Problem set 1 - Week 3 - DSGE models

Thomas Pellet

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Exercise 1

The Euler equation of the Brock-Mirman model is:

$$\frac{1}{e^{z_t}K_t^{\alpha} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^{\alpha} - K_{t+2}} \right\} \tag{0.1}$$

We guess the policy function to be $K_{t+1} = Ae^{z_t}K_t^{\alpha}$ and replace K_{t+2} in the Euler equation. We then have:

$$\frac{1}{\frac{1}{A}K_{t+1} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^{\alpha} - Ae^{z_{t+1}} K_{t+1}^{\alpha}} \right\} \tag{0.2}$$

$$\Leftrightarrow A^2 - (\beta\alpha + 1)A - \beta\alpha = 0 \tag{0.3}$$

The two solutions to this equation are $\alpha\beta, 1$.

Exercise 2

The functional forms of the utility and the production functions are given by:

$$u\left(c_t, \ell_t\right) = \ln c_t + a \ln\left(1 - \ell_t\right) \tag{0.4}$$

$$F\left(K_t, L_t, z_t\right) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} \tag{0.5}$$

We can replace these in the baseline model to get the following seven equations:

$$c_t = (1-\tau)\left[w_t \ell_t + \left(r_t - \delta\right) k_t\right] + k_t + T_t - k_{t+1} \tag{0.6}$$

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left[\left(r_{t+1} - \delta\right)\left(1 - \tau\right) + 1\right] \right\} \tag{0.7}$$

$$-\frac{a}{1 - l_t} = \frac{1}{c_t} w_t (1 - \tau) \tag{0.8}$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} \tag{0.9}$$

$$w_t = (1-\alpha) e^{z_t} K_t^{\alpha} L_t^{-\alpha} \tag{0.10}$$

$$\tau\left[w_t \ell_t + (r_t - \delta) k_t\right] = T_t \tag{0.11}$$

We cannot use the same technic as in the first exercise because we know have consumption and leisure. The random term does not cancel out if we guess a functional form, making the integral impossible to solve analytically.

1 Exercise 3

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \ln(1 - \ell_t) \tag{1.1}$$

$$F(K_t, L_t, z_t) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} \tag{1.2}$$

and the characterizing equations are now:

$$c_t = (1-\tau)\left[w_t \ell_t + (r_t - \delta) k_t\right] + k_t + T_t - k_{t+1} \tag{1.3}$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} \left[(r_{t+1} - \delta)(1-\tau) + 1\right] \right\} \tag{1.4}$$

$$-\frac{a}{1 - l_t} = c_t^{-\gamma} w_t (1-\tau) \tag{1.5}$$

$$r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} \tag{1.6}$$

$$w_t = (1-\alpha) e^{z_t} K_t^{\alpha} L_t^{-\alpha} \tag{1.7}$$

$$\tau\left[w_t \ell_t + (r_t - \delta) k_t\right] = T_t \tag{1.8}$$

2 Exercise 4

The functional forms are now

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a \frac{(1 - \ell_t)^{1-\xi} - 1}{1-\xi} \tag{2.1}$$

$$F(K_t, L_t, z_t) = e^{z_t} \left[\alpha K_t^{\eta} + (1-\alpha) L_t^{\eta}\right]^{\frac{1}{\eta}} \tag{2.2}$$

and the characterizing equations are now:

$$c_t = (1-\tau)\left[w_t \ell_t + (r_t - \delta) k_t\right] + k_t + T_t - k_{t+1} \tag{2.3}$$

$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma} \left[(r_{t+1} - \delta)(1-\tau) + 1\right] \right\} \tag{2.4}$$

$$\frac{a}{(1 - \ell_t)^{\xi}} = c_t^{-\gamma} w_t (1-\tau) \tag{2.5}$$

$$r_t = \alpha K_t^{\eta-1} e^{z_t} \left[\alpha K_t^{\eta} + (1-\alpha) L_t^{\eta}\right]^{\frac{1}{\eta}-1} \tag{2.6}$$

$$w_t = (1-\alpha)L_t^{\eta-1}e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1}{\eta}-1} \tag{2.7}$$
$$\tau\left[w_t\ell_t + (r_t-\delta)\,k_t\right] = T_t \tag{2.8}$$

3 Exercise 5

The functional forms are now

$$u\left(c_t,\ell_t\right) = \frac{c_t^{1-\gamma}-1}{1-\gamma} \tag{3.1}$$
$$F\left(K_t,L_t,z_t\right) = K_t^{\alpha}\left(e^{z_t}L_t\right)^{1-\alpha} \tag{3.2}$$

and the characterizing equations are now:

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)\,k_t\right] + k_t + T_t - k_{t+1} \tag{3.3}$$
$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[(r_{t+1}-\delta)\,(1-\tau)+1\right]\right\} \tag{3.4}$$
$$r_t = \alpha K_t^{\alpha-1}\left(e^{z_t}\right)^{1-\alpha} \tag{3.5}$$
$$w_t = (1-\alpha)K_t^{\alpha}e^{(1-\alpha)z_t} \tag{3.6}$$
$$\tau\left[w_t\ell_t + (r_t-\delta)\,k_t\right] = T_t \tag{3.7}$$
$$z_t = (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z \tag{3.8}$$

Solving the model at the steady state, we have:

$$c = (1-\tau)\left[w + (r-\delta)\,k\right] + T \tag{3.9}$$
$$1 = \beta\left\{\left[(r-\delta)\,(1-\tau)+1\right]\right\} \tag{3.10}$$
$$1 = c_t^{-\gamma}w(1-\tau) \tag{3.11}$$
$$r = \alpha K^{\alpha-1}\left(e^{z}\right)^{1-\alpha} \tag{3.12}$$
$$w = (1-\alpha)K^{\alpha}e^{(1-\alpha)z} \tag{3.13}$$
$$\tau\left[w + (r-\delta)\,k\right] = T_t \tag{3.14}$$
$$z^* = \overline{z} \tag{3.15}$$

Simplyfying, we get:

$$c = \left((1-\tau)(1-\alpha)\left(\frac{1-\beta}{\beta(1-\tau)\alpha}+\delta\right)^{\frac{1}{\alpha-1}}\right)^{\frac{1}{\gamma}} \tag{3.16}$$
$$r = \frac{1-\beta}{\beta(1-\tau)} + \delta \tag{3.17}$$

$$w = (1-\alpha)\left(\frac{1-\beta}{\beta(1-\tau)\alpha} + \delta\right)^{\frac{1}{\alpha-1}} \tag{3.18}$$

$$k = \left(\frac{1-\beta}{\beta(1-\tau)\alpha} + \delta\right)^{\frac{1}{\alpha-1}} \tag{3.19}$$

$$T = \tau\left[w + (r-\delta)\,k\right] \tag{3.20}$$

$$z = \overline{z} \tag{3.21}$$

4 Exercise 6

The functional forms are now

$$u\left(c_t, \ell_t\right) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{\left(1-\ell_t\right)^{1-\xi} - 1}{1-\xi} \tag{4.1}$$

$$F\left(K_t, L_t, z_t\right) = K_t^{\alpha}\left(e^{z_t}L_t\right)^{-\alpha} \tag{4.2}$$

and the characterizing equations are now:

$$c_t = (1-\tau)\left[w_t\ell_t + \left(r_t-\delta\right)k_t\right] + k_t + T_t - k_{t+1} \tag{4.3}$$

$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[\left(r_{t+1}-\delta\right)(1-\tau) + 1\right]\right\} \tag{4.4}$$

$$\frac{a}{(1-\ell_t)^{\xi}} = c_t^{-\gamma}w_t(1-\tau) \tag{4.5}$$

$$r_t = \alpha K_t^{\alpha-1}e^{(1-\alpha)z_t}L_t^{-\alpha} \tag{4.6}$$

$$w_t = (1-\alpha)K_t^{\alpha}e^{(1-\alpha)z_t}L_t^{-\alpha} \tag{4.7}$$

$$\tau\left[w_t\ell_t + \left(r_t-\delta\right)k_t\right] = T_t \tag{4.8}$$

Solving for the steady state, we get the following equations:

$$c = (1-\tau)\left[w\ell + (r-\delta)\,k\right] + T \tag{4.9}$$

$$1 = \beta E\left\{\left[(r-\delta)(1-\tau) + 1\right]\right\} \tag{4.10}$$

$$\frac{a}{(1-\ell)^{\xi}} = c^{-\gamma}w(1-\tau) \tag{4.11}$$

$$r = \alpha K^{\alpha-1}e^{(1-\alpha)z}L^{1-\alpha} \tag{4.12}$$

$$w = (1-\alpha)K^{\alpha}e^{(1-\alpha)z}L^{1-\alpha} \tag{4.13}$$

$$\tau\left[w\ell + (r-\delta)\,k\right] = T \tag{4.14}$$

and therefore:

$$c = w\ell + (r-\delta)k \tag{4.15}$$

$$r = \frac{1-\beta}{\beta(1-\tau)} + \delta \tag{4.16}$$

$$r = \alpha K^{\alpha-1} e^{(1-\alpha)z} L^{1-\alpha} \tag{4.17}$$

$$K = \frac{\alpha w}{(1-\alpha) r} \tag{4.18}$$

$$c = \left(-\frac{w}{a}(1-\ell)^{-\xi}(1-\tau)\right)^{\frac{1}{\gamma}} \tag{4.19}$$

Exercise 7

The model is now defined by:

$$\begin{aligned}
\bar{c} &= (1-\tau)[\overline{w}\overline{\ell} + (\bar{r}-\delta)\bar{k}] + \overline{T} \\
u_c(\bar{c},\overline{\ell}) &= \beta E_t \left\{ u_c(\bar{c},\overline{\ell})[(\bar{r}-\delta)(1-\tau)+1] \right\} \\
-u_\ell(\bar{c},\overline{\ell}) &= u_c(\bar{c},\overline{\ell})\overline{w}(1-\tau) \\
\bar{r} &= f_K(\bar{k},\overline{\ell},\bar{z}) \\
\overline{w} &= f_L(\bar{k},\overline{\ell},\bar{z}) \\
\tau[\overline{w}\overline{\ell} + (\bar{r}-\delta)\bar{k}] &= \overline{T}
\end{aligned}$$

For the numerical differentiation, we need to express every variable as a function of the parameters and other variables of the system:

$$\begin{aligned}
\bar{k} &= \left(\frac{\bar{r}}{\alpha} e^{(1-\alpha)z} L^{\alpha-1}\right)^{\frac{1}{\alpha-1}} \\
\overline{\ell} &= 1 - \left(\frac{\bar{c}^{\gamma}}{\overline{w}(1-\tau)}\right)^{\frac{1}{\xi}} \\
\bar{y} &= F(\bar{k},\bar{l},\bar{z}) \\
\overline{w} &= f_L(\bar{k},\overline{\ell},\bar{z}) \\
\bar{r} &= \frac{1-\beta}{\beta(1-\tau)} + \delta \\
\overline{T} &= \tau[\overline{w}\overline{\ell} + (\bar{r}-\delta)\bar{k}] \\
\bar{c} &= (1-\tau)[\overline{w}\overline{\ell} + (\bar{r}-\delta)\bar{k}] + \overline{T}
\end{aligned}$$