The University of Sydney

Faculties of Arts, Economics, Education,
Engineering and Science

MATH2021: Vector Calculus and Differential Equations

Lecturer: Zhou Zhang

Time allowed: Take-Home Quiz

This booklet contains **4** pages.

SID: ....

Total Marks out of 30

- Open book.
- Material in Weeks 1–5 lectures and practice classes (Tutorial Sheets 1–5).
- Show all necessary work, which will be marked.
- There is no need to print out the question paper.
- Write down solutions in pen by hand, scan and use turnitin for submission.
- Question sheet available at 3:30PM Wednesday Week 7.
- Turnitin link available at 5PM Wednesday Week 7.
- Due at 5PM Friday Week 7.
- All times for Sydney, Australia.

1. A curve $\mathcal{C}$ is given by the following parametrisation

$$\boldsymbol{\alpha}(t) = 3\cos t\,\boldsymbol{i} + 5\sin t\,\boldsymbol{j} - 4\cos t\,\boldsymbol{k}, \quad 0 \le t \le 2\pi.$$

(a) Calculate the velocity vector and acceleration vector of $\boldsymbol{\alpha}$ for $t \in [0, 2\pi]$.

(b) Write down the arc length parameter $s(t)$, as a function of t.

(c) Calculate the scalar curvature of $\mathcal{C}$, $k(t)$, at as a function of t.

(d) Write down the equation of the tangent line of $\boldsymbol{\alpha}(t)$ at $t = 1$.

(e) Calculate the integral of the vector field $\boldsymbol{F} = (x, z, y)$ along $\boldsymbol{\alpha}$ for $t \in [0, 2\pi]$.

2. Consider the vector field

$$\boldsymbol{F} = (yz - e^{y+z}\sin x)\boldsymbol{i} + \big(xz + e^{y+z}\cos x\big)\boldsymbol{j} + \big(xy + e^{y+z}\cos x\big)\boldsymbol{k}.$$

(a) Show that $\boldsymbol{F}$ is a gradient vector field by finding a function ϕ such that $\boldsymbol{F} = \nabla\phi$.

(b) Show that $\boldsymbol{F}$ is conservative by showing for any loop $\mathcal{C}$, which is $\boldsymbol{\alpha(t)}$ for $t \in [a, b]$ satisfying $\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b)$,

$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot dr = \oint_{\mathcal{C}} \nabla\phi \cdot dr = 0.$$

Hint: the explicit ϕ from (a) is not needed.

(c) By direct calculation, show that $\operatorname{curl}\boldsymbol{F} = \nabla \times \boldsymbol{F} = 0$.

3. In the xy-plane, consider the loop $\mathcal{C}$: the union of the unit upper half circle centred at the origin and the interval between $[-1, 0]$ and $[1, 0]$, going around in the counter-clockwise direction.

(a) Use the definition of line integral to calculate

$$\oint_{C} (-y + e^x)\,dx + (2x + e^y)\,dy$$

Hint: $\mathcal{C}$ can be parametrised in pieces for line integral, with no need for a single parameter.

(b) Apply Green's Theorem/Formula to calculate the loop integral in Part (a). You need to work out the double integral directly **without** using any area formula.

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Formula Sheet

Most of the formulas and theorems provided are stated without the conditions under which they apply. The notations used are the same as those used in lectures.

Line Integral

$$\int_{\mathcal{C}} \phi(x,y,z)\,ds = \int_a^b \phi(x(t),y(t),z(t))\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2+\left(\tfrac{dz}{dt}\right)^2}\,dt.$$
$$\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{\mathcal{C}} F_1\,dx + F_2\,dy + F_3\,dz = \text{Work done by } \boldsymbol{F} \text{ along } \mathcal{C}.$$

Grad

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\phi}{\partial z}\,\boldsymbol{k}.$$

$\nabla\phi$ is normal to the tangent plane of the level surface $\{\phi(x,y,z)=k\}$ for constant k.

If $\boldsymbol{F} = \nabla\phi$ for smooth function ϕ, then $\boldsymbol{F}$ is a conservative field, ϕ is a potential function of $\boldsymbol{F}$, and $\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r}$ is path-independent.

Curl

$$\text{curl } \boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If the domain of $\boldsymbol{F}$ is simply connected (no need to worry about this) and $\nabla\times\boldsymbol{F} = \boldsymbol{0}$ then $\boldsymbol{F}$ is conservative.

Double Integral

$$\text{Area of } R = \iint_R dxdy.$$
$$\text{Volume under the surface } z = f(x,y) \text{ over } R = \iint_R f(x,y)\,dxdy.$$
$$\text{In polar coordinates: } \iint_R f(x,y)\,dxdy = \iint_R f(r\cos\theta, r\sin\theta) r\,d\theta dr.$$

Green's Formula (Theorem)

$$\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_{\mathcal{C}} F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\,dxdy.$$

Using "curl $= \nabla\times$", $\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R (\nabla\times\boldsymbol{F})\cdot\boldsymbol{k}\,dxdy.$

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Table of Standard Integrals

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\int \frac{dx}{x} = \ln|x| + C$

3. $\int e^x\,dx = e^x + C$

4. $\int \sin x\,dx = -\cos x + C$

5. $\int \cos x\,dx = \sin x + C$

6. $\int \sec^2 x\,dx = \tan x + C$

7. $\int \sinh x\,dx = \cosh x + C$

8. $\int \cosh x\,dx = \sinh x + C$

9. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

10. $\int x\,\sin x\,dx = -x\cos x + \sin x + C$

11. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

12. $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C'$

13. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C \quad (x > a)$$
$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C' \quad (x > a \text{ or } x < -a)$$

This is the end of the examination paper