STAT2911

Probability and Statistical Models

Assignment 1

Due by 11:59pm Friday April 3 2020

Late submissions are penalized as per standard university policy: Deduction of 5% of the maximum mark for each calendar day after the due date. After ten calendar days late, a mark of zero will be awarded.

1. The negative binomial RV X models the number of trials until the rth success in a sequence of independent Bernoulli trials with probability of success p in each trial. So, if $q = 1 - p$,
$$P(X = k) = \binom{k-1}{r-1} p^r q^{k-r}, \ k = r, r+1, \cdots .$$
Let $Y = X - r$. What does Y measure?
Suppose that $r \to \infty$ and $q \to 0$ so that $rq \to \lambda$. Show that, for fixed $m = 0, 1, \ldots$
$$P(Y = m) \longrightarrow \frac{\lambda^m}{m!} e^{-\lambda}.$$

2. Let
$$\Omega = \{0,1\}^{\mathbb{N}} = \{\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots) \, : \, \alpha_i \in \{0,1\}\} .$$
Fact. There exists a σ-algebra $\mathcal{F}$ s.t. for every $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, if we define the set $E_{\boldsymbol{\beta}} \subset \Omega$ as
$$E_{\boldsymbol{\beta}} = \{\boldsymbol{\alpha} \in \Omega \, : \, (\alpha_1, \alpha_2, \ldots, \alpha_n) = \boldsymbol{\beta}\} ,$$
then $E_{\boldsymbol{\beta}} \in \mathcal{F}$.
For example, with $\boldsymbol{\beta} = (0,1,1,0)$, $E_{\boldsymbol{\beta}}$ consists of all sequences $(0,1,1,0,\alpha_5,\alpha_6,\ldots)$, where for $i \geq 5$, $\alpha_i \in \{0,1\}$.
Let $p \in [0,1]$ then define the function P on sets $E_{\boldsymbol{\beta}}$ as above as
$$P(E_{\boldsymbol{\beta}}) = p^{\sum_{i=1}^n \beta_i} (1-p)^{n - \sum_{i=1}^n \beta_i} .$$
Fact. P can be extended to a probability measure on $(\Omega, \mathcal{F})$.
   - (i) What does $(\Omega, \mathcal{F}, P)$ model in this particular case?
   - (ii) Show that with $\mathbf{0} = (0,0,\ldots) \in \Omega$, the set $\{\mathbf{0}\} \in \mathcal{F}$. Justify every step in your proof, that is refer precisely to any result you are using in your derivation.
   - (iii) We will show later on that if B_n is a decreasing sequence of events, i.e., $B_n \supset B_{n+1}$ then
   $$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n).$$
   Use this to show that if $p > 0$ then $P(\{\mathbf{0}\}) = 0$.
   - (iv) For $\boldsymbol{\alpha} \in \Omega$ and $n \in \mathbb{N}$ let $X_n(\boldsymbol{\alpha}) = \alpha_n$. What kind of a RV is X_n?
   - (v) For $\boldsymbol{\alpha} \in \Omega$ let $X(\boldsymbol{\alpha}) = \inf\{n : \alpha_n = 1\}$, where for $A \subset \mathbb{R}$, $\inf(A)$ is the infimum of the set A or its greatest lower bound. What kind of a RV is X?
   - (vi) What is $X(\mathbf{0})$?
   - (vii) Conclude that if $p > 0$ then $P(X = \infty) = 0$ but $\infty \in X(\Omega)$ (∞ is in the range of X).