Problem Set Econ - Week 3 - DSGE models

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Exercise 1

In the Brock-Mirman model, there is only one equation, as there is no labor-leisure choice. The one Euler Equation (EE) is:

$$\frac{1}{e^{z_t}K_t^{\alpha} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^{\alpha} - K_{t+2}} \right\} \tag{0.1}$$

This time we want to rewrite the equation as the Γ function (using Uhlig's notation) to analytically find the values of the matrices F, G, H, L, M, N and consequently P and Q.

To do so we rewrite the above equation:

$$E_t \left\{ \beta \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} (e^{z_t} K_t^{\alpha} - K_{t+1})}{e^{z_{t+1}} K_{t+1}^{\alpha} - K_{t+2}} - 1 \right\} = 0 \tag{0.2}$$

This is equivalent to :

$$E_t \left\{ \Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t) \right\} = 0 \tag{0.3}$$

With $X_{t+1} = K_{t+1}$, $X_t = K_t$, $X_{t-1} = K_{t-1}$, $Z_{t+1} = Z_{t+1}$, $Z_t = Z_t$ and our Γ function having only 1 dimension.

We differentiate Γ wrt all parameters, then evaluate them at the steady state for z == 0.
For F:

$$\frac{d\Gamma}{dX_{t+1}} = \frac{d\Gamma}{dK_{t+2}} = E \left\{ \beta \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} (e^{z_t} K_t^{\alpha} - K_{t+1})}{(e^{z_{t+1}} K_{t+1}^{\alpha} - K_{t+2})^2} \right\} \tag{0.4}$$

$$\frac{d\Gamma}{dX} = \left\{ \beta \frac{\alpha K^{\alpha-1} (K^{\alpha} - K)}{(K^{\alpha} - K)^2} \right\} \tag{0.5}$$

$$F = \left\{ \beta \frac{\alpha K^{\alpha-1}}{K_{\alpha}.K} \right\} \tag{0.6}$$

For G (evaluated at the steady state):

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})((\alpha-1)K^{\alpha-1} - \alpha)(K^{\alpha} - K) - \alpha K^{\alpha-1}(K^{\alpha} - 1)}{(K^{\alpha} - K)^2} \tag{0.7}$$

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})(K^\alpha - K)(-K^{\alpha-1} - \alpha)}{K^\alpha - 1)}(K^\alpha - K)^2 \tag{0.8}$$

$$\frac{d\Gamma}{dX_t} = \frac{\beta(\alpha K^{\alpha-1})(-K^{\alpha-1} - \alpha)}{K^\alpha - 1)}(K^\alpha - K) \tag{0.9}$$

$$G = -\frac{\beta(\alpha K_{\alpha-1})(K^{\alpha-1} + \alpha)}{K^\alpha - 1)}(K^\alpha - K) \tag{0.10}$$

For H (evaluated at the steady state):

$$\frac{d\Gamma}{dX_{t-1}} = \frac{\beta\alpha K^{\alpha-1}\alpha K^{\alpha-1}}{K^\alpha - K} \tag{0.11}$$

$$\frac{d\Gamma}{dX_{t-1}} = \frac{\beta\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.12}$$

$$H = \frac{\beta\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.13}$$

$$\tag{0.14}$$

For L (evaluated at the steady state):

$$\frac{d\Gamma}{dZ_{t+1}} = L = -\frac{\beta\alpha K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.15}$$

$$\tag{0.16}$$

For M (evaluated at the steady state):

$$\frac{d\Gamma}{dZ_t} = M = -\frac{\beta\alpha^2 K2(\alpha-1)}{K^\alpha - K} \tag{0.17}$$

$$\tag{0.18}$$

Moreover, we have

$$GP^2 + GP + H = 0 \tag{0.19}$$

$$FQN + (FP + G)Q + (LN + M) = 0 \tag{0.20}$$

Rearranging (see exercise 3 for details)

$$H = \frac{\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.21}$$

$$P = \frac{-G + / - (G^2 - 4FH)^{0.5}}{2F} \tag{0.22}$$

$$Q = -\frac{LN + M}{FN + FP + G} \tag{0.23}$$

Fitting in our calibrated parameter values from the DSGE lecture: $\beta = .98$, $\alpha = .40$, $z = 0$, and having :

$$F = \left\{ -\frac{\alpha K^{\alpha-1}}{K_\alpha . K} \right\} \tag{0.24}$$

$$G = -\frac{(\alpha K_{\alpha-1})(K^{\alpha-1}+\alpha)}{K^\alpha - 1)}(K^\alpha - K) \tag{0.25}$$

$$H = \frac{\alpha^2 K^{2(\alpha-1)}}{K^\alpha - K} \tag{0.26}$$

$$L = -\frac{\alpha K_{2(\alpha-1)}}{K^\alpha - K} \tag{0.27}$$

$$M = -\frac{\alpha^2 K_{2(\alpha-1)}}{K^\alpha - K} \tag{0.28}$$

$$P = \frac{-G \pm (G^2 - 4FH)^{0.5}}{2F} \tag{0.29}$$

$$Q = -\frac{LN + M}{FN + FP + G} \tag{0.30}$$

becomes

$$F = \left\{ -\frac{.40K^{-0.60}}{K_{.40} . K} \right\} \tag{0.31}$$

$$G = -\frac{(\alpha K_{-0.60})(K^{-0.60}+0.40)}{K^{.40} - 1)}(K^{.40} - K) \tag{0.32}$$

$$H = \frac{0.40^2 K^{-1.2}}{K^(0.40) - K} \tag{0.33}$$

$$L = -\frac{\alpha K_{-1.2}}{K^{.40} - K} \tag{0.34}$$

$$M = -\frac{\alpha^2 K_{-1.2}}{K^{.40} - K} \tag{0.35}$$

$$SolvePQ \tag{0.36}$$

See Notebook for more the values of P & Q

Exercise 2

Now we redo the exercise with $k = \ln(K)$. Using the previous equations, we can replace K by e^k and take the logarithm of the gamma equation to get:

$$\ln\left(E_t\left\{\beta\frac{\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}}(e^{z_t+\alpha k_t} - e^{k_{t+1}})}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}}\right\}\right) = 0 \tag{0.37}$$

$$\ln\beta + \ln\left(\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - \alpha e^{z_{t+1}+\alpha k_{t+1}}\right) - \ln\left(e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}\right) = 0 \tag{0.38}$$

We then have the following equations:

$$
\begin{aligned}
F &= \frac{e^{z_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
G &= \frac{(\alpha-1)e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - \alpha e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} - \frac{\alpha e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
H &= \frac{\alpha e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} \\
L &= 1 - \frac{e^{z_{t+1}+\alpha k_{t+1}}}{e^{z_{t+1}+\alpha k_{t+1}} - e^{k_{t+2}}} \\
M &= \frac{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t}}{e^{z_{t+1}+(\alpha-1)k_{t+1}+z_t+\alpha k_t} - e^{z_{t+1}+\alpha k_{t+1}}} \\
P &= \frac{-G \pm \left(G^2 - 4FH\right)^{\frac{1}{2}}}{2F} \\
Q &= -\frac{LN+M}{FN+FP+G}
\end{aligned}
$$

Exercise 3

$$
\begin{aligned}
E_t\left\{F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right\} &= 0 && (0.39) \\
E_t\left\{F\left(P\tilde{X}_t + Q\tilde{Z}_{t+1}\right) + G\left(P\tilde{X}_{t-1} + Q\tilde{Z}_t\right) + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right\} &= 0 && (0.40) \\
F\left[P\left(P\tilde{X}_{t-1} + Q\tilde{Z}_t\right) + QN\tilde{Z}_t\right] + G\left(P\tilde{X}_{t-1} + Q\tilde{Z}_t\right) + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t &= 0 && (0.41) \\
[(FP+G)P+H]\tilde{X}_{t-1} + [(FQ+L)N + (FP+G)Q + M]\tilde{Z}_t &= 0 && (0.42)
\end{aligned}
$$

Exercise 6

The functional forms are now

$$
\begin{aligned}
u\left(c_t, \ell_t\right) &= \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{\left(1-\ell_t\right)^{1-\xi} - 1}{1-\xi} && (0.43) \\
F\left(K_t, L_t, z_t\right) &= K_t^{\alpha}\left(e^{z_t}L_t\right)^{1-\alpha} && (0.44)
\end{aligned}
$$

and the characterizing equations are now:

$$
\begin{aligned}
c_t &= (1-\tau)\left[w_t\ell_t + \left(r_t - \delta\right)k_t\right] + k_t + T_t - k_{t+1} && (0.45) \\
c_t^{-\gamma} &= \beta E_t\left\{c_{t+1}^{-\gamma}\left[\left(r_{t+1} - \delta\right)(1-\tau) + 1\right]\right\} && (0.46) \\
\frac{a}{(1-\ell_t)^{\xi}} &= c_t^{-\gamma} w_t (1-\tau) && (0.47) \\
r_t &= \alpha K_t^{\alpha-1} e^{(1-\alpha)z_t} L_t^{-\alpha} && (0.48) \\
w_t &= (1-\alpha) K_t^{\alpha} e^{(1-\alpha)z_t} L_t^{-\alpha} && (0.49)
\end{aligned}
$$

$$\tau \left[w_t \ell_t + \left(r_t - \delta \right) k_t \right] = T_t \tag{0.50}$$

Dropping the definition equations, we have the following $\Gamma(k_{t+2}, k_{t+1}, k_t, l_{t+1}, l_t, z_{t+1}, z_t)$ function:

$$E_t \left\{ \beta c_{t+1}^{-\gamma} \left[\left(r_{t+1} - \delta \right) \left(1 - \tau \right) + 1 \right] - c_t^{-\gamma} \right\} = 0 \tag{0.51}$$

$$\frac{a}{(1-\ell_t)^{\xi}} = c_t^{-\gamma} w_t (1-\tau) \tag{0.52}$$

Denoting $X_t = \{K_{t+1}, \ell_t\}$, we have $\Gamma(X_{t+1}, X_t, X_{t-1}, z_{t+1}, z_t)$:

$$E_t \left\{ \begin{array}{l} \beta c_{t+1}^{-\gamma} \left[\left(r_{t+1} - \delta \right) \left(1 - \tau \right) + 1 \right] - c_t^{-\gamma} \\ \frac{a}{(1-\ell_t)^{\xi}} - c_t^{-\gamma} w_t (1-\tau) \end{array} \right\} = 0 \tag{0.53}$$