

# Problem set 1 - Week 3 - DSGE models

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## Exercise 1

The Euler equation of the Brock-Mirman model is:

$$\frac{1}{e^{z_t} K_t^\alpha - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^\alpha - K_{t+2}} \right\} \quad (0.1)$$

We guess the policy function to be  $K_{t+1} = A e^{z_t} K_t^\alpha$  and replace  $K_{t+2}$  in the Euler equation. We then have:

$$\frac{1}{\frac{1}{A} K_{t+1} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^\alpha - A e^{z_{t+1}} K_{t+1}^\alpha} \right\} \quad (0.2)$$

$$\Leftrightarrow A^2 - (\beta\alpha + 1)A - \beta\alpha = 0 \quad (0.3)$$

The two solutions to this equation are  $\alpha\beta, 1$ .

## Exercise 2

The functional forms of the utility and the production functions are given by:

$$u(c_t, \ell_t) = \ln c_t + a \ln(1 - \ell_t) \quad (0.4)$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^\alpha L_t^{1-\alpha} \quad (0.5)$$

We can replace these in the baseline model to get the following seven equations:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (0.6)$$

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \quad (0.7)$$

$$-\frac{a}{1 - \ell_t} = \frac{1}{c_t} w_t (1 - \tau) \quad (0.8)$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} \quad (0.9)$$

$$w_t = (1 - \alpha) e^{z_t} K_t^\alpha L_t^{-\alpha} \quad (0.10)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (0.11)$$

We cannot use the same technic as in the first exercise because we know have consumption and leisure. The random term does not cancel out if we guess a functional form, making the integral impossible to solve analytically.

## 1 Exercise 3

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \ln(1 - \ell_t) \quad (1.1)$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^\alpha L_t^{1-\alpha} \quad (1.2)$$

and the characterizing equations are now:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (1.3)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \quad (1.4)$$

$$-\frac{a}{1 - \ell_t} = c_t^{-\gamma} w_t (1 - \tau) \quad (1.5)$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} \quad (1.6)$$

$$w_t = (1 - \alpha) e^{z_t} K_t^\alpha L_t^{-\alpha} \quad (1.7)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (1.8)$$

## 2 Exercise 4

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1-\xi} \quad (2.1)$$

$$F(K_t, L_t, z_t) = e^{z_t} [\alpha K_t^\eta + (1 - \alpha) L_t^\eta]^{\frac{1}{\eta}} \quad (2.2)$$

and the characterizing equations are now:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (2.3)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \quad (2.4)$$

$$\frac{a}{(1 - \ell_t)^\xi} = c_t^{-\gamma} w_t (1 - \tau) \quad (2.5)$$

$$r_t = \alpha K_t^{\eta-1} e^{z_t} [\alpha K_t^\eta + (1 - \alpha) L_t^\eta]^{\frac{1}{\eta}-1} \quad (2.6)$$

$$w_t = (1 - \alpha) L_t^{\eta-1} e^{z_t} [\alpha K_t^\eta + (1 - \alpha) L_t^\eta]^{\frac{1}{\eta}-1} \quad (2.7)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (2.8)$$

### 3 Exercise 5

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} \quad (3.1)$$

$$F(K_t, L_t, z_t) = K_t^\alpha (e^{z_t} L_t)^{1-\alpha} \quad (3.2)$$

and the characterizing equations are now:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (3.3)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta) (1 - \tau) + 1] \right\} \quad (3.4)$$

$$1 = c_t^{-\gamma} w_t (1 - \tau) \quad (3.5)$$

$$r_t = \alpha K_t^{\alpha-1} (e^{z_t})^{1-\alpha} \quad (3.6)$$

$$w_t = (1 - \alpha) K_t^\alpha e^{(1-\alpha)z_t} \quad (3.7)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (3.8)$$

$$z_t = (1 - \rho_z) \bar{z} + \rho_z z_{t-1} + \epsilon_t^z \quad (3.9)$$

Solving the model at the steady state, we have:

$$c = (1 - \tau) [w + (r - \delta) k] + T \quad (3.10)$$

$$1 = \beta \{ [(r - \delta) (1 - \tau) + 1] \} \quad (3.11)$$

$$1 = c^{-\gamma} w (1 - \tau) \quad (3.12)$$

$$r = \alpha K^{\alpha-1} (e^z)^{1-\alpha} \quad (3.13)$$

$$w = (1 - \alpha) K^\alpha e^{(1-\alpha)z} \quad (3.14)$$

$$\tau [w + (r - \delta) k] = T_t \quad (3.15)$$

$$z^* = \bar{z} \quad (3.16)$$

Simplyfying, we get:

$$c = \left( (1 - \tau)(1 - \alpha) \left( \frac{1 - \beta}{\beta(1 - \tau)\alpha} + \delta \right)^{\frac{1}{\alpha-1}} \right)^{\frac{1}{\gamma}} \quad (3.17)$$

$$r = \frac{1 - \beta}{\beta(1 - \tau)} + \delta \quad (3.18)$$

$$w = (1 - \alpha) \left( \frac{1 - \beta}{\beta(1 - \tau)\alpha} + \delta \right)^{\frac{1}{\alpha-1}} \quad (3.19)$$

$$k = \left( \frac{1 - \beta}{\beta(1 - \tau)\alpha} + \delta \right)^{\frac{1}{\alpha-1}} \quad (3.20)$$

$$T = \tau [w + (r - \delta) k] \quad (3.21)$$

$$z = \bar{z} \quad (3.22)$$

## 4 Exercise 6

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1 - \xi} \quad (4.1)$$

$$F(K_t, L_t, z_t) = K_t^\alpha (e^{z_t} L_t)^{-\alpha} \quad (4.2)$$

and the characterizing equations are now:

$$c_t = (1 - \tau) [w_t \ell_t + (r_t - \delta) k_t] + k_t + T_t - k_{t+1} \quad (4.3)$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} [(r_{t+1} - \delta)(1 - \tau) + 1] \right\} \quad (4.4)$$

$$\frac{a}{(1 - \ell_t)^\xi} = c_t^{-\gamma} w_t (1 - \tau) \quad (4.5)$$

$$r_t = \alpha K_t^{\alpha-1} e^{(1-\alpha)z_t} L_t^{-\alpha} \quad (4.6)$$

$$w_t = (1 - \alpha) K_t^\alpha e^{(1-\alpha)z_t} L_t^{-\alpha} \quad (4.7)$$

$$\tau [w_t \ell_t + (r_t - \delta) k_t] = T_t \quad (4.8)$$

Solving for the steady state, we get the following equations:

$$c = (1 - \tau) [w \ell + (r - \delta) k] + T \quad (4.9)$$

$$1 = \beta E \{ [(r - \delta)(1 - \tau) + 1] \} \quad (4.10)$$

$$\frac{a}{(1 - \ell)^\xi} = c^{-\gamma} w (1 - \tau) \quad (4.11)$$

$$r = \alpha K^{\alpha-1} e^{(1-\alpha)z} L^{1-\alpha} \quad (4.12)$$

$$w = (1 - \alpha) K^\alpha e^{(1-\alpha)z} L^{1-\alpha} \quad (4.13)$$

$$\tau [w \ell + (r - \delta) k] = T \quad (4.14)$$

and therefore:

$$c = w \ell + (r - \delta) k \quad (4.15)$$

$$r = \frac{1 - \beta}{\beta(1 - \tau)} + \delta \quad (4.16)$$

$$r = \alpha K^{\alpha-1} e^{(1-\alpha)z} L^{1-\alpha} \quad (4.17)$$

$$K = \frac{\alpha w}{(1 - \alpha)r} \quad (4.18)$$

$$c = \left( -\frac{w}{a} (1 - \ell)^{-\xi} (1 - \tau) \right)^{\frac{1}{\gamma}} \quad (4.19)$$

## Exercise 7

The model is now defined by:

$$\begin{aligned} \bar{c} &= (1 - \tau)[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k}] + \bar{T} \\ u_c(\bar{c}, \bar{\ell}) &= \beta E_t \left\{ u_c(\bar{c}, \bar{\ell}) [(\bar{r} - \delta)(1 - \tau) + 1] \right\} \\ -u_\ell(\bar{c}, \bar{\ell}) &= u_c(\bar{c}, \bar{\ell}) \bar{w} (1 - \tau) \\ \bar{r} &= f_K(\bar{k}, \bar{\ell}, \bar{z}) \\ \bar{w} &= f_L(\bar{k}, \bar{\ell}, \bar{z}) \\ \tau[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k}] &= \bar{T} \end{aligned}$$

For the numerical differentiation, we need to express every variable as a function of the parameters and other variables of the system:

$$\begin{aligned} \bar{k} &= \left( \frac{\bar{r}}{\alpha} e^{(1-\alpha)z} L^{\alpha-1} \right)^{\frac{1}{\alpha-1}} \\ \bar{\ell} &= 1 - \left( \frac{\bar{c}^\gamma}{\bar{w}(1 - \tau)} \right)^{\frac{1}{\xi}} \\ \bar{y} &= F(\bar{k}, \bar{l}, \bar{z}) \\ \bar{w} &= f_L(\bar{k}, \bar{\ell}, \bar{z}) \\ \bar{r} &= \frac{1 - \beta}{\beta(1 - \tau)} + \delta \\ \bar{T} &= \tau[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k}] \\ \bar{c} &= (1 - \tau)[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k}] + \bar{T} \end{aligned}$$

$$\left| \left[ \left( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \right)^{-1} \right] \right| \tag{4.20}$$

Unfinished.

**Theorem 4.1** (Test Theorem). *Hello.*

**Definition 4.1** (Test Definition). Hello.

**Example 4.1** (Test Examples). My examples are pink-ish.

$$\mathbb{E}[X_{i,j}] \in \mathbb{R} \quad \mathbb{P}(X_{i,j}) \in \mathbb{R} \tag{4.21}$$

Better math.

Rebekah is amazing!!!!!!

$$\sum_{n=1}^N \sum_{t=1}^T \sum_{j=1}^J d_{njt} \left\{ \log [p_{jt} (x_{nt})] + \sum_{x=1}^X / \{x_{n,t+1} = x\} \log [f_{jt} (x|x_{nt})] \right\} \tag{4.22}$$

**Empirical Results.** test

$$\{1,\dots,5\} \overset{p}{\rightarrow} 5\mathbf{x} \operatorname{col} A$$

**Todo list**

<div> <div></div> <div>Unfinished.</div> </div>	6
<div> <div></div> <div>Better math.</div> </div>	6
<div> <div></div> <div>Rebekah is amazing!!!!!!</div> </div>	6