MATH2022 Take Home Quiz 2

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Question 1 Answer: B.

We have $\mathbb{Z}$, $\mathbb{Z}_5$, and $\mathbb{Z}_6$ are cyclic groups under addition since 1 and -1 always generate $\mathbb{Z}$ and $\mathbb{Z}_n$ with respect to addition.

We have $\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(a, b) | a \in \mathbb{Z}_2, b \in \mathbb{Z}_3\} = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$ and that $\langle (1,1) \rangle$ generates everything in $\mathbb{Z}_2 \times \mathbb{Z}_3$ which makes this group cyclic under addition.

Lastly, we have $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(a, b) | a, b \in \mathbb{Z}_2\} = \{(0,0), (0,1), (1,0), (1,1)\}$. Each of the elements generate themselves and the identity element:

$$\langle (0,0) \rangle = \{(0,0)\}$$
$$\langle (0,1) \rangle = \{(0,1), (0,0)\}$$
$$\langle (1,0) \rangle = \{(1,0), (0,0)\}$$
$$\langle (1,1) \rangle = \{(1,1), (0,0)\}$$

None of the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ generates $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic under addition.

Question 2 Answer: C.

Consider the group G of symmetries of a regular pentagon, generated by a rotation $\alpha = (1\ 2\ 3\ 4\ 5)$ and a reflection along the vertical axis $\beta = (2\ 5)(3\ 4)$.

We have the facts that $\alpha^5 = \beta^2 = 1$ and $\alpha\beta = \beta\alpha^{-1} = \beta\alpha^4$. Furthermore, $\beta^{-1}\alpha^i\beta = \alpha^{-i}$ for all i and $\beta^{-1} = \beta$,

$$\begin{aligned}
\beta\alpha^3\beta^3\alpha^{-3}\beta\alpha^7 &= \beta\alpha^3\beta^3\beta^{-1}\alpha^3\beta\beta\alpha^7 \\
&= \beta\alpha^3\alpha^3\alpha^7 \\
&= \beta\alpha^{13} \\
&= \beta\alpha^3 \\
&= \beta\alpha^4\alpha^{-1} \\
&= \alpha\beta\alpha^{-1} \\
&= \alpha\alpha\beta \\
&= \alpha^2\beta.
\end{aligned}$$

Question 3 Answer: E.

Write the system of linear equations as an augmented matrix and work over $\mathbb{Z}_3 = \{0, 1, 2\}$:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

It appears that the system is inconsistent over $\mathbb{Z}_3$. Therefore, there is no solutions for (x, y, z, w).

Question 4 Answer: A.

We have $R^6_{\pi/3} = R_{6\pi/3} = R_{2\pi} = I$ while $T^6_{\pi/3} = (T^3_{\pi/3})^2 = I$.

Question 5 Answer: A.

Performing row reduction on matrix M over $\mathbb{R}$:

$\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ row operation corresponds to $N_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ row operation corresponds to $N_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ row operation corresponds to $N_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

We have $I = N_3N_2N_1M = M^{-1}M$. It turns out that $N_3 = E_2, N_2 = E_1, N_1 = E_3$.

Therefore, $I = N_3N_2N_1M = E_2E_1E_3M$.

Question 6 Answer: C.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Observe that

$det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 2 \end{vmatrix} = (\lambda-1)(\lambda-2)-6 = \lambda^2-3\lambda-4 = (\lambda+1)(\lambda-4)$

yielding eigenvalues -1 and 4 over $\mathbb{R}$, which is 6 and 4 (respectively) over $\mathbb{Z}_7$.

Question 7 Answer: E.

$M = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ with entries from $\mathbb{Z}_3 = \{0, 1, 2\}$. Observe that

$det(\lambda I - M) = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda-1)(\lambda-2)$ yielding eigenvalues 1 and 2.

Eigenspace for $\lambda = 1$

$$I - M = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ yielding } \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| t \in \mathbb{Z}_3 \right\}.$$

Eigenspace for $\lambda = 2$

$$2I - M = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \text{ yielding } \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| t \in \mathbb{Z}_3 \right\}.$$

We choose eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to eigenvalues 2 and 1 respectively. So $M = PDP^{-1}$ with D = $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and P = $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Question 8 Answer: A.

We have $M = PDP^{-1}$ with D $= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and P $= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Observe that $P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Thus, for all positive k,

$$\begin{aligned} M^k = PD^kP^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3^k \\ 2^k & 3^k \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3^k & 0 \\ 3^k - 2^k & 2^k \end{bmatrix}. \end{aligned}$$

Question 9 Answer: B.

We have matrix $M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. The characteristic polynomial of M is:

$$\begin{aligned} det(\lambda I - M) = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 1 \end{vmatrix} &= \lambda \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & -1 \\ \lambda - 1 & 0 \end{vmatrix} \\ &= \lambda(\lambda - 1)(\lambda - 1) - (\lambda - 1) \\ &= (\lambda - 1)(\lambda^2 - \lambda - 1) \\ &= \lambda^3 - 2\lambda^2 + 1 \end{aligned}$$

By the Cayley-Hamilton Theorem, $\chi(M) = M^3 - 2M^2 + I = 0$. That is, $M^3 - 2M^2 = -I \iff M^2(M - 2I) = -I \iff M^2(2I - M) = I$ which implies

$$M^{-1} = M(2I - M) = -M^2 + 2M.$$

Question 10 Answer: D.

Consider matrix $D = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$. Observe that

$det(\lambda I - D) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda - 2 \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ yielding two non-distinct eigenvalues $\lambda_1 = \lambda_2 = 1$.

Eigenspace for $\lambda = 1$:

$$I - D = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \text{ yielding } \left\{ \begin{bmatrix} -t \\ t \end{bmatrix} \middle| t \in \mathbb{C} \right\}.$$

Possible eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$, etc. However, if we attempt to form matrix P with these two eigenvectors then P $= \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$ has zero determinant. This implies P^{-1} does not exist and D is not diagonalisable over $\mathbb{C}$.

Question 11 Answer: D.

Consider the function $f(x, y) = (y - x, x - y)$

Let $\mathbf{v_1} = (x_1, y_1)$ and $\mathbf{v_2} = (x_2, y_2)$

$$\begin{aligned} f(\mathbf{v_1} + \mathbf{v_2}) &= f((x_1, y_1) + (x_2, y_2)) \\ &= f(x_1 + x_2, y_1 + y_2) \\ &= ((y_1 + y_2) - (x_1 + x_2), (x_1 + x_2) - (y_1 + y_2)) \\ &= (-x_1 - x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2) \\ &= (-x_1 + y_1 - x_2 + y_2, x_1 - y_1 + x_2 - y_2) \\ &= (-x_1 + y_1, x_1 - y_1) + (-x_2 + y_2, x_2 - y_2) \\ &= f(x_1, y_1) + f(x_2, y_2) \\ &= f(\mathbf{v_1}) + f(\mathbf{v_2}) \end{aligned}$$

which verifies f preserves addition. Furthermore, let $\mathbf{v} = (x, y)$ and $\lambda \in \mathbb{R}$:

$$\begin{aligned} f(\lambda \mathbf{v}) &= f(\lambda(x, y)) \\ &= f(\lambda x, \lambda y) \\ &= (\lambda y - \lambda x, \lambda x - \lambda y) \\ &= (\lambda(y - x), \lambda(x - y)) \\ &= \lambda(y - x, x - y) \\ &= \lambda f(x, y) \\ &= \lambda f(\mathbf{v}) \end{aligned}$$

which verifies f preserves scalar multiplication. Therefore, $f(x, y) = (y - x, x - y)$ defines a linear combination.

Question 12 Answer: C.

We have the effect of $f(x, y) = (5x-y, 2x+y, y-x)$ on standard basis vector is as follows:

$$f(1, 0) = (5, 2, -1) \text{ and } f(0, 1) = (-1, 1, 1)$$

transposing into columns, we get the matrix corresponding to the linear transformation $\begin{bmatrix} 5 & -1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix}$.

Question 13 Answer: E.

$M_{gf} = M_g M_f = \begin{bmatrix} 0 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 7 & -4 & 7 \end{bmatrix}$

so $M_{gf} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 7 & -4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + 3z \\ 7x - 4y + 7z \end{bmatrix}$

Hence $(gf)(x, y, z) = (-x + 3z, 7x - 4y + 7z)$.

Question 14 Answer: B.

Given $\alpha = (1\ 3\ 2)(4\ 6\ 5)(7\ 8)$ and $\beta = \gamma^{-1}\alpha\gamma = (1\ 4\ 2)(8\ 5\ 6)(3\ 7)$, our goal is to find the permutation γ such that β is the conjugate of α by γ.

Consider $\gamma = (5\ 8\ 3\ 4)$, for the first cycle of α, we have:

The image of 1 under γ is 1.
The image of 3 under γ is 4.
The image of 2 under γ is 2.

For the second cycle of α:

The image of 4 under γ is 5.
The image of 6 under γ is 6.
The image of 5 under γ is 8.

For the third cycle of α:

The image of 7 under γ is 7.
The image of 8 under γ is 3.

Thus, $\alpha^\gamma = (1\ 4\ 2)(5\ 6\ 8)(7\ 3) = (1\ 4\ 2)(8\ 5\ 6)(3\ 7) = \beta$.

Question 15 Answer: D.

We have the configuration D

	3	8	15
5	2	10	6
13	7	11	12
9	4	1	14

can be transformed into

3	8	15	6
5	2	10	12
13	7	11	14
9	4	1	

corresponds to (1 15 3)(2 6 4 14 12 8)(7 10)(13 9) which is a product of an odd number of transpositions (2+5+1+1=9). Hence the configuration of D is impossible to reach from the 15-puzzle square.