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Assignment 1

5. $z^4 - 5z^3 + 10z^2 - 10z + 4 = 0 \quad (*)$

Since it is given that $1+i$ is a root, and $(*)$ is a polynomial with all real coefficients
$\Rightarrow$ $1-i$ is also a root.

$$\begin{aligned} z^4 - 5z^3 + 10z^2 - 10z + 4 &= [x-(1+i)][x-(1-i)]\, g(x) = 0 \\ &= [(x-1)-i][(x-1)+i]\, g(x) = 0 \\ &= [(x-1)^2 - i^2]\, g(x) = 0 \\ &= (x^2 - 2x + 2)\, g(x) = 0 \end{aligned}$$

Using long division, we have:

$$\begin{array}{r|l} & z^2 - 3z + 2 \\ \hline z^2 - 2z + 2 & z^4 - 5z^3 + 10z^2 - 10z + 4 \\ & \underline{z^4 - 2z^3 + 2z^2} \\ & -3z^3 + 8z^2 - 10z \\ & \underline{-3z^3 + 6z^2 - 6z} \\ & 2z^2 - 4z + 4 \\ & \underline{2z^2 - 4z + 4} \\ & 0 \end{array}$$

$$\begin{aligned} \Rightarrow g(x) = z^2 - 3z + 2 &= 0 \\ z^2 - z - 2z + 2 &= 0 \\ z(z-1) - 2(z-1) &= 0 \end{aligned}$$

$$\left[\begin{aligned} z - 2 &= 0 \\ z - 1 &= 0 \end{aligned}\right. \Leftrightarrow \left[\begin{aligned} z &= 2 \\ z &= 1 \end{aligned}\right.$$

Therefore, $(*)$ has 4 complex roots $z = 1-i$, $z = 1+i$, $z = 1$ and $z = 2$.

1. $z = \dfrac{(1-i)^4}{i^{47}}$

$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\tan\theta = \dfrac{-1}{1} = -1 \Rightarrow \theta = \dfrac{-\pi}{4}$

$(1-i)^4$ can be written as $\left[\sqrt{2}\left(\cos\dfrac{-\pi}{4} + i.\sin\dfrac{-\pi}{4}\right)\right]^4$

$$\begin{aligned} &= (\sqrt{2})^4 (\cos -\pi + i.\sin -\pi) \\ &= 4.(-1 + i.0) \\ &= -4 \end{aligned}$$

while $i^{47} = i^{44}.i^3 = (i^4)^{11}.i^3 = 1.(-i) = -i$

hence $z = \dfrac{(1-i)^4}{i^{47}} = \dfrac{-4}{-i} = \dfrac{4}{i} = \dfrac{4i}{i^2} = -4i$, which when written in Cartesian form will be $0 - 4i$, with $a = 0$ and $b = -4i$.

2. $z = 1 + i$. Express $\frac{z^9}{(\bar{z})^8}$

$$\frac{z^9}{(\bar{z})^8} = \frac{(1+i)^9}{(1-i)^8}$$

- Solve for $(1+i)^9$:

$|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\arg(1+i) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$(1+i)^9$ can be written as: $\left[\sqrt{2}\cdot\left(\cos\frac{\pi}{4} + i.\sin\frac{\pi}{4}\right)\right]^9$

$$= (\sqrt{2})^9 \cdot \left(\cos\frac{9\pi}{4} + i.\sin\frac{9\pi}{4}\right)$$

$$= 16\sqrt{2}\left(\cos\frac{9\pi}{4} + i.\sin\frac{9\pi}{4}\right)$$

- Solve for $(1-i)^8$:

$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\arg(1-i) = \tan^{-1}\left(\frac{-1}{1}\right) = \frac{-\pi}{4}$

$(1-i)^8$ can be written as: $\left[\sqrt{2}\cdot\left(\cos\frac{-\pi}{4} + i.\sin\frac{-\pi}{4}\right)\right]^8$

$$= (\sqrt{2})^8 \cdot (\cos -2\pi + i.\sin -2\pi)$$

$$= 16 \cdot (\cos -2\pi + i.\sin -2\pi)$$

$$\frac{z^9}{(\bar{z})^8} = \frac{16\sqrt{2}\,(\cos 9\pi/4 + i.\sin 9\pi/4)}{16\,(\cos -2\pi + i.\sin -2\pi)} = \sqrt{2}\left[\cos\left(\frac{9\pi}{4} + 2\pi\right) + i.\sin\left(\frac{9\pi}{4} + 2\pi\right)\right]$$

$$= \sqrt{2}\left(\cos\frac{17\pi}{4} + i.\sin\frac{17\pi}{4}\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i.\sin\frac{\pi}{4}\right)$$

Therefore $\frac{z^9}{(\bar{z})^8} = \sqrt{2}\left(\cos\frac{\pi}{4} + i.\sin\frac{\pi}{4}\right)$

3. a. $2z - z^2 - 10 = 0$

$$2z - z^2 - 10 = 0 \Leftrightarrow -z^2 + 2z - 10 = 0$$

Using the quadratic function, we have:

$$z = \frac{-2 \pm \sqrt{2^2 + 4.(-1).(-10)}}{2.(-1)}$$

$$= \frac{-2 \pm \sqrt{-36}}{-2} = \frac{-2 \pm \sqrt{36i^2}}{-2}$$

$$= \frac{-2 \pm 6i}{-2} = 1 \pm 3i$$

Therefore $z = 1 + 3i$ and $z = 1 - 3i$.

b, $z^4 = 16$

Using the De Moivre 'theorem, we have:

$z^4 = r^4 (\cos 4\theta + i.\sin 4\theta)$

also $16 = 16 [\cos(0 + 2k\pi) + i.\sin(0 + 2k\pi)]$

hence we must have $r^4 = 16$ and $4\theta = 0 + 2k\pi$

$\Leftrightarrow r = 2 \qquad \Leftrightarrow \theta = \frac{2k\pi}{4} = \frac{k\pi}{2}$

$\Rightarrow z = 2\left(\cos \frac{k\pi}{2} + i.\sin \frac{k\pi}{2}\right)$

since z^4 has the degree of 4, it has 4 complex fourth roots

when $k = 0$,

$z = 2(\cos 0 + i.\sin 0) = 2(1 + i.0) = 2$

when $k = 1$,

$z = 2\left(\cos \frac{\pi}{2} + i.\sin \frac{\pi}{2}\right) = 2(0 + i.1) = 2i$

when $k = 2$,

$z = 2\left(\cos \frac{2\pi}{2} + i.\sin \frac{2\pi}{2}\right)$

$= 2(\cos \pi + i.\sin \pi) = 2(-1 + i.0) = -2$

when $k = 3$,

$z = 2\left(\cos \frac{3\pi}{2} + i.\sin \frac{3\pi}{2}\right)$

$= 2\left(\cos \frac{-\pi}{2} + i.\sin \frac{-\pi}{2}\right) = 2.(0 - i.1) = -2i$

Therefore, $z = 2$, $z = -2$, $z = 2i$, and $z = -2i$.

4. Find the fourth roots of $1 + \sqrt{3}i$.

let $z^4 = 1 + \sqrt{3}i$

$|1 + \sqrt{3}i| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\arg(1 + \sqrt{3}i) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

Using the De Moivre's theorem, we have:

$z^4 = r^4.(\cos 4\theta + i.\sin 4\theta)$

also $z^4 = 2.\left[\cos\left(\frac{\pi}{3} + k2\pi\right) + i.\sin\left(\frac{\pi}{3} + k2\pi\right)\right]$

hence we must have $r^4 = 2$ and $4\theta = \frac{\pi}{3} + k2\pi$

$r = \sqrt[4]{2} \qquad \theta = \frac{\pi}{12} + \frac{k\pi}{2}$

$$\Rightarrow z = \sqrt[4]{2} \cdot \left[\cos\left(\frac{\pi}{12} + \frac{k\pi}{2}\right) + i.\sin\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)\right]$$

Since z^4 has the degree of 4, it has 4 complex fourth roots

when $k = 0$,

$$z = \sqrt[4]{2}\left(\cos\frac{\pi}{12} + i.\sin\frac{\pi}{12}\right)$$

when $k = 1$,

$$z = \sqrt[4]{2}\left[\cos\left(\frac{\pi}{12} + \frac{\pi}{2}\right) + i.\sin\left(\frac{\pi}{12} + \frac{\pi}{2}\right)\right]$$

$$= \sqrt[4]{2}\left(\cos\frac{7\pi}{12} + i.\sin\frac{7\pi}{12}\right)$$

when $k = 2$,

$$z = \sqrt[4]{2}\left[\cos\left(\frac{\pi}{12} + \pi\right) + i.\sin\left(\frac{\pi}{12} + \pi\right)\right]$$

$$= \sqrt[4]{2}\left(\cos\frac{13\pi}{12} + i.\sin\frac{13\pi}{12}\right)$$

$$= \sqrt[4]{2}\left(\cos\frac{-11\pi}{12} + i.\sin\frac{-11\pi}{12}\right)$$

when $k = 3$,

$$z = \sqrt[4]{2}\left[\cos\left(\frac{\pi}{12} + \frac{3\pi}{2}\right) + i.\sin\left(\frac{\pi}{12} + \frac{3\pi}{2}\right)\right]$$

$$= \sqrt[4]{2}\left(\cos\frac{19\pi}{12} + i.\sin\frac{19\pi}{12}\right)$$

$$= \sqrt[4]{2}\left(\cos\frac{-5\pi}{12} + i.\sin\frac{-5\pi}{12}\right)$$