Assignment 6

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1 Problem Set 1

1.1 Question 1

First we find the equation for this graph.

$f(t) = t-(t-3)H(t-3)-3H(t-3)+H(t-3)-H(t-4)+(-(t-4)+1)H(t-4)+(t-5)H(t-5)+2(t-5)H(t-5)$

Then we take the Laplace transform.

$$\begin{aligned} L\{f(t)\} &= \frac{1}{s^2} - \frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s} + \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{e^{-4s}}{s} + \frac{e^{-5s}}{s^2} + 2\frac{e^{-5s}}{s^2} \\ &= \frac{1-e^{-3s}-e^{-4s}+3e^{-5s}}{s^2} - \frac{2e^{-3s}}{s} \end{aligned}$$

1.2 Question 2

We can use the theorem for periodic functions,

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt$$

Substituting the function $f(t) = \sin(t)$ with period $T = 2\pi$, we get

$$L\{\sin(t)\} = \frac{1}{1-e^{-s2\pi}} \int_0^{2\pi} \sin(t)e^{-st}dt$$

Using integration by parts, we can simplify the integral.

$$\begin{aligned} \int_0^{2\pi} \sin(t)e^{-st}dt &= \frac{e^{-st}\sin(t)}{-s}\Big|_0^{2\pi} + \frac{1}{s}\int_0^{2\pi} e^{-st}\cos(t)dt \\ \int_0^{2\pi} \sin(t)e^{-st}dt &= \frac{e^{-st}\sin(t)}{-s}\Big|_0^{2\pi} + \frac{1}{s}\left(\frac{e^{-st}\cos(t)}{-s}\Big|_0^{2\pi} - \frac{1}{s}\int_0^{2\pi} e^{-st}\sin(t)dt\right) \\ \left(1+\frac{1}{s^2}\right)\int_0^{2\pi} \sin(t)e^{-st}dt &= \frac{e^{-st}\sin(t)}{-s}\Big|_0^{2\pi} + \frac{1}{s}\frac{e^{-st}\cos(t)}{-s}\Big|_0^{2\pi} \\ \frac{s^2+1}{s^2}\int_0^{2\pi} \sin(t)e^{-st}dt &= \frac{1}{s}\left(\frac{e^{-s2\pi}}{-s} - \frac{e^0}{-s}\right) \\ \int_0^{2\pi} \sin(t)e^{-st}dt &= \frac{1-e^{-s2\pi}}{s^2+1} \end{aligned}$$

Substituting into the origin expression,

$$\begin{aligned} L\{\sin(t)\} &= \frac{1}{1-e^{-s2\pi}} \int_0^{2\pi} \sin(t)e^{-st}dt \\ &= \frac{1}{1-e^{-s2\pi}} \frac{1-e^{-s2\pi}}{s^2+1} \\ &= \frac{1}{s^2+1} \end{aligned}$$

Therefore the Laplace transform of $\sin(t)$ is $\frac{1}{s^2+1}$.

2 Problem Set 2

2.1 Question 1

$$cos(4t), 0 < t < 2\pi$$

We know that

$$\begin{aligned} L\{cos(4t)\} &= \frac{1}{1-e^{-sT}} \int_0^T cos(4t)e^{-st}dt \\ &= \frac{1}{1-e^{-s2\pi}} \int_0^{2\pi} cos(4t)e^{-st}dt \end{aligned}$$

Use integration by parts to find the integral

$$\begin{aligned} \int_0^{2\pi} \cos(4t)e^{-st}dt &= \frac{e^{-st}\cos(4t)}{-s}\Big|_0^{2\pi} - \frac{4}{s}\int_0^{2\pi} e^{-st}\sin(4t)dt \\ &= \frac{e^{-st}\cos(4t)}{-s}\Big|_0^{2\pi} - \frac{4}{s}\left(\frac{e^{-st}\sin(4t)}{-s}\Big|_0^{2\pi} + \frac{4}{s}\int_0^{2\pi} e^{-st}\cos(4t)dt\right) \\ \left(1+\frac{16}{s^2}\right)\int_0^{2\pi} \cos(4t)e^{-st}dt &= \frac{e^{-st}\cos(4t)}{-s}\Big|_0^{2\pi} - \frac{4}{s}\frac{e^{-st}\sin(4t)}{-s}\Big|_0^{2\pi} \\ \left(\frac{s^2+16}{s^2}\right)\int_0^{2\pi} \cos(4t)e^{-st}dt &= \frac{e^{-s2\pi}}{-s} - \frac{1}{-s} - 0 + 0 \\ &= \frac{1-e^{-s2\pi}}{s} \\ \int_0^{2\pi} \cos(4t)e^{-st}dt &= \frac{s(1-e^{-s2\pi})}{s^2+16} \end{aligned}$$

Substitute back into the original equation

$$\begin{aligned} L\{cos(4t)\} &= \frac{1}{1-e^{-s2\pi}} \int_0^{2\pi} cos(4t)e^{-st}dt \\ &= \frac{1}{1-e^{-s2\pi}} * \frac{s(1-e^{-s2\pi})}{s^2+16} \\ &= \frac{s}{s^2+16} \end{aligned}$$

So the Laplace transform of $cos(4t)$ is $\frac{s}{s^2+16}$.

2.2 Question 2

Assuming $f(t) = 0$ when $t \leq 0$ and $t \geq 4$. As the question is unclear.

$$x'' - x = f(t), x(0) = 0, x'(0) = 0, f(t) = twhen0 < t < 4$$

Turn into a heaviside function

$$x'' - x = t - 4u(t-4) - (t-4)u(t-4)$$

Take the Laplace of both sides

$$
\begin{aligned}
&x'' - x = t - 4u(t-4) - (t-4)u(t-4)\\
&s^2X(s) - X(s) = \frac{1}{s^2} - \frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2}\\
&(s^2-1)X(s) = \frac{1}{s^2} - \frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2}\\
&X(s) = \frac{1}{s^2(s-1)(s+1)} - \frac{4e^{-4s}}{s(s-1)(s+1)} - \frac{e^{-4s}}{s^2(s-1)(s+1)}
\end{aligned}
$$

Use partial fraction decomposition.

$$
\begin{aligned}
&\frac{1}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}\\
&1 = As^3 - As + Bs^2 - B + Cs^3 + Cs^2 + Ds^3 - Ds^2\\
&A + C + D = 0, B + C - D = 0, -A = 0, -B = 1\\
&A = 0, B = -1, C = \frac{1}{2}, D = -\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}\\
&1 = As^2 - A + Bs^2 + Bs + Cs^2 - Cs\\
&A + B + C = 0, B - C = 0, -A = 1\\
&A = -1, B = \frac{1}{2}, C = \frac{1}{2}
\end{aligned}
$$

Putting it all together

$$
\begin{aligned}
&X(s) = \frac{1}{s^2(s-1)(s+1)} - \frac{4e^{-4s}}{s(s-1)(s+1)} - \frac{e^{-4s}}{s^2(s-1)(s+1)}\\
&X(s) = -\frac{1}{s^2} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + 4e^{-4s}\left(-\frac{1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}\right) + e^{-4s}\left(-\frac{1}{s^2} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right)\\
&X(s) = -\frac{1}{s^2} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + e^{-4s}\left(-\frac{4}{s} - \frac{1}{s^2} + \frac{5}{2}\frac{1}{s-1} + \frac{3}{2}\frac{1}{s+1}\right)\\
&x(t) = -t + \frac{e^t}{2} - \frac{e^{-t}}{2} + u(t-4)\left(-4 - (t-4) + \frac{5e^{t-4}}{2} + \frac{3e^{-(t-4)}}{2}\right)
\end{aligned}
$$