MATH2022 Take Home Quiz 3

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1 Answer: C

We have $\langle 1 \rangle$ and $\langle -1 \rangle$ generate all of $\mathbb{Z}$ and so $\mathbb{Z}$ is cyclic under addition.

2 Answer: B

We can rewrite the system of equations as an augmented matrix and first work over $\mathbb{Z}_3$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

It is clear that the system has no solutions over $\mathbb{Z}_3$.
Now, we work over $\mathbb{Z}_5$:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 4 \end{array}\right]$$

It appears that we have two free variables z and w, making each has 5 choices over $\mathbb{Z}_5$. Therefore, there are 25 solutions over $\mathbb{Z}_5$.

3 Answer: E

We have $R_{\pi/3}T_{\pi/2}R_{\pi/2} = T_{5\pi/6}R_{\pi/2} = T_{\pi/3}$, not $T_{2\pi/3}$.

4 Answer: C

Consider the group G of symmetries of a regular hexagon, generated by a rotation $\alpha = (1\ 2\ 3\ 4\ 5\ 6)$ and a reflection $\beta = (1\ 6)(2\ 5)(3\ 4)$. We have the facts that $\alpha^6 = \beta^2 = 1$ and $\alpha\beta = \beta\alpha^{-1} = \beta\alpha^5$. Furthermore, $\beta^{-1}\alpha^i\beta = \alpha^{-i}$ for all i and $\beta^{-1} = \beta$,

$$\begin{aligned}
\beta\alpha^3\beta^3\big(\alpha^{-2}\big)\beta^{-3}\alpha^5 &= \beta\alpha^3\beta^3\big(\beta^{-1}\alpha^2\beta\big)\beta^{-3}\alpha^5 \\
&= \beta\alpha^3\big(\beta^2\alpha^2\beta^{-2}\big)\alpha^5 \\
&= \beta\alpha^5\beta^{-2}\alpha^5 \\
&= \beta\alpha^{-1}\beta^{-2}\alpha^5 \\
&= \alpha\beta\beta^{-2}\alpha^5 \\
&= \alpha\beta^{-1}\alpha^5 \\
&= \alpha\beta\alpha^5 \\
&= \alpha\alpha\beta \quad = \alpha^2\beta.
\end{aligned}$$

5 Answer: E

We perform row reduction on M over $\mathbb{Z}_2$:

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(M) = the number of rows with leading pivots = 4. Using the Rank-Nullity Theorem, the nullity of this matrix is equals to number of columns - rank(M) $= 4 - 4 = 0$. Hence, rank(M) = 4 and nulltiy(M) = 0.

6 Answer: D

Let $C = \{(1,0),(0,1)\}$ be the standard basis of $\mathbb{R}^2$, then it is easy to write each element of B as a linear combination of the standard basis vectors:

$$(1,-1) = 1(1,0) - 1(0,1) \text{ and } (2,-3) = 2(1,0) - 3(0,1),$$

$$\therefore \big[id\big]_C^B = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \text{ and } \big[id\big]_B^C = \left(\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$$

so

$$\big[\mathbf{v}\big]_B = \big[id\big]_C^B\big[\mathbf{v}\big]_C = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

7 Answer: A

Let M be a 3 x 3 matrix with columns the vectors of the subset $\{(1,1,0),(0,1,1),(1,0,1)\}$ and row reduce over $\mathbb{Z}_2$:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the vectors from this subset are linearly dependent, they cannot form a basis for the vector space $\mathbb{Z}_2^3$.

8 Answer: D

We have the ordered basis $B = \{(1,1),(1,-2)\}$ and the linear operator $L(x,y) = (x+y, x-y)$. Observe that:

$$L(1,1) = (1+1, 1-1) = (2,0) = \frac{4}{3}(1,1) + \frac{2}{3}(1,-2),$$
$$L(1,-2) = (1-2, 1+2) = (-1,3) = \frac{1}{3}(1,1) - \frac{4}{3}(1,-2).$$

Hence, $\left[L\right]_B^B = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{4}{3} \end{bmatrix}$.

9 Answer: E

We have the linear operator $L(x,y) = (x+2y, 3x+2y)$. Consider the ordered basis $B = \{(-1,1),(2,3)\}$. Observe that:

$$L(-1,1) = (-1+2, -3+2) = (1,-1) = -1(-1,1) + 0(2,3)$$
$$L(2,3) = (2+6, 6+6) = (8,12) = 0(-1,1) + 4(2,3)$$

We have found that $\left[L(-1,1)\right]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\left[L(2,3)\right]_B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, so $\left[L\right]_B^B = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

10 Answer: D

We have V be the vector space of functions spanned by $B = \{cos(x)+sin(x), cos(x)-sin(x)\}$ and the linear operator $D : V \to V$ that maps a function to its derivative. Observe that

$$\begin{aligned} D(cos(x) + sin(x)) &= (cos(x) + sin(x))' = cos(x) - sin(x) \\ &= 0(cos(x) + sin(x)) + 1(cos(x) - sin(x)) \\ D(cos(x) - sin(x)) &= (cos(x) - sin(x))' = -(cos(x) + sin(x)) \\ &= -1(cos(x) + sin(x)) + 0(cos(x) - sin(x)) \end{aligned}$$

Hence, $\big[D\big]_B^B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

11 Answer: A

Performing row reduction on M over $\mathbb{Z}_5$:

$$M = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x + 4y + 3z = 0 \\ \qquad y \qquad = y \\ \qquad\qquad z = z \end{cases} \iff \begin{cases} x = y + 2z \\ y = y \\ z = z \end{cases} \quad \therefore null(M) = \left\{ y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \middle| \quad y, z \in \mathbb{Z}_5 \right\},$$

Hence the basis of $null(M) = \{(1, 1, 0), (2, 0, 1)\}$.

12 Answer: B

Consider $B = \{(1, 1), (2, 3)\}$ and $D = \{(2, 1), (3, 4)\}$ ordered bases of $\mathbb{Z}_7^2$. We need to write each elements of B as a linear combination of the elements of D. We solve the following equations over $\mathbb{Z}_7^2$.

$$\begin{cases} (1, 1) = \alpha_1(2, 1) + \alpha_2(3, 4) \quad \therefore \quad \alpha_1 = 3, \alpha_2 = 3 \\ (2, 3) = \beta_1(2, 1) + \beta_2(3, 4) \quad \therefore \quad \beta_1 = 4, \beta_2 = 5 \end{cases}$$

Hence, $\big[(1, 1)\big]_D = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\big[(2, 3)\big]_D = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \therefore \big[id\big]_D^B = \begin{bmatrix} 3 & 4 \\ 3 & 5 \end{bmatrix}$.

13 Answer: E

We have matrix $M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ with entries over $\mathbb{Z}_5$.

Observe that

$$\begin{aligned} det(\lambda I - M) &= (\lambda - 1)\begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 1 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ \lambda - 2 & 0 \end{vmatrix} \\ &= (\lambda - 1)^2(\lambda - 2) - 4(\lambda - 2) \\ &= \lambda^3 - 4\lambda^2 + \lambda + 6. \end{aligned}$$

Applying the Cayley-Hamilton Theorem, we have:

$$\begin{aligned} \chi(M) = M^3 - 4M^2 + M + 6I &= 0 \\ M^3 - 4M^2 + M &= -6I \\ M(M^2 - 4M + I) &= -6I \\ M\frac{-1}{6}(M^2 - 4M + I) &= I \end{aligned}$$

Hence, over $\mathbb{R}$, $M^{-1} = \frac{-1}{6}(M^2 - 4M + I)$. Working over $\mathbb{Z}_5$:

$$M^{-1} = 4M^2 + 4M + 4I$$

14 Answer: B

We have

$$\begin{aligned} \alpha &= (1\ 3)(2\ 4)(5\ 6), \\ \beta &= (\text{a b})(\text{c d})(\text{e f}). \end{aligned}$$

Firstly, the product of disjoint cycles are commutative so we can arrange the 2-cycles in $3! = 6$ ways. Furthermore, each of the can be written in two ways, irrespective of the cycle notation. In total, there are 3! x 2 x 2 x 2 = 48 permutations of γ such that β is the conjugate of α by γ.

15 Answer: C

We learned from the concepts of 8-puzzle is that one configuration is reachable from another if they are an even number of permutations away. Considering each given statements and their corresponding configuration

$$a=1, c=2 \text{ gives A} = \begin{array}{|c|c|c|}\hline 4 & 8 & 1 \\ \hline 6 & & 3 \\ \hline 5 & 7 & 2 \\ \hline \end{array} \qquad c=1, b=2 \text{ gives B} = \begin{array}{|c|c|c|}\hline 4 & 8 & 3 \\ \hline 6 & & 2 \\ \hline 5 & 7 & 1 \\ \hline \end{array}$$

$$c=2, b=1 \text{ gives C} = \begin{array}{|c|c|c|}\hline 4 & 8 & 3 \\ \hline 6 & & 1 \\ \hline 5 & 7 & 2 \\ \hline \end{array} \qquad a=2, c=3 \text{ gives D} = \begin{array}{|c|c|c|}\hline 4 & 8 & 2 \\ \hline 6 & & 1 \\ \hline 5 & 7 & 3 \\ \hline \end{array}$$

$$c=3, b=1 \text{ gives E} = \begin{array}{|c|c|c|}\hline 4 & 8 & 2 \\ \hline 6 & & 1 \\ \hline 5 & 7 & 3 \\ \hline \end{array}$$

It appears that configurations A, B, D, and E can be reached within each other since they are all an even number of permutations away from each other. Indeed,

- A can reach B by performing (1 3)(1 2) (reverse can also be done),
- A can reach D and E by performing (2 3)(2 1) since D and E are the same configuration (reverse can also be done),
- B can reach D and E by performing (1 3)(1 2) (reverse can also be done),
- D and E are identical which we can also say identity permutation has been performed (which is even).

This means that, if one of the configurations from the set {A, B, D, E} has been reached by moving squares in and out, the other configurations from this set can also be reached. However, this would let a, b, and c attain more than one value from {1, 2, 3} at once, which is impossible. Therefore, the only possible statement from above is $c = 2 \implies b = 1$. We can check the configuration of C

$$\begin{array}{|c|c|c|}\hline 4 & 8 & 3 \\ \hline 6 & & 1 \\ \hline 5 & 7 & 2 \\ \hline \end{array} \quad \text{can be transformed into} \quad \begin{array}{|c|c|c|}\hline 4 & 8 & 3 \\ \hline 6 & 1 & 2 \\ \hline 5 & 7 & \\ \hline \end{array}$$

which corresponds to (1 5 7 8 2 6 4) which is a product of 6 transpositions.