Assignment 2

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1 Problem Set 1

1.1 Problem 1

a) $y' + 3x^2y^2 = 0, y(0) = 1$

$$\begin{aligned}
y' + 3x^2y^2 &= 0, y(0) = 1 \\
y' &= -3x^2y^2 \\
\frac{dy}{dx} &= -3x^2y^2 \\
\frac{dy}{y^2} &= -3x^2dx \\
\int \frac{dy}{y^2} &= \int -3x^2dx \\
\frac{-1}{y} &= -x^3 + C \\
-1 &= (-x^3 + C)y \\
y &= \frac{-1}{-x^3 + C}
\end{aligned}$$

Particular solution:

$$\begin{aligned}
y &= \frac{-1}{-x^3 + C} \\
1 &= \frac{-1}{0 + C} \\
1 &= \frac{-1}{C} \\
C &= -1
\end{aligned}$$

$$\therefore y = \frac{-1}{-x^3 - 1} = \frac{1}{x^3 + 1}$$

b) $x^2y' = 1 + y^2$

$$\begin{aligned}
x^2y' &= 1 + y^2 \\
\frac{dyx^2}{dx} &= 1 + y^2 \\
\frac{dy}{1 + y^2} &= \frac{1}{x^2}dx \\
\int \frac{dy}{1 + y^2} &= \int \frac{1}{x^2}dx \\
\arctan y &= \frac{-1}{x} + C \\
y &= \tan(\frac{-1}{x} + C)
\end{aligned}$$

1.2 Problem 2

$$
\begin{aligned}
&(x-1)y' + \frac{2(x-1)y}{x} = (x-1)(x+1) \\
&y' + \frac{2y}{x} = x+1 && (x \neq 1) \\
&\sigma y' + \sigma\frac{2y}{x} = \sigma(x+1)
\end{aligned}
$$

Use integrating factor method:

$$
\begin{aligned}
&(\sigma y)' = \sigma y' + \sigma' y = \sigma y' + \sigma\frac{2y}{x} \\
&\sigma' y = \sigma\frac{2y}{x} \\
&\sigma' = \sigma\frac{2}{x} \\
&\frac{d\sigma}{dx} = \sigma\frac{2}{x} \\
&\frac{d\sigma}{\sigma} = \frac{2}{x}dx \\
&\int\frac{d\sigma}{\sigma} = \int\frac{2}{x}dx \\
&\ln|\sigma| = 2\ln|x| + C \\
&|\sigma| = e^C|x|^2 && (x \neq 0) \\
&\sigma = Dx^2
\end{aligned}
$$

Calculate y:

$$
\begin{aligned}
&\frac{dDx^2y}{dx} = Dx^2(x+1) \\
&\frac{dx^2y}{dx} = x^2(x+1) \\
&dx^2y = x^2(x+1)dx \\
&dx^2y = x^3 + x^2dx \\
&\int dx^2y = \int x^3 + x^2dx \\
&x^2y = \frac{x^4}{4} + \frac{x^3}{3} + C \\
&y = \frac{x^2}{4} + \frac{x}{3} + \frac{C}{x^2} && (x \neq 0)
\end{aligned}
$$

In addition, $x \neq 1$, since we divided the original equation by $(x-1)$.
Find particular solution:

$$y = \frac{x^2}{4} + \frac{x}{3} + \frac{C}{x^2}$$
$$\frac{7}{3} = \frac{2^2}{4} + \frac{2}{3} + \frac{C}{2^2}$$
$$\frac{7}{3} = 1 + \frac{2}{3} + \frac{C}{4}$$
$$\frac{7}{3} = \frac{5}{3} + \frac{C}{4}$$
$$\frac{2}{3} = \frac{C}{4}$$
$$\frac{8}{3} = C$$

$$\therefore y = \frac{x^2}{4} + \frac{x}{3} + \frac{8}{3x^2}$$

2 Problem Set 2

2.1 Problem 1

Given the linear equation $xy' - y = x - 1$, solve for $y(x)$.
a) $y(e) = 1$

Using the integrating factor method, first multiply all terms by the integrating factor σ:

$$xy' - y = x - 1$$
$$y' - \frac{y}{x} = 1 - \frac{1}{x}$$
$$\sigma y' - \sigma\frac{y}{x} = \sigma - \sigma\frac{1}{x}$$

Set the left hand side to $(\sigma y)'$:

$$(\sigma y)' = \sigma y' - \sigma\frac{y}{x}$$
$$\sigma y' + \sigma' y = \sigma y' - \sigma\frac{y}{x}$$
$$\sigma' y = -\sigma\frac{y}{x}$$
$$\frac{d\sigma}{dx} = -\frac{\sigma}{x}$$
$$\frac{d\sigma}{\sigma} = -\frac{dx}{x}$$
$$\ln|\sigma| = -\ln|x|$$
$$\sigma = \frac{1}{x}$$

Then set the right hand side to $\frac{d}{dx}(\sigma y)$:

$$\begin{aligned}
\frac{d}{dx}(\frac{y}{x}) &= \frac{1}{x} - \frac{1}{x}\frac{1}{x} \\
\int d(\frac{y}{x}) &= \int \frac{1}{x} - \frac{1}{x^2} dx \\
\frac{y}{x} &= \ln|x| + \frac{1}{x} + c \\
y &= x\ln|x| + cx + 1
\end{aligned}$$

Substituting initial conditions,

$$\begin{aligned}
1 &= e\ln|e| + ce + 1 \\
0 &= (c+1)e \\
c &= -1
\end{aligned}$$

$$\therefore y = x\ln|x| - x + 1$$

b) $y(-1) = 1$
Using the same general solution, substitute initial conditions.

$$\begin{aligned}
1 &= (-1)\ln|(-1)| + c(-1) + 1 \\
c &= 0
\end{aligned}$$

$$\therefore y = x\ln|x| + 1$$

c) $y(0) = 1$

$$1 = (0)\ln|(-1)| + c(-1) - 1$$

2.2 Problem 2

Solve for $y(x)$ given initial condition y(0) = -2:

$$\begin{aligned}
y' &= \frac{e^x}{2y} \\
\frac{dy}{dx} &= \frac{e^x}{2y} \\
\int 2y dy &= \int e^x dx \\
y^2 &= e^x + c \\
y &= e^{\frac{x}{2}} + c
\end{aligned}$$

Substituting initial conditions,

$$\begin{aligned}
-2 &= e^{\frac{0}{2}} + c \\
c &= -2
\end{aligned}$$

Thus $y(x) = e^{\frac{x}{2}} - 2$