MATH2021 Quiz 2

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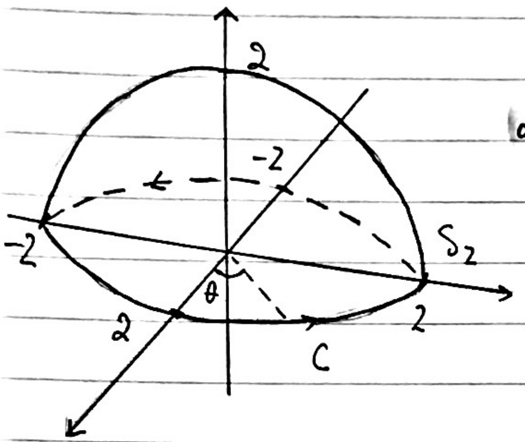
Question 1.



$$S_2 = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 4,\ z \geq 0 \}$$

a) We can define our upper unit sphere S_2 by setting $g(x, y, z) = x^2 + y^2 + z^2 - 4 = 0$.

Gradient of g:

$$\nabla g = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right] = [2x, 2y, 2z]$$

We also have: $\| \nabla g \| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$

$$= \sqrt{4(x^2 + y^2 + z^2)} = \sqrt{4 \cdot 4} = 4$$

Hence the unit normal vector is given by:

$$\vec{n} = \frac{\nabla g}{\| \nabla g \|} = \frac{[2x, 2y, 2z]}{4} = \frac{1}{2} [x, y, z]$$

Note that $\vec{n}$ is a position vector points radially outward from the ~~sphere~~ upper hemisphere, so $\vec{n}$ is the outward unit normal.

b) We can compute the flux through S_2 by calculating its surface integral.
Given that it is a solid region with the boundary surface of

$$S_2 = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 4 \},\ z \geq 0$$

We can apply Gauss' theorem with the region V for the triple integral is then the region enclosed by the surface S_2 and its projection of the xy-plane which is the circle of radius 2.

$\Rightarrow$ Upper hemisphere can be defined by the following inequalities:

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 2 \quad \text{with } r^2 = x^2 + y^2 \text{ because } \begin{cases} x = r.\cos\theta \\ y = r.\sin\theta \end{cases}$$

so $r^2 + z^2 \leq 4 \Rightarrow 0 \leq z \leq \sqrt{4 - r^2}$ (considering the upper hemisphere)

With $\vec{F} = [\underset{P}{-y}, \underset{Q}{x}, \underset{R}{z}] \Rightarrow \text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$$= 0 + 0 + 1 = 1$$

Using Gauss's Theorem:

$$\iint_{S_2} \vec{F} \cdot \vec{n} \, dS = \iiint_V \text{div } \vec{V} \, dV = \iiint_V 1 \, dV$$

$$= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[rz \right]_{z=0}^{z=\sqrt{4-r^2}} dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r . \sqrt{4-r^2} \, dr \, d\theta$$

Inner integral: let $u = 4 - r^2 \Rightarrow \frac{du}{dr} = -2r \Rightarrow du = -2r . dr$

so $\int_0^2 r . \sqrt{4-r^2} \, dr = \frac{-1}{2} \int_0^2 \sqrt{4-r^2} \, (-2r) \, dr = \frac{-1}{2} \int_0^2 \sqrt{u} \, du$

$$= \frac{-1}{2} \left[\frac{2}{3} u^{3/2} \right]_{r=0}^{r=2} = \frac{-1}{2} \left[\frac{2}{3} (4-r^2)^{3/2} \right]_{r=0}^{r=2} = \frac{8}{3}$$

Finally: $\int_0^{2\pi} \frac{8}{3} \, d\theta = \left[\frac{8}{3} \theta \right]_0^{2\pi} = \frac{8}{3} . 2\pi = \frac{16\pi}{3}$.

c) From part a) we have concluded that $\vec{n} = \frac{1}{2}[x, y, z]$ is a unit normal vector to S_2 that points radially outward. Since the intersection of S_2 with the xy-plane is the circle $x^2 + y^2 = 4$, S_2 is thus bounded by this curve we denote C.

Using the right hand rule (thumb pointing in the direction of $\vec{n}$), we can quickly see that c is counter-clockwise (i.e. positively oriented).

Using Stoke's Theorem, we can find $\iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} \, dS$ by finding its equivalence $\int_C \vec{F} \cdot d\vec{r}$.

Parameterise curve C: $\vec{r}(t) = [2\cos(t), 2\sin(t)]$, $0 \leq t \leq 2\pi$

$$\vec{r}'(t) = [-2\sin(t), 2\cos(t)]$$

Since the curve C is on the xy-plane the z-component of the vector field $\vec{F}$ will not affect the total work done to move an object along the curve C.

$$\Rightarrow \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_0^{2\pi} [-2\sin(t), 2\cos(t)] \cdot [-2\sin(t), 2\cos(t)] \, dt$$

$$= \int_0^{2\pi} 4\sin^2(t) + 4\cos(t) \, dt$$

$$= \int_0^{2\pi} 4 \, dt = [4t]_0^{2\pi} = 4.2\pi = 8\pi.$$

Question 2.

a) $y' - y = -1, \quad y(0) = 0$

This is a first order ODE with the form $\frac{dy}{dx} + p(x).y = q(x)$

with $p(x) = -1$ and $q(x) = -1$.

Integrating factor: $\exp\left(\int p(x)\,dx\right) = e^{\int -1\,dx} = e^{-x}$.

$\Rightarrow y(x) = \frac{1}{r(x)} . \int r(x)\, q(x)\, dx + \frac{C}{r(x)}$

$= e^{x} . \int e^{-x}(-1)\,dx + C.e^{x}$

$= e^{x} . e^{-x} + C.e^{x}$

$= 1 + Ce^{x}$

Given that $y(0) = 0$, $1 + C.e^{0} = 0 \Leftrightarrow 1 + C = 0 \Leftrightarrow C = -1$

Hence, the particular solution is: $y(x) = 1 - e^{x}$.

b) $y' + 2xy = -x, \quad y'(0) = 0$.

This is also a first order ODE with the form $\frac{dy}{dx} + p(x).y = q(x)$

with $p(x) = 2x$ and $q(x) = -x$.

Integrating factor : ~~exx~~ $\exp\left(\int p(x)\,dx\right) = e^{\int 2x\,dx} = e^{x^2}$

$\Rightarrow y(x) = \frac{1}{r(x)} . \int r(x)\, q(x)\, dx + \frac{C}{r(x)}$

$= e^{-x^2} . \int e^{x^2}(-x)\,dx + C.e^{-x^2}$

. For $-\int e^{x^2}.x\,dx$, let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$, $du = 2x\,dx$

$\Rightarrow -\int e^{x^2}.x\,dx = \frac{-1}{2}\int e^{x^2} 2x\,dx = \frac{-1}{2}\int e^{u}\,du = \frac{-1}{2} e^{x^2}$

$\Rightarrow$ General solution: $y(x) = e^{-x^2} . \left(\frac{-1}{2}\right) e^{x^2} + C.e^{-x^2} = \frac{-1}{2} + C.e^{-x^2}$

Given that $y'(0) = 0$, we have $y'(x) = C.e^{-x^2}.(-2x)$

$\Rightarrow y'(0) = C.e^{-0}(-2.0) = 0$

It appears that, regardless of the value of the constant C, $y'(0) = 0$.
Indeed, let $C = 1$: $y(x) = \frac{-1}{2} + e^{-x^2} \Rightarrow y'(x) = -2x \cdot e^{-x^2}$

$$\Rightarrow y'(0) = -2.0.e^{-0} = 0.$$

let $C = \frac{-3}{4}$: $y(x) = \frac{-1}{2} - \frac{3}{4} e^{-x^2} \Rightarrow y'(x) = \frac{-3}{4}(-2x) e^{-x^2}$

$$\Rightarrow y'(0) = \frac{-3}{4}(-2.0).e^{-0} = 0$$

Thus, there are infinitely many particular solution that satisfies the initial condition and we can choose one of them such as
$y(x) = \frac{-1}{2} + e^{-x^2}$.

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c) $y'' + y = 0 \quad , \quad y(1) = y'(1) = 1$

Characteristic equation: $\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda_1, \lambda_2 = \pm i$

$\Rightarrow y(x) = C_1 . e^{ix} + C_2 e^{-ix} = E . \cos(x) + F . \sin(x)$

By the initial condition, we have $y(1) = E . \cos(1) + F . \sin(1) = 1 \quad (1)$

Also, $y'(x) = -E . \sin(x) + F . \cos(x)$

$\Rightarrow y'(1) = -E . \sin(1) + F . \cos(1) = 1 \quad (2)$

From (1), we have $F = \dfrac{1 - E . \cos(1)}{\sin(1)}$, sub this into (2):

$\Rightarrow -E . \sin(1) + \dfrac{1 - E . \cos(1)}{\sin(1)} . \cos(1) = 1$

$\Leftrightarrow \dfrac{-E . \sin^2(1)}{\sin(1)} + \dfrac{\cos(1) - E . \cos^2(1)}{\sin(1)} = \dfrac{\sin(1)}{\sin(1)}$

$\Leftrightarrow -E . [\sin^2(1) + \cos^2(1)] + \cos(1) = \sin(1)$

$\Leftrightarrow -E = \sin(1) - \cos(1)$

$\Leftrightarrow E = \cos(1) - \sin(1)$

Now, we can solve for F

$$F = \frac{1 - [\cos(1) - \sin(1)] . \cos(1)}{\sin(1)}$$

$$= \frac{1 - \cos^2(1) + \sin(1) . \cos(1)}{\sin(1)}$$

$$= \frac{\sin^2(1) + \sin(1) . \cos(1)}{\sin(1)} = \frac{\sin(1) [\sin(1) + \cos(1)]}{\sin(1)} = \sin(1) + \cos(1)$$

Thus, the particular solution is:

$$y(x) = [\cos(1) - \sin(1)] . \cos(x) + [\sin(1) + \cos(1)] . \sin(x)$$

d) $y'' + y' - 2y = 0$, $y(0) = y'(0) = 1.$

Characteristic equation: $\lambda^2 + \lambda - 2 = 0 \Leftrightarrow (\lambda-1)(\lambda+2) = 0$

$$\Leftrightarrow \left[\begin{array}{l} \lambda = 1 \\ \lambda = -2. \end{array}\right.$$

$\Rightarrow$ General solution: $y(x) = Ae^{x} + B.e^{-2x}$

Given the initial conditions, we have:

$y(0) = A.e^{0} + B.e^{-2.0} = A + B = 1$ (1)

Also, $y'(x) = A.e^{x} - 2B.e^{-2x}$

$\Rightarrow y'(0) = A.e^{0} - 2.Be^{-2.0} = A - 2B = 1.$ (2)

From (1) and (2), we have

$$\begin{cases} A + B = 1 \\ A - 2B = 1 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = 0 \end{cases}$$

$\Rightarrow$ Particular solution:

$y(x) = e^{x}$

Question 3.

a) $y'' + y = \cos(x) - \sin(x)$

We have the complementary solution:

$$y'' + y = 0 \Rightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda^2 = -1, \Leftrightarrow \lambda = \pm i$$

$$\Rightarrow y_c(x) = C_1 . e^{ix} + C_2 . e^{-ix}$$

$$= E.\cos(x) + F.\sin(x) \quad \text{with } y_1 = \cos(x) \text{ and } y_2 = \sin(x)$$

The Wronskian of these two functions are:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = 1$$

Particular solution is then

$$y_p(x) = -y_1 \int \frac{y_2 . g(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1\, g(x)}{W(y_1, y_2)} dx$$

$$= -\cos(x) \int \sin(x) [\cos(x) - \sin(x)] dx + \sin(x) \int \cos(x) [\cos(x) - \sin(x)] dx$$

$$= -\cos(x) \int \sin(x).\cos(x) - \sin^2(x)\, dx + \sin(x) \int \cos^2(x) - \sin(x).\cos(x)\, dx$$

. With $\int \sin(x).\cos(x)\, dx$, let $u = \sin(x)$ ~~cos(x)~~ $\Rightarrow \frac{du}{dx} = \cos(x)$, $du = \cos(x).dx$

$$\Rightarrow \int \sin(x).\cos(x)\, dx = \int u.\, du = \left[\frac{u^2}{2}\right] = \frac{\sin^2(x)}{2}$$

. With quick calculation, we also have

$$\int \sin^2(x)\, dx = \frac{x}{2} - \frac{\sin(x).\cos(x)}{2} \quad \text{and}$$

$$\int \cos^2(x)\, dx = \frac{x}{2} + \frac{\sin(x).\cos(x)}{2}$$

Continue with the calculation for $y_p(x)$:

$$y_p(x) = -\cos(x).\left[\frac{\sin^2(x)}{2} - \frac{x}{2} + \frac{\sin(x).\cos(x)}{2}\right]$$

$$+ \sin(x).\left[\frac{x}{2} + \frac{\sin(x).\cos(x)}{2} - \frac{\sin^2(x)}{2}\right]$$

And so the general solution is:

$$y(x) = y_c(x) + y_p(x)$$
$$= E.\cos(x) + F.\sin(x) - \frac{1}{2}.\cos(x)\left[\sin^2(x) - x + \sin(x).\cos(x)\right]$$
$$+ \frac{1}{2}\sin(x)\left[x + \sin(x).\cos(x) - \sin^2(x)\right]$$

b) $y'' - y = |x|$ over the x- interval $[-1, 1]$

With $L=1$, $a_0 = \frac{1}{2} \cdot \int_{-1}^{1} f(x)\,dx = \frac{1}{2} \cdot 2\int_0^1 x\,dx = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$

(since $f(x) = |x|$ is an even function)

$$a_n = \frac{1}{1} \cdot \int_{-1}^{1} f(x) \cdot \cos(n\pi x)\,dx \text{ for } n > 0$$

$$= 1 \cdot 2\int_0^1 |x| \cdot \cos(n\pi x)\,dx = 2\int_0^1 x \cdot \cos(n\pi x)\,dx$$

$$= 2 \cdot \left[\frac{\cos(n\pi)}{n^2\pi^2} + \frac{\sin(n\pi)}{n \cdot \pi} - \frac{1}{n^2\pi^2}\right]$$

$$\cancel{b_n = \frac{1}{1} \cdot \int_{-1}^{1} f(x) \cdot \sin(n\pi x)\,dx \text{ for } n > 0}$$

$$\cancel{= 1 \cdot 2\int_0^1 x \cdot \sin(n\pi x)\,dx}$$

$$\cancel{= 2\left[\frac{\sin(n\pi)}{n^2\pi^2} - \frac{\cos(n\pi)}{n\pi}\right]}$$

Since $f(x) = |x|$ is an even function, its Fourier sine coefficients all vanish.

Hence the Fourier series of function $f(x)$ on interval $[-1, 1]$ is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos(n\pi x)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} 2 \cdot \left(\frac{\cos(n\pi)}{n^2\pi^2} + \frac{\sin(n\pi)}{n\pi} - \frac{1}{n^2\pi^2}\right) \cdot \cos(n\pi x)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} 2 \cdot \left[\frac{(-1)^n}{n^2\pi^2} + 0 - \frac{1}{n^2\pi^2}\right] \cdot \cos(n\pi x)$$

(since $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$)

$$= \frac{1}{2} + \sum_{n=1}^{\infty} 2 \cdot \left[\frac{(-1)^n - 1}{n^2\pi^2}\right] \cdot \cos(n\pi x)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1) \cdot \cos(n\pi x)}{n^2\pi^2}$$

Since the original ODE is linear and so a solution $y(x)$ is of the form

$$y(x) = y_c(x) + y_p(x)$$

We have the complementary solution:

$$y'' - y = 0 \Rightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda^2 = 1 \Leftrightarrow \lambda = \pm 1$$

$$\Rightarrow y_c(x) = c_1 . e^x + c_2 . e^{-x}$$

We now construct $y_p(x)$ by assuming

$$y_p = \frac{-1}{2} - \sum_{n=1}^{\infty} C . \cos(n\pi x), \quad \text{and thus}$$

$$y'_p = \sum_{n=1}^{\infty} C . n\pi . \sin(n\pi x)$$

$$y''_p = \sum_{n=1}^{\infty} C . n^2\pi^2 \cos(n\pi x)$$

Substitution of y_p and y''_p into the original ODE gives

$$y'' - y = |x|$$

$$\Rightarrow \sum_{n=1}^{\infty} C . n^2\pi^2 \cos(n\pi x) - \left[\frac{-1}{2} - \sum_{n=1}^{\infty} C . \cos(n\pi x)\right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi^2} . \cos(n\pi x)$$

$$\Leftrightarrow \frac{1}{2} + \left[\sum_{n=1}^{\infty} C . n^2\pi^2 \cos(n\pi x) + C . \cos(n\pi x)\right] = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi^2} . \cos(n\pi x)$$

and so

$$Cn^2\pi^2 + C = \frac{2[(-1)^n - 1]}{n^2\pi^2} \Rightarrow C(n^2\pi^2 + 1) = \frac{2[(-1)^n - 1]}{n^2\pi^2}$$

$$\Rightarrow C = \frac{2[(-1)^n - 1]}{n^2\pi^2(n^2\pi^2 + 1)}$$

Thus, $$y_p(x) = \frac{-1}{2} - \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi^2(n^2\pi^2 + 1)} . \cos(n\pi x)$$

$$= \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n^2\pi^2(n^2\pi^2 + 1)} . \cos(n\pi x)$$

Finally, the general solution is:

$$y(x) = c_1 . e^x + c_2 . e^{-x} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n^2\pi^2(n^2\pi^2 + 1)} . \cos(n\pi x)$$

c) $y'' - y = 1$, $y(0) = y'(0) = 2$

Take the transform of every term:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$\Rightarrow (s^2 Y(s) - s.y(0) - y'(0)) - Y(s) = \frac{1}{s}$$

$$\Leftrightarrow s^2.Y(s) - 2s - 2 - Y(s) = \frac{1}{s}$$

$$\Leftrightarrow Y(s)[s^2 - 1] - 2s - 2 = \frac{1}{s}$$

$$\Leftrightarrow Y(s) = \frac{\frac{1}{s} + 2s + 2}{s^2 - 1} = \frac{1 + 2s^2 + 2s}{s(s^2-1)}$$

Using partial fraction decomposition:

$$Y(s) = \frac{1 + 2s^2 + 2s}{s(s^2-1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$= \frac{A(s-1)(s+1) + Bs(s+1) + Cs(s-1)}{s(s-1)(s+1)}$$

Taking the numerators only:

$$A(s^2-1) + B(s^2+s) + C(s^2-s) = 2s^2 + 2s + 1$$

$$\Leftrightarrow s^2(A+B+C) + s(B-C) - A = 2s^2 + 2s + 1$$

$$\Rightarrow \begin{cases} A = -1 \\ B - C = 2 \\ A + B + C = 2 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ B = 5/2 \\ C = 1/2 \end{cases}$$

Thus,

$$Y(s) = \frac{-1}{s} + \frac{5/2}{s-1} + \frac{1/2}{s+1} = -1.\frac{1}{s} + \frac{5}{2}.\frac{1}{s-1} + \frac{1}{2}.\frac{1}{s+1}$$

Taking the inverse transform then gives

$$y(t) = -1 + \frac{5}{2}.e^{t} + \frac{1}{2}e^{-t}$$