STAT2911 Assignment 1

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1. We have $X \sim$ Negative Binomial (r, p)

$$\Rightarrow P(X = k) = \binom{k-1}{r-1} p^r q^{k-r}, \quad k = r, r+1, \dots$$

Let $Y = X - r$. Y measures the number of failures before the r^{th} success.

$$\Rightarrow P(Y = m) = \binom{r+m-1}{r-1} p^r q^m, \quad m = 0, 1, 2, \dots$$

We want to prove that $\lim\limits_{\substack{r \to \infty \\ q \to 0 \\ rq \to \lambda}} P(Y = m) = \dfrac{\lambda^m}{m!} e^{-\lambda}$

Let $q = \frac{\lambda}{r}$, we can rewrite $P(Y=m)$ as

$$P(Y = m) = \binom{r+m-1}{r-1}\left(1 - \frac{\lambda}{r}\right)^r \left(\frac{\lambda}{r}\right)^m$$

$$= \frac{\lambda^m}{m!} \underbrace{\frac{(r+m-1)!}{(r-1)!\, r^m}\left(1 - \frac{\lambda}{r}\right)^r}_{\text{want to prove that this} \to e^{-\lambda} \text{ as } r \to \infty}$$

$$\frac{(r+m-1)!}{(r-1)!\, r^m} = \frac{(r+m-1)(r+m-2)\dots(r+1)r}{r^m} \longrightarrow \text{there are } m \text{ terms of this}$$

$$= \frac{r+m-1}{r} \cdot \frac{r+m-2}{r} \cdots \frac{r+1}{r} \cdot \frac{r}{r}$$

$$= \left(1 + \frac{m-1}{r}\right)\left(1 + \frac{m-2}{r}\right) \cdots \left(1 + \frac{1}{r}\right) \cdot 1$$

Substitute back to $P(Y=m)$

$$P(Y=m) = \frac{\lambda^m}{m!}\left(1 + \frac{m-1}{r}\right)\left(1 + \frac{m-2}{r}\right) \cdots \left(1 + \frac{1}{r}\right)\left(1 - \frac{\lambda}{r}\right)^r$$

$$\Rightarrow \lim_{r\to\infty} P(Y=m) = \frac{\lambda^m}{m!} \times \lim_{r\to\infty} \underbrace{\left(1 + \frac{m-1}{r}\right)}_{1}\underbrace{\left(1 + \frac{m-2}{r}\right)}_{1} \cdots \underbrace{\left(1 + \frac{1}{r}\right)}_{1} \times \lim_{r\to\infty} \underbrace{\left(1 - \frac{\lambda}{r}\right)^r}_{e^{-\lambda}}$$

using the equality: $e^x = \lim\limits_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$

Thus, $\lim_{r \to \infty} P(Y = m) = \frac{\lambda^m}{m!} \cdot e^{-\lambda}$

2.

(i) Given that each $\alpha_i \in \{0, 1\}$ and Ω is the set of all α, (Ω, F, P) models sequences of infinite Bernoulli trials. We can also interpret this problem as an infinite coin toss model.

In addition, we are particularly interested in the first n tosses. Let head $= 1$ and tail $= 0$.

For example, with $\beta = (0, 1, 1, 0) \Rightarrow E_\beta$ is the set of all sequences of coin toss such that the first 4 tosses follow the sequence of (T, H, H, T).

(ii) Let γ_i be a sequence of length $i + 1$ starts with i 0's and ends with a 1.

For example, $\gamma_0 = (1) \Rightarrow E_{\gamma_0} = (1, \alpha_2, \alpha_3, \dots) \in F$

$\gamma_1 = (0, 1) \Rightarrow E_{\gamma_1} = (0, 1, \alpha_3, \alpha_4, \dots) \in F$

$\gamma_2 = (0, 0, 1) \quad E_{\gamma_2} = (0, 0, 1, \alpha_4, \alpha_5, \dots) \in F$

Let $E_\gamma = \bigcup_{i=0}^{\infty} E_{\gamma_i}$

Since each $E_{\gamma_i} \in F$, by definition of a σ-algebra, $E_\gamma \in F$.

$\Rightarrow$ The set $\{\underset{\sim}{0}\} = \Omega \setminus E_\gamma \in F$.

(iii) We recognise that $\{(0, 0, \dots)\} = \bigcap_{n=1}^{\infty} E_{(0)_n}$

$= E_{(0)} \cap E_{(0,0)} \cap E_{(0,0,0)} \cap \dots$

(with $E_{(0)} \supset E_{(0,0)} \supset E_{(0,0,0)} \supset \dots$)

$\Rightarrow P(\{\underset{\sim}{0}\}) = P\left(\bigcap_{n=1}^{\infty} E_{(0)_n}\right) = \lim_{n \to \infty} P\left(E_{(0)_n}\right)$

Using the probability measure given, $P(E_{(0)_n}) = (1 - p)^n$

Since $p \in (0, 1]$, $1 - p \in [0, 1) \Rightarrow \lim_{n \to \infty} P\left(E_{(0)_n}\right) = \lim_{n \to \infty} (1 - p)^n \to 0$

(iv) For $\alpha \in \Omega$ and $n \in \mathbb{N}$, let $X_n(\alpha) = \alpha_n$.

For example, let $\alpha = (0, 1, 0, 1)$ then $X_4(\alpha) = \alpha_4 = 1$.

Since the values X_n can attain are 0 and 1 and we are only interested in the n^{th} element of a sequence $\Rightarrow$ X_n is a Bernoulli random variable.

(v) For $\alpha \in \Omega$ and let $X(\alpha) = \inf\{n : a_n = 1\}$.

That is, in a sequence α of n Bernoulli trials, $X(\alpha)$ will return the index of α_n where the first time a "1" appears. For example: $\alpha = (0, 0, 0, 1)$ then $X(\alpha) = 4$

Therefore, X is a Geometric random variable.

(vi) Since $\underset{\sim}{0}$ is a continuation of an infinite number of Bernoulli trials result in a "0".

$X(\underset{\sim}{0}) = \infty$.

(vii) $P(X = \infty) = P(E_\beta : X(\alpha \in E_\beta) = \infty)$

$= P(\underset{\sim}{0} = (0, 0, \dots)) = 0$ (as proved in part (iii)).

We have $X(\Omega) = \cup \{X(\alpha), \alpha \in \Omega\}$

since $\underset{\sim}{0} \in \Omega$ and $X(\underset{\sim}{0}) = \infty \Rightarrow \infty \in X(\Omega)$.