The University of Sydney

MATH2022 Linear and Abstract Algebra

Semester 1 **First Quiz (Take Home)** 2020

This take-home quiz comprises 15 multiple-choice questions, worth three marks each.

Exactly one alternative is correct in each question.

The total marks available for this quiz is 45 marks.

Instructions:

You should write your answers to this quiz on pages (or type them if you wish), and include your SID (student identification number) on each page.

You should then scan your pages as a single pdf document and upload this into Canvas **well before midnight on Thursday 26 March 2020**.

Anonymous marking will be employed, so you should not include your name. (If you do happen to include your name then there is no penalty, so please do not worry about it.)

Please be considerate of the marker and write your answers neatly in the same numerical order as the questions, with spacing between each of your 15 answers.

For each question, first indicate which alternative, from (a), (b), (c), (d), (e), you believe is correct, and then provide reasoning to justify your answer.

For each question, the alternative that you choose is worth 1 mark, and the reasoning you provide to support that alternative is worth 2 marks.

1. Which one of the following is not a field with respect to addition and multiplication?

(a) $\mathbb{Z}$ (b) $\mathbb{Z}_2$ (c) $\mathbb{Q}$ (d) $\mathbb{Z}_{13}$ (e) $\mathbb{C}$

2. Which one of the following statements is false?

(a) $\frac{3}{4} = 4$ in $\mathbb{Z}_{13}$. (b) $\frac{2}{3} = 8$ in $\mathbb{Z}_{11}$. (c) $\frac{3}{4} = 2$ in $\mathbb{Z}_5$.
(d) $\frac{2}{3} = 6$ in $\mathbb{Z}_{13}$. (e) $\frac{3}{4} = 9$ in $\mathbb{Z}_{11}$.

3. If today is Monday, what day of the week will it be after 2019^{2022} days have elapsed?

(a) Monday (b) Tuesday (c) Wednesday
(d) Thursday (e) Friday

4. Consider the following matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

with entries from $\mathbb{Z}_5$. Working over $\mathbb{Z}_5$, which of the following is true?

(a) $\det M = 0$ (b) $\det M = 1$ (c) $\det M = 2$
(d) $\det M = 3$ (e) $\det M = 4$

5. Find the unique solution to the following matrix equation

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

working over $\mathbb{Z}_2$.

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
(d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

6. Find the value of λ such that the system

$$\begin{array}{rcrcrcr} x & & & + & z & = & 1 \\ x & + & y & + & \lambda z & = & 2 \\ -2x & + & \lambda y & + & 4z & = & -4 \end{array}$$

is inconsistent over $\mathbb{R}$, but has more than one solution over $\mathbb{Z}_5$.

(a) $\lambda = 0$ (b) $\lambda = 1$ (c) $\lambda = 2$
(d) $\lambda = 3$ (e) $\lambda = 4$

7. Consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

with entries from $\mathbb{Z}_2$. Which of the following is row equivalent to M and in reduced row echelon form?

(a) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

8. Consider the following system of equations over $\mathbb{Z}_3$:

$$\begin{array}{rcrcrcrcl} x & + & y & & & + & w & = & 1 \\ & & y & + & z & + & w & = & 1 \\ x & + & 2y & + & z & + & 2w & = & 2 \\ 2x & + & y & + & 2z & + & w & = & 1 \end{array}$$

Working over $\mathbb{Z}_3$, how many distinct solutions are there for (x, y, z, w)?

(a) no solutions (b) exactly one (c) exactly three

(d) exactly nine (e) infinitely many

9. Consider the real matrix

$$M = \begin{bmatrix} 2 & 8 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}.$$

Use the chain of equivalences above, or otherwise, to find a correct expression for M as a product of these elementary matrices.

(a) $M = E_2E_4E_1E_3$ (b) $M = E_4E_2E_3E_1$ (c) $M = E_4E_3E_1E_2$

(d) $M = E_2E_1E_3E_4$ (e) $M = E_3E_1E_4E_2$

10. Consider the following matrices over $\mathbb{R}$, where θ is a real number:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Which one of the following statements is false?

(a) $R_\pi^2 = I = T_\pi^4$ (b) $R_{2\pi/3}^3 = I = T_{2\pi/3}^2$ (c) $R_{\pi/2}^4 = I = T_{\pi/2}^2$

(d) $T_{\pi/2}R_{2\pi/3}T_{\pi/2} = R_{4\pi/3}$ (e) $R_{\pi/2}T_{2\pi/3}R_{\pi/2} = T_{4\pi/3}$

11. Suppose that a, b, c, x are elements of a group G such that

$$a^{-1}xc^{-1}ba \;=\; b\,.$$

Which one of the following is a correct expression for x?

(a) $x = abc(ba)^{-1}$ (b) $x = (ba)^{-1}cba$ (c) $x = ab(ba)^{-1}c$

(d) $x = c(ba)^{-1}ab$ (e) $x = (ba)^{-1}abc$

12. Consider the permutation

$$\alpha \;=\; (1\;2)(3\;2\;1)(4\;3\;1)(3\;6\;5)$$

of $\{1, 2, 3, 4, 5, 6\}$ expressed in cycle notation where we compose from left to right. Which one of the following is a correct equivalent expression?

(a) $\alpha = (1\;4\;6\;5\;3\;2)$ (b) $\alpha = (1\;2\;3\;5\;6\;4)$ (c) $\alpha = (1\;4)(3\;6\;5)$

(d) $\alpha = (1\;3\;2\;4)(5\;6)$ (e) $\alpha = (1\;4\;2\;3)(5\;6)$

13. Consider the permutations

$$\alpha \;=\; (1\;3\;2)(4\;6\;5) \qquad \text{and} \qquad \gamma \;=\; (4\;2\;5)(6\;1\;3)$$

of $\{1, 2, 3, 4, 5, 6\}$ expressed in cycle notation. Which one of the following is a correct expression for a permutation β with the property

$$\gamma \;=\; \beta^{-1}\alpha\beta$$

where we compose from left to right?

(a) $\beta = (1\;4\;6)(2\;3\;5)$ (b) $\beta = (1\;4\;2)(3\;6\;5)$ (c) $\beta = (1\;6\;2\;3)$

(d) $\beta = (1\;3\;6\;4)(3\;5)$ (e) $\beta = (1\;3\;2\;6)$

14. Consider the permutations

$$\alpha \;=\; (1\;2\;3\;4\;5)\,, \qquad \beta \;=\; (1\;3)(2\;4)(6\;5)\,, \qquad \gamma \;=\; (1\;2\;3)(4\;5\;6)$$

of $\{1, 2, 3, 4, 5, 6\}$ expressed in cycle notation. Which one of the following is correct?

(a) α and γ are odd, and β is even. (b) α and γ are even, and β is odd.

(c) α and β are even, and γ is odd. (d) α and β are odd, and γ is even.

(e) β and γ are even, and α is odd.

15. Which one of the following configurations is impossible to reach from the 8-puzzle

1	2	3
4	5	6
7	8	

by moving squares in and out of the space?

(a)

2	4	
8	1	5
7	6	3

(b)

1	2	3
6		8
7	5	4

(c)

7	1	2
3		6
4	8	5

(d)

	1	4
3	2	7
6	5	8

(e)

3	1	4
	8	2
6	5	7