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Tutorial: 10:00 Monday Room 452
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Assignment 2

Question 1:

- From the question, we know that: - The circumference of the equilateral triangle is x.
  - The circumference of the circle is $x - 20$.

- Denote each side of the equilateral triangle by s, $s = \frac{x}{3}$.
  - Height: $h = \frac{\sqrt{3}}{2} s = \frac{\sqrt{3}}{2} \cdot \frac{x}{3} = \frac{x}{2\sqrt{3}}$
  - Area:
    $S_{triangle} = \frac{1}{2} sh = \frac{1}{2} \cdot \frac{x}{3} \cdot \frac{x}{2\sqrt{3}} = \frac{x^2}{12\sqrt{3}}$

- Denote the radius of the circle by r
  - $C_{circle} = 2\pi r = 20 - x$
    $r = \frac{20 - x}{2\pi r} = \frac{10}{\pi} - \frac{x}{2\pi}$
  - Area of the circle:
    $$S_{circle} = \pi r^2 = \pi \left(\frac{10}{\pi} - \frac{x}{2\pi} \right)^2$$
    $$= \pi \left(\frac{100}{\pi^2} - \frac{10x}{\pi^2} + \frac{x^2}{4\pi^2} \right)$$
    $$= \frac{x^2 - 40x + 400}{4\pi}$$

- Total area (in terms of x):

$$A(x) = S_{triangle} + S_{circle} = \frac{x^2}{12\sqrt{3}\pi} + \frac{x^2 - 40x + 400}{4\pi}$$
$$= \frac{x^2}{12\sqrt{3}\pi} + \frac{3\sqrt{3}x^2 - 120\sqrt{3}x + 1200\sqrt{3}}{12\sqrt{3}\pi}$$
$$= \frac{1 + 3\sqrt{3}}{12\sqrt{3}} x^2 - \frac{120\sqrt{3}}{12\sqrt{3}} x + \frac{1200\sqrt{3}}{12\sqrt{3}}$$

- Since this is a quadratic expression with a positive leading coefficient, we know that $A(x)$ can have an absolute minimum value where its first derivative is zero.

$$A'(x) = \frac{d}{dx}\left[\frac{1}{12\sqrt{3}} x^2 + \frac{1}{4\pi}(x^2 - 40x + 400) \right] = 0$$
$$\frac{1}{12\sqrt{3}} \cdot 2x + \frac{1}{4\pi}(2x - 40) = 0$$
$$\frac{x}{6\sqrt{3}} + \frac{x - 20}{2\pi} = 0$$
$$\frac{x\pi + 3\sqrt{3}x - 60\sqrt{3}}{6\sqrt{3}\pi} = 0$$
$$(\pi + 3\sqrt{3})x = 60\sqrt{3}$$
$$x = \frac{60\sqrt{3}}{\pi + 3\sqrt{3}}$$

- Therefore, $x = \frac{60\sqrt{3}}{\pi + 3\sqrt{3}}$ will minimise the total area enclosed by the shapes.

Question 2. $f(x) = \dfrac{e^x}{x-1}$

a). For $f(x)$ to be defined, requires $x - 1 \neq 0 \iff x \neq 1$.

Natural domain of $f(x)$ is $x \in (-\infty, 1), (1, +\infty)$.

. $f(x)$ is undefined at $x = 1$. Thus, the vertical line $x = 1$ is the vertical asymptote of $f(x)$.

. Horizontal asymptote:

- $\lim\limits_{x \to +\infty} \dfrac{e^x}{x-1} = \dfrac{\infty}{\infty} \to$ indeterminate form

using L'Hôpital rules, we have: $\lim\limits_{x \to \infty} \dfrac{e^x}{1} = 1 . \lim\limits_{x \to \infty} e^x = \infty$

- $\lim\limits_{x \to -\infty} \dfrac{e^x}{x-1} = \lim\limits_{x \to -\infty} e^x . \lim\limits_{x \to -\infty} \dfrac{1}{x-1} = 0$

$\Rightarrow y = 0$ is a horizontal asymptote of $f(x)$.

b). First derivative of $f(x)$: $f'(x) = \dfrac{e^x(x-1) - e^x.1}{(x-1)^2} = \dfrac{e^x(x-2)}{(x-1)^2}$

. We know that critical points are where $f'(x) = 0$ or is undefined.

$f'(x) = \dfrac{e^x(x-2)}{(x-1)^2} = 0 \iff x = 2$

$f'(x)$ is undefined at $x = 1$, but $f(x)$ is also undefined at $x = 1 \to$ not critical point.

. Critical points: $x = 2$

. Sign diagram:

x	$-\infty$		2		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$		$\searrow$		$\nearrow$	

$\Rightarrow$ $f(x)$ is decreasing on the intervals $(-\infty, 1)$ and $(1, 2)$.

$f(x)$ is increasing on the interval $(2, +\infty)$.

c). Second derivative of $f(x)$: $f''(x) = \dfrac{(x^2 - 4x + 5)e^x}{(x-1)^2}$

. We know that points of inflections are where $f''(x) = 0$ or is undefined.

$f''(x) = \dfrac{(x^2 - 4x + 5)e^x}{(x-1)^3} = 0 \iff x^2 - 4x + 5 = 0 \to$ no real solutions.

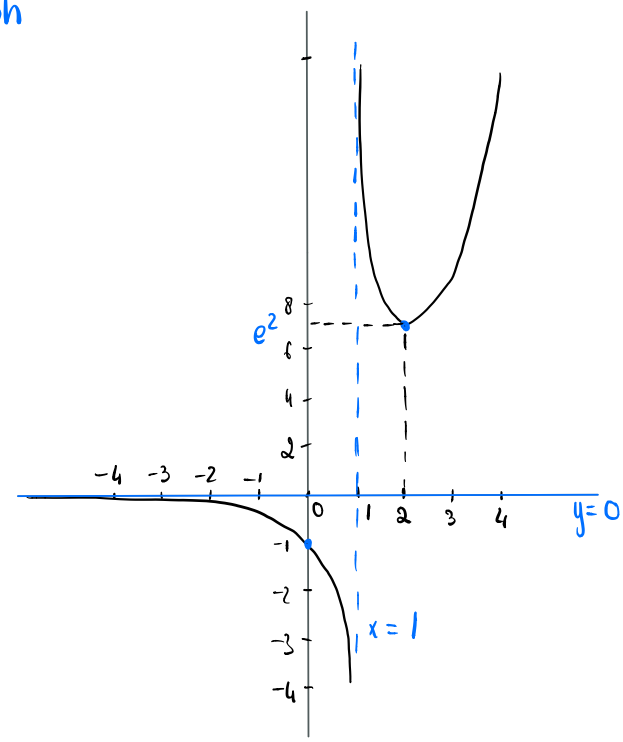
$f''(x)$ is undefined at $x = 1$, but $f(x)$ is also undefined at $x = 1 \to$ not point of inflection.

. $f(x)$ does not have any point of inflection.

· Sign diagram:

x	$-\infty$	1	$+\infty$
$f''(x)$	$-$		$+$
$f(x)$	concave down		concave up

d) Graph



y-intercept: $f(0) = \dfrac{e^0}{0-1} = \dfrac{1}{-1} = -1$

$f(2): \dfrac{e^2}{2-1} = e^2 \approx 7.4$