Assignment 7

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Problem Set 1

Question 1

1.1

For part 1.1, I'm going to assume you meant the inverse laplace, instead of the laplace of the two rational expressions.

I shall prove this using induction, consider the base case: $n = 0$. We know that $L\{t^0\} = L\{1\} = \frac{1}{s} = \frac{0!}{s^{0+1}}$. So the statement is true for the base case.

Assume that the statement is true for some integer $k \geq 0$. Consider $k + 1$.

Consider the following equation.

$$Y(s) = \frac{(k+1)!}{s^{k+2}}$$

This is equivalent to

$$Y(s) = \frac{k!}{s^{k+1}} \frac{(k+1)}{s}$$

We know from the induction hypothesis that

$$L\{t^k\} = \frac{k!}{s^{k+1}}$$

So therefore

$$Y(s) = L\{t^k\}L\{k+1\}$$

Therefore, by convolution

$$y(t) = \int_0^t (k+1)(\tau^k)d\tau = t^{k+1}$$

So by the principle of mathematical induction,

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

1.2

Also, again, assuming that you mean inverse laplace.

$$\frac{1}{(s+2)(s^2+1)} + \frac{3}{s^2+9}$$

Firstly, we know that

$$L^{-1}\{\frac{3}{s^2+9}\} = \sin(3t)$$

$$L^{-1}\{\frac{1}{s+2}\} = e^{-2t}$$

$$L^{-1}\{\frac{1}{s^2+1}\} = \sin(t)$$

So,

$$
\begin{aligned}
L^{-1}\{\frac{1}{(s+2)(s^2+1)}+\frac{3}{s^2+9}\} &= \int_0^t \sin(\tau)e^{-2(t-\tau)}d\tau + \sin(3t) \\
&= e^{-2t}\int_0^t \sin(\tau)e^{2\tau}d\tau + \sin(3t)
\end{aligned}
$$

Now we solve the integral.

$$
\begin{aligned}
\int_0^t \sin(\tau)e^{2\tau}d\tau &= \frac{e^{2\tau}}{2}\sin(\tau)\Big|_0^t - \frac{1}{2}\int_0^t \cos(\tau)e^{2\tau}d\tau \\
&= \frac{e^{2\tau}}{2}\sin(\tau)\Big|_0^t - \frac{1}{2}\left(\frac{e^{2\tau}}{2}\cos(\tau)\Big|_0^t + \frac{1}{2}\int_0^t \sin(\tau)e^{2\tau}d\tau\right) \\
&= \frac{e^{2t}}{2}\sin(t) - \frac{1}{2}\left(\frac{e^{2t}}{2}\cos(t) - \frac{1}{2} + \frac{1}{2}\int_0^t \sin(\tau)e^{2\tau}d\tau\right) \\
2\int_0^t \sin(\tau)e^{2\tau}d\tau &= e^{2t}\sin(t) - \frac{e^{2t}}{2}\cos(t) + \frac{1}{2} - \frac{1}{2}\int_0^t \sin(\tau)e^{2\tau}d\tau \\
4\int_0^t \sin(\tau)e^{2\tau}d\tau &= 2e^{2t}\sin(t) - e^{2t}\cos(t) + 1 - \int_0^t \sin(\tau)e^{2\tau}d\tau \\
5\int_0^t \sin(\tau)e^{2\tau}d\tau &= 2e^{2t}\sin(t) - e^{2t}\cos(t) - 1 \\
\int_0^t \sin(\tau)e^{2\tau}d\tau &= \frac{e^{2t}}{5}(2\sin(t) - \cos(t)) - \frac{1}{5}
\end{aligned}
$$

So therefore,

$$
\begin{aligned}
L^{-1}\{\frac{1}{(s+2)(s^2+1)}+\frac{3}{s^2+9}\} &= e^{-2t}\int_0^t \sin(\tau)e^{2\tau}d\tau + \sin(3t) \\
&= e^{-2t}\left(\frac{e^{2t}}{5}(2\sin(t) - \cos(t)) - \frac{1}{5}\right) + \sin(3t) \\
&= \frac{1}{5}(2\sin(t) - \cos(t)) - \frac{e^{-2t}}{5} + \sin(3t)
\end{aligned}
$$

Question 2

$$x'' + x' - 6x = 10 - u(t-4), x(0) = x'(0) = 0$$

First we take the Laplace of both sides

$$
\begin{aligned}
s^2X(s) + sX(s) - 6X(s) &= \frac{10 - e^{-4s}}{s} \\
X(s) &= \frac{10 - e^{-4s}}{s(s^2+s-6)} \\
&= \frac{10 - e^{-4s}}{s(s+3)(s-2)}
\end{aligned}
$$

Consider $\frac{1}{s(s+3)(s-2)}$.

$$L^{-1}\{\frac{1}{s(s+3)(s-2)}\} = L^{-1}\{\frac{1}{s}\} * L^{-1}\{\frac{1}{s+3}\} * L^{-1}\{\frac{1}{s-2}\}$$

Consider $L^{-1}\{\frac{1}{s}\} * L^{-1}\{\frac{1}{s+3}\}$.

$$\begin{aligned} L^{-1}\{\frac{1}{s}\} * L^{-1}\{\frac{1}{s+3}\} &= \int_0^t e^{-3\tau} d\tau \\ &= -\frac{e^{-3t}}{3} + \frac{1}{3} \end{aligned}$$

We use this value in the second convolution equation.

$$\begin{aligned} L^{-1}\{\frac{1}{s}\} * L^{-1}\{\frac{1}{s+3}\} * L^{-1}\{\frac{1}{s-2}\} &= \frac{1}{3}\int_0^t (1-e^{-3\tau})e^{2t-2\tau} d\tau \\ &= \frac{e^{2t}}{3}\int_0^t e^{-2\tau} - e^{-5\tau} d\tau \\ &= \frac{e^{2t}}{3}\left(-\frac{e^{-2\tau}}{2} + \frac{e^{-5\tau}}{5}\right)\Big|_0^t \\ &= \frac{e^{2t}}{3}\left(-\frac{e^{-2t}}{2} + \frac{e^{-5t}}{5} + \frac{3}{10}\right) \\ &= \frac{1}{3}\left(-\frac{1}{2} + \frac{e^{-3t}}{5} + \frac{3e^{2t}}{10}\right) \end{aligned}$$

So in total, the final solution is.

$$x(t) = \frac{10}{3}\left(-\frac{1}{2} + \frac{e^{-3t}}{5} + \frac{3e^{2t}}{10}\right) - \frac{u(t-4)}{3}\left(-\frac{1}{2} + \frac{e^{-3(t-4)}}{5} + \frac{3e^{2(t-4)}}{10}\right)$$

Problem Set 2

Question 1

Question 2