Assignment 3

Jerry Jiang
TianQi Shi

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1 Problem Set 1

1.1 Problem 1

a) $y'' + 3y' - 10y = 0$

$$\begin{aligned}
&\lambda^2 + 3\lambda - 10 = 0 \\
&(\lambda + 5)(\lambda - 2) = 0 \\
&\therefore \lambda = 2, -5
\end{aligned}$$

So $y = c_1 e^{2x} + c_2 e^{-5x}$.
b) $y'' + 2y' + 3y = 0$

$$\begin{aligned}
&\lambda^2 + 2\lambda + 3 = 0 \\
&\lambda = \frac{-2 \pm \sqrt{2^2 - 4 * 1 * 3}}{2} \\
&\lambda = \frac{-2 \pm \sqrt{4 - 12}}{2} \\
&\lambda = \frac{-2 \pm i\sqrt{8}}{2} \\
&\lambda = -1 \pm i\sqrt{2}
\end{aligned}$$

So $y = e^{-x}(c_1 \sin(\sqrt{2}x) + c_1 \cos(\sqrt{2}x))$.

1.2 Problem 2

$$y'' - 3y' = e^x + x$$

First find homogeneous solution.

$$\begin{aligned}
&\lambda^2 - 3\lambda = 0 \\
&\lambda(\lambda - 3) = 0 \\
&\lambda = 0, 3
\end{aligned}$$

So $y_h = c_1 e^{0x} + c_2 e^{3x}$.
Second find particular solution. Guess $y_p = Ae^x + Bx + C$.
C is part of homogeneous solution. Guess $y_p = Ae^x + Bx^2 + Cx$ instead.

$$\begin{aligned}
&y_p' = Ae^x + 2Bx + C \\
&y_p'' = Ae^x + 2B
\end{aligned}$$

$$\begin{aligned}
&y'' - 3y' = e^x + x \\
&Ae^x + 2B - 3(Ae^x + 2Bx + C) = e^x + x \\
&(A - 3A)e^x - 6Bx + (2B - 3C) = e^x + x
\end{aligned}$$

Find A.

$$\begin{aligned} A - 3A &= 1 \\ -2A &= 1 \\ A &= -\frac{1}{2} \end{aligned}$$

Find B.

$$\begin{aligned} -6B &= 1 \\ B &= -\frac{1}{6} \end{aligned}$$

Find C.

$$\begin{aligned} 2B - 3C &= 0 \\ -\frac{2}{6} &= 3C \\ C &= -\frac{1}{9} \end{aligned}$$

So $y_p = -\frac{1}{2}e^x - \frac{1}{6}x^2 - \frac{1}{9}x$.
So $y = -\frac{1}{2}e^x - \frac{1}{6}x^2 - \frac{1}{9}x + c_1 + c_2e^{3x}$.

2 Problem Set 2

2.1 Problem 1

We can show that a set of equations are linearly dependent if there exists a set of non-zero coefficients for the terms such that the sum is equal to 0.

a) $\{0, e^{x^2}\cos(\arctan x)\}$

$$1(0) + 0(e^{x^2}\cos(\arctan x)) = 0$$

b) $\{e^x, e^{x+\pi}, \sin x\}$

$$ce^x + e^{x+\pi} + 0(\sin x) = 0$$

This statement holds where $c = -e^{\pi}$.

c) $\{e^{-x}, \sinh x, \cosh x\}$

$$\sinh x - \cosh x + e^{-x} = 0$$

We know this statement holds by expanding the hyperbolic functions.

$$\begin{aligned} \sinh x - \cosh x + e^{-x} &= \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2} + e^{-x} \\ &= -e^{-x} + e^{-x} \\ &= 0 \end{aligned}$$

d) $\{2x^2-1, 5, 1-x^2\}$

$$\begin{aligned}&(2x^2-1)+2(1-x^2)-\frac{1}{5}(5)\\&=2x^2-1+2-2x^2-1\\&=0\end{aligned}$$

2.2 Problem 2

First, we solve the homogeneous case using the characteristic equation,

$$\lambda^2+3\lambda-2=0$$

Using the quadratic formula,

$$\lambda=\frac{-3\pm\sqrt{17}}{2}$$

Thus,

$$y_h=c_1e^{\frac{-3+\sqrt{17}}{2}}+c_2e^{\frac{-3-\sqrt{17}}{2}}$$

For the particular solution, we use the method of undetermined coefficients. Derivatives of $e^x(x^2+1)$ gives us $\{x^2e^x, xe^x, e^x\}$, so we try

$$\begin{aligned}y_p&=Ax^2e^x+Bxe^x+Ce^x\\y_p'&=Ax^2e^x+(2A+B)xe^x+(B+C)e^x\\y_p''&=Ax^2e^x+(4A+B)xe^x+(2A+2B+C)e^x\end{aligned}$$

Substituting into ODE,

$$\begin{aligned}&Ax^2e^x+(4A+B)xe^x+(2A+2B+C)e^x+3(Ax^2e^x+(2A+B)xe^x+(B+C)e^x)-2(Ax^2e^x+Bxe^x+Ce^x)\\&=e^x((A+3A-2A)x^2+(4A+B+6A+3B-2B)x+(2A+2B+C+3B+3C-2C))\end{aligned}$$

Comparing coefficients with RHS $e^x(x^2+1)$, we get

$$\begin{aligned}A&=\frac{1}{2}\\B&=\frac{-5}{2}\\C&=\frac{25}{4}\end{aligned}$$

Thus $y_p=e^x(\frac{1}{2}x^2-\frac{5}{2}x+\frac{25}{4})$. So the general solution is:

$$y=y_h+y_p=c_1e^{\frac{-3+\sqrt{17}}{2}}+c_2e^{\frac{-3-\sqrt{17}}{2}}+e^x(\frac{1}{2}x^2-\frac{5}{2}x+\frac{25}{4})$$