The University of Sydney
School of Mathematics and Statistics

Assignment 2

MATH1002: Linear Algebra	Semester 1, 2019

Web Page: http://www.maths.usyd.edu.au/u/UG/JM/MATH1002/

Lecturers: Zahra Afsar, Anna Aksamit, Dzmitry Badziahin, Alexander Fish, Alexander Kerschl and Bregje Pauwels

This assignment is due by **11:59pm Monday 13th May 2019**, via Turnitin. A PDF copy of your answers must be uploaded in the Learning Management System (Canvas) at https://canvas.sydney.edu.au/courses/15250. Please submit only a PDF document (scan or convert other formats). It should include your SID, your tutorial time, day, room and Tutor's name. **Do not include your name in the PDF document**, since anonymous marking will be implemented. It is your responsibility to preview each page of your assignment after uploading to ensure each page is included in correct order and is legible (not sideways or upside down) before confirming your submission. After submitting you can go back and view your submission to check it. **You will not be emailed a receipt, so please download one after submitting.** The School of Mathematics and Statistics encourages some collaboration between students when working on problems, but students must write up and submit their own version of the solutions.

This assignment is worth 10% of your final assessment for this course. Your answers should be well written, neat, thoughtful, mathematically concise, and a pleasure to read. Please cite any resources used and show all working. Present your arguments clearly using words of explanation and diagrams where relevant. After all, mathematics is about communicating your ideas. This is a worthwhile skill which takes time and effort to master. The marker will give you feedback and allocate an overall letter grade and mark to your assignment using the following criteria:

Mark	**Grade**	**Criterion**
8	A	Outstanding and scholarly work, answering all parts correctly, with clear accurate explanations and all relevant diagrams and working.
7	B	Very good work, making excellent progress, but with a few substantial errors, misunderstandings or omissions throughout the assignment.
6	C	Good work, making good progress, but making more than a few substantial errors, misunderstandings or omissions throughout the assignment.
5	D	A reasonable attempt, but making more substantial errors than for 'C', misunderstandings or omissions throughout the assignment.
4	E+	Some attempt with progress made
3	E	Some attempt, with limited progress made.
2	F+	Limited attempt.
1	F	Extremely limited attempt.
0	Z	No credit awarded.

1. Consider the vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ a+1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -a \end{bmatrix} \in \mathbb{R}^3$.

   (a) Find the condition on a that ensures that

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ a+1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -a \end{bmatrix} \right) = \mathbb{R}^3.$$

   (b) Use your working in part (a), or otherwise, to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

2. (a) Use **Octave as a Calculator**[1] to answer this question.

   Suppose that A and B are two 8×9 matrices. The (i, j)-entry of the matrix B is given by $i * j - 1$. The (i, j)-entry of the matrix A equals 0 if $i + j$ is divisible by 5 and equals the (i, j)-entry of the matrix B otherwise.

   i. What are the rank and nullity of matrices A and B?

   ii. Is vector $\mathbf{u} = [9, 64, -71, 42, 49, 59, 234, -196, 97]$ in the row space of A? If yes, write $\mathbf{u}$ as a suitable linear combination of the rows of A (rounding coefficients to two decimal places). Here, denote the k-th row of $A = (a_{i,j})_{8\times 9}$ by $\mathbf{a_k} := [a_{k,1}, a_{k,2}, ..., a_{k,9}]$ and use this notation if necessary.

   iii. Is vector $\mathbf{v} = [21, 11, 32, 23, -115, 141, 41, 92]$ in the row space of A? Justify your answer.

   (b) Let D be a $m \times n$ matrix. Show that every vector in $\text{null}(D)$ is orthogonal to every vector in $\text{row}(D)$. For $k \in \{1, 2, ..., m\}$ you may denote the k-th row of $D = (d_{i,j})_{m\times n}$ by $\mathbf{d_k} := [d_{k,1}, d_{k,2}, ..., d_{k,n}]$.

3. (a) Use **Octave as a Calculator**[2] to answer this question.

   i. Calculate the number N equal to the sum of all digits of your SID.

   ii. For the $N \times N$ matrix $A = (a_{i,j})$ defined as $a_{i,j} = 1$ if $i = j$, or $i + 1 = j$, and $a_{i,j} = 0$ for all other values of (i, j).

   - Compute $(A^{-1})^T\mathbf{x}$ for $\mathbf{x} = [1, 1, \ldots, 1]^T \in \mathbb{R}^N$,
   - Compute $(A^{-1})^T\mathbf{x}$ for $\mathbf{x} = [1, 2, 3, \ldots, N]^T \in \mathbb{R}^N$.
   - Compute $\mathbf{x} \cdot A(A^{-1})^T\mathbf{x}$ for $\mathbf{x} = [1, 1, \ldots, 1]^T \in \mathbb{R}^N$.

   (b) Let A and B be $n \times n$ matrices. Prove that:

   i. If A is invertible then A^T is invertible (Provide an explicit inverse and make the computation that shows that your guess of the inverse is correct).

   ii. If A and B are invertible then A^2B is invertible (Use similar method as in (i)).

4. (a) Let $a, b \in \mathbb{R}$.

[1] **Octave as a Calculator** is available at https://edstem.org/courses/3269/challenges/. Please do **NOT** include your code here, just answers and justification if you are asked for any.

[2] See footnote 1.

i. Let U_1 be the set of solutions for the equation

$$x_1 + (1-a)x_2^{-1} + 2x_3 + b^2x_4 = 0.$$

For which values of a and b is U_1 a subspace of $\mathbb{R}^4$?

ii. Let U_2 be the set of solutions for the equation

$$ax_1 + x_2 - 3x_3 + (a-a^2)|x_4| = a^3 - a.$$

For which values of a is U_2 a subspace of $\mathbb{R}^4$?

iii. Let U_3 be the set of solutions for the equation

$$x_1 + (a-b)x_2 + x_3 + 2a^2x_4 = b.$$

For which values of a and b is U_3 a subspace of $\mathbb{R}^4$?

Justify your answer in each case.

(b) Let V be the set of solutions for the set of equations

$$\begin{array}{rcrcrcrcr} x_1 & + & (1-a)x_2^{-1} & + & 2x_3 & + & b^2x_4 & = & 0 \\ ax_1 & + & x_2 & - & 3x_3 & + & (a-a^2)|x_4| & = & a^3-a \\ x_1 & + & (a-b)x_2 & + & x_3 & + & 2a^2x_4 & = & b. \end{array}$$

For which values of $a, b \in \mathbb{R}$ is V a subspace of $\mathbb{R}^4$? Use part (a) or otherwise to justify your answer.