

# Lecture 2: Benchmark Heterogeneous Firm Model and Overview of Solution Methods

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# Motivating Facts: Doms and Dunne (1998)

# Measurement

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- Use [Census data](#) from LRD, 1972 - 1988
  - After 1988, stopped collecting data on sales and retirements
- Construct capital stock using perpetual inventory method
  - Focus on balanced panel
- Analyze the [growth rate of capital](#) for plant  $i$  at time  $t$

$$GK_{it} = \frac{i_{it} - \delta k_{it-1}}{0.5 \times (k_{it-1} + k_{it})}$$

# Plant-Level Investment is Lumpy Across Plants

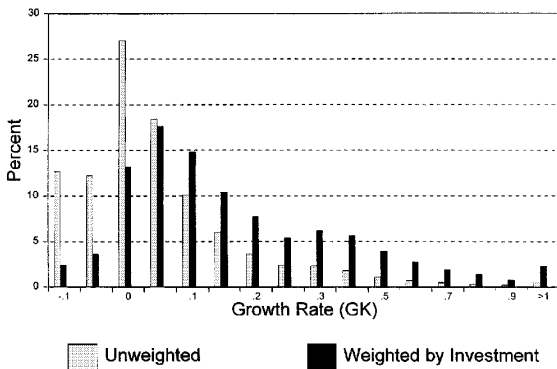
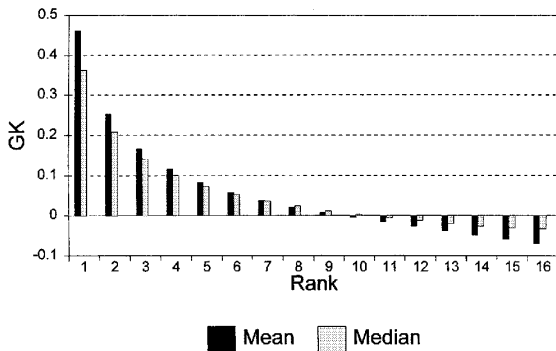


FIG. 1. Capital growth rate (GK) distributions: Unweighted and weighted by investment.

- 51.9% of plants increase capital  $\leq 2.5\%$
- 11% of plants increase capital  $\geq 20\%$

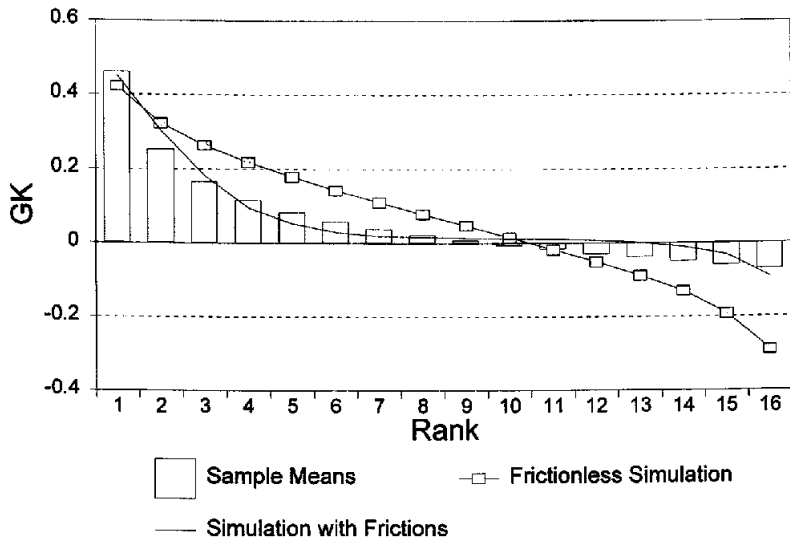
# Plant-Level Investment is Lumpy Within Plants

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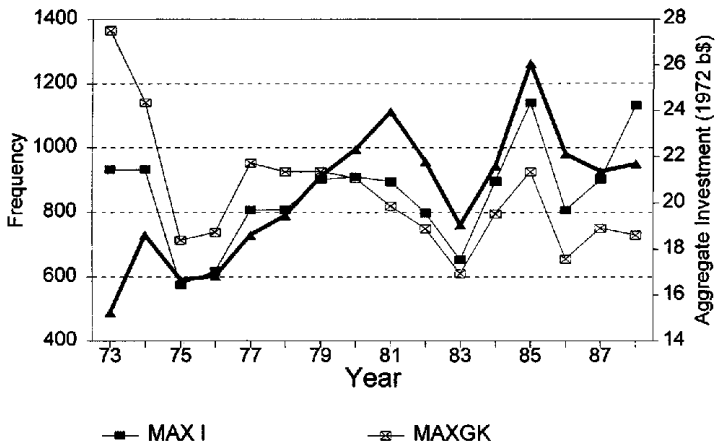
- Capital growth in largest investment episode nearly 50%
- In median investment episode approximately 0%

# Plant-Level Investment is Lumpy Within Plants



# Frequency of Spikes Correlated with Aggregate Investment

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# Benchmark Model: Khan and Thomas (2008)



# Model Overview

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## **Heterogeneous Firms**

- Fixed mass
- Idiosyncratic + aggregate productivity shocks
- Fixed capital adjustment costs

## **Representative Household**

- Owns firms
- Supplies labor
- Complete markets

# Heterogeneous Firms

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**Production technology**  $y_{jt} = e^{z_t} e^{\varepsilon_{jt}} k_{jt}^{\theta} n_{jt}^{\nu}$ ,  $\theta + \nu < 1$

- Idiosyncratic productivity shock  $\varepsilon_{jt+1} = \rho_{\varepsilon} \varepsilon_{jt} + \omega_{jt+1}^{\varepsilon}$  where  $\omega_{jt+1}^{\varepsilon} \sim N(0, \sigma_{\varepsilon}^2)$
- Aggregate productivity shock  $z_{t+1} = \rho_z z_t + \omega_{t+1}^z$  where  $\omega_{t+1}^z \sim N(0, \sigma_z^2)$

**Firms accumulate capital** according to  $k_{jt+1} = (1 - \delta)k_{jt} + i_{jt}$

- If  $\frac{i_{jt}}{k_{jt}} \notin [-a, a]$ , pay fixed cost  $\xi_{jt}$  in units of labor
- Fixed cost  $\xi_{jt} \sim U[0, \bar{\xi}]$

# Firm Optimization Problem: Recursive Formulation

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$$v(\varepsilon, k, \xi; \mathbf{s}) = \max_n e^z e^\varepsilon k^\theta n^\nu - w(\mathbf{s})n \\ + \max \left\{ v^A(\varepsilon, k; \mathbf{s}) - w(\mathbf{s})\xi, v^N(\varepsilon, k; \mathbf{s}) \right\}$$

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$$v^A(\varepsilon, k; \mathbf{s}) = \max_{i \in \mathbb{R}} -i + \mathbb{E}[\Lambda(\mathbf{s}'; \mathbf{s}) v(\varepsilon', k', \xi'; \mathbf{s}') | \varepsilon, k; \mathbf{s}]$$



$$v^N(\varepsilon, k; \mathbf{s}) = \max_{i \in [-ak, ak]} -i + \mathbb{E}[\Lambda(\mathbf{s}'; \mathbf{s}) v(\varepsilon', k', \xi'; \mathbf{s}') | \varepsilon, k; \mathbf{s}]$$

# Firm Optimization Problem: Recursive Formulation

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$$\hat{v}(\varepsilon, k; \mathbf{s}) = \max_n e^z e^\varepsilon k^\theta n^\nu - w(\mathbf{s})n \\ + \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\bar{\xi}} \left( v^A(\varepsilon, k; \mathbf{s}) - w(\mathbf{s}) \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{2} \right) \\ + \left( 1 - \frac{\hat{\xi}(\varepsilon, k; \mathbf{s})}{\bar{\xi}} \right) v^N(\varepsilon, k; \mathbf{s})$$

# Household

---

**Representative household** who owns all firms in the economy

$$\max_{C(\mathbf{s}), N(\mathbf{s})} \frac{C(\mathbf{s})^{1-\sigma} - 1}{1-\sigma} - \chi \frac{N(\mathbf{s})^{1+\alpha}}{1+\alpha} \text{ such that}$$
$$C(\mathbf{s}) = w(\mathbf{s})N(\mathbf{s}) + \Pi(\mathbf{s})$$

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$$C(\mathbf{s}) = w(\mathbf{s})N(\mathbf{s}) + \Pi(\mathbf{s})$$

**Complete markets** implies that  $\Lambda(\mathbf{s}'; \mathbf{s}) = \beta \left( \frac{C(\mathbf{s}')}{C(\mathbf{s})} \right)^{-\sigma}$

- Firms maximize their market value
- Market value given by expected present value of dividends using [stochastic discount factor](#)
- With complete markets, SDF is household's [intertemporal marginal rate of substitution](#)

# Defining Recursive Competitive Equilibrium

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What is the aggregate state  **$\mathbf{s}$** ?



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- Aggregate shock  $z$

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 $\implies$  need distribution of firms  $g(\varepsilon, k)$

What is the law of motion for the  $\mathbf{s}$ ?

$$g'(\varepsilon', k') = \int \left[ \mathbb{1} \{ \rho_\varepsilon \varepsilon + \sigma_\varepsilon \omega'_\varepsilon = \varepsilon' \} \right. \\ \left. \times \int \mathbb{1} \{ k'(\varepsilon, k, \xi; \mathbf{s}) = k' \} dG(\xi) \right] \\ \times p(\omega'_\varepsilon) g(\varepsilon, k) d\omega'_\varepsilon d\varepsilon dk$$

# Recursive Competitive Equilibrium

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A set of  $\hat{v}(\varepsilon, k; z, g)$ ,  $C(z, g)$ ,  $N(z, g)$ ,  $w(z, g)$ ,  $\Lambda(z'; z, g)$ , and  $g'(z, g)$  s.t.

1. **Firm optimization:** Taking  $\Lambda(z'; z, g)$  and  $w(z, g)$  as given,  $\hat{v}(\varepsilon, k; z, g)$  solves Bellman equation
2. **Household optimization:**  $w(z, g)C(z, g)^{-\sigma} = \chi N(z, g)^\alpha$
3. **Market clearing:**

$$N(z, g) = \int n(\varepsilon, k; z, g)g(\varepsilon, k)d\varepsilon dk$$

$$C(z, g) = \int (y(\varepsilon, k, \xi; z, g) - i(\varepsilon, k, \xi; z, g))dG(\xi)g(\varepsilon, k)d\varepsilon dk$$

$$\Lambda(z'; z, g) = \beta \left( \frac{C(z', g'(z, g))}{C(z, g)} \right)^{-\sigma}$$

4. **Consistency:**  $g'(\varepsilon, k)$  satisfies law of motion for distribution

# Model Parameterization

Desc.	Value	Desc.	Value
$\beta$ Discount factor	.961	$\rho_z$ Aggregate TFP AR(1)	.859
$\sigma$ Utility curvature	1	$\sigma_z$ Aggregate TFP AR(1)	.014
$\alpha$ Inverse Frisch	$\lim \alpha \rightarrow 0$	$\bar{\xi}$ Fixed cost	.0083
$\chi$ Labor disutility	$\Rightarrow N^* = \frac{1}{3}$	$a$ No fixed cost region	.011
$\nu$ Labor share	.64	$\rho_\epsilon$ Idiosyncratic TFP AR(1)	.859
$\theta$ Capital share	.256	$\sigma_\epsilon$ Idiosyncratic TFP AR(1)	.022
$\delta$ Capital depreciation	.085		

# Overview of Computational Methods



# Computing Equilibrium

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- Key challenge: aggregate state  $g$  is infinite-dimensional

# Computing Equilibrium

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- Key challenge: aggregate state  $g$  is infinite-dimensional
- Two steps:
  1. Compute steady state without aggregate shocks  $\implies$  distribution constant at  $g^*$
  2. Compute full model with aggregate shocks  $\implies$  distribution varies over time

# Steady State Recursive Competitive Equilibrium

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A set of  $v^*(\varepsilon, k)$ ,  $C^*$ ,  $N^*$ ,  $w^*$ , and  $g^*(\varepsilon, k)$  such that

1. **Firm optimization:** Taking  $w^*$  as given:  $v^*(\varepsilon, k)$  solves Bellman equation
2. **Household optimization:** Taking  $w^*$  as given:  $w^*(C^*)^{-\sigma} = \chi(N^*)^\alpha$
3. **Market clearing:**

$$N^* = \int n(\varepsilon, k) g(\varepsilon, k) d\varepsilon dk$$

$$C^* = \int (y(\varepsilon, k, \xi) - i(\varepsilon, k, \xi)) dG(\xi) g^*(\varepsilon, k) d\varepsilon dk$$

4. **Consistency:**  
 $g^*(\varepsilon, k)$  satisfies law of motion for distribution given  $g^*$

# Hopenhayn-Rogerson (1993) Algorithm

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**Start with guess of  $w^*$**

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**Start with guess of  $w^*$**

- Solve **firm optimization** problem  $\implies v^*(\varepsilon, k), n^*(\varepsilon, k), k'(\varepsilon, k, \xi)$

# Hopenhayn-Rogerson (1993) Algorithm

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## Start with guess of $w^*$

- Solve **firm optimization** problem  $\implies v^*(\varepsilon, k), n^*(\varepsilon, k), k'(\varepsilon, k, \xi)$
- Use  $k'(\varepsilon, k, \xi)$  to compute **stationary distribution**  $g^*(\varepsilon, k)$  by iterating on law of motion
- Compute implied labor demand  $N^d = \int n^*(\varepsilon, k) g^*(\varepsilon, k) d\varepsilon dk$

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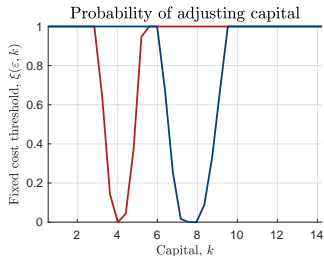
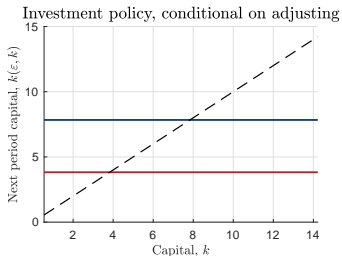
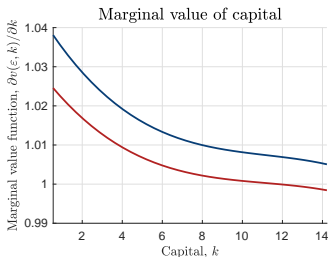
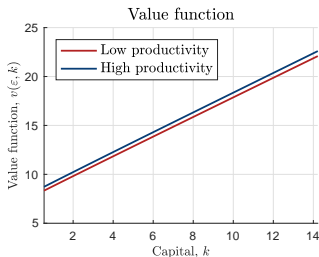
- Solve **firm optimization** problem  $\implies v^*(\varepsilon, k), n^*(\varepsilon, k), k'(\varepsilon, k, \xi)$
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## Update guess of $w^*$ based on $N^d - N^s$

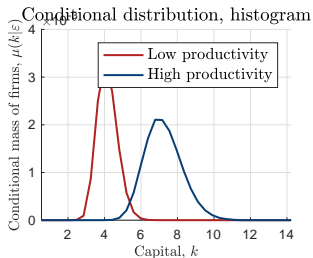
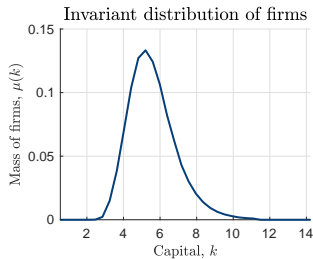
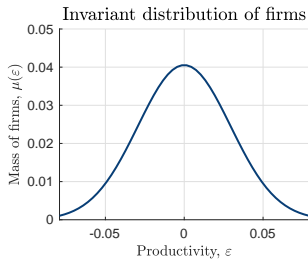
## Iterate to convergence



# Steady State Decisions



# Steady State Distribution



# Full Model with Aggregate Shocks

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- Outside of steady state,
  - Distribution  $g$  varies over time  $\implies$   
how to approximate distribution?
  - Law of motion for  $g$  is complicated  $\implies$   
how to approximate law of motion?

# Full Model with Aggregate Shocks

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  - Distribution  $g$  varies over time  $\implies$   
how to approximate distribution?
  - Law of motion for  $g$  is complicated  $\implies$   
how to approximate law of motion?
- Will provide overview of three approaches in the literature:
  1. **Krusell-Smith**: approximate distribution with moments (used in Khan and Thomas 2008)
  2. **Reiter methods**: perturbation w.r.t. the distribution (used in Winberry 2018; will discuss details on Friday if time)
  3. **MIT shocks**: perfect foresight transition paths (used in Ottonello-Winberry 2018; will discuss on Friday)

# Krusell and Smith (1998)

---

- Typical approach to dealing with challenge: Krusell-Smith (1998)
  - Approximate distribution with **moments**, e.g.  $g(\varepsilon, k) \approx \bar{K}$
  - Approximate law of motion with **parametric form**  
 $\log \bar{K}' = \alpha_0 + \alpha_1 z + \alpha_2 \log \bar{K}$
  - Approximate prices with **parametric form**  
 $\log C = \gamma_0 + \gamma_1 z + \gamma_2 \log \bar{K}$  and  $\log w = \eta_0 + \eta_1 z + \eta_2 \log \bar{K}$

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 $\log C = \gamma_0 + \gamma_1 z + \gamma_2 \log \bar{K}$  and  $\log w = \eta_0 + \eta_1 z + \eta_2 \log \bar{K}$
- **Overview of the algorithm:**
  - Start with guess of  $\alpha$ ,  $\gamma$ , and  $\eta$ 
    - Solve **firm optimization**  $\Rightarrow \hat{v}(\varepsilon, k; z, \bar{K}), n(\varepsilon, k; z, \bar{K}), k'(\varepsilon, k, \xi; z, \bar{K})$
    - **Simulate** large panel of firms using  $n(\varepsilon, k; z, \bar{K}), k'(\varepsilon, k, \xi; z, \bar{K})$
    - Compute aggregate time series  $z_t, \bar{K}_t, C_t, w_t$
  - Update  $\alpha$ ,  $\gamma$ , and  $\eta$  using OLS on simulated series
  - Iterate to convergence on  $\alpha$ ,  $\gamma$ , and  $\eta$
- Assess accuracy using, e.g.,  **$R^2$  of forecasting rules**

- Approximate distribution with [parametric family](#):

$$g(\varepsilon, k) \cong g_0 \exp\{g_1^1 (\varepsilon - m_1^1) + g_1^2 (\log k - m_1^2) + \\ \sum_{i=2}^{n_g} \sum_{j=0}^i g_i^j \left[ (\varepsilon - m_1^1)^{i-j} (\log k - m_1^2)^j - m_i^j \right]\}$$

→ Aggregate state approximated by  $(z, g(\varepsilon, k)) \approx (z, \mathbf{m})$

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- Compute law of motion + prices directly by [integration](#)



- Approximate distribution with **parametric family**:

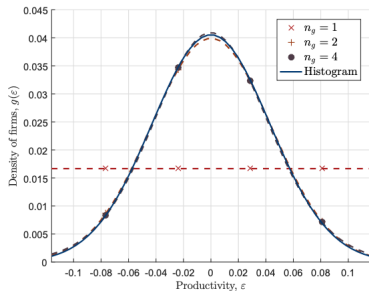
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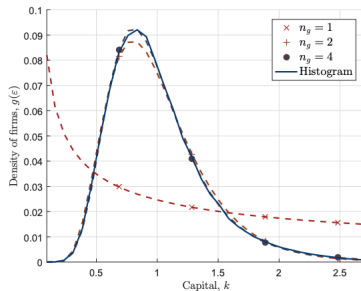
- Compute law of motion + prices directly by **integration**
- Compute aggregate dynamics using **perturbation methods**
  - Solve for steady state in **Matlab**
  - Solve for aggregate dynamics using **Dynare**

# Winberry (2018)

(a) Marginal distribution of productivity

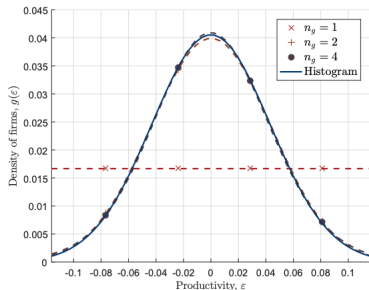


(b) Marginal distribution of capital

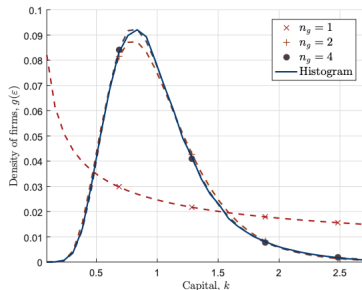


# Winberry (2018)

(a) Marginal distribution of productivity



(b) Marginal distribution of capital



- Run time  $\approx 20 - 40$  seconds for accurate approximation
- Fast enough for [likelihood-based estimation](#)
- Codes at my [website](#)

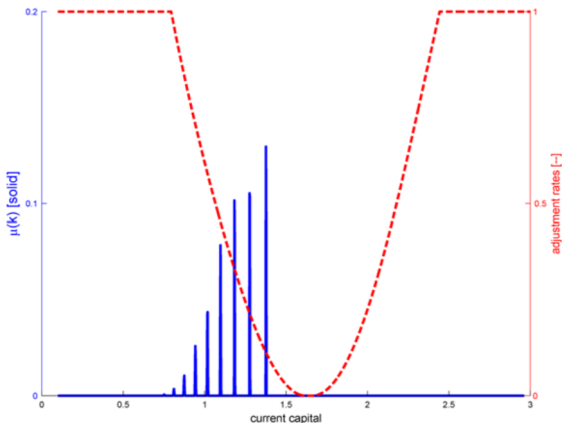
# MIT Shocks (aka Transition Paths)

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- **MIT shock** = unexpected innovation  $\Delta z_0$  at  $t = 0$   
+ perfect foresight transition path back to steady state
- In this model, characterized by  $\{w_t\}_{t=0}^{\infty}$  and  $\{C_t\}_{t=0}^{\infty}$ 
  1. Solve **firm's problem** by backward iteration
  2. Given  $\{\hat{v}_t(\varepsilon, k)\}_{t=0}^{\infty}$ , **simulate decisions** to get  $n_t^d(\varepsilon, k)$ ,  $y_t(\varepsilon, k) - i_t(\varepsilon, k)$ , and  $g_t(\varepsilon, k) \implies$  get aggregates  
 $N_t^d = \int n_t^d(\varepsilon, k) g_t(\varepsilon, k) d\varepsilon dk$  and  
 $C_t^s = \int (y_t(\varepsilon, k) - i_t(\varepsilon, k)) g_t(\varepsilon, k) d\varepsilon dk$
  3. **Equilibrium**:  $N_t^d = N_t^s$  and  $C_t^s = C_t$
- Computational method: set  $T = \text{large enough}$  and **iterate over path of  $\{w_t\}_{t=0}^T$  and  $\{C_t\}_{t=0}^T$**
- Converges to true IRF as  $\Delta z_0 \rightarrow 0$

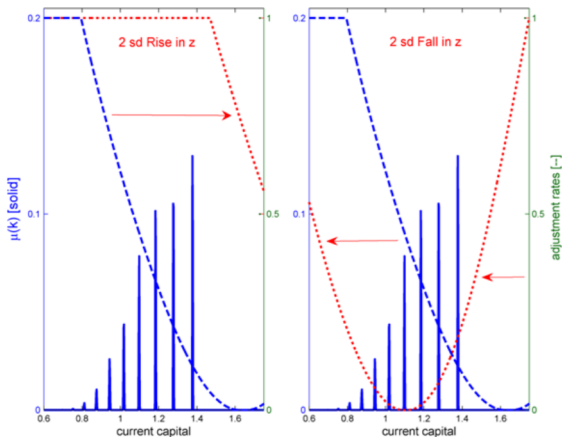
# Results

# Complicated Impulse Responses with Fixed Prices



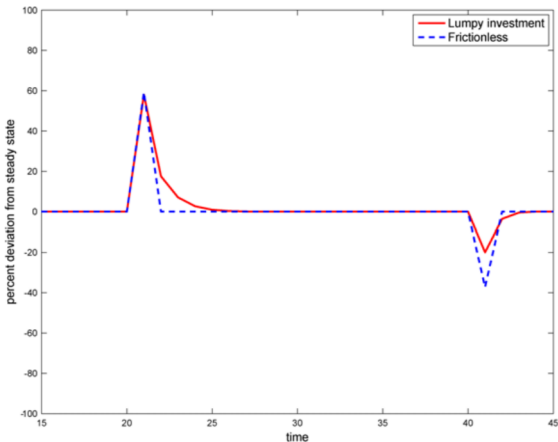
Distribution in model with no idiosyncratic productivity shocks  
Investment decision characterized by adjustment hazard

# Complicated Impulse Responses with Fixed Prices



Response of aggregate investment to shock depends on interaction of initial distribution and adjustment hazards

# Implication: Sign Dependence

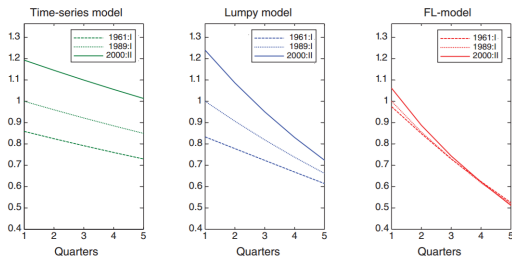


Aggregate investment more responsive to positive than negative shocks

Not true in frictionless model



# Implication: State Dependence



From Bachmann, Caballero, and Engel (2013)

$$\frac{I_t}{K_t} = \sum_{j=1}^p \phi_j \frac{I_{t-j}}{K_{t-j}} + \sigma_t e_t$$

$$\sigma_t = \alpha_1 + \eta_1 \frac{1}{p} \sum_{j=1}^p \frac{I_{t-j}}{K_{t-j}}$$

# Aggregate Nonlinearities with Fixed Prices

---

- Both of these are examples of nonlinear aggregate dynamics
  - Linear model has constant loading on aggregate shock

# Aggregate Nonlinearities with Fixed Prices

---

- Both of these are examples of nonlinear aggregate dynamics
  - Linear model has constant loading on aggregate shock
- Some evidence in aggregate data
  - Sign and state dependence  $\rightarrow$  distribution of  $\frac{I_t}{K_t}$  positively skewed
  - State dependence  $\rightarrow$  dynamics of  $\frac{I_t}{K_t}$  feature conditional heteroskedasticity

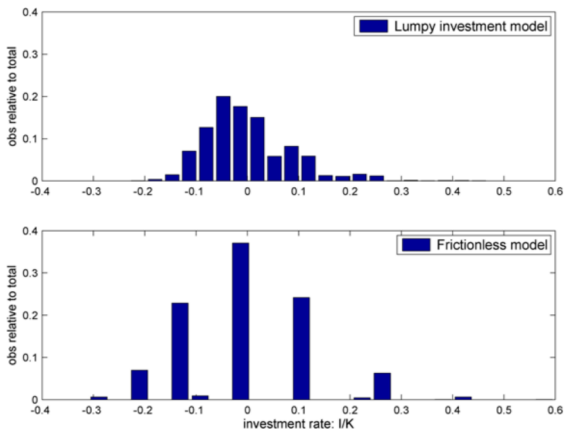
# Aggregate Nonlinearities with Fixed Prices

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- Both of these are examples of **nonlinear aggregate dynamics**
  - Linear model has constant loading on aggregate shock
- Some evidence in aggregate data
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  - State dependence  $\rightarrow$  dynamics of  $\frac{I_t}{K_t}$  feature **conditional heteroskedasticity**
- My view: time series evidence is suggestive at best
  - Predictions are about extreme states, which are rare
  - But that is exactly when we care about these predictions!  
 $\Rightarrow$  **rely on cross-sectional data + carefully specified general equilibrium model**

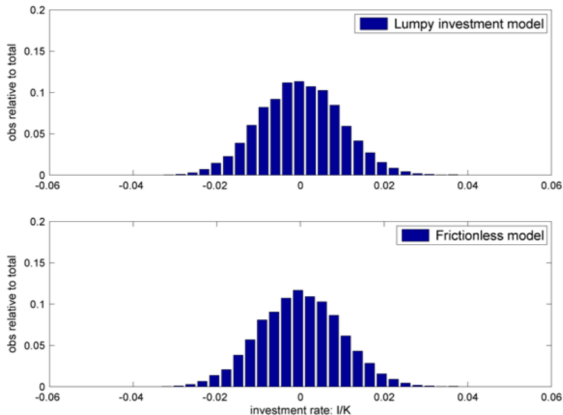
# Distribution of Aggregate $\frac{I_t}{K_t}$ with Fixed Prices

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# Distribution of Aggregate $\frac{I_t}{K_t}$ in General Equilibrium

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# Distribution of Aggregate $\frac{I_t}{K_t}$ in General Equilibrium

TABLE III  
ROLE OF NONCONVEXITIES IN AGGREGATE INVESTMENT RATE DYNAMICS

	Persistence	Standard Deviation	Skewness	Excess Kurtosis
<i>Postwar U.S. data</i> <sup>a</sup>	0.695	0.008	0.008	-0.715
A. Partial equilibrium models				
PE frictionless	-0.069	0.128	0.358	0.140
PE lumpy investment	0.210	0.085	1.121	2.313
B. General equilibrium models				
GE frictionless	0.659	0.010	0.048	0.048
GE lumpy investment	0.662	0.010	0.067	-0.074

<sup>a</sup>Data are annual private investment-to-capital ratio, 1954–2005, computed using Bureau of Economic Analysis tables.

# Business Cycles Nearly Identical to Representative Firm

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TABLE IV  
AGGREGATE BUSINESS CYCLE MOMENTS

	Output	TFP <sup>a</sup>	Hours	Consump.	Invest.	Capital
A. Standard deviations relative to output <sup>b</sup>						
GE frictionless	(2.277)	0.602	0.645	0.429	3.562	0.494
GE lumpy	(2.264)	0.605	0.639	0.433	3.539	0.492
B. Contemporaneous correlations with output						
GE frictionless		1.000	0.955	0.895	0.976	0.034
GE lumpy		1.000	0.956	0.900	0.976	0.034

<sup>a</sup>Total factor productivity.

<sup>b</sup>The logarithm of each series is Hodrick–Prescott-filtered using a weight of 100. The output column of panel A reports percent standard deviations of output in parentheses.



# Why Do the Nonlinearities Disappear?

---

## General equilibrium price movements

- Time-varying elasticity comes from large movements in adjustment hazard
- [Procyclical real interest rate](#) and wage restrain those movements

$$1 + r_t = \frac{1}{\mathbb{E}_t[\Lambda_{t,t+1}]}$$

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## Specification of adjustment costs

- Calibrated adjustment costs [small](#)

# Why Do the Nonlinearities Disappear?

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## **General equilibrium price movements** [Winberry (2018)]

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## **Specification of adjustment costs** [Bachmann, Caballero, and Engel (2013)]

- Calibrated adjustment costs **small**

## Bachmann, Caballero, and Engel (2013)

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- Argue Khan and Thomas' calibration of adjustment costs responsible for irrelevance result
- Calibrate larger adjustment costs and recover aggregate nonlinearities

# Bachmann, Caballero, and Engel (2013)

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- Argue Khan and Thomas' calibration of adjustment costs responsible for irrelevance result
- Calibrate larger adjustment costs and recover aggregate nonlinearities
- Argument based on decomposition between AC smoothing and PR smoothing
  - Frictionless partial equilibrium model excessively volatile
  - AC smoothing: dampening due to adjustment costs
  - PR smoothing: dampening due to price movements
- Measure AC smoothing in data and target in calibration → higher adjustment costs

# Model

---

**Production technology**  $y_{jt} = e^{z_t} e^{\epsilon_{st}} e^{\epsilon_{jt}} k_{jt}^{\theta} n_{jt}^{\nu}, \theta + \nu < 1$

- Idiosyncratic productivity shock  $\epsilon_{jt+1} = \rho_{\epsilon} \epsilon_{jt} + \omega_{jt+1}^{\epsilon}$  where  $\omega_{jt+1}^{\epsilon} \sim N(0, \sigma_{\epsilon}^2)$
- Aggregate productivity shock  $z_{t+1} = \rho_z z_t + \omega_{t+1}^z$  where  $\omega_{t+1}^z \sim N(0, \sigma_z^2)$
- Sectoral productivity shock  $\epsilon_{st+1} = \rho_{\epsilon} \epsilon_{st} + \omega_{st+1}^{\epsilon}$  where  $\omega_{st+1}^{\epsilon} \sim N(0, \sigma_{\epsilon_s}^2)$

# Model

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Firms **accumulate capital** according to  $k_{jt+1} = (1 - \delta)k_{jt} + i_{jt}$

- If don't pay fixed cost, must undertake maintenance investment  $\chi \times \delta k_{jt}$
- Otherwise, pay fixed cost  $\xi_{jt}$  in units of labor
- Fixed cost  $\xi_{jt} \sim U[0, \bar{\xi}]$

# Calibration

---

Set most parameters **exogenously**

Choose  $\sigma_z$ ,  $\bar{\xi}$ , and  $\chi$  to match degree of **AC-smoothing**

- Identify AC-smoothing using **volatility of sectoral investment rates**
  - Aggregated enough to capture interaction of distribution and hazards
  - Small enough to not generate price response



# Calibration

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Set most parameters **exogenously**

Choose  $\sigma_z$ ,  $\bar{\xi}$ , and  $\chi$  to match degree of **AC-smoothing**

- Identify AC-smoothing using **volatility of sectoral investment rates**
  - Aggregated enough to capture interaction of distribution and hazards
  - Small enough to not generate price response
- Targets:
  1. Volatility of aggregate investment rate
  2. Average volatility of sectoral investment rates
  3. **Amount of conditional heteroskedasticity**

# AC vs. PR Smoothing Decomposition

TABLE 6—SMOOTHING DECOMPOSITION

Model	AC smoothing/total smoothing (in percent)		
	LB	UB	Average
Khan-Thomas-lumpy annual	0.0	16.1	8.0
Khan-Thomas-lumpy annual, our $\bar{\xi}$	8.1	59.2	33.7
Our model annual ( $\chi = 0$ ), Khan and Thomas' $\bar{\xi}$	0.8	16.0	8.4
Our model annual ( $\chi = 0$ )	18.9	75.3	47.0
Our model annual ( $\chi = 0.25$ )	19.1	75.7	47.4
Our model annual ( $\chi = 0.50$ )	19.9	76.6	48.3
Our model quarterly ( $\chi = 0$ )	14.5	80.9	47.7
Our model quarterly ( $\chi = 0.25$ )	15.4	80.9	48.2
Our model quarterly ( $\chi = 0.5$ )	15.4	81.0	48.2

$$UB = \log [\sigma(\text{none})/\sigma(\text{AC})] / \log [\sigma(\text{none})/\sigma(\text{both})]$$

$$LB = 1 - \log [\sigma(\text{none})/\sigma(\text{PR})] / \log [\sigma(\text{none})/\sigma(\text{both})]$$

# Calibrated Adjustment Costs

TABLE 4—THE ECONOMIC MAGNITUDE OF ADJUSTMENT COSTS—ANNUAL

Model	Adjustment costs/ unit's output (in percent) (1)	Adjustment costs/ unit's wage bill (in percent) (2)
This paper ( $\chi = 0$ )	38.9	60.9
This paper ( $\chi = 0.25$ )	12.7	19.8
This paper ( $\chi = 0.50$ )	3.6	5.6
Caballero-Engel (1999)	16.5	—
Cooper-Haltiwanger (2006)	22.9	—
Bloom (2009)	35.4	—
Khan-Thomas (2008)	0.5	0.8
Khan-Thomas (2008) "Huge Adj. Costs"	3.7	5.8

*Notes:* This table displays the average adjustment costs paid, conditional on adjustment, as a fraction of output (left column) and as a fraction of the wage bill (right column), for various models. Rows 4–6 are based on table IV in Bloom (2009). For Cooper and Haltiwanger (2006) and Bloom (2009) we report the sum of costs associated with two sources of lumpy adjustment: fixed adjustment costs and partial irreversibility. The remaining models only have fixed adjustment costs.

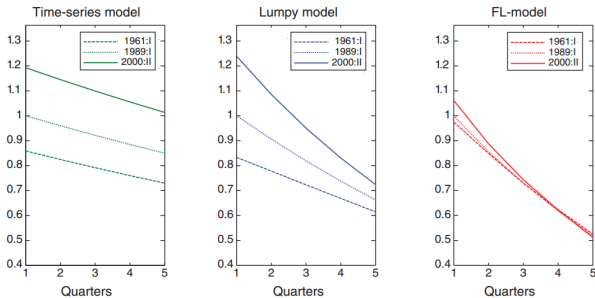
# Aggregate Nonlinearities

TABLE 5—HETEROSCEDASTICITY RANGE

Model	$\log(\sigma_{95}/\sigma_5)$
<i>Data</i>	<i>0.3021</i>
This paper ( $\chi = 0$ )	0.1830
This paper ( $\chi = 0.25$ )	0.2173
This paper ( $\chi = 0.50$ )	0.2901
Quadratic adj. costs ( $\chi = 0$ )	0.0487
Quadratic adj. costs ( $\chi = 0.25$ )	0.0411
Quadratic adj. costs ( $\chi = 0.50$ )	0.0321
Frictionless	0.0539
Khan-Thomas (2008)	0.0468

*Notes:* This table displays heteroscedasticity range ( $\log(\sigma_{95}/\sigma_5)$ ) for the data (row 1) and various model specifications that vary in terms of the maintenance parameter  $\chi$  and the adjustment technology for capital: fixed adjustment costs (rows 2–4), quadratic adjustment costs (rows 5–7), a frictionless model, and the Khan-Thomas (2008) model. The adjustment costs for the models in rows 2–7 have been calibrated to match aggregate and sectoral investment rate volatilities.

# Aggregate Nonlinearities



$$\frac{l_t}{K_t} = \sum_{j=1}^p \phi_j \frac{l_{t-j}}{K_{t-j}} + \sigma_t e_t$$

$$\sigma_t = \alpha_1 + \eta_1 \frac{1}{p} \sum_{j=1}^p \frac{l_{t-j}}{K_{t-j}}$$

# Aggregate Nonlinearities

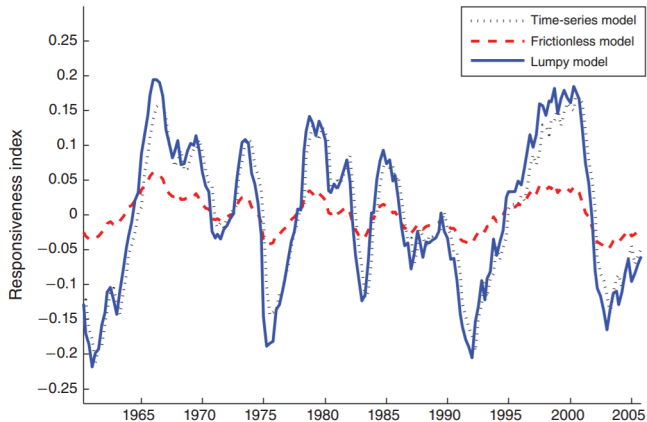


FIGURE 3. TIME PATHS OF THE RESPONSIVENESS INDEX

# Aggregate Nonlinearities

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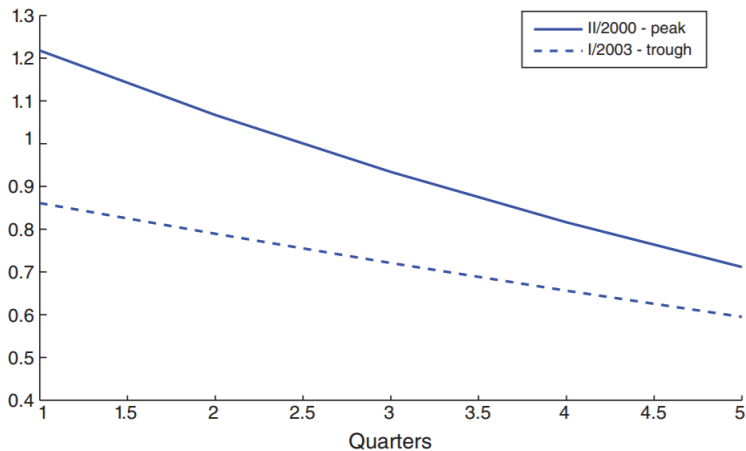


FIGURE 7. IMPULSE RESPONSES OF THE AGGREGATE INVESTMENT RATE  
IN THE 2000 BOOM-BUST CYCLE

# Winberry (2018)

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- Argues that procyclical interest rate inconsistent with data
- When consistent with data recover aggregate nonlinearities



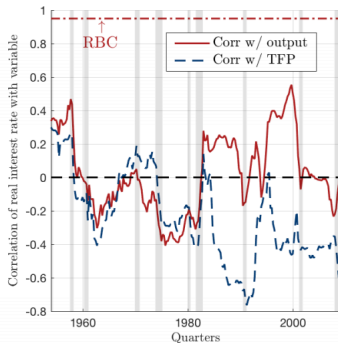
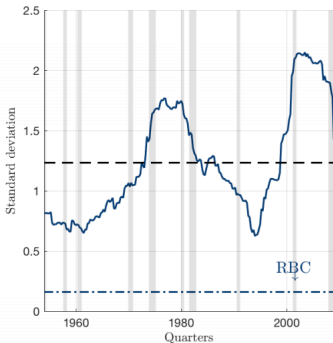
# Winberry (2018)

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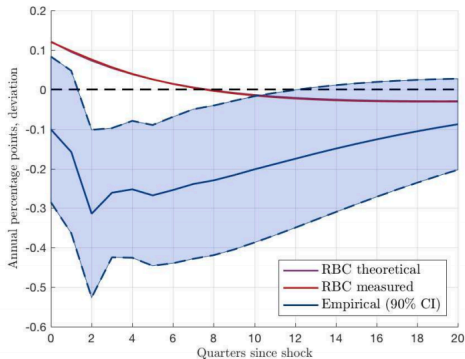
- Argues that procyclical interest rate inconsistent with data
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	$\sigma(r_t)$	$\rho(r_t, y_{t-1})$	$\rho(r_t, y_t)$	$\rho(r_t, y_{t+1})$
<i>T-bill</i>	2.18%	-0.08	-0.17	-0.251
<i>AAA</i>	2.34%	-0.29	-0.37	-0.40
<i>BAA</i>	2.43%	-0.32	-0.41	-0.45
<i>Stock</i>	24.7%	-0.24	-0.14	0.02
<i>RBC</i>	0.16%	0.61	0.97	0.74

# Rolling Windows of $r_t$ Dynamics



# IRF of $r_t$ to TFP Shock



# Takeaways from this Lecture

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- Specification of **benchmark heterogeneous firm model**
  - Individual vs. aggregate states
  - Role of the distribution in the aggregate state variable
  - Steady state: constant aggregates, but lots of churning at individual level
- Overview of how people **solve heterogeneous agent models**
- Response of aggregate investment to shocks **depends on distribution of firms**, which changes over the business cycle
  - Less responsive in recessions