THE UNIVERSITY OF SYDNEY

MATH2022 LINEAR AND ABSTRACT ALGEBRA

Semester 1	**Second Quiz (Take Home)**	2020

This take-home quiz comprises 15 multiple-choice questions, worth three marks each.

Exactly one alternative is correct in each question.

The total marks available for this quiz is 45 marks.

Instructions:

You should write your answers to this quiz on pages (or type them if you wish), and include your SID (student identification number) on each page.

You should then scan your pages as a single pdf document and upload this into Canvas **well before midnight on Thursday 30 April 2020**.

Anonymous marking will be employed, so you should not include your name. (If you do happen to include your name then there is no penalty, so please do not worry about it.)

Please be considerate of the marker and write your answers neatly in the same numerical order as the questions, with spacing between each of your 15 answers.

For each question, first indicate which alternative, from (a), (b), (c), (d), (e), you believe is correct, and then provide reasoning to justify your answer.

For each question, the alternative that you choose is worth 1 mark, and the reasoning you provide to support that alternative is worth 2 marks.

1. Which one of the following groups, under addition, is not cyclic?

(a) $\mathbb{Z}$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (c) $\mathbb{Z}_2 \times \mathbb{Z}_3$ (d) $\mathbb{Z}_5$ (e) $\mathbb{Z}_6$

2. Consider the group G of symmetries of a regular pentagon, generated by a rotation α and a reflection β. Simplify the following expression in G:

$$\beta\alpha^3\beta^3\alpha^{-3}\beta\alpha^7 \;=$$

(a) α (b) $\alpha\beta$ (c) $\alpha^2\beta$ (d) α^2 (e) β

3. Consider the following system of equations over $\mathbb{Z}_3$:

$$\begin{array}{rcrcrcrcl} x & + & 2y & + & z & & & = & 1 \\ 2x & + & y & & & + & w & = & 2 \\ x & + & 2y & + & 2z & + & w & = & 0 \end{array}$$

Working over $\mathbb{Z}_3$, how many distinct solutions are there for (x, y, z, w)?

(a) infinitely many (b) exactly nine (c) exactly one

(d) exactly three (e) no solutions

4. Consider the following matrices over $\mathbb{R}$, where θ is a real number:

$$R_\theta \;=\; \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad , \quad T_\theta \;=\; \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Which one of the following statements is true?

(a) $R^6_{\pi/3} = I = T^6_{\pi/3}$ (b) $R^3_{2\pi/3} = I = T^3_{2\pi/3}$ (c) $R^4_{\pi/4} = I = T^4_{\pi/4}$

(d) $R_{\pi/2}T_{2\pi/3}R_{\pi/2} = T_{4\pi/3}$ (e) $T_{\pi/2}R_{2\pi/3}T_{\pi/2} = R_{2\pi/3}$

5. Consider the real matrix

$$M \;=\; \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and elementary matrices

$$E_1 \;=\; \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad E_2 \;=\; \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad E_3 \;=\; \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Express $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as a product of these elementary matrices with M.

(a) $I = E_2E_1E_3M$ (b) $I = E_1E_2E_3M$ (c) $I = E_3E_2E_1M$

(d) $I = E_2E_3E_1M$ (e) $I = E_1E_3E_2M$

6. Working over $\mathbb{Z}_7$, the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ are

(a) 1 and 4. (b) 2 and 6. (c) 4 and 6.

(d) 4 only. (e) 6 only.

7. Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ with entries from $\mathbb{Z}_3$. Then $M = PDP^{-1}$ where

(a) $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

(c) $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(e) $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

8. Working over $\mathbb{R}$, suppose that $M = PDP^{-1}$ where $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then, for any positive integer k, we have that M^k is

(a) $\begin{bmatrix} 3^k & 0 \\ 3^k - 2^k & 2^k \end{bmatrix}$ (b) $\begin{bmatrix} 3^k & 2^k - 3^k \\ 0 & 2^k \end{bmatrix}$ (c) $\begin{bmatrix} 2^k & 3^k - 2^k \\ 0 & 3^k \end{bmatrix}$

(d) $\begin{bmatrix} 3^k & 0 \\ 2^k - 3^k & 2^k \end{bmatrix}$ (e) $\begin{bmatrix} 2^k & 0 \\ 2^k - 3^k & 3^k \end{bmatrix}$

9. Which one of the following expressions describes M^{-1} where $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and I is the 3×3 identity matrix, working over $\mathbb{R}$?

(a) $M^2 - 2M - 2I$ (b) $-M^2 + 2M$ (c) $M^2 - 2M + 2I$

(d) $M^2 + 2M - 2I$ (e) $-M^2 + 2M + 2I$

10. Which one of the following matrices is not diagonalisable, working over $\mathbb{C}$?

(a) $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

11. Which one of the following rules for $f : \mathbb{R}^2 \to \mathbb{R}^2$ defines a linear transformation?

(a) $f(x, y) = (x + 1, y - 1)$ (b) $f(x, y) = (y - x^2, x - y^2)$ (c) $f(x, y) = (y + 1, x + y)$

(d) $f(x, y) = (y - x, x - y)$ (e) $f(x, y) = (2x, 3y + 4)$

12. Find the matrix corresponding to the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ with the following rule:

$$f(x, y) = (5x - y, 2x + y, y - x) .$$

(a) $\begin{bmatrix} 5 & -1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 5 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & 2 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & 2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

13. Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations such that

$$M_f = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix} \qquad \text{and} \qquad M_g = \begin{bmatrix} 0 & 1 \\ 4 & 1 \end{bmatrix}.$$

Find the rule for the linear transformation $gf : \mathbb{R}^3 \to \mathbb{R}^2$.

(a) $(gf)(x, y, z) = (7x - 4y + 7z\,,\ x - z)$

(b) $(gf)(x, y, z) = (x - y - 3z\,,\ -x + 3y + z)$

(c) $(gf)(x, y, z) = (x + y + 3z\,,\ 7x + y - 7z)$

(d) $(gf)(x, y, z) = (x - 3z\,,\ -7x + 4y - 7z)$

(e) $(gf)(x, y, z) = (-x + 3z\,,\ 7x - 4y + 7z)$

14. Consider the permutations

$$\alpha = (1\ 3\ 2)(4\ 6\ 5)(7\ 8) \qquad \text{and} \qquad \beta = (1\ 4\ 2)(8\ 5\ 6)(3\ 7)$$

of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ expressed in cycle notation. Which one of the following is a correct expression for a permutation γ with the property

$$\beta = \gamma^{-1}\alpha\gamma$$

where we compose from left to right?

(a) $\gamma = (5\ 7\ 8\ 4\ 3)$ (b) $\gamma = (5\ 8\ 3\ 4)$ (c) $\gamma = (3\ 8\ 5\ 7\ 4)$

(d) $\gamma = (1\ 8\ 7\ 5\ 2\ 3\ 6\ 4)$ (e) $\gamma = (1\ 8\ 2\ 3\ 6)$

15. Which one of the following configurations is impossible to reach from the 15-puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

by moving squares in and out of the space?

(a)

1	3	2	4
5	8	7	6
9	10	11	12
13	14	15	

(b)

	2	3	4
5	6	7	8
9	10	11	12
13	15	14	1

(c)

1	8	9	13
2	7	10	14
3	6	11	15
4	5	12	

(d)

	3	8	15
5	2	10	6
13	7	11	12
9	4	1	14

(e)

	15	14	13
12	11	10	9
8	7	6	5
4	3	2	1