MATH2022 Linear and Abstract Algebra

Semester 1	**Third Quiz (Take Home)**	2020

This take-home quiz comprises 15 multiple-choice questions, worth three marks each.

Exactly one alternative is correct in each question.

The total marks available for this quiz is 45 marks.

Instructions:

You should write your answers to this quiz on pages (or type them if you wish), and include your SID (student identification number) on each page.

You should then scan your pages as a single pdf document and upload this into Canvas **well before midnight on Thursday 28 May 2020**.

Anonymous marking will be employed, so you should not include your name. (If you do happen to include your name then there is no penalty, so please do not worry about it.)

Please be considerate of the marker and write your answers neatly in the same numerical order as the questions, with spacing between each of your 15 answers.

For each question, first indicate which alternative, from (a), (b), (c), (d), (e), you believe is correct, and then provide reasoning to justify your answer.

For each question, the alternative that you choose is worth 1 mark, and the reasoning you provide to support that alternative is worth 2 marks.

1. Which one of the following groups, under addition, is cyclic?

(a) $\mathbb{Z}_3 \times \mathbb{Z}_9$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (c) $\mathbb{Z}$ (d) $\mathbb{R}$ (e) $\mathbb{C}$

2. Consider the following system of equations:

$$\begin{array}{rcrcrcrcl} x & + & 2y & + & z & + & 2w & = & 1 \\ 2x & + & y & + & 2z & + & w & = & 0 \end{array}$$

Which one of the following statements is true, for the number of distinct solutions for (x, y, z, w) working over $\mathbb{Z}_3$ and working over $\mathbb{Z}_5$?

(a) There are exactly 27 solutions over $\mathbb{Z}_3$, and exactly 125 solutions over $\mathbb{Z}_5$.

(b) There are exactly 25 solutions over $\mathbb{Z}_5$, but there are no solutions over $\mathbb{Z}_3$.

(c) There are exactly 5 solutions over $\mathbb{Z}_5$, but there are no solutions over $\mathbb{Z}_3$.

(d) There are exactly 9 solutions over $\mathbb{Z}_3$, and exactly 5 solutions over $\mathbb{Z}_5$.

(e) There are exactly 25 solutions over $\mathbb{Z}_5$, and exactly 9 solutions over $\mathbb{Z}_3$.

3. Consider the following matrices over $\mathbb{R}$, where θ is a real number:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad , \quad T_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Which one of the following statements is false?

(a) $R^8_{\pi/4} = I = T^8_{\pi/3}$ (b) $T_{\pi/3}R_{\pi/2}T_{\pi/2} = R_{4\pi/3}$ (c) $R^4_\pi = I = T^4_{\pi/2}$

(d) $T_\pi R_{\pi/3} T_{\pi/2} = R_{\pi/6}$ (e) $R_{\pi/3}T_{\pi/2}R_{\pi/2} = T_{2\pi/3}$

4. Consider the group G of symmetries of a regular hexagon, generated by a rotation α and a reflection β. Simplify the following expression in G:

$$\beta\alpha^3\beta^3\alpha^{-2}\beta^{-3}\alpha^5 =$$

(a) α (b) $\alpha\beta$ (c) $\alpha^2\beta$ (d) α^2 (e) β

5. Let M be the following matrix over $\mathbb{Z}_2$:

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Which one of the following is correct:

(a) $\text{rank}(M) = 3$ and $\text{nullity}(M) = 1$. (b) $\text{rank}(M) = 2$ and $\text{nullity}(M) = 2$.

(c) $\text{rank}(M) = 3$ and $\text{nullity}(M) = 2$. (d) $\text{rank}(M) = 4$ and $\text{nullity}(M) = 1$.

(e) $\text{rank}(M) = 4$ and $\text{nullity}(M) = 0$.

6. Let B be the following ordered basis for $\mathbb{R}^2$:

$$B = \{(1,-1),(2,-3)\}\ .$$

Find the coordinate vector of $\mathbf{v} = (1,0)$ with respect to B.

(a) $[\mathbf{v}]_B = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ (b) $[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ (c) $[\mathbf{v}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(d) $[\mathbf{v}]_B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ (e) $[\mathbf{v}]_B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

7. Which one of the following subsets does not form a basis for the vector space $\mathbb{Z}_2^3$?

(a) $\{(1,1,0),(0,1,1),(1,0,1)\}$ (b) $\{(1,1,1),(1,1,0),(0,1,1)\}$

(c) $\{(1,1,0),(0,1,1),(1,0,0)\}$ (d) $\{(1,1,1),(0,1,1),(1,0,1)\}$

(e) $\{(1,1,0),(0,1,1),(0,1,0)\}$

8. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by the rule

$$L(x,y) = (x+y, x-y)\ .$$

and let B be the following ordered basis for $\mathbb{R}^2$:

$$B = \{(1,1),(1,-2)\}\ .$$

Then the matrix $[L]_B^B$ of L with respect to B is which one of the following?

(a) $[L]_B^B = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{4}{3} \end{bmatrix}$ (b) $[L]_B^B = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ (c) $[L]_B^B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(d) $[L]_B^B = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{4}{3} \end{bmatrix}$ (e) $[L]_B^B = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$

9. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by the rule

$$L(x,y) = (x+2y, 3x+2y)\ .$$

Find an ordered basis B for $\mathbb{R}^2$ such that

$$[L]_B^B = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\ .$$

(a) $B = \{(2,3),(-1,1)\}$ (b) $B = \{(3,2),(1,-1)\}$ (c) $B = \{(1,1),(-2,3)\}$

(d) $B = \{(-1,3),(2,1)\}$ (e) $B = \{(-1,1),(2,3)\}$

10. Let V be the vector space of functions spanned by $B = \{\cos x + \sin x, \cos x - \sin x\}$. Find the matrix $[D]_B^B$ of the linear operator $D : V \to V$ that maps a function to its derivative.

(a) $[D]_B^B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) $[D]_B^B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (c) $[D]_B^B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(d) $[D]_B^B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (e) $[D]_B^B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

11. Let $M = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ with entries from $\mathbb{Z}_5$. Then a basis B for the null space $M^\perp$ of M is which one of the following?

(a) $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ (b) $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ (d) $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(e) $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

12. Let $B = \{(1,1),(2,3)\}$ and $D = \{(2,1),(3,4)\}$, both of which are ordered bases for $\mathbb{Z}_7^2$. Let $M = [\mathrm{id}]_D^B$ be a change of basis matrix. Which one of the following is true?

(a) $M = \begin{bmatrix} 3 & 3 \\ 4 & 5 \end{bmatrix}$ (b) $M = \begin{bmatrix} 3 & 4 \\ 3 & 5 \end{bmatrix}$ (c) $M = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$

(d) $M = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ (e) $M = \begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix}$

13. Which of the following expressions describes M^{-1} where $M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and I is the 3×3 identity matrix, working over $\mathbb{Z}_5$?

(a) $4M^2 + 3M + I$ (b) $4M^2 + 3M + 4I$ (c) $M^2 + M + I$

(d) $3M^2 + 4M + I$ (e) $4M^2 + 4M + 4I$

14. Consider the permutations

$$\alpha = (1\,3)(2\,4)(5\,6) \qquad \text{and} \qquad \beta = (1\,4)(2\,6)(3\,5)$$

expressed in cycle notation. How many permutations γ of $\{1,2,3,4,5,6\}$ exist with the property $\beta = \gamma^{-1}\alpha\gamma$ where we compose from left to right?

(a) 8 (b) 48 (c) 6 (d) 24 (e) 3

15. You are given that $\{a,b,c\} = \{1,2,3\}$ and, from the 8-puzzle

1	2	3
4	5	6
7	8	

, the following configuration has been reached by moving squares in and out of the space:

4	8	a
6		b
5	7	c

Which one of the following statements is true?

(a) If $a = 1$ then $c = 2$. (b) If $c = 1$ then $b = 2$. (c) If $c = 2$ then $b = 1$.

(d) If $a = 2$ then $c = 3$. (e) If $c = 3$ then $b = 1$.