sigma y = \sigma(4x + 5x^2)$$

Let

$$(\sigma y)' = \sigma y' + \sigma' y = LHS = \sigma y' + \sigma y$$

(0.5 marks)

So

$$\begin{aligned} \sigma' y &= \sigma y \\ \sigma' &= \sigma \end{aligned}$$

So $\sigma = e^x$. (1 mark)

We know

$$(\sigma y)' = RHS = \sigma(4x + 5x^2)$$

(0.5 marks)

Substituting $\sigma = e^x$

$$\begin{aligned} \frac{de^x y}{dx} &= e^x(4x + 5x^2) \\ de^x y &= e^x(4x + 5x^2)dx \\ \int de^x y &= \int e^x(4x + 5x^2)dx \\ e^x y &= \int 4xe^x dx + \int 5x^2 e^x dx \\ e^x y &= 4\int xe^x dx + 5\int x^2 e^x dx \\ e^x y &= 4(xe^x - \int e^x dx) + 5(x^2 e^x - \int 2xe^x dx) \\ e^x y &= 4(xe^x - e^x + C) + 5(x^2 e^x - 2(xe^x - e^x + D)) \\ e^x y &= 4xe^x - 4e^x + 5x^2 e^x - 10xe^x + 10e^x + E \\ y &= 4x - 4 + 5x^2 - 10x + 10 + \frac{E}{e^x} \\ y &= 5x^2 - 6x + 6 + \frac{E}{e^x} \end{aligned}$$

(2 marks)

So $y = 5x^2 - 6x + 6 + \frac{E}{e^x}$ is the general solution. (1 mark)

Question 2

Use seperation of variables to find the general solution. Then obtain the particular solution satisfying the given initial condition. Sketch the graph of the solution, showing the key feaures, and label any key values.

$$y' = \frac{y}{3x+1}, y(0) = 2$$

Separate variables to find the general solution:

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{3x+1} \\ \frac{dy}{y} &= \frac{dx}{3x+1} \\ \int \frac{dy}{y} &= \int \frac{dx}{3x+1} \\ \ln|y| &= \frac{1}{3}\ln|3x+1| + c \\ y &= ce^{\frac{1}{3}\ln|3x+1|} \\ y &= c\,|3x+1|^{\frac{1}{3}}\end{aligned}$$

(2 marks)

Substituting the initial condition, we get,

$$y = 2\,|3x+1|^{\frac{1}{3}}$$

(1 mark)

The equation is plotted below. Special points to note include: x-intercept at $\frac{-1}{3}$, the given initial condition $y(0) = 2$, symmetric about $x = \frac{-1}{3}$,and $y- > \infty$ as $x- > \infty$ and $x- > -\infty$. (2 marks)