Assignment 2

SID: 480048691
Tutorial: Time: 9.00 Friday Room 356
Tutor's name: Irina Choubina

1. a, Find the condition on a that ensures that

$$\text{span}\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\a+1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\a\end{bmatrix}\right) = \mathbb{R}^3.$$

Let the three given vectors as $\underset{\sim}{u}$, $\underset{\sim}{v}$, and $\underset{\sim}{w}$, respectively. Construct a 3x3 matrix A with $\underset{\sim}{u}$, $\underset{\sim}{v}$, and $\underset{\sim}{w}$ as the column vectors.

Since $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$, it follows from the Rank-Nullity Theorem that $\operatorname{rank}(A) = n$ if and only if $\operatorname{nullity}(A) = 0$. Assume that $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 0$, which means the original column vectors of A form a basis for $\mathbb{R}^3$. Then, they are linearly independent.

Thus, the given vectors span $\mathbb{R}^3$ if and only if they are linearly independent and $\operatorname{rank}(A) = 3$.

We have $A = \begin{bmatrix}1 & 2 & 1\\1 & a+1 & 0\\-1 & -1 & a\end{bmatrix}$

Now, performing row reducing for A:

$$\begin{array}{l} R_2 - R_1 \to R_2 \\ R_3 + R_1 \to R_3 \end{array} \begin{bmatrix}1 & 2 & 1\\0 & a-1 & -1\\0 & 1 & -a+1\end{bmatrix}$$

$$\begin{array}{l} R_3 \to R_2 \\ R_2 \to R_3 \end{array} \begin{bmatrix}1 & 2 & 1\\0 & 1 & 1-a\\0 & a-1 & -1\end{bmatrix}$$

$$-R_3 \to R_3 \begin{bmatrix}1 & 2 & 1\\0 & 1 & 1-a\\0 & 1-a & 1\end{bmatrix}$$

$$\frac{R_3}{1-a} \to R_3 \begin{bmatrix}1 & 2 & 1\\0 & 1 & 1-a\\0 & 1 & \frac{1}{1-a}\end{bmatrix}$$

$$\begin{array}{l} R_1 - 2R_2 \to R_1 \\ R_3 - R_2 \to R_3 \end{array} \begin{bmatrix}1 & 0 & 2a-1\\0 & 1 & 1-a\\0 & 0 & \frac{2a-a^2}{1-a}\end{bmatrix}$$

For $\operatorname{rank}(A)$ to be 3, we need row 3 to be not all zeroes. Thus,

$$\frac{2a-a^2}{1-a} \neq 0 \quad \therefore \quad a(2-a) \neq 0 \quad \therefore \quad \left[\begin{array}{l} a \neq 0 \\ 2-a \neq 0 \end{array}\right. \quad \therefore \quad \left[\begin{array}{l} a \neq 0 \\ a \neq 2 \end{array}\right.$$

b. Find $c_1, c_2, c_3 \in \mathbb{R}^3$ such that

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The linear combination can be written as the matrix equation :

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

To find the solution of the equation, we apply Gauss - Jordan elimination.

The augmented matrix can be row reduced by elementary row operations.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & -1 & 1 \end{array}\right]$$

$$\begin{array}{l} -R_3 \to R_1 \\ R_1 \to R_2 \\ R_2 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & 2 & 1 & -3 \\ 1 & 2 & 0 & 2 \end{array}\right]$$

$$\begin{array}{l} R_2 - R_1 \to R_2 \\ -(R_3 - R_2) \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$R_1 - R_2 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$R_1 - R_3 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

Thus, $c_1 = 6$, $c_2 = -2$, and $c_3 = -5$.

Substituting these values into (*), we have

$$\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

2. a) Use Octave as calculator, we have

Matrix A

0	1	2	0	4	5	6	7	0
1	3	0	7	9	11	13	0	17
2	0	8	11	14	17	0	23	26
0	7	11	15	19	0	27	31	35
4	9	14	19	0	29	34	39	44
5	11	17	0	29	35	41	47	0
6	13	0	27	34	41	48	0	62
7	0	23	31	39	47	0	63	71

Matrix B

$$\begin{array}{ccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\
2 & 5 & 8 & 11 & 14 & 17 & 20 & 23 & 26 \\
3 & 7 & 11 & 15 & 19 & 23 & 27 & 31 & 35 \\
4 & 9 & 14 & 19 & 24 & 29 & 34 & 39 & 44 \\
5 & 11 & 17 & 23 & 29 & 35 & 41 & 47 & 53 \\
6 & 13 & 20 & 27 & 34 & 41 & 48 & 55 & 62 \\
7 & 15 & 23 & 31 & 39 & 47 & 55 & 63 & 71
\end{array}$$

From the Rank - Nullity Theorem, we have:

n = rank + nullity, with n as the numbers of columns

Using Octave, we have rank $(A) = 8$. Therefore, nullity $(A) = 9 - 8 = 1$.

Also, rank $(B) = 2$. Thus, nullity $(B) = 9 - 2 = 7$.

b, Is vector $\underset{\sim}{y} = [9, 64, -71, 42, 49, 59, 234, -196, 97]$ in the row space of A?

. Consider matrix A =

$$\begin{array}{ccccccccc}
0 & 1 & 2 & 0 & 4 & 5 & 6 & 7 & 0 \\
1 & 3 & 0 & 7 & 9 & 11 & 13 & 0 & 17 \\
2 & 0 & 8 & 11 & 14 & 17 & 0 & 23 & 26 \\
0 & 7 & 11 & 15 & 19 & 0 & 27 & 31 & 35 \\
4 & 9 & 14 & 19 & 0 & 29 & 34 & 39 & 44 \\
5 & 11 & 17 & 0 & 29 & 35 & 41 & 47 & 0 \\
6 & 13 & 0 & 27 & 34 & 41 & 48 & 0 & 62 \\
7 & 0 & 23 & 31 & 39 & 47 & 0 & 63 & 71
\end{array}$$

Performing row reducing by elementary row operations, we have

$$R = \text{rref}(A) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

→ Since there are 8 rows with leading 1's in rref (A), and the 8 row vectors are linear independent, hence the original row vectors of A form the row space for A.

. Suppose $\underset{\sim}{y}$ is in row (A), $\underset{\sim}{y}$ can be written as a linear combination of the row vectors of A.

Consider the linear combination

$$x_1 a_1 + x_2 a_2 + x_3 a_3 + x_4 a_4 + x_5 a_5 + x_6 a_7 + x_8 a_8 = \underset{\sim}{y}$$

or $$x_1 \underset{\sim}{a_1} + x_2 \underset{\sim}{a_2} + x_3 \underset{\sim}{a_3} + x_4 \underset{\sim}{a_4} + x_5 \underset{\sim}{a_5} + x_6 \underset{\sim}{a_7} + x_8 \underset{\sim}{a_8} + x_9 \underset{\sim}{y} = 0$$

with variables $x_1, x_2, \ldots, x_9 \in \mathbb{R}$.

The linear combination is written as the matrix equation

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 4 & 5 & 6 & 7 & 9 \\ 1 & 3 & 0 & 7 & 9 & 11 & 13 & 0 & 64 \\ 2 & 0 & 8 & 11 & 14 & 17 & 0 & 23 & -71 \\ 0 & 7 & 11 & 15 & 19 & 0 & 27 & 31 & 42 \\ 4 & 9 & 14 & 19 & 0 & 29 & 34 & 39 & 49 \\ 5 & 11 & 17 & 0 & 29 & 35 & 41 & 47 & 59 \\ 6 & 13 & 0 & 27 & 34 & 41 & 48 & 0 & 234 \\ 7 & 0 & 23 & 31 & 39 & 47 & 0 & 63 & -196 \\ 0 & 17 & 26 & 35 & 44 & 0 & 62 & 71 & 97 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \underset{\sim}{0} \qquad (*)$$

To find the solutions of the equation,

The augmented matrix $[C \mid \underset{\sim}{0}]$ can be reduced by elementary row operations.

$$\left[\begin{array}{ccccccccc|c} 0 & 1 & 2 & 0 & 4 & 5 & 6 & 7 & 9 & 0 \\ 1 & 3 & 0 & 7 & 9 & 11 & 13 & 0 & 64 & 0 \\ 2 & 0 & 8 & 11 & 14 & 17 & 0 & 23 & -71 & 0 \\ 0 & 7 & 11 & 15 & 19 & 0 & 27 & 31 & 42 & 0 \\ 4 & 9 & 14 & 19 & 0 & 29 & 34 & 39 & 49 & 0 \\ 5 & 11 & 17 & 0 & 29 & 35 & 41 & 47 & 59 & 0 \\ 6 & 13 & 0 & 27 & 34 & 41 & 48 & 0 & 234 & 0 \\ 7 & 0 & 23 & 31 & 39 & 47 & 0 & 63 & -196 & 0 \\ 0 & 17 & 26 & 35 & 44 & 0 & 62 & 71 & 97 & 0 \end{array}\right]$$

$$\sim \operatorname{rref}(C) = \left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the general solution is given by $\begin{cases} x_1 = x_9 \\ x_7 = -5x_9 \\ x_8 = 3x_9 \end{cases}$ where x_9 is a free variable

If we let $x_9 = 1$, then we have a non-zero solution $x_1 = 1$, $x_7 = -5$, $x_8 = 3$, $x_9 = 1$.

Thus the set of vectors is linear dependent. In other words, $\underset{\sim}{u}$ is in the row space of A.

Substituting these values into $(*)$, we have

$$\underset{\sim}{a_1} - 5\underset{\sim}{a_7} + 3\underset{\sim}{a_8} + \underset{\sim}{u} = 0$$

or $\underset{\sim}{u} = -\underset{\sim}{a_1} + 5\underset{\sim}{a_7} - 3\underset{\sim}{a_8}$

iii. Is vector $\underset{\sim}{v} = [21, 11, 32, 23, -115, 141, 41, 92]$ in the row space of A?

For $\underset{\sim}{v}$ to be in row(A), $\underset{\sim}{v}$ has to be in the spanning set of the row vectors of A.

That is, $\underset{\sim}{v}$ can be written as a linear combination of the row vectors of A.

Since A is a 8×9 matrix, every row vectors of A has 9 components. Thus, any linear combination of the row vectors of A will result in a vector with 9 components because only 2 vectors of the same size can be added.

Meanwhile, $\underset{\sim}{v}$ is a vector with 8 components. Hence, $\underset{\sim}{v}$ cannot be in row(A).

b, Let D be a $m \times n$ matrix. Show that every vector in null (D) is orthogonal to every vector in row(D).

- Let D be a $m \times n$ matrix. The null space of D is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogenous linear system $D.\underset{\sim}{x} = \underset{\sim}{0}$.

Let $d_1, d_2, \dots d_m$ be the row vectors of D. So that D can be written as $\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$

Since $row(D) = col(D^T)$, $d_1, d_2, \dots, d_m$ are also the column vectors of D^T.

$$D^T = [d_1 \ d_2 \ \dots \ d_m]$$

Suppose $\underset{\sim}{u}$ is a vector in null(D). Then $D\underset{\sim}{u} = 0$ and this system of equation can be written as columns of D^T multiplying $\underset{\sim}{u}$:

$$D^T . \underset{\sim}{u} = [d_1 \ d_2 \ \dots \ d_m] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \underset{\sim}{0}$$

By definition of matrix multiplication, the dot product of every column vectors of D^T and $\underset{\sim}{u}$ is zero. That is, $\underset{\sim}{u}$ is orthogonal to every column of D^T, or every row of D.

Let $\underset{\sim}{w}$ be a vector in row(D). That is, $\underset{\sim}{w}$ is any linear combination of the row vectors in D

$\underset{\sim}{w} = c_1 d_1 + c_2 d_2 + \dots + c_m d_m$ where $c_1, c_2, \dots c_m \in \mathbb{R}$ and $d_1, d_2, \dots d_m$ are the row vectors of D.

Consider $$\underset{\sim}{u} . \underset{\sim}{w} = c_1 . \underbrace{(u_1 . r_1)}_{0} + c_2 \underbrace{(u_2 . r_2)}_{0} + \dots + c_m \underbrace{(u_m . r_m)}_{0}$$

Therefore, $\underset{\sim}{u}$ is orthogonal to every combination of the row vectors. Each $\underset{\sim}{u}$ in the null space is orthogonal to each $\underset{\sim}{w}$ of the row space.

3. a,

i. N = sum of SID number = 40

ii. . Compute $(A^{-1})^T . x$ for $x = [1, 1, \dots, 1]^T \in \mathbb{R}^N$

$$(A^{-1})^T . x = [1, 0, 1, 0, \dots, 0]^T \in \mathbb{R}^{40}$$

iii. . Compute $(A^{-1})^T . x$ for $x = [1, 2, 3, \dots, N]^T \in \mathbb{R}^N$

$$(A^{-1}) . x = [1, 1, 2, 2, \dots, 20]^T \in \mathbb{R}^{40}$$

b, Let A and B be $n \times n$ matrices. Prove that:

i. If A is invertible then A^T is invertible.

Suppose that A is invertible. Then there exists A^{-1} such that $A . A^{-1} = I$ and $A^{-1} . A = I$.

(A^{-1} is the inverse of A)

Note that since I is a square diagonal matrix. Hence, it is symmetric which implies $I^T = I$.

Since $(A^{-1})^T . A^T = \underbrace{(A . A^{-1})^T}_{\text{properties of the transpose}} = I^T = I$

Similarly $A^T . (A^{-1})^T = (A^{-1} . A)^T = I^T = I$

We see that multiplying A^T and $(A^{-1})^T$ gives the identity matrix, which corresponds to the definition of an invertible matrix. Hence A^T is invertible and its inverse is $(A^{-1})^T$.

ii. If A and B are invertible then $A^2 B$ is invertible.

Suppose that A and B are invertible. We have $\det(A) \neq 0$ and $\det(B) \neq 0$. (1)

For $n \times n$ matrix A, we have $\det(A^2) = \det(A)^2 \neq 0$ (since A is invertible). (2)

It follows from the multiplicative property of the determinant,

$$\det(A^2 B) = \det(A^2) . \det(B) \neq 0 \text{ (from point 1 and 2).}$$

Since the determinant of the product A^2B is not zero, we conclude that A^2B is invertible.

4. a) Let $a, b \in \mathbb{R}$.

i. Let U_1 be the set of solutions for the equation

$$x_1 + (1-a) x_2^{-1} + 2x_3 + b^2 x_4 = 0 \quad (*)$$

For which values of a, b is U_1 a subspace of $\mathbb{R}^4$?

For a set of vectors to be a subspace of $\mathbb{R}^n$, it must satisfy the following conditions:

- contains the zero vector (1)
- closed under vector addition (2)
- closed under scalar multiplication (3)

The $(*)$ can be rewrite as: $x_1 + (1-a) \cdot \frac{1}{x_2} + 2x_3 + b^2 x_4 = 0$

- For U_1 to satisfy condition (1), requires $x_1 = x_2 = x_3 = x_4 = 0$

However, if $x_2 = 0$, the equation becomes undefined.

Thus, we set a condition that $1 - a = 0 \Leftrightarrow -a = -1 \Leftrightarrow a = 1$.

With $a = 1$, x_2 becomes obsolete and the equation can be rewrite as

$$x_1 + 2x_3 + b^2 x_4 = 0$$

Since the zero vector satisfies the defining relation, it is in U_1 regardless of the value of b. That is, $b \in \mathbb{R}$.

- Let $\underset{\sim}{u}, \underset{\sim}{v} \in U_1$, $\underset{\sim}{u} = (u_1, u_2, u_3)$ and $u_1 + 2u_2 + b^2 u_3 = 0$

$\underset{\sim}{v} = (v_1, v_2, v_3)$ and $v_1 + 2v_2 + b^2 v_3 = 0$

now $\underset{\sim}{u} + \underset{\sim}{v} = y_1 + z_1 + 2y_2 + 2z_2 + b^2 y_3 + b^2 z_3 = 0 + 0 = 0$

$= (y_1 + z_1) + 2(y_2 + z_2) + b^2(y_3 + z_3) = 0$

$\triangleq x_1 + 2x_3 + b^2 x_4 = 0 \in U_1$

Hence, condition 2 is met.

- Let $\underset{\sim}{u} \in U_1$ and $c \in \mathbb{R}$, $\underset{\sim}{u} = (u_1, u_2, u_3)$ and $u_1 + 2u_2 + b^2 u_3 = 0$

Now $c\underset{\sim}{u} = (cx, cy, cz)$

From the defining relation, we have $cu_1 + 2(cu_2) + b^2(cu_3) = 0$

$\triangleq x_1 + 2x_3 + b^2 x_4 = 0 \in U_1$.

Hence, condition 3 is met. U_1 is a subspace of $\mathbb{R}^4$ with $a = 1$ and $b \in \mathbb{R}$.

ii, Let U_2 be the set of solutions for the equation

$$ax_1 + x_2 - 3x_3 + (a - a^2)|x_4| = a^3 - a.$$

For which values of a is U_2 a subspace of $\mathbb{R}^4$?

- For U_2 to satisfy the first condition requires: $x_1 = x_2 = x_3 = x_4 = 0$.

  Substituting the values to the defining relation, we have:

  $$a.0 + 0 - 3.0 + (a - a^2).0 = a^3 - a$$

  that is $a^3 - a = 0 \Leftrightarrow a(a^2 - 1) = 0 \Leftrightarrow \left[\begin{array}{l} a = 0 \\ a^2 = 1 \end{array}\right. \quad \left[\begin{array}{l} a = 0 \\ a = \pm 1 \end{array}\right.$

- Let $a = 0$, the defining relation can be written as:

  $0. x_1 + x_2 - 3x_3 + (0 - 0^2).|x_4| = 0^3 - 0$ or $x_2 - 3x_3 = 0$

  - Consider $\underset{\sim}{u}$ and $\underset{\sim}{v} \in U_2$. $\underset{\sim}{u} = (u_1, u_2)$ and $u_1 - 3u_2 = 0$ or $u_1 = 3u_2$

    $\underset{\sim}{v} = (v_1, v_2)$ and $v_1 - 3v_2 = 0$ or $v_1 = 3v_2$

  Then $u_1 = 3u_2$ and $v_1 = 3v_2$. Now $\underset{\sim}{u} + \underset{\sim}{v} = (u_1 + v_1, u_2 + v_2)$.

  We have $u_1 + v_1 = 3u_2 + 3v_2 = 3(u_2 + v_2) \in U_2$.

  Hence, condition 2 is met.

  - Consider $\underset{\sim}{u} \in U_2$ and $c \in \mathbb{R}$, $\underset{\sim}{u} = (u_1, u_2)$ and $u_1 - 3u_2 = 0$

  Now $c\underset{\sim}{u} = (cu_1, cu_2)$ and $cu_1 - 3(cu_2) = 0 \triangleq x_2 - 3x_3 = 0 \quad \in U_2$.

  Condition 3 is met. U_2 is a subspace of $\mathbb{R}^4$ when $a = 0$.

- Let $a = 1$, the defining relation can be written as:

  $1. x_1 + x_2 - 3x_3 + (1 - 1^2)|x_4| = 1^3 - 1$

  or $x_1 + x_2 - 3x_3 = 0$

  - Consider $\underset{\sim}{u}, \underset{\sim}{v} \in U_2$, $\underset{\sim}{u} = (u_1, u_2, u_3)$ and $u_1 + u_2 - 3u_3 = 0$

    $\underset{\sim}{v} = (v_1, v_2, v_3)$ and $v_1 + v_2 - 3v_3 = 0$

  Now $\underset{\sim}{u} + \underset{\sim}{v} = u_1 + v_1 + u_2 + v_2 - 3u_3 - 3v_3 = 0 + 0 = 0$

  $= (u_1 + v_1) + (u_2 + v_2) - 3(u_3 + v_3) = 0$

$\triangleq \quad x_1 + x_2 - 3x_3 = 0 \in U_2.$

Hence, condition 2 is met.

- Consider $\underset{\sim}{u} \in U_2$ and $c \in \mathbb{R}$. $\underset{\sim}{u} = (u_1, u_2, u_3)$ and $u_1 + u_2 - 3u_3 = 0$

  We have $c\underset{\sim}{u} = (cu_1, cu_2, cu_3)$ and $cu_1 + cu_2 - 3cu_3 = 0$

  $\triangleq \quad x_1 + x_2 - 3x_3 = 0 \in U_2.$

  Condition 3 is met. U_2 is a subspace of $\mathbb{R}^4$ when $a = 1$.

- Let $a = -1$, the defining relation can be written as:

  $-1.x_1 + x_2 - 3x_3 + [(-1) - (-1)^2]\,|x_4| = (-1)^3 - (-1)$

  or $-x_1 + x_2 - 3x_3 - 2|x_4| = 0$

  - Consider $[1, 1, -2, -3]$ and $[-3, -3, -4, 6] \in U_2$

    We have $[1, 1, -2, -3] + [-3, -3, -4, 6] = [-2, -2, -6, 3]$ is not in U_2 since $12 \neq 0$.

    Hence, condition 2 is not met, and thus U_2 is not a subspace of $\mathbb{R}^4$ when $a = -1$.

- Therefore, U_2 is a subspace of $\mathbb{R}^4$ with $a = 0$ or $a = 1$.

ii, Let U_3 be the set of solutions for the equation

$$x_1 + (a-b)x_2 + x_3 + 2a^2 x_4 = b.$$

For which values of a and b is U_3 a subspace of $\mathbb{R}^4$?

- For U_3 to satisfy the first condition requires: $x_1 = x_2 = x_3 = x_4 = 0$.

  Substituting the values to the defining relation, we have:

  $0 + (a-b).0 + 0 + 2a^2.0 = b$, that is $b = 0$.

  With $b = 0$, the defining relation can be written as:

  $x_1 + ax_2 + x_3 + 2a^2 x_4 = 0$

  Since the zero vector satisfies the defining equation, it is in U_3 regardless of the value of a.

  That is, a can be any number $\in \mathbb{R}$.

  - Consider $\underset{\sim}{u}, \underset{\sim}{v}$ in U_3. $\underset{\sim}{u} = (u_1, u_2, u_3, u_4)$ and $u_1 + au_2 + u_3 + 2a^2u_4 = 0$

    $\underset{\sim}{v} = (v_1, v_2, v_3, v_4)$ and $v_1 + av_2 + v_3 + 2a^2v_4 = 0$

Now $\underset{\sim}{u} + \underset{\sim}{v} = u_1 + v_1 + au_2 + av_2 + u_3 + v_3 + 2a^2u_4 + 2a^2v_4 = 0$

$= (u_1 + v_1) + a(u_2 + v_2) + (u_3 + v_3) + 2a^2(u_4 + v_4) = 0$

$\triangleq x_1 + a x_2 + x_3 + 2a^2 x_4 = 0 \in U_3.$

Hence, condition 2 is met.

- Consider $\underset{\sim}{u}$ in U_3 and c in $\mathbb{R}$. $\underset{\sim}{u} = (u_1, u_2, u_3, u_4)$ and $u_1 + au_2 + u_3 + 2a^2u_4 = 0$

We have $c.\underset{\sim}{u} = (cu_1, cu_2, cu_3, cu_4)$ and $(cu_1) + a(cu_2) + (cu_3) + 2a^2(cu_4) = 0$

$\triangleq x_1 + ax_2 + x_3 + 2a^2x_4 = 0 \in U_3.$

Condition 3 is met. U_3 is a subspace of $\mathbb{R}^4$ when $b = 0$ and $a \in \mathbb{R}$.

b, let V be the set of solutions for the set of equations

$$\begin{aligned} x_1 &+ (1-a)x_2^{-1} &+ 2x_3 &+ b^2x_4 &= 0 \quad &(1) \\ ax_1 &+ x_2 &- 3x_3 &+ (a-a^2)|x_4| &= a^3 - a \quad &(2) \\ x_1 &+ (a-b)x_2 &+ x_3 &+ 2a^2x_4 &= b \quad &(3) \end{aligned}$$

For which values of $a, b \in \mathbb{R}$ is V a subspace of $\mathbb{R}^4$?

From works shown above in section 4.i, ii, and iii, we have found conditions for each set of vectors defined by each equation to be a subspace of $\mathbb{R}^4$.

And since V is a set of solutions for the systems containing (1), (2), and (3), the conditions for a and b must simultaneously making each subset a subspace of $\mathbb{R}^4$.

From section 4. a. i., for (1) to be a subspace of $\mathbb{R}^4$, we have $a = 1$ and $b \in \mathbb{R}$.

for (2), we have $a = 0$ and $a = 1$

for (3), we have $a \in \mathbb{R}$ and $b = 0$.

Therefore, finding common conditions for the given subsets, we can conclude that V is a subspace for $\mathbb{R}^4$ if and only if $a = 1$ and $b = 0$.