Measure Theory - Problem set 1 - Week 3

Thomas Pellet

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1 Measure Spaces

Exercise 1.3

$\mathcal{G}_1 = \{A : A \subset \mathbb{R}, A \text{ open }\}$ is the set of open sets. It therefore includes the empty set and its complement $\mathbb{R}$ which are both open. The complements of open sets are open. $\mathcal{G}_1$ is therefore closed under complement. Union of open sets are also open. This set is therefore an algebra. Countable unions of open sets are also open. $\mathcal{G}_1$ is therefore a σ-algebra.
$\mathcal{G}_2 = \{A : A \text{ is a finite union of intervals of the form } (\mathrm{a}, \mathrm{b}], (-\infty, \mathrm{b}], \text{ and } (\mathrm{a}, \infty)\}$ is an algebra because it includes the empty set for $a = b$, includes complements of intervals and by construction includes finite unions of intervals. It is not a σ-algebra because it does not include countable unions.

$\mathcal{G}_2 = \{A : A \text{ is a countable union of intervals of the form } (\mathrm{a}, \mathrm{b}], (-\infty, \mathrm{b}], \text{ and } (\mathrm{a}, \infty)\}$ is an algebra because it includes the empty set for $a = b$, includes complements of intervals and by construction includes finite unions of intervals. It is a σ-algebra because it does include countable unions.

Exercise 1.7

Suppose $\exists B$ a σ-algebra in X such that $B \not\subset \mathcal{P}(X)$. Then $\exists\ A \in B$ s.t. $A \notin \mathcal{P}(X)$.
$\therefore A \notin X$, contradiction.

Suppose $\exists / S$ a σ-algebra in X such that $\{\emptyset, X\} \not\subset S$. If the empty set is not in S we have a contradiction. If $X \notin S$ Then $\emptyset^c \notin S$ and we have another contradiction.

Exercise 1.10

Let $\{S_\alpha\}$ be a family of σ-algebras on X. The empty set is included in all S_α and is therefore part of the intersection. Let $A \in \cap S_\alpha$, A is in all the elements of the family $\{S_\alpha\}$. It is therefore closed under countable union and complement in each S_α, which are therefore part of their intersection.

Exercise 1.22

- monotonicity $\mu(B) = \mu(A \cup A^c \cap B) = \mu(A) + \mu(A^c \cap B) \geq \mu(A)$
- subadditivity: For n = 2

$$\mu(A_1) + \mu(A_2) = \mu(A_1 \setminus A_2) + \mu(A_2 \setminus A_1) + 2\mu(A_1 \cap A_2) \quad (1.1)$$

$$\mu\left(\cup A_i\right) = \mu(A_1 \setminus A_2) + \mu(A_2 \setminus A_1) + \mu(A_1 \cap A_2) \quad (1.2)$$

$$\Rightarrow \mu\left(\cup_1^2 A_i\right) \leq \mu(A_1) + \mu(A_2) \quad (1.3)$$

By iteration, one can prove that it is true for a countable union of sets.

Exercise 1.23

Let $A, B \in S$ such that $\lambda(A) = \mu(A \cap B)$

- $\lambda(\emptyset) = \mu(\emptyset \cap B) = 0$
- Let $\{A_i\}$ be a family of disjoint sets in S:

$$\begin{aligned}
\lambda(\cup_1^\infty A_i) &= \mu((\cup_1^\infty A_i) \cap B) \\
&= \mu(\cup_1^\infty A_i \cap B) \\
&= \sum_1^\infty \mu(A_i \cap B) \\
&= \sum_1^\infty \lambda(A_i)
\end{aligned}$$

Exercise 1.26

$$\begin{aligned}
\mu(\cap_{i=1}^\infty A_i) &= \mu\left((\cup_1^\infty A_i^c)^c\right) \\
&= \mu(X) - \mu\left(\cup_1^\infty A_i^c\right) \\
&= \mu(X) - \lim_{n\to\infty} \mu\left(A_n^c\right) \quad \text{using } i) \\
&= \mu(X) - \mu(X) + \lim_{n\to\infty} \mu\left(A_n\right) \\
&= \lim_{n\to\infty} \mu\left(A_n\right)
\end{aligned}$$

$$\left\lfloor \left\lfloor \left(\left(\frac{1}{2} \right) \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \right)^{-1} \right\rceil \right\rceil \tag{1.4}$$

Unfinished.

Theorem 1.1 (Test Theorem). *Hello.*

Definition 1.1 (Test Definition). Hello.

Example 1.1 (Test Examples). My examples are pink-ish.

$$\mathbb{E}[X_{i,j}] \in \mathbb{R} \quad \mathbb{P}(X_{i,j}) \in \mathbb{R} \tag{1.5}$$

Better math.

Rebekah is amazing!!!!!!

$$\sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{j=1}^{J} d_{njt} \left\{ \log \left[p_{jt} \left(x_{nt} \right) \right] + \sum_{x=1}^{X} / \left\{ x_{n,t+1} = x \right\} \log \left[f_{jt} \left(x | x_{nt} \right) \right] \right\} \tag{1.6}$$

Empirical Results. test

$\{1, \ldots, 5\} \xrightarrow{p} 5\mathbf{x} \operatorname{col} A$

Todo list