MATH2021 Quiz 1 (Student ID: 480048691)

1. $\vec{\alpha}(t) = [3\cos t, 5\sin t, -4\cos t] \quad t \in [0, 2\pi]$

a) Velocity vector: $\vec{\alpha}'(t) = [-3\sin t, 5\cos t, 4\sin t]$
Acceleration vector: $\vec{\alpha}''(t) = [-3\cos t, -5\sin t, 4\cos t]$

b) $$\|\vec{\alpha}'(t)\| = \sqrt{(-3\sin t)^2 + (5\cos t)^2 + (4\sin t)^2}$$
$$= \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t}$$
$$= \sqrt{25\sin^2 t + 25\cos^2 t}$$
$$= \sqrt{25} = 5$$

We have the arc length function

$$s(t) = \int_0^t \|\vec{\alpha}'(u)\|\,du = \int_0^t 5\,du = [5u]_0^t = 5t$$
$$\Rightarrow s(t) = 5t$$

c) Scalar curvature $K = \dfrac{\|\vec{T}'(t)\|}{\|\vec{\alpha}'(t)\|}$ where $\vec{T}(t)$ is the unit tangent vector

$$\vec{T}(t) = \frac{\vec{\alpha}'(t)}{\|\vec{\alpha}'(t)\|} = \left[\frac{-3}{5}\sin t, \cos t, \frac{4}{5}\sin t\right]$$
$$\vec{T}'(t) = \left[\frac{-3}{5}\cos t, -\sin t, \frac{4}{5}\cos t\right]$$
$$\Rightarrow \|\vec{T}'(t)\| = \sqrt{\left(\frac{-3}{5}\cos t\right)^2 + (-\sin t)^2 + \left(\frac{4}{5}\cos t\right)^2}$$
$$= \sqrt{\frac{9}{25}\cos^2 t + \sin^2 t + \frac{16}{25}\cos^2 t} = \sqrt{1} = 1.$$

Therefore, $K = \dfrac{\|\vec{T}'(t)\|}{\|\vec{\alpha}'(t)\|} = \dfrac{1}{5}$

d) The vector function of the tangent line at $t = 1$ of $\vec{\alpha}(t)$:
$$\vec{r}(t) = \vec{\alpha}(1) + t.\vec{\alpha}'(1)$$
$$= [3\cos(1), 5\sin(1), -4\cos(1)] + t.[-3\sin(1), 5.\cos(1), 4\sin(1)]$$

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e) Line integral of the vector field $\vec{F} = (x, z, y)$ along $\vec{\alpha}(t)$ for $t \in [0, 2\pi]$

$$\int_C \vec{F} . d\vec{r} = \int_C [3\cos t, -4\cos t, 5\sin t] . [-3\sin t, 5\cos t, 4\sin t]\, dt$$

$$= \int_C (-9\cos t . \sin t - 20\cos^2 t + 20\sin^2 t)\, dt$$

$$= \int_C \frac{-9}{2}\sin(2t) - 20\cos(2t) \quad dt$$

$$= \left[\frac{9}{4} . \cos(2t) - 10 . \sin(2t) \right]_0^{2\pi}$$

$$= \frac{9}{4} . \cos(2.2\pi) - 10 . \sin(2.2\pi) - \frac{9}{4}\cos(0) + 10 . \sin(0)$$

$$= \frac{9}{4} . 1 - \frac{9}{4} . 1 = 0$$

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$$\vec{F} = [\overbrace{yz - e^{y+z}\sin x}^{P}, \overbrace{xz + e^{y+z}\cos x}^{Q}, \overbrace{xy + e^{y+z}\cos x}^{R}]$$

$$\vec{F} = \nabla\phi \text{ for some } \phi = \left[\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right]$$

$$\frac{\partial\phi}{\partial x} = yz - e^{y+z}\sin x \Rightarrow \phi = \int yz\,dx - \int e^{y+z}\sin x\,dx + h(y,z)$$

$$= yzx + e^{y+z}\cos x + h(y,z)$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left[xyz + e^{y+z}\cos x + h(y,z)\right] = Q$$

$$\Leftrightarrow xz + e^{y+z}\cos x + h'(y) = xz + e^{y+z}\cos x$$

$$\Leftrightarrow h'(y) = 0$$

$$\Leftrightarrow h(y) = C : \text{constant}$$

$$\Rightarrow \phi = yzx + e^{y+z}\cos x + C + g(z)$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial z}\left[yzx + e^{y+z}\cos x + C + g(z)\right] = R$$

$$\Leftrightarrow xy + e^{y+z}\cos x + 0 + g'(z) = xy + e^{y+z}\cos x$$

$$\Leftrightarrow g'(z) = 0$$

$$\Leftrightarrow g(z) = D : \text{constant}$$

$$\Rightarrow \phi = xyz + e^{y+z}\cos x + E \quad (E = C + D)$$

$\Rightarrow \vec{F}$ is a gradient vector field

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2.b) We have $\vec{F} = \nabla\phi$ for $\phi = xyz + e^{y+z}\cos x + E$

From the Fundamental Theorem of line Integral, we know ~~f~~ that for a smooth curve C given by $\vec{\alpha}(t)$, $t \in [a, b]$, and that ϕ is continuous on C

Then

$$\int_C \nabla\phi \cdot dr = \phi(\vec{\alpha}(b)) - \phi(\vec{\alpha}(a))$$

For any loop C, we have the starting point is equal to the ending point

$$\Rightarrow \quad \vec{\alpha}(b) = \vec{\alpha}(a)$$

$$\Leftrightarrow \quad \phi(\vec{\alpha}(b)) = \phi(\vec{\alpha}(a))$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \oint_C \nabla\phi \cdot d\vec{r} = \phi(\vec{\alpha}(b)) - \phi(\vec{\alpha}(a)) = 0.$$

$\therefore$ $\vec{F}$ is ~~con~~ a conservative vector field (by definition).

c) $$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz - e^{y+z}\sin x & xz + e^{y+z}\cos x & xy + e^{y+z}\cos x \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}\left(xy + e^{y+z}\cos x\right) - \frac{\partial}{\partial z}\left(yz - e^{y+z}\cos x\right)\right]$$

$$- \hat{\jmath}\left[\frac{\partial}{\partial x}\left(xy + e^{y+z}\cos x\right) - \frac{\partial}{\partial x}\left(yz - e^{y+z}\sin x\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(xz + e^{y+z}\cos x\right) - \frac{\partial}{\partial y}\left(yz - e^{y+z}\sin x\right)\right]$$

$$= \hat{\imath}\left[\left(x + e^{y+z}\cos x\right) - \left(x + e^{y+z}\cos x\right)\right]$$

$$- \hat{\jmath}\left[\left(y - e^{y+z}\sin x\right) - \left(y - e^{y+z}\sin x\right)\right]$$

$$+ \hat{k}\left[\left(z - e^{y+z}\sin x\right) - \left(z - e^{y+z}\sin x\right)\right]$$

$$= \hat{\imath}\cdot 0 - \hat{\jmath}\cdot 0 + \hat{k}\cdot 0 = 0$$

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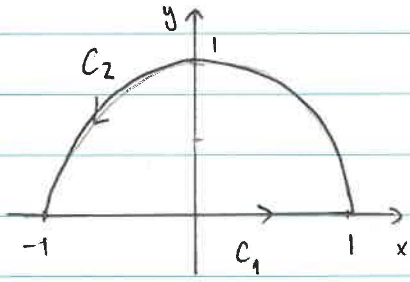

We can divide the curve C into two curves C_1 & C_2

C_1: a straight line segment from $[-1,0]$ to $[1,0]$

$$\Rightarrow \vec{\alpha}_1(t) = (1-t)[-1,0] + t[1,0] \quad , \quad 0 \le t \le 1$$
$$= [t-1, 0] + [t, 0]$$
$$= [2t-1, 0] \quad , \quad t \in [0,1]$$

C_2 : half a unit circle, i.e. radius $= 1$

$$\Rightarrow \vec{\alpha}_2(t) = [\cos t, \sin t] \quad , \quad t \in [0, \pi]$$

3a. $\oint_C (-y + e^x)\,dx + (2x + e^y)\,dy = \oint_C \vec{F} \cdot d\vec{r}$ with $\vec{F} = [-y + e^x, 2x + e^y]$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

. We have $\vec{\alpha}_1'(t) = [2, 0]$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} [e^{2t-1}, 4t-1] \cdot [2, 0]\, dt$$
$$= \int_{C_1} 2 . e^{2t-1}\, dt = \left[e^{2t-1}\right]_0^1 = e^{2.1-1} - e^{2.0-1} = e - e^{-1}$$

. We have $\vec{\alpha}_2'(t) = [-\sin t, \cos t]$

$$\Rightarrow \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} [-\sin t + e^{\cos t}, 2\cos t + e^{\sin t}] . [-\sin t, \cos t]\, dt$$
$$= \int_{C_2} (\sin^2 t - \sin t . e^{\cos t} + 2\cos^2 t + \cos t . e^{\sin t})\, dt$$
$$= \int_C (1 + \cos^2 t - \sin t . e^{\cos t} + \cos t . e^{\sin t})\, dt$$
$$= \left[t + \left(\frac{1}{4} \sin(2t) + \frac{1}{2} t \right) + e^{\cos t} + e^{\sin t} \right]_0^{\pi}$$
$$= \left[\frac{1}{4} \sin(2t) + \frac{3}{2} t + e^{\cos t} + e^{\sin t} \right]_0^{\pi}$$

$$= \left(\frac{1}{4}\sin(2\pi) + \frac{3}{2}\pi + e^{\cos\pi} + e^{\sin\pi}\right) - \left(\frac{1}{4}\sin(0) + \frac{3}{2}.0 + e^{\cos(0)} + e^{\sin(0)}\right)$$

$$= \left(0 + \frac{3}{2}\pi + e^{-1} + e^{0}\right) - \left(0 + 0 + e + e^{0}\right) = \frac{3}{2}\pi + e^{-1} - e$$

$$\Rightarrow \oint_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r} = (e - e^{-1}) + \left(\frac{3}{2}\pi + e^{-1} - e\right) = \frac{3}{2}\pi.$$

3b. Green Theorem : $\oint \vec{F}.d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

We have $\vec{F} = [\underbrace{-y + e^x}_{P}, \underbrace{2x + e^y}_{Q}] \Rightarrow \frac{\partial Q}{\partial x} = 2$ and $\frac{\partial P}{\partial y} = -1$

$$\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - (-1) = 3$$

Since D is the region bounded by the upper half of a unit circle and the horizontal line $y = 0$, D can be defined in polar coordinates by the following inequalities $\begin{cases} 0 \le \theta \le \pi \\ 0 \le r \le 1 \end{cases}$

$$\Rightarrow \oint_C \vec{F}.d\vec{r} = \iint_D 3\, dA = \int_0^{\pi}\int_0^1 3r\, dr\, d\theta$$

Inner integral: $\int_0^1 3r\, dr = \left[\frac{3}{2}r^2\right]_0^1 = \frac{3}{2}.1 - \frac{3}{2}.0 = \frac{3}{2}$

$$\Rightarrow \oint_C \vec{F}.d\vec{r} = \int_0^{\pi} \frac{3}{2}\, d\theta = \left[\frac{3}{2}\theta\right]_0^{\pi} = \frac{3}{2}\pi.$$