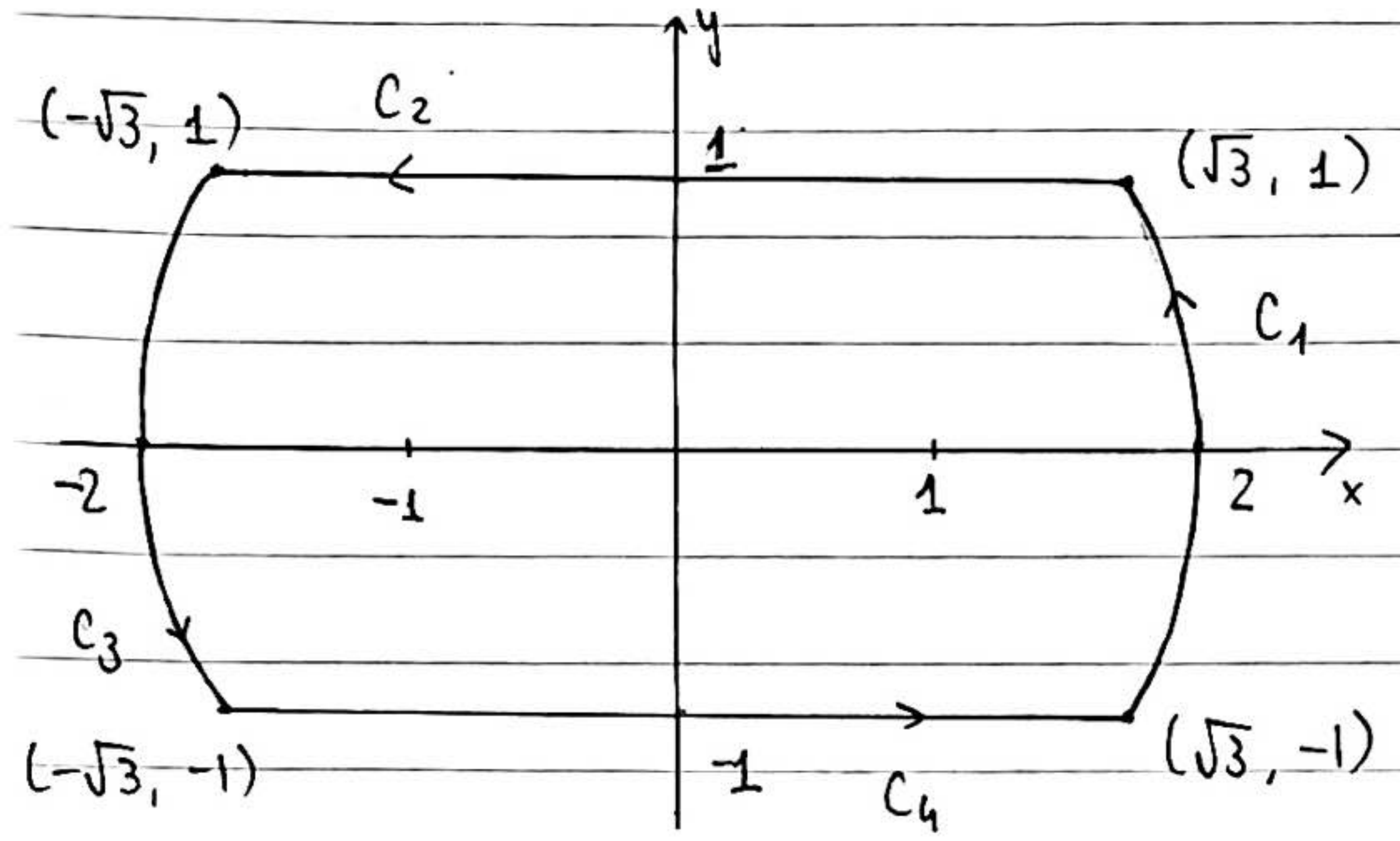
MATH2021 Assignment 1 480048691



We divide the curve C into 4 individual curves C_1, C_2, C_3, and C_4.

1. (ii) Parameterisation

. C_1: $\vec{\alpha_1}(t) = [2\cos(t), 2\sin(t)]$, $t \in \left[\frac{-\pi}{6}, \frac{\pi}{6}\right]$

$\rightarrow$ need to re-parameterise it with respect to arc length

$\vec{\alpha_1}'(t) = [-2\sin(t), 2\cos(t)] \Rightarrow \|\vec{\alpha_1}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = \sqrt{4} = 2$

$s = s(t) = \int_a^t \|\vec{\alpha_1}'(u)\|\, du = \int_{-\pi/6}^{t} 2\, du = \left[2u\right]_{-\pi/6}^{t}$

$= 2t - 2\left(\frac{-\pi}{6}\right) = 2t + \frac{\pi}{3} \quad \therefore \; t = \frac{1}{2}s - \frac{\pi}{6}$

Thus $\vec{\alpha_1}(s(t)) = \left[2.\cos\left(\frac{1}{2}s - \frac{\pi}{6}\right), 2\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right]$ with $s \in \left[0, \frac{2\pi}{3}\right]$

. C_2: C_2 is a straight line segment from $(\sqrt{3}, 1)$ to $(-\sqrt{3}, 1)$

$\vec{\alpha_2}(t) = [\sqrt{3} - t, 1]$, $t \in [0, 2\sqrt{3}]$

since $\vec{\alpha_2}'(t) = [-1, 0] \Rightarrow \|\vec{\alpha_2}'(t)\| = 1 \rightarrow$ it's already in the unit speed form

. C_3: $\vec{\alpha_3}(t) = [2\cos(t), 2\sin(t)]$, $t \in \left[\frac{5\pi}{6}, \frac{7\pi}{6}\right]$

$s = s(t) = \int_{5\pi/6}^{t} 2\, du = \left[2u\right]_{\frac{5\pi}{6}}^{t} = 2t - 2.\frac{5\pi}{6} \quad \therefore \; t = \frac{1}{2}s + \frac{5\pi}{6}$

Thus $\vec{\alpha_3}(s(t)) = \left[2\cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right), 2\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right]$, $s \in \left[0, \frac{2\pi}{3}\right]$

. C_4: C_4 is a straight line segment from $(-\sqrt{3}, -1)$ to $(\sqrt{3}, -1)$

$\vec{\alpha}_4(t) = [-\sqrt{3} + t, -1]$, $t \in [0, 2\sqrt{3}]$ → also in unit speed form

2. (i) Calculate the total arc length

We will find the individual arc length and find the total.

Since all 4 curves are in their unit speed form, $\|\vec{\alpha}_i'(t)\| = 1$.

C_1. $L_1 = \int_0^{2\pi/3} \|\vec{\alpha}_1'(s)\| \, ds = \int_0^{2\pi/3} 1 \, ds = [s]_0^{2\pi/3} = \frac{2\pi}{3}$.

C_2. $L_2 = \int_0^{2\sqrt{3}} \|\vec{\alpha}_2'(t)\| \, dt = \int_0^{2\sqrt{3}} 1 \, dt = [t]_0^{2\sqrt{3}} = 2\sqrt{3}$.

C_3. $L_3 = \int_0^{2\pi/3} \|\vec{\alpha}_3'(s)\| \, ds = \int_0^{2\pi/3} 1 \, ds = [s]_0^{2\pi/3} = \frac{2\pi}{3}$.

C_4. $L_4 = \int_0^{2\sqrt{3}} \|\vec{\alpha}_4'(t)\| \, dt = \int_0^{2\sqrt{3}} 1 \, dt = [t]_0^{2\sqrt{3}} = 2\sqrt{3}$.

Thus, the total arc length $L = L_1 + L_2 + L_3 + L_4 = 2 \cdot \frac{2\pi}{3} + 2 \cdot 2\sqrt{3} = \frac{4\pi}{3} + 4\sqrt{3}$

2. (ii) Calculate the line integral $\int_C \vec{V} \cdot d\vec{r}$ for the vector field $\vec{V} = (-y, x)$

Ignoring the corners, we can evaluate the line integral over each of the pieces such that

$$\int_C \vec{V} \cdot d\vec{r} = \sum_{i=1}^{n} \int_{C_i} \vec{V} \cdot d\vec{r}$$

. C_1: $\vec{\alpha}_1(s(t)) = \left[2 \cdot \cos\left(\frac{1}{2}s - \frac{\pi}{6}\right), 2\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right]$ with $s \in \left[0, \frac{2\pi}{3}\right]$

We have $\vec{\alpha}_1'(s(t)) = \left[-\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right), \cos\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right]$

$$\Rightarrow \int_{C_1} \vec{V} \cdot d\vec{r} = \int_{C_1} \left[-2\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right), 2\cos\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right] \cdot \left[-\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right), \cos\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right] ds$$

$$= \int_{C_1} \left(2\sin^2\left(\frac{1}{2}s - \frac{\pi}{6}\right) + 2\cos^2\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right) ds$$

$$= \int_{C_1} 2 \, ds = \left[2s\right]_0^{2\pi/3} = 2 \cdot \frac{2\pi}{3} = \frac{4\pi}{3}.$$

. C_2: $\vec{\alpha}_2(t) = [\sqrt{3} - t, 1]$ with $t \in [0, 2\sqrt{3}]$

$\vec{\alpha}_2'(t) = [-1, 0]$

$$\Rightarrow \int_{C_2} \vec{V} \cdot d\vec{r} = \int_{C_2} [-1, \sqrt{3} - t] \cdot [-1, 0] \, dt = \int_{C_2} 1 \, dt = [t]_0^{2\sqrt{3}} = 2\sqrt{3}.$$

. C_3: $\vec{\alpha}_3(s(t)) = \left[2 \cdot \cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right), 2\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right]$ with $s \in \left[0, \frac{2\pi}{3}\right]$

We have $\vec{\alpha}_3'(s(t)) = \left[-\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right), \cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right]$

$$\Rightarrow \int_{C_3} \vec{V} \cdot d\vec{r} = \int_{C_3} \left[-2\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right), 2\cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right] \cdot \left[-\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right), \cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right] ds$$

$$= \int_{C_3} \left(2\sin^2\left(\frac{1}{2}s + \frac{5\pi}{6}\right) + 2\cos^2\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right) ds$$

$$= \int_{C_3} 2 \, ds = \left[2s\right]_0^{2\pi/3} = 2 \cdot \frac{2\pi}{3} = \frac{4\pi}{3}.$$

- C_4: $\vec{\alpha}_4(t) = [-\sqrt{3}+t, -1]$ with $t \in [0, 2\sqrt{3}]$
  $\vec{\alpha}_4'(t) = [1, 0]$

$$\Rightarrow \int_{C_4} \vec{V} \cdot d\vec{r} = \int_{C_4} [1, -\sqrt{3}+t] \cdot [1, 0]\, dt = \int_{C_4} 1\, dt = [t]_0^{2\sqrt{3}} = 2\sqrt{3}.$$

Finally, the line integral over curve C is :

$$\int_C \vec{V} \cdot d\vec{r} = \sum_{i=1}^{4} \int_{C_i} \vec{V} \cdot d\vec{r} = 2 \cdot 2\sqrt{3} + 2 \cdot \frac{4\pi}{3} = 4\sqrt{3} + \frac{8\pi}{3}.$$

2. (iii) Calculate the line integral $\int_C \vec{W} \cdot d\vec{r}$ for the vector field $\vec{W} = (x, y)$

- C_1: $\vec{\alpha}_1(s(t)) = \left[2 \cdot \cos\left(\frac{1}{2}s - \frac{\pi}{6}\right), 2 \cdot \sin\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right]$ with $s \in \left[0, \frac{2\pi}{3}\right]$

  We have $\vec{\alpha}_1'(s(t)) = \left[-\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right), \cos\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right]$

$$\Rightarrow \int_{C_1} \vec{W} \cdot d\vec{r} = \int_{C_1} \left[2 \cdot \cos\left(\frac{1}{2}s - \frac{\pi}{6}\right), 2\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right] \cdot \left[-\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right), \cos\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right] ds$$

$$= \int_{C_1} \left(-2 \cdot \cos\left(\frac{1}{2}s - \frac{\pi}{6}\right)\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right) + 2\cos\left(\frac{1}{2}s - \frac{\pi}{6}\right)\sin\left(\frac{1}{2}s - \frac{\pi}{6}\right)\right) ds$$

$$= \int_{C_1} 0\, ds = 0.$$

- C_2: $\vec{\alpha}_2(t) = [\sqrt{3} - t, 1]$ with $t \in [0, 2\sqrt{3}]$
  $\vec{\alpha}_2'(t) = [-1, 0]$

$$\Rightarrow \int_{C_3} \vec{W} \cdot d\vec{r} = \int_{C_3} [\sqrt{3} - t, 1] \cdot [-1, 0]\, dt = \int_{C_2} (t - \sqrt{3})\, dt$$

$$= \left[\frac{t^2}{2} - \sqrt{3}t\right]_0^{2\sqrt{3}} = \frac{(2\sqrt{3})^2}{2} - \sqrt{3}(2\sqrt{3}) = \frac{12}{2} - 6 = 0.$$

- C_3 : $\vec{\alpha}_3(s(t)) = \left[2.\cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right), 2\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right]$ with $s \in \left[0, \frac{2\pi}{3}\right]$

We have $\vec{\alpha}_3'(s(t)) = \left[-\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right), \cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right]$

$$\Rightarrow \int_{C_3} \vec{W} . d\vec{r} = \int_{C_3} \left[2.\cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right), 2\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right]\left[-\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right), \cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right] ds$$

$$= \int_{C_3}\left(-2\cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right).\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right) + 2\cos\left(\frac{1}{2}s + \frac{5\pi}{6}\right).\sin\left(\frac{1}{2}s + \frac{5\pi}{6}\right)\right) ds$$

$$= \int_{C_3} 0 \; ds = 0.$$

- C_4 : $\vec{\alpha}_4(t) = [-\sqrt{3} + t, 1]$ with $t \in [0, 2\sqrt{3}]$

$\vec{\alpha}_4'(t) = [1, 0]$

$$\Rightarrow \int_{C_4} \vec{W} . d\vec{r} = \int_{C_4} [-\sqrt{3} + t, 1].[1, 0] \; dt = \int_{C_4} (-\sqrt{3} + t) \; dt$$

$$= \left[-\sqrt{3}t + \frac{t^2}{2}\right]_0^{2\sqrt{3}} = -3(2\sqrt{3}) + \frac{(2\sqrt{3})^2}{2} = -6 + \frac{12}{2} = 0$$

2. (iv) Decide whether $\vec{V}$ and $\vec{W}$ are conservative

- $\vec{V} = (-y, x) = P\,dx + Q\,dy$ with $P = -y$ and $Q = x \rightarrow \frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1.$

Thus, $\vec{V}$ is not a conservative vector field (also its line integral over C is not zero).

- $\vec{W} = (x, y)$ is conservative since its line integral over C is zero.

Indeed, let a function $g = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 1$

$$\Rightarrow \frac{\partial g}{\partial x} = x, \quad \frac{\partial g}{\partial y} = y \Rightarrow \nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} = x\hat{i} + y\hat{j} = \vec{W}.$$

3. (i) Calculate the total mass of the piece, given its uniform density $\frac{1}{2}$

For the region R in problem 1, it is bounded by the circle $x^2 + y^2 = 4$ along the x-axis, which can be defined by the following inequality

$$-\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2}$$

Accordingly for the y-axis, it is bounded between $y = -1$ and $y = 1$.

$$-1 \leq y \leq 1.$$

Given that density is mass per unit area, we can divide the piece into many infinitesimally rectangles with width dx and height dy. We then have the mass of one tiny rectangle is $\underbrace{\rho(x,y)}_{\text{density}} \; \underbrace{dx\,dy}_{\text{area}}$.

And thus the total mass of the piece can be determined as

$$\text{Mass} = \iint_R \rho(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{2}\,dx\,dy.$$

Inner integral:

$$\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{2}\,dx = \left[\frac{1}{2}x\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} = \frac{1}{2}\sqrt{4-y^2} + \frac{1}{2}\sqrt{4-y^2} = \sqrt{4-y^2}$$

$$\Rightarrow M = \int_{-1}^{1} \sqrt{4-y^2}\,dy$$

Let $y = 2\sin\theta$, $\frac{dy}{d\theta} = 2\cos\theta$, $dy = 2\cos\theta\,d\theta$

$$\Rightarrow M = \int_{-1}^{1}\sqrt{4-y^2}\,dy = \int_{-\pi/6}^{\pi/6}\sqrt{4-4\sin^2\theta}\,.\,2\cos\theta\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\sqrt{4(1-\sin^2\theta)}\,.\,2\cos\theta\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\sqrt{4.\cos^2\theta}\,.\,2\cos\theta\,d\theta = \int_{-\pi/6}^{\pi/6} 4\cos^2\theta\,d\theta$$

Using the trigonometric identity $\cos(2\theta) = 2\cos^2(\theta) - 1$

$$\Rightarrow M = \int_{-\pi/6}^{\pi/6} 2\cos(2\theta) + 2\,d\theta = \left[\sin(2\theta) + 2\theta\right]_{-\pi/6}^{\pi/6}$$

$$= \sin\left(\frac{\pi}{3}\right) + \frac{\pi}{3} - \sin\left(\frac{-\pi}{3}\right) + \frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3} + \frac{2\pi}{3}$$

3(ii). We have the x-coordinate of the centre of mass for the piece is

$$x_c = \frac{M_y}{M} = \frac{\iint_R x \cdot \frac{1}{2}\, dx\, dy}{\iint_R \frac{1}{2}\, dx\, dy}$$

Indeed, we can calculate the moment with respect to the y-axis

$$M_y = \int_{-1}^{1} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{2} x \, dx\, dy$$

Inner integral

$$\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{1}{2} x \, dx = \left[\frac{x^2}{4}\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} = \frac{(\sqrt{4-y^2})^2}{4} - \frac{(-\sqrt{4-y^2})^2}{4} = 0.$$

$$\Rightarrow M_y = \int_{-1}^{1} 0 \, dy = 0.$$

since $x_c = \dfrac{M_y}{M} = \dfrac{0}{\sqrt{3} + 2\pi/3} = 0$ which explains the value of x_c.