STAT2911
Probability and Statistical Models
Assignment 2
Due by 11:59pm Friday 22/5/2020

You can quote any result that was stated in class or given as a tutorial problem. The marks awarded for each problem are specified in brackets.

1. [60] The RVs X, Y have a joint density

$$f_{XY}(x, y) = 2e^{-(x+y)} \quad 0 < x < y.$$

   (i) Find the marginal CDF of X.
   (ii) Compute $P(Y < 1|X < 1)$.
   (iii) Find the conditional density of Y given $X = x$, $f_{Y|X}(y|x)$.
   (iv) Find $E(Y|X)$.
   (v) Compute $P(Y < 1|X = 1)$.
   (vi) Find the joint CDF F_{XY}.

2. [40] Let $X_1, \ldots, X_n$ be be a sample from a $U(0, \theta)$ distribution where $\theta > 0$ is an unknown parameter.

   (i) Find the expectation and variance of the sample mean $\bar{X}$.
   (ii) Find the expectation and variance of the nth order statistics $X_{(n)}$.
   (iii) Construct two unbiased estimators by applying a linear function to each of $\bar{X}$ and $X_{(n)}$.
   (iv) Which estimator is better in the MSE sense?

3. This is a *bonus* problem. You would only get extra points if your correct answer is *crystal clear*. Murky arguments would not be awarded by any partial credit.
   Suppose the CDF F is a continuous real function and note that this does *not* imply F is differentiable. Assume for simplicity that $F(0) = 0$ and $F(1) = 1$ (this does not change the major statements below but makes the proof a bit cleaner).

   (i) [2] Show that if X, Y are independent F-distributed RVs then for any $n \in \mathbb{N}$

$$P(X = Y) \leq \sum_{k=1}^{n} \left[F\left(\frac{k}{n}\right) - F\left(\frac{k-1}{n}\right) \right]^2.$$

   (ii) [4] Conclude that $P(X = Y) = 0$ (remember that you *cannot* assume F has a density). Hint: n above can be arbitrarily large.
   (iii) [2] Suppose $X_1, \ldots, X_n$ is a sample drawn from F. Show that the probability that all X_i differ from one another is 1, in other words that

$$P\left(\text{exist } i < j \text{ with } X_i = X_j\right) = 0.$$