Problem Set 1 - 411-3

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1 Solving a Dynamic Programming Problem

[11]:
```
import numpy as np
# import scipy
from scipy.optimize import brentq    #equation solver
from scipy import optimize
from scipy.stats import norm
from scipy.optimize import minimize
import matplotlib.pyplot as plt
import numpy.polynomial.chebyshev as chebyshev
import numba
from scipy import interpolate
```

1.1 Value Function Iteration

[12]:
```
alpha  = 0.33
beta   = 0.98
delta  = 0.13
A      = 2

@numba.njit
def f(k):
    return A*k**alpha + (1-delta)*k
@numba.njit
def fk(k):
    return A*alpha*k**(alpha-1) + (1-delta)

@numba.njit
def util(c):
    return np.log(c)
```

[13]:
```
kstar = ((1/beta - 1 + delta)/A/alpha)**(1/(alpha-1))
kstar
```

[13]: 9.091084887868014

[14]:
```
assert np.isclose(beta*fk(kstar), 1)
```

[15]:
```python
# grid
k = np.linspace(0.01, 15, 1500)
```

[16]:
```python
# Compute value function today, from guess Value function tomorrow
@numba.njit
def Vendog(kplus, k, Vplus):
        value = util(max(f(k) - kplus, 1E-10)) + beta * Vplus # avoid negative␣
 ↪consumption
        return  value

# Backward iteration
@numba.njit
def backward_iterate(maxindex, V, k):

    maxindex = np.empty_like(k)
    # loop over the grid k
    for ik, k_cur in enumerate(k):

        value = np.empty_like(k)

        for ik2, k_cur2 in enumerate(k):

            #compute value function for all availabe saving choices
            value[ik2] = Vendog(k[ik2], k_cur, V[ik2])

        # find optimal saving decision
        index         = np.argmax(value)
        maxindex[ik]  = index
        # compute new value function
        V[ik]     = value[index]

    return maxindex, V
```

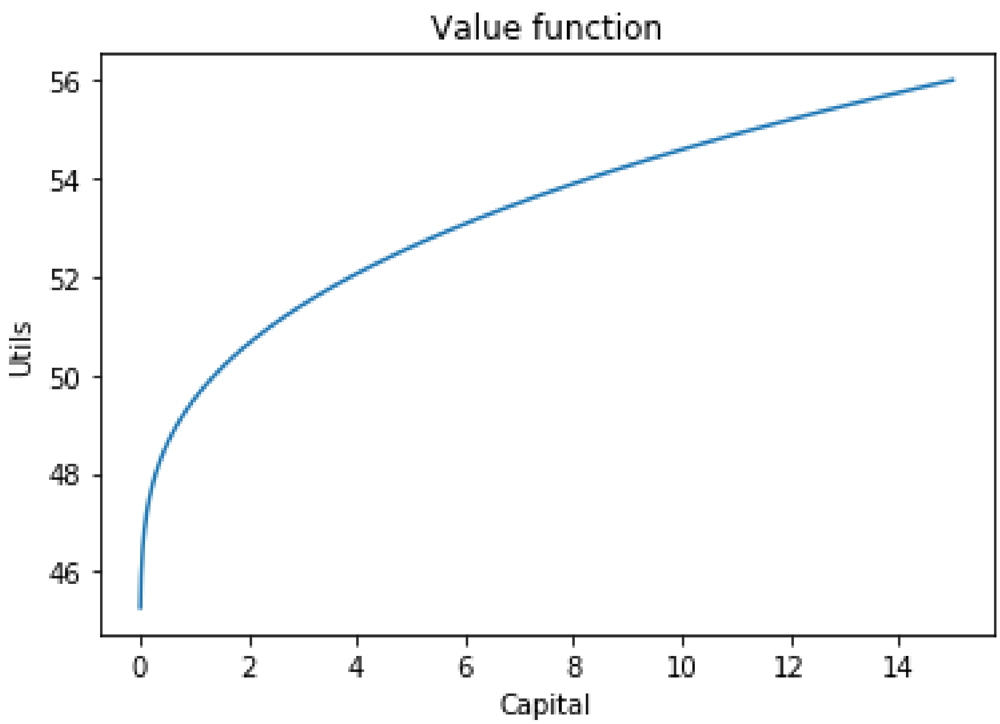
[17]:
```python
# Iteration algorithm
@numba.njit
def ss_policy(k):

    maxindex = np.zeros_like(k)
    V    = np.zeros_like(k)
    Vold    = np.zeros_like(k)

    for it in range(2000):

        maxindex, V = backward_iterate(maxindex, V, k)

        if it % 10 == 1 and  np.linalg.norm(V - Vold) < 1E-5:
```

```
            print("convergence in", it, " iterations!")
            return maxindex.astype(np.int32), V


        Vold = np.copy(V) # avoid updating both Vold and V by making a copy
```

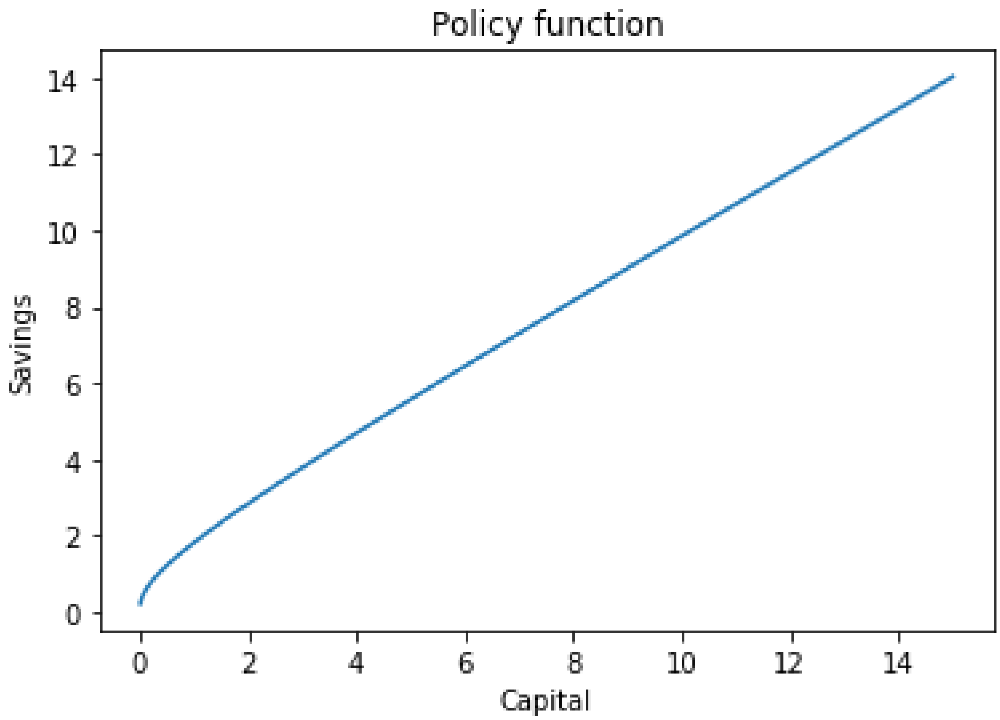
[78]:
```
%time maxindex, V = ss_policy(k)
```

```
convergence in 281  iterations!
CPU times: user 2.33 s, sys: 0 ns, total: 2.33 s
Wall time: 2.33 s
```

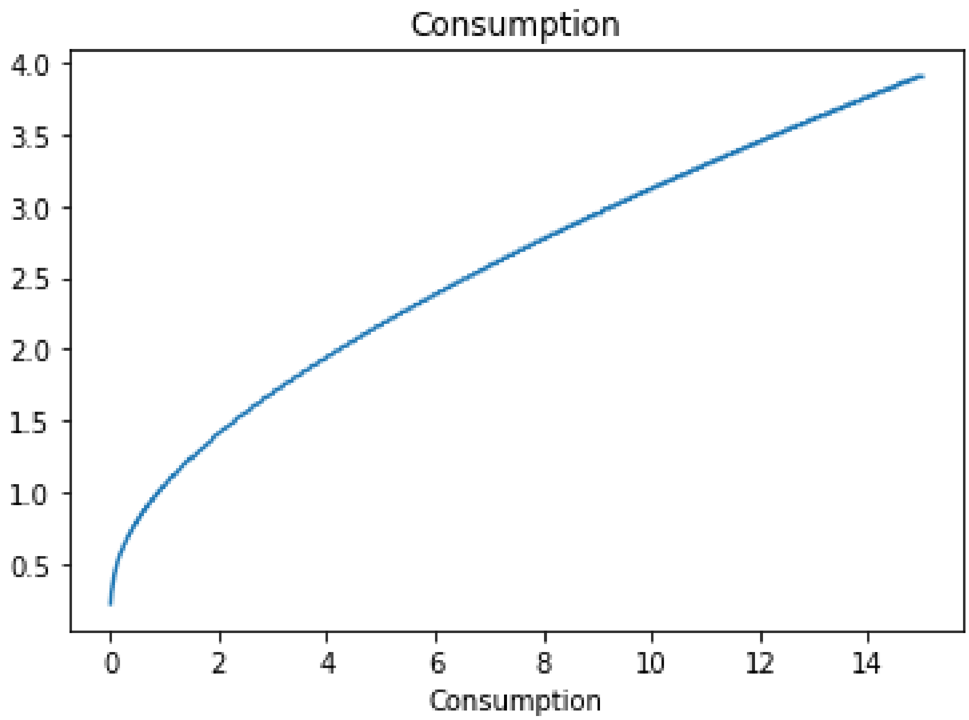
[37]:
```
plt.plot(k, V)
plt.xlabel("Capital")
plt.ylabel('Utils')
plt.title('Value function');
```



[38]:
```
plt.plot(k, k[maxindex.astype(int)])
plt.xlabel("Capital")
plt.ylabel("Savings")
plt.title('Policy function');
```



[39]:
```
c = f(k) - k[maxindex.astype(int)]
plt.plot(k, c)
plt.xlabel("Capital")
plt.xlabel("Consumption")
plt.title('Consumption');
```



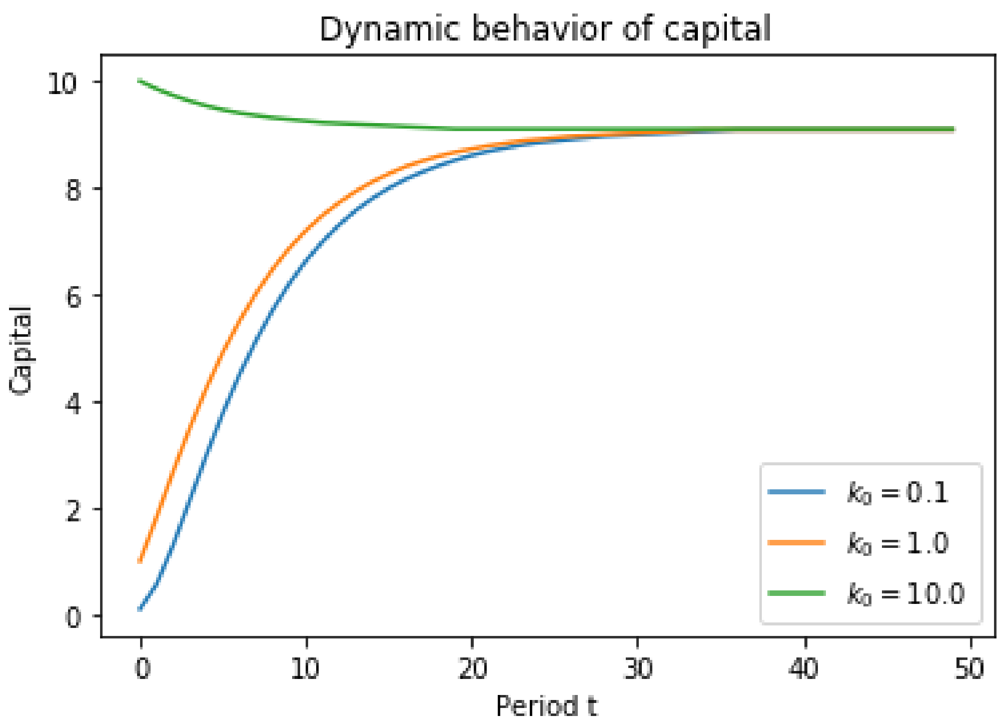
[40]:
```
## Steady state level of capital
kstar = k[k[maxindex] == k]
kstar
```

[40]: array([9.08802803, 9.10303303])

[41]:
```
@numba.njit
def policy_interp(x, k, maxindex):
    policy = np.interp(x, k, k[maxindex])
    return policy

@numba.njit
def timeseries(start, periods, k, maxindex):
    series = np.empty(periods)
    series[0] = start
    for t in range(periods-1):
        series[t+1]  = policy_interp(series[t], k, maxindex)
    return series
```

[44]:
```
start = np.array([0.1, 1, 10])
T = 50

for element in start:
```

```
    series = timeseries(element, T, k, maxindex)
    plt.plot(range(T),
    series, label=r'$k_0 = {j} $'.format(j=element))
plt.legend()
plt.xlabel('Period t')
plt.ylabel('Capital')
plt.title('Dynamic behavior of capital');
```



This model predicts rapid convergence of capital towards the steady state over the medium run. It would predict rapid convergence of wealth across countries.

1.1.1 Howard policy improvement

[6]:
```
# Compute value function today, from guess Value function tomorrow
@numba.njit
def Vendog(kplus, k, Vplus):
        value = util(max(f(k) - kplus, 1E-10)) + beta * Vplus # avoid negative␣
 ↪consumption
        return  value

# Backward iteration
@numba.njit
def backward_iterate(maxindex, V, k):
```

```
    maxindex = np.empty_like(k)
    # loop over the grid k
    for ik, k_cur in enumerate(k):

        value = np.empty_like(k)

        for ik2, k_cur2 in enumerate(k):

            #compute value function for all availabe saving choices
            value[ik2] = Vendog(k[ik2], k_cur, V[ik2])

        # find optimal saving decision
        index         = np.argmax(value)
        maxindex[ik]  = index
        # compute new value function
        V[ik]     = value[index]

    return maxindex, V

@numba.njit
def U_pol(maxindex, k):
    U = np.empty_like(k)

    for ik, k_cur in enumerate(k):
        U[ik] = util(max(f(k_cur) - k[maxindex[ik]], 1E-6))
    return U


# Howard Value iteration for given policy
@numba.njit
def Howard(maxindex, k, V_guess, UC):

    V_update = np.empty_like(V_guess)

    for ik, k_cur in enumerate(k):

        V_update[ik] = UC[ik] + beta * V_guess[ik]

    return V_update
```

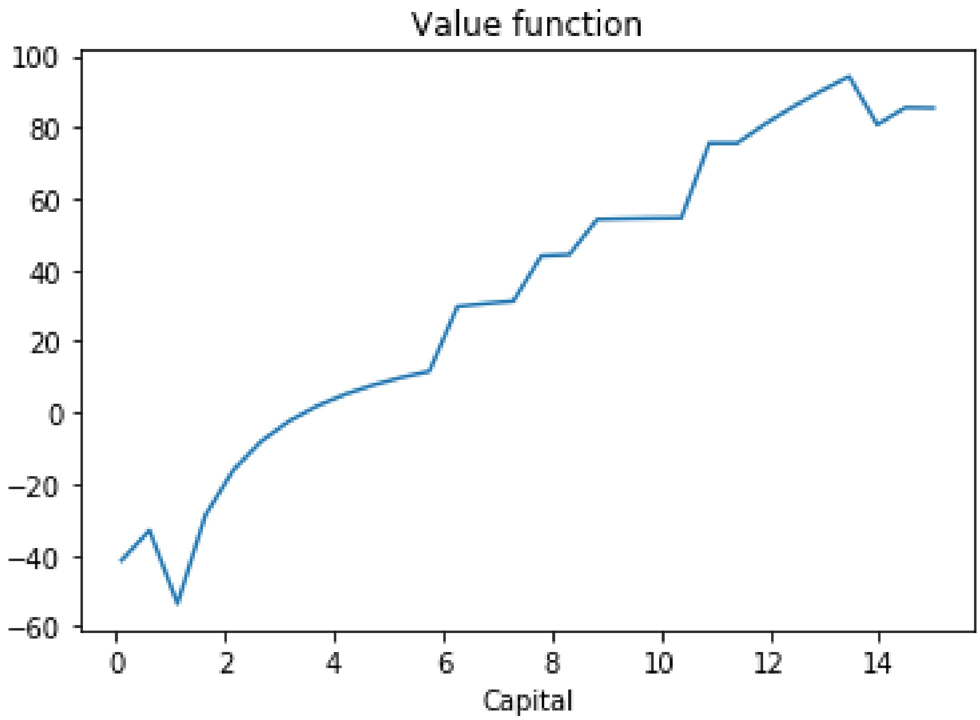
[7]:
```
# Iteration algorithm
@numba.njit
def ss_policy_howard(k, howard=1):

    maxindex = np.zeros_like(k)
    V    = np.zeros_like(k)
```

```
    Vold    = np.zeros_like(k)

    for it in range(20000):

        maxindex, V = backward_iterate(maxindex, V, k)
        print(np.linalg.norm(V - Vold))
        if it % 10 == 1 and  np.linalg.norm(V - Vold) < 1E-5:

            print("convergence in", it, " iterations!")
            return maxindex.astype(np.int32), V

        Vold = np.copy(V) # avoid updating both Vold and V by making a copy



        if howard == 1:
            V_h = np.copy(V)
            V_hold = np.copy(V)
            ind = maxindex.astype(np.int32)
            UC = U_pol(ind, k)        # Compute utility conditional on grid and␣
↪policy function

            for it2 in range(2000):

                V_h = Howard(ind, k, V_h, UC)
#                 print(np.linalg.norm(V_h - V_hold))
                if it % 10 == 1 and  np.linalg.norm(V_h - V_hold) < 1E-10:
#                     print("convergence in", it2, " iterations!")
                    V = np.copy(V_h)
                    Vold = np.copy(V_h)
                    break

                elif it2 == 2000:

                        print('Howard Improvement did not converge!');

                V_hold = np.copy(V_h)
```

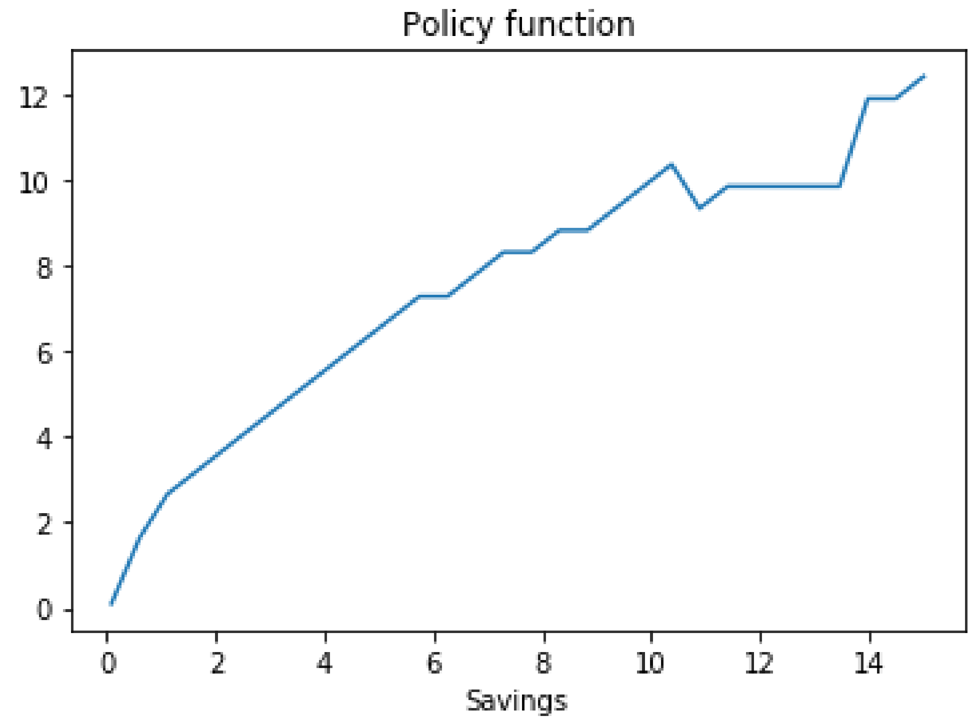
[ ]:
```
%time maxindex_howard, V_howard = ss_policy_howard(k)
```

The algorithm is not converging with Howard improvement

[118]:
```
plt.plot(k, V_howard)
plt.xlabel("Capital")
plt.title('Value function');
```



[119]:
```
plt.plot(k, k[maxindex_howard.astype(int)])
plt.xlabel("Capital")
plt.xlabel("Savings")
plt.title('Policy function');
```



1.2 Policy Function Iteration - Method I

1.2.1 Euler equation

First-order and envelope conditions

$$u'(c(k)) = \beta[V_k(k_+)]$$
$$V_k(k) = f_k(k)u'(c(k))$$

can be combined to obtain a Euler equation

$$u'(c(k)) = \beta\mathbb{E}[f_k(k_+(k))u'(c(k_+(k)))]$$

[27]:
```
@numba.njit
def up(c):
    return 1/c
```

[76]:
```
# @numba.jit
def Euler_err_cubic(kplus, tck, k):
    err = up(f(k) - kplus) - beta * fk(kplus) * up(interpolate.splev(kplus, tck))
    return err

# @numba.jit
```

```
def backward_iterate_cubic(cplus, k):

    kplus = np.empty(len(k))
    cendog = np.empty(len(k))
    tck = interpolate.splrep(k, cplus)
#     fun = lambda x: Euler_err_cubic(x, tck, k)

    for ik, k_cur in enumerate(k):

        res = optimize.root(Euler_err_cubic, cplus[ik], args=(tck, k_cur),␣
↪method='hybr')
        kplus[ik] = res.x                      # Value of the root
        cendog[ik] = f(k_cur) - kplus[ik]     # Budget constraint

    return cendog
```

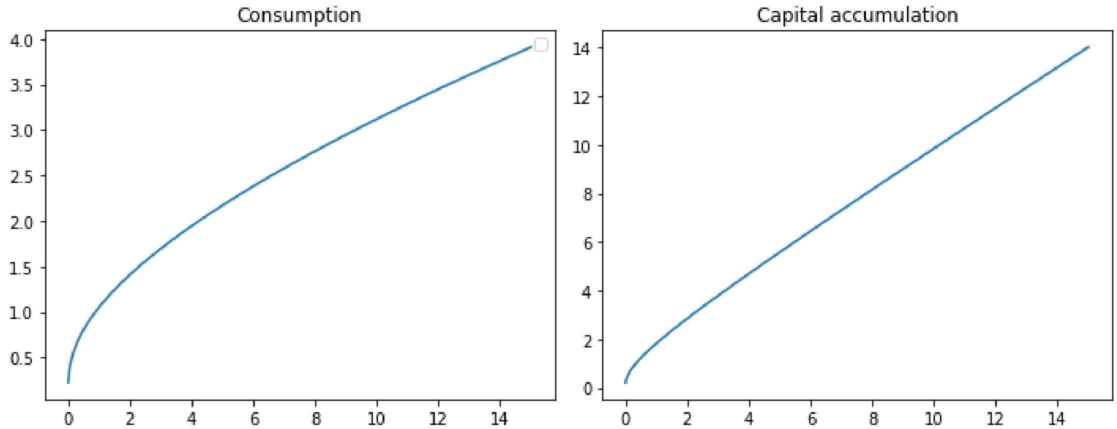
[77]:
```
def ss_policy_cubic(k):
    c = np.empty(len(k))
    cplus = 0.2*k
    for it in range(1000):

        c = backward_iterate_cubic(cplus, k)
#         print(np.linalg.norm(cplus - c))
        if it % 10 == 1 and  np.linalg.norm(cplus - c) < 1E-10:
#             print(f'convergence in {it} iterations!')
            return interpolate.splrep(k, c)
        cplus = np.copy(c)
```

[78]:
```
%time tck = ss_policy_cubic(k)
```

```
CPU times: user 24.7 s, sys: 48.1 ms, total: 24.7 s
Wall time: 24.5 s
```

[62]:
```
c = interpolate.splev(k, tck)
savings = f(k) - c
_, (ax1, ax2) = plt.subplots(1, 2, figsize=(10, 4))
ax1.set_title('Consumption')
ax2.set_title('Capital accumulation')
ax1.plot(k, c)
ax2.plot(k, savings)
ax1.legend()
plt.tight_layout()
```

```
No handles with labels found to put in legend.
```



[124]:
```
#Find a fixed point
fun = lambda x: f(x) - interpolate.splev(x, tck)
kstar = optimize.fixed_point(fun, [7, 10])
kstar
```

[124]: array([9.09108489, 9.09108489])

1.2.2 Policy Function Iteration - Method II

[21]:
```
def BC(kplus, kpplus):
    #Get consumption from budget constraint
    cons = f(kplus) - kpplus
    if cons < 0:
        return 10E-10
    else:
        return cons

def Euler(cplus, kplus):
    # Get consumption from Euler
    euler = cplus / (beta * fk(kplus))
    return euler

def backward_iterate_II(kplus, k, kbar):
    # init
    kpplus = np.empty(len(k))
    cendog = np.empty(len(k))
    kendog = np.empty(len(k))
    cplus = np.empty(len(k))

    # Use policy function to get K''
    tck = interpolate.splrep(k, kplus)
```

```python
    kpplus = interpolate.splev(kplus, tck)
#     print(kpplus)

    # Use BC tomorrow, Euler today and BC today to get back policy func
    for ik, k_cur in enumerate(k):

        cplus[ik]  = BC(kplus[ik], kpplus[ik])    #Budget constraint tomorrow
        cendog[ik] = Euler(cplus[ik], kplus[ik])  # Euler equation today
        kendog[ik] = min(max(f(k[ik]) - cendog[ik], 0), kbar) # Budget␣
↪constraint today lower bar = 0, upper bar to define

    return kendog
```

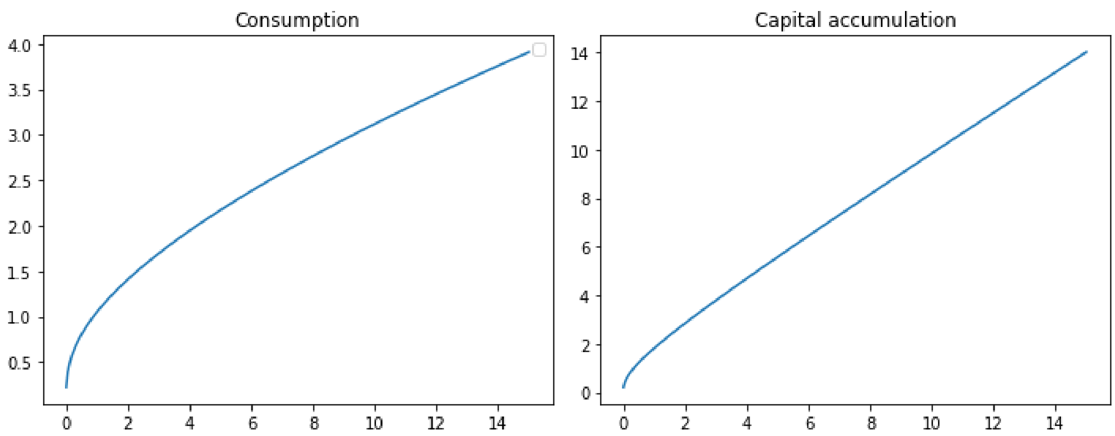
[22]:
```python
def ss_policy_II(k, kbar):

    knew = np.empty(len(k))
    kplus = 0.5*k

    for it in range(1000):
#         print("Iteration number:", it)
        knew = backward_iterate_II(kplus, k, kbar)

        if it % 10 == 1 and  np.linalg.norm(knew - kplus) < 1E-10:
            print(f'convergence in {it} iterations!')
            tck_out = interpolate.splrep(k, knew)
            return tck_out

        kplus = np.copy(knew)
```

[6]:
```python
%time tck = ss_policy_II(k, 5*kstar)
```

```
convergence in 111 iterations!
CPU times: user 217 ms, sys: 3.89 ms, total: 221 ms
Wall time: 220 ms
```

[7]:
```python
savings = interpolate.splev(k, tck)
c = f(k) - savings
_, (ax1, ax2) = plt.subplots(1, 2, figsize=(10, 4))
ax1.set_title('Consumption')
ax2.set_title('Capital accumulation')
ax1.plot(k, c)
ax2.plot(k, savings)
ax1.legend()
plt.tight_layout()
```

```
No handles with labels found to put in legend.
```



[50]:
```
#Find a fixed point
fun = lambda x: interpolate.splev(x, tck)
kstar = optimize.fixed_point(fun, [7, 10])
kstar
```

[50]:
```
array([9.09108489, 9.09108489])
```

1.2.3 Endogenous Grid Points Method

[23]:
```
def up_inv(c):
    return 1/c
def up(c):
    return 1/c
```

[24]:
```
def get_cendog(cplus, fk, up, up_inv, beta, k):
    return up_inv(beta * (fk(k) *up(cplus)))

def backward_iterate_endog(cplus, k):                 # Should interpret k as k␣
 ↪tomorrow here
    c_endog = up_inv(beta * (fk(k) *up(cplus)))       # if save k tomorrow, what␣
 ↪would be the optimal c today
    c = np.empty_like(c_endog)
    tck = interpolate.splrep(c_endog + k, c_endog)    # cendog + k is production␣
 ↪today
    c = interpolate.splev(f(k), tck)
    return c
```

[25]:
```
def ss_policy_endog(k):
    c = 0.3*k
    for it in range(2000):
        c = backward_iterate_endog(c, k)
```

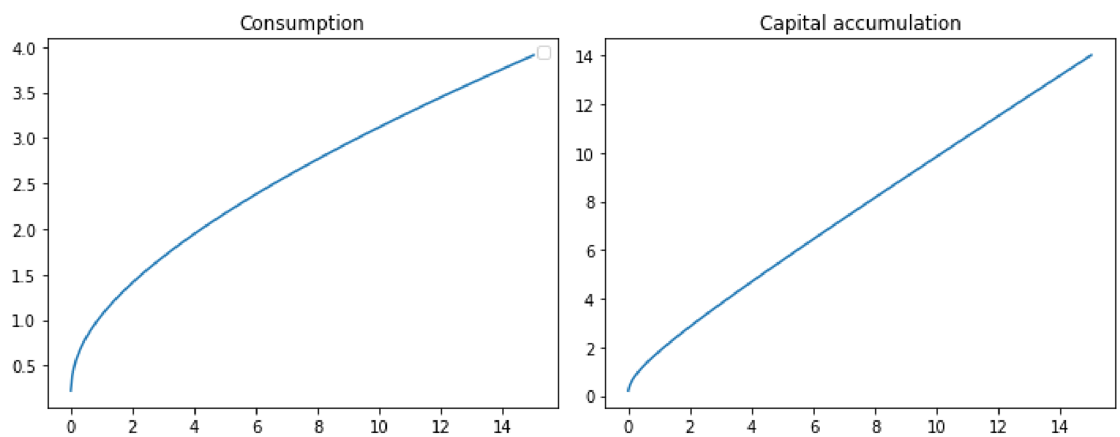
```
        if it % 10 == 1 and  np.max(np.abs(c - cold)) < 1E-10:
            print(f'convergence in {it} iterations!')
            return c
        cold = c
```

[56]:
```
%prun c = ss_policy_endog(k)
```

```
convergence in 111 iterations!
```

[27]:
```
savings = f(k) - c
_, (ax1, ax2) = plt.subplots(1, 2, figsize=(10, 4))
ax1.set_title('Consumption')
ax2.set_title('Capital accumulation')
ax1.plot(k, c)
ax2.plot(k, savings)
ax1.legend()
plt.tight_layout()
```

```
No handles with labels found to put in legend.
```



1.2.4 Comparing methods

[26]:
```
%time maxindex, V = ss_policy(k)
%time tck          = ss_policy_cubic(k)
%time tck          = ss_policy_II(k, 5*kstar)
%time c            = ss_policy_endog(k)
```

```
convergence in 281  iterations!
CPU times: user 2.36 s, sys: 0 ns, total: 2.36 s
Wall time: 2.36 s
convergence in 141 iterations!
CPU times: user 25.2 s, sys: 25.4 ms, total: 25.2 s
```

```
Wall time: 25 s
convergence in 111 iterations!
CPU times: user 382 ms, sys: 0 ns, total: 382 ms
Wall time: 382 ms
convergence in 111 iterations!
CPU times: user 132 ms, sys: 0 ns, total: 132 ms
Wall time: 132 ms
```

Using optimize.root for policy function iteration is very costly.

2 Problem 2

2.1 a) Stating the problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to:

$$c_t + a_{t+1} \leq R a_t + y_t, \; \forall t$$

The First Order Condition yields the usual Euler equation, which in this case, corresponds to:

$$exp(-\sigma c_t) = R\beta \, E_t\{exp(-\sigma c_{t+1})\}$$

2.2 b) Computing allocation: Planner's problem

When \$ $\epsilon = 0$\$

we are back to the deterministic case, and we can take the expectation operators from our equations.

The euler equation then becomes:

\$ exp(-σc_t) $= R\beta\{exp(-\sigma c\_\{t+1\})\}$\$

Now we can substitute in our guess for $c_{t+1} = B(Ra_{t+1} + y_t) + D$:

$$exp(-\sigma c_t) = R\beta\{exp(-\sigma(B(Ra_{t+1} + y_{t+1}) + D))\} = R\beta\{exp(-\sigma B(Ra_{t+1} + \bar{y}) - \sigma D)\}$$

applying ln() at both sides:

$$-\sigma c_t = \ln(R\beta) - \sigma B(Ra_{t+1} + \bar{y}) - \sigma D$$

and now replacing a_{t+1} from the budget constraint:

$$-\sigma c_t = \ln(R\beta) - \sigma B(R(a_t + y_t - c_t) + \bar{y}) - \sigma D$$

So we can leave c_t on one side yielding:

$$c_t = \frac{BR}{1+BR}(Ra_t + y_t) + \frac{\sigma D - \ln(\beta R) + \sigma B\bar{y}}{\sigma(1+BR)}$$

Now, using the method of undetermined coefficients and our guess we get:

$$B = \frac{R-1}{R}$$

and

$$D = -\frac{\ln(\beta R)}{\sigma(R-1)} + \frac{\bar{y}}{R}$$

2.3 c) Introducing exogenous market incompleteness

When $\epsilon \neq 0$ we go back to the original case in which:

$$exp(-\sigma c_t) = R\beta\, E_t\{exp(-\sigma c_{t+1})\}$$

Again, by substituting our guess we get:

$$exp(-\sigma c_t) = R\beta\, E_t\{exp(-\sigma(B(Ra_{t+1} + y_{t+1}) + D))\}$$

And manipulating this equation we get:

$$1 = R\beta exp(\sigma c_t) exp(-\sigma BRa_{t+1} - \sigma D)\, E_t\{exp(-\sigma y_{t+1})\}$$

Now, we can use the additional fact that ϵ_t is an i.i.d. random variable and leverage the fact that this assumption implies that:

$$exp(\sigma c_t) exp(-\sigma BRa_{t+1}) = exp(\sigma(c_t - BRa_{t+1}))$$

is also independent of t since $E_t\{exp(-\sigma y_{t+1})\}$ is constant in time. Now using these two facts we get:

$$\begin{aligned} c_t - BRa_{t+1} &= c_t - BR(Ra_t + y_t - c_t) \\ &= c_t(1+BR) - BR(Ra_t + y_t) = c_t(1+BR) - R(c_t - D) \\ &= c_t(1 + BR - R) + RD \end{aligned}$$

And now, from the fact that the path of consumption is constant as we've discussed before we know that: $1 + BR - R = 0$ so that we get the same expression for B as in the deterministic case. Using this and plugging it back in the previous equations yields:

$$1 = R\beta exp(\sigma(R-1)D)\, E_t[exp(-\sigma B y_{t+1})]$$

so that:

$$D = -\tfrac{\ln(R\beta)\, E_t[exp(-\sigma B y_{t+1})]}{\sigma(R-1)}$$

Now, to compare this value of D with the previous one, notice that using Jensen's Inequality:

$$E_t[exp(-\sigma B y_{t+1})] > exp(-\sigma B\, E_t[y_{t+1}]) = exp(-\sigma B \bar{y})$$

Meaning that by going from $\epsilon = 0$ to $\epsilon \neq 0$ the value of D is lower. The intuition is given by the fact that this utility function exhibits *prudence* so there is a precautionary motive for savings. Recall that D represents the constant level of consumption that all individuals wish to maintain, since it does not vary with (a,y). So that if uncertainty is higher, this value is going to be lower to increase precautionary savings.

3 Problem 3

3.1 a) Arrow - Debreu Equilibrium

For this setting, we need to keep track of the state space. In the given setting, we only have two possible states: $s \in \{H, L\}$. Where $s = L$ means that the individual is on the low endowment trajectory, and analogously for the $s = H$ case. Let $c_t(s)$ be the consumption that individual with state s gets at time t. Let p_t be the price of the consumption good at time t. Note that since the aggregate endowment at every period is 1 (i.e. no aggregate uncertainty) the price does not depend on the state.

Individuals solve:

$$\max_{\{c_t(s)\}_{t=0}^{\infty}} E_{-1}\left[\sum_{t=0}^{\infty} \beta^t u(c_t(s))\right]$$

subject to:

$$\sum_{t=0}^{\infty}\sum_{s} p_t c_t(s) \leq \sum_{t=0}^{\infty}\sum_{s} p_t y_t(s)$$

In this context, an Arrow-Debreu equilibrium is defined as a sequence of consumptions $\{c_t(s)\}_{t=0, s\in\{L,H\}}^{\infty}$ and a sequence of prices $\{p_t\}_{t=0}^{\infty}$ such that they solve the individuals problem and market clears: $c_t(H) + c_t(L) = 1$, for every t.

3.2 b) Computing allocations: Planner's problem

Since there is no aggregate uncertainty and there is not any other friction by solving the planners problem for two equally weighted agents we would get a constant level of consumption across time/states. Now let's solve more formally:

$$\max_{\{c_t(s)\}_{t=0}^{\infty}} E_{-1}\left[\sum_{t=0}^{\infty} \beta^t u(c_t(s))\right]$$

subject to: $\sum_{t=0}^{\infty} p_t c_t(H) + \sum_{t=0}^{\infty} p_t c_t(L) \leq \sum_{t=0}^{\infty} p_t[0.75 - (0.25)^{t+1}] + \sum_{t=0}^{\infty} p_t[0.25 + (0.25)^{t+1}]$.\

The First Order Conditions for t and $t = 0$ yield:

$$[c_t(s)]: \quad \beta^t \frac{1}{2c_t(s)} - \lambda p_t = 0$$

$$[c_0(s)]: \quad \frac{1}{2c_0(s)} - \lambda = 0$$

Together, and with $c_t(s) = c_0$ they imply that $p_t = \beta^t$. From the Budget Constraint we get that:

$$\sum_{t=0}^{\infty} \beta^t \bar{c} + \sum_{t=0}^{\infty} \beta^t \bar{c} = \sum_{t=0}^{\infty} \beta^t$$

so that $\bar{c} = \frac{1}{2}$.

3.3 c) Sequential incomplete market

Now we need to be carefull with the timing of the problem. At $t = -1$ agents don't know their type, and since assets can't depend on the realization of the state of the world, there is no trade. Only once the uncertainty of types has been resolved there can be trade. In this case, individual of type s solve the following problem:

$$\max_{\{c_t(s), a(s)_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t(s))$$

subject to:

$$\begin{aligned} c_t(s) + a_{t+1}(s) &\leq (1 + r_t) a_t(s) + y_t(s) \\ a_{t+1}(s) &\geq -b_{t+1} \quad \text{for all t} \end{aligned}$$

In this context, a sequence of markets equilibrium is defined as a sequence of consumptions $\{c(H)_t, c(L)_t\}_{t=0}^{\infty}$, assets $\{a(H)_{t+1}, a(L)_{t+1}\}_{t=0}^{\infty}$, and prices $\{r_t\}_{t=0}^{\infty}$ such that they solve individuals' problems and markets clear: $c_t(H) + c_t(L) = 1$ and $a_{t+1}(H) + a_{t+1}(L) = 0$, for every t.

3.4 d) Computing allocation under incomplete market

We know that the FOC for the individual's problem in this case yields the regular Euler equation:

$$u'(c(s)_t) = \beta u'(c(s)_{t+1})(1 + r_{t+1}) \Leftrightarrow c(s)_{t+1} = \beta(1 + r_{t+1}) c(s)_t$$

Now if we sum up both euler equations and impose market clearing in the goods market we get that: $1 = \beta(1 + r_{t+1})$, for all t. Also from this condition and going back to the individual's euler equations we get that: $c(s)_t = c(s)_{t+1}$. With this, we can go back to the individual's intertemporal budget constraint and compute $\bar{c}_H$ and $\bar{c}_L$:

$$\sum_{t=0}^{\infty} \frac{\bar{c}_H}{(1 + r)^t} = \sum_{t=0}^{\infty} \frac{[0.75 - (0.25)^{t+1}]}{(1 + r)^t}$$

$$\sum_{t=0}^{\infty} \frac{\bar{c}_L}{(1 + r)^t} = \sum_{t=0}^{\infty} \frac{[0.25 + (0.25)^{t+1}]}{(1 + r)^t}$$

After some manipulations we get:

$$\frac{\bar{c}_H}{r} = \frac{0.75}{r} - 0.25\frac{1}{1-\frac{0.25}{1+r}}$$
$$= \frac{\frac{1}{2}+\beta\frac{1}{16}}{1-0.25\beta}$$

and $\bar{c}_L = 1 - \bar{c}_H$ from the goods' market clearing condition.

We can tell that $\bar{c}_H \in (0.5, 0.75)$ and is increasing in β as we would expect. Since the $s = H$ has an increasing income trajectory, to consume a constant amount he'll have to borrow at the beginning, whereas the situation is the exact opposite for the $s = L$ type.

Of course the borrowing constraint holds with slack for both agents since the way it is constructed (natural debt limit) imposes zero consumption forever and that's clearly not optimal with the given utility functions. Also note that here we do not get that consumption is across types as we've discussed previously. The reason is that in this context it is not possible to self-insure against states, and agents can only smooth consumption throughout time.

3.5 e) No borrowing constraint

Now we impose the restriction of $a(s)_t = 0$, for all s and t. Under this scenario, we know that the Euler condition will hold with inequality:

$$u'(c(s)_t) \geq \beta(1+r)u'(c(s)_{t+1})$$

Since individuals cannot save, the budget constraint implies that they'll have to consume their endowments every period, so that $c(s)_t = y(s)_t$. This means that:

$$\beta(1+r) \leq \frac{y(s)_{t+1}}{y(s)_t}$$

Now, since the individuals of type $s = L$ are the ones who wanted to save, we can expect that this condition will be "tighter" for them (also because they are the ones with a decreasing income trajectory). So replacing, we get:

$$\beta(1+r) \leq \frac{0.25 + (0.25)^{t+2}}{0.25 + (0.25)^{t+1}} = \frac{5}{8}.$$

This last bound is obtained by replacing at $t = 0$ since that's when we get the "tightest" bound for the parameters.