MATH2022
Take Home Quiz

1. Answer: A.

   For the elements in $\mathbb{Z}$, only $(-1)^{-1} = -1$ and $1^{-1} = 1$. Other elements do not have multiplicative inverses and therefore $\mathbb{Z}$ is not a field with respect to multiplication.

2. Answer: D.

   We have 5 x 3 = 15 (over $\mathbb{R}$), which is 2 over $\mathbb{Z}_{13}$. Therefore, $\frac{2}{3} = 5$ in $\mathbb{Z}_{13}$, not 6.

   For other options:

   A. 4 x 4 = 16 (over $\mathbb{R}$), which is 3 over $\mathbb{Z}_{13} \rightarrow \frac{3}{4} = 4$ in $\mathbb{Z}_{13}$. True.

   B. 8 x 3 = 24 (over $\mathbb{R}$), which is 2 over $\mathbb{Z}_{11} \rightarrow \frac{2}{3} = 8$ in $\mathbb{Z}_{11}$. True.

   C. 2 x 4 = 8 (over $\mathbb{R}$), which is 3 over $\mathbb{Z}_5 \rightarrow \frac{3}{4} = 2$ in $\mathbb{Z}_5$. True.

   E. 9 x 4 = 36 (over $\mathbb{R}$), which is 3 over $\mathbb{Z}_{11} \rightarrow \frac{3}{4} = 9$ in $\mathbb{Z}_{11}$. True.

3. Answer: B.

$$(2019)^{2022} = 3^{2022}\ (mod\ 7) = (3^2)^{1011}$$
$$= 9^{1011} = 2^{1011}\ (mod\ 7)$$
$$= (2^3)^{337} = 8^{337}$$
$$= 1^{337}\ (mod\ 7) = 1.$$

   It will be Tuesday (1 day after Monday).

4. Answer: A.

   Expand the matrix M over the second column. We have

$$\det M = -2\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = 5 \text{ over } \mathbb{R}\text{, which is 0 over } \mathbb{Z}_5.$$

5. Answer: E.

   We write the matrix equation into an augmented matrix and performing row reduction over $\mathbb{Z}_2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right] \quad (R_3 \to R_3 + R_1) \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad (R_2 \to R_2 + R_3)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad (R_1 \to R_1 + R_2) \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, $x = 1$, $y = 0$, and $z = 1$.

6. Answer: D.

Rewrite the system of linear equations into an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & \lambda & 2 \\ -2 & \lambda & 4 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & \lambda - 1 & 1 \\ 0 & \lambda & 6 & -2 \end{array}\right] \quad (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 + 2R_1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & \lambda - 1 & 1 \\ 0 & 0 & 6 - \lambda^2 + \lambda & -2 - \lambda \end{array}\right] \quad (R_3 \to R_3 - \lambda R_2)$$

The system is inconsistent over $\mathbb{R}$ when it has a pivot at the rightmost column, that is when $6 - \lambda^2 + \lambda = 0$ and $-2 - \lambda \neq 0$.

We have $6 - \lambda^2 + \lambda = 0$ when $\lambda = -2$ or $\lambda = 3$. On the other hand, $-2 - \lambda \neq 0$ when $\lambda \neq -2$.

Thus, for the system to be inconsistent over $\mathbb{R}$, we need $\lambda = 3$.
Let $\lambda = 3$, our augmented matrix in $\mathbb{Z}_5$ would be

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = t$, we can write x and y in terms of t:

$$\begin{cases} x = 1 - t = 1 + 4t \\ y = 1 - 2t = 1 + 3t \end{cases}$$

The solution set over $\mathbb{Z}_5$:

$$\{(1 + 4t, 1 + 3t, t), t \in \mathbb{R}\}$$

Thus with $\lambda = 3$, the system is consistent over $\mathbb{R}$, but has 5 solutions over $\mathbb{Z}_5$ (5 choices for t).

7. Answer: B.

Performing row reduction in $\mathbb{Z}_2$.

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \ (R_3 \to R_3 + R_1)$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ (R_3 \to R_3 + R_2 \text{ and } R_1 \to R_1 + R_2)$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ (R_2 \to R_2 + R_3)$$

8. Answer: D.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let z = t, w = s

$$\begin{cases} x = -2t = t \\ y = 1 - t - s = 1 + 2t + 2s \end{cases}$$

The solution set over $\mathbb{Z}_3$:

$$\{(t, 1 + 2t + 2s, t, s) \ with \ t, s \in \mathbb{R}\}$$

Thus, there are $3^2 = 9$ solutions over $\mathbb{Z}_3$ (3 choices for t and 3 choices for s).

9. Answer: E.

We have matrix M

$\begin{bmatrix} 2 & 8 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ row operation corresponds to $N_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 \\ 0 & -5 \end{bmatrix}$ row operation corresponds to $N_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$\begin{bmatrix}1 & 4\\0 & -5\end{bmatrix} \sim \begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}$ row operation corresponds to $N_3 = \begin{bmatrix}1 & 0\\0 & -1/5\end{bmatrix}$

$\begin{bmatrix}1 & 4\\0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ row operation corresponds to $N_4 = \begin{bmatrix}1 & -4\\0 & 1\end{bmatrix}$

Since

$$I = N_4N_3N_2N_1M = M^{-1}M$$

$$M^{-1} = N_4N_3N_2N_1$$

$$(M^{-1})^{-1} = (N_4N_3N_2N_1)^{-1}$$

We can express M in terms of ${N_4}^{-1}$, …, ${N_1}^{-1}$:

$$M = {N_1}^{-1}{N_2}^{-1}{N_3}^{-1}{N_4}^{-1}$$

It turns out that ${N_1}^{-1} = E_3$, repeating the same process for the remaining elementary matrices, we have

$$\mathrm{M} = {N_1}^{-1}\,{N_2}^{-1}\,{N_3}^{-1}\,{N_4}^{-1} = E_3E_1E_4E_2$$

10. Answer: E.

For the right hand side, we have:

$$R_{\pi/2}T_{2\pi/3}R_{\pi/2} = \begin{bmatrix}\cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right)\\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right)\end{bmatrix}\begin{bmatrix}\cos\left(\frac{2\pi}{3}\right) & sin\left(\frac{2\pi}{3}\right)\\ sin\left(\frac{2\pi}{3}\right) & -cos\left(\frac{2\pi}{3}\right)\end{bmatrix}\begin{bmatrix}\cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right)\\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right)\end{bmatrix}$$

$$= \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

While for the right hand side:

$$T_{4\pi/3} = \begin{bmatrix}\cos\left(\frac{4\pi}{3}\right) & sin\left(\frac{4\pi}{3}\right)\\ sin\left(\frac{4\pi}{3}\right) & -cos\left(\frac{4\pi}{3}\right)\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}.$$

11. Answer: C.

Substituting x $= ab(ba)^{-1}c$, we have:

$$\begin{aligned} a^{-1}xc^{-1}ba &= a^{-1}ab(ba)^{-1}cc^{-1}ba \\ &= (a^{-1}a)b(ba)^{-1}(cc^{-1})ba \\ &= b(a^{-1}b^{-1})(ba) \\ &= ba^{-1}(b^{-1}b)a \\ &= b\,(a^{-1}a) \\ &= b \end{aligned}$$

12. Answer: A

$$\begin{aligned} \alpha &= (12)(321)(431)(365) \\ &= (23)(431)(365) \\ &= (1432)(365) \\ &= (146532) \end{aligned}$$

13. Answer: C.

Given $\alpha = (1\ 3\ 2)(4\ 6\ 5)$ and $\gamma = \ \beta^{-1}\alpha\beta = (4\ 2\ 5)(6\ 1\ 3)$, our goal is to find the permutation β such that γ is the conjugate of α by β.

Consider $\beta = (1\ 6\ 2\ 3)$:

For the first 3-cycle of α, we have:

the image of 1 under β is 6

the image of 3 under β is 1

the image of 2 under β is 3

While for the second 3-cycle of α:

the image of 4 under β is 4

the image of 6 under β is 2

the image of 5 under β is 5

Thus, $\alpha^{\beta} = (6\ 1\ 3)(4\ 2\ 5)$. Since the product of disjoint cycles are commutative, $(6\ 1\ 3)(4\ 2\ 5) = (4\ 2\ 5)(6\ 1\ 3) = \ \gamma$.

14. Answer: B.

$$\alpha = (1\ 2\ 3\ 4\ 5) = (1\ 2)(1\ 3)(1\ 4)(1\ 5)\text{: even}$$

$$\beta = (1\ 3)(2\ 4)(6\ 5)\text{: odd}$$

$$\gamma = (1\ 2\ 3)(4\ 5\ 6) = (1\ 2)(1\ 3)(4\ 5)(4\ 6)\text{: even}$$

15. Answer: C.

7	1	2
3		6
4	8	5

can be transformed into

7	1	2
3	5	8
4	6	

corresponds to (1 7 4 3 2)(6 8) or (1 7)(1 4)(1 3)(1 2)(6 8), which is a product of an odd number of transpositions.

Hence, the configuration C is impossible to reach from the 8-puzzle square.