Measure Theory - Problem set 1 - Week 3

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1 Measure Spaces

Exercise 1.3

$\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open }\}$ is the set of open sets. It therefore includes the empty set and its complement $\mathbb{R}$ which are both open. The complements of open sets are open. $\mathcal{G}_1$ is therefore closed under complement. Union of open sets are also open. This set is therefore an algebra. Countable unions of open sets are also open. $\mathcal{G}_1$ is therefore a σ-algebra.
$\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (\mathrm{a}, \mathrm{b}], (-\infty, \mathrm{b}], \text{ and } (\mathrm{a}, \infty)\}$ is an algebra because it includes the empty set for $a = b$, includes complements of intervals and by construction includes finite unions of intervals. It is not a σ-algebra because it does not include countable unions.

$\mathcal{G}_2 = \{A : A \text{ is a countable union of intervals of the form } (\mathrm{a}, \mathrm{b}], (-\infty, \mathrm{b}], \text{ and } (\mathrm{a}, \infty)\}$ is an algebra because it includes the empty set for $a = b$, includes complements of intervals and by construction includes finite unions of intervals. It is a σ-algebra because it does include countable unions.

Exercise 1.7

Suppose $\exists B$ a σ-algebra in X such that $B \not\subset \mathcal{P}(X)$. Then $\exists\ A \in B$ s.t. $A \notin \mathcal{P}(X)$.
$\therefore A \notin X$, contradiction.

Suppose $\exists / S$ a σ-algebra in X such that $\{\emptyset, X\} \not\subset S$. If the empty set is not in S we have a contradiction. If $X \notin S$ Then $\emptyset^c \notin S$ and we have another contradiction.

Exercise 1.10

Let $\{S_\alpha\}$ be a family of σ-algebras on X. The empty set is included in all S_α and is therefore part of the intersection. Let $A \in \cap S_\alpha$, A is in all the elements of the family $\{S_\alpha\}$. It is therefore closed under countable union and complement in each S_α, which are therefore part of their intersection.

Exercise 1.22

- monotonicity $\mu(B) = \mu(A \cup A^c \cap B) = \mu(A) + \mu(A^c \cap B) \geq \mu(A)$
- subadditivity: For n = 2

$$\mu(A_1)+\mu(A_2)=\mu(A_1\setminus A_2)+\mu(A_2\setminus A_1)+2\mu(A_1\cap A_2) \tag{1.1}$$
$$\mu\left(\cup A_i\right)=\mu(A_1\setminus A_2)+\mu(A_2\setminus A_1)+\mu(A_1\cap A_2) \tag{1.2}$$
$$\Rightarrow \mu\left(\cup_1^2 A_i\right)\le \mu(A_1)+\mu(A_2) \tag{1.3}$$

By iteration, one can prove that it is true for a countable union of sets.

Exercise 1.23

Let $A, B \in S$ such that $\lambda(A)=\mu(A\cap B)$

- $\lambda(\emptyset)=\mu(\emptyset\cap B)=0$
- Let $\{A_i\}$ be a family of disjoint sets in S:

$$\begin{aligned}\lambda(\cup_1^\infty A_i) &= \mu((\cup_1^\infty A_i)\cap B)\\ &= \mu(\cup_1^\infty A_i\cap B)\\ &= \sum_1^\infty \mu(A_i\cap B)\\ &= \sum_1^\infty \lambda(A_i)\end{aligned}$$

Exercise 1.26

$$\begin{aligned}\mu(\cap_{i=1}^\infty A_i) &= \mu\left((\cup_1^\infty A_i^c)^c\right)\\ &= \mu(X)-\mu\left(\cup_1^\infty A_i^c\right)\\ &= \mu(X)-\lim_{n\to\infty}\mu\left(A_n^c\right) \quad \text{using } i)\\ &= \mu(X)-\mu(X)+\lim_{n\to\infty}\mu\left(A_n\right)\\ &= \lim_{n\to\infty}\mu\left(A_n\right)\end{aligned}$$

Lebesgue Measure

Exercise 2.10

$$\mu\left(B\right)=\mu\left(B\cap E\cup B\cap E^c\right)\le \mu\left(B\cap E\right)+\left(B\cap E^c\right) \quad \text{by subadditivity. Equality holds} \tag{1.4}$$

2.14

Using Cathéodory extension, we need to prove that $\mathcal{B}(\mathbb{R})=\sigma(\mathcal{O})=\sigma(\mathsf{A})$

Measurable functions

Exercise 3.1

Let $\{x\} \subset \mathbb{R}$ be a singleton set. It is therefore a countable set. Suppose that $\exists\ \varepsilon$ such that $\mu(x) > \varepsilon$ with μ the Lebesgue measure. By construction: $\{x\} \subset \left[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right]$. Using subadditivity we have that :

$$\mu(x) \leq \mu\left(\left[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right]\right) \qquad \leq \varepsilon \tag{1.5}$$

This is a contradiction. Therefore, the measure of a singleton set is zero and by the countable sets are measure zero as union of singleton sets.

Exercise 3.7

Suppose $f : X \to \mathbb{R}$ is measurable and define $g(x) = -f(x)$.

For any a in $\mathbb{R}$ We have that $\{x \in X : f(x) < a\}$ is measurable. This is equivalent to $\{x \in X : -g(x) < a\} = \{x \in X : -g(x) > a\}$ being measurable.

Statements with non-strict equalities hold because $\mathcal{M}$ is a σ-algebra so that $f^{-1}((-\infty, a)) = f^{-1}([a, \infty))$ is also in $\mathcal{M}$ and therefore also measurable. Hence, $\mathcal{M}$ contains closed sets and for any a $\{x \in X : -g(x) \leq a\}$ is measurable.

$$\left\lfloor \left[\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \right)^{-1} \right] \right\rfloor \tag{1.6}$$

Unfinished.

Theorem 1.1 (Test Theorem)**.** *Hello.*

Definition 1.1 (Test Definition)**.** Hello.

Example 1.1 (Test Examples)**.** My examples are pink-ish.

$$\mathbb{E}[X_{i,j}] \in \mathbb{R} \quad \mathbb{P}(X_{i,j}) \in \mathbb{R} \tag{1.7}$$

Better math.

Rebekah is amazing!!!!!!

$$\sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{j=1}^{J} d_{njt} \left\{ \log \left[p_{jt} \left(x_{nt} \right) \right] + \sum_{x=1}^{X} / \left\{ x_{n,t+1} = x \right\} \log \left[f_{jt} \left(x | x_{nt} \right) \right] \right\} \tag{1.8}$$

Empirical Results. test

$\{1, \ldots, 5\} \xrightarrow{p} 5\mathbf{x} \operatorname{col} A$

Todo list