MATH2021 Assignment 2

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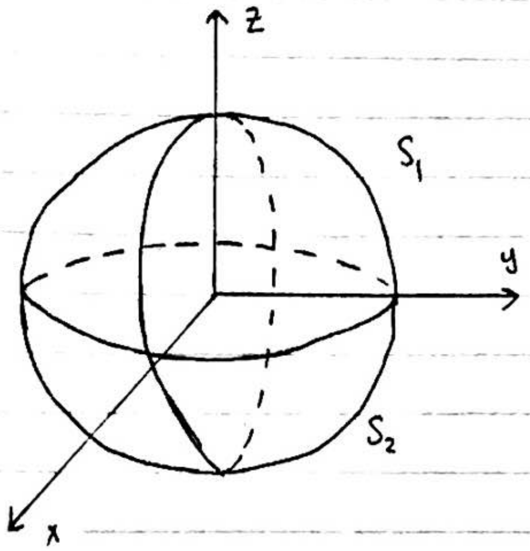
Question 1.



$S = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$

a) We can define our unit sphere S by setting

$$g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$$

We have the gradient of g:

$$\nabla g = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right] = [2x, 2y, 2z]$$

Also, $\|\nabla g\| = \sqrt{(2x)^2 + (2y)^2 + (2z)^2}$

$$= \sqrt{4(x^2+y^2+z^2)} = \sqrt{4.1} = \sqrt{4} = 2$$

Hence, a unit normal vector is given by:

$$\vec{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{[2x, 2y, 2z]}{2} = [x, y, z]$$

Note that $\vec{n}$ is a position vector points radially outward from the sphere, so $\vec{n}$ is the outward unit normal.

b) We want to express the surface integral $\iint_{S_1} \vec{V}.\vec{n}\, dS$ with $\vec{V} = [x, y, z^2]$ and $\vec{n} = [x, y, z]$

. Rewrite the upper hemisphere in terms of x and y as:

$$f(x,y) = z = \sqrt{1 - x^2 - y^2} = (1 - x^2 - y^2)^{1/2}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{1}{2}(1 - x^2 - y^2)^{-1/2} . (-2x) = \frac{-x}{\sqrt{1 - x^2 - y^2}}$$

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$$\frac{\partial f}{\partial y} = \frac{1}{2}(1-x^2-y^2)^{-1/2}\cdot(-2y) = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow \sqrt{f_x^2+f_y^2+1} = \sqrt{\left(\frac{-x}{\sqrt{1-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)^2+1}$$

$$= \sqrt{\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}+\frac{1-x^2-y^2}{1-x^2-y^2}}$$

$$= \sqrt{\frac{1}{1-x^2-y^2}} = (1-x^2-y^2)^{-1/2}$$

. Thus the surface integral over S_1 is:

$$\iint_{S_1} \vec{V}\cdot\vec{n}\, dS = \iint_{S_1} \vec{V}\cdot\vec{n}\cdot\sqrt{f_x^2+f_y^2+1}\; dA$$

$$= \iint_{S_1} [x,y,z^2]\cdot[x,y,z]\cdot(1-x^2-y^2)^{-1/2}\; dA$$

$$= \iint_{S_1} (x^2+y^2+z^3)\cdot(1-x^2-y^2)^{-1/2}\; dA$$

$$= \iint_{S_1} \left[x^2+y^2+(1-x^2-y^2)^{3/2}\right]\cdot(1-x^2-y^2)^{-1/2}\; dA$$

$$= \cancel{\iint_{S_1}} \iint_{S_1} \frac{x^2+y^2}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2)^2\; dA$$

By considering S_1 over the xy-plane, we know the projection of the upper hemisphere S_1 on the xy-plane is the unit circle defined by the inequalities $-1 \le y \le 1$ and $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

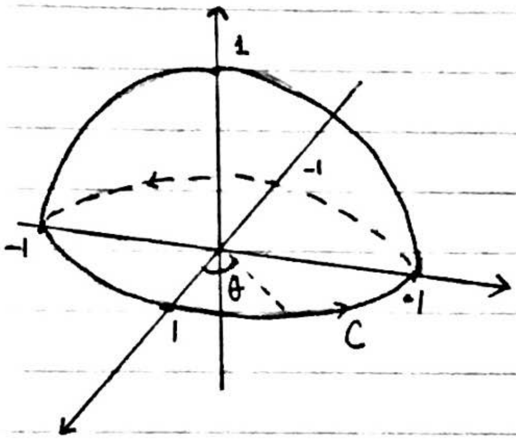
Hence, $$\iint_{S_1} \vec{V}\cdot\vec{n}\, dS = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{x^2+y^2}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2)^2\; dx\, dy$$

c) Gauss's theorem helps us transform a surface integral $\iint_{S_1} \vec{V} \cdot \vec{n}\, dS$ into a volume integral $\iiint_V \operatorname{div} V \, dv$

With $\vec{V} = [\underset{P}{x}, \underset{Q}{y}, \underset{R}{z^2}] \Rightarrow \operatorname{div} \vec{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$= 1 + 1 + 2z = 2 + 2z$

Hence $\iint_{S_1} \vec{V} \cdot \vec{n}\, dS = \iiint_V \operatorname{div} \vec{V}\, dv = \iiint_V (2 + 2z)\, dv$



Using cylindrical coordinates, the upper hemisphere can be defined by the following inequalities:

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 1$

with $r^2 = x^2 + y^2$ because $\begin{cases} x = r.\cos\theta \\ y = r.\sin\theta \end{cases}$

so $r^2 + z^2 \leq 1$

$\Rightarrow \quad 0 \leq z \leq \sqrt{1-r^2}$ (considering the upper hemisphere)

$\Rightarrow \iint_{S_1} \vec{V} \cdot \vec{n}\, dS = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} (2 + 2z)\, r\, dz\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^1 [2r\sqrt{1-r^2} + r(1-r^2)]\, dr\, d\theta$ (full calculation next page)

$= \int_0^{2\pi} \frac{11}{12}\, d\theta = \left[\frac{11}{12}\theta\right]_0^{2\pi} = \frac{11\pi}{6}.$

By symmetry, $\iint_{S_1} \vec{V} \cdot \vec{n}\, dS = \iint_{S_2} \vec{V} \cdot \vec{n}\, dS$

$\Rightarrow \iint_S \vec{V} \cdot \vec{n}\, dS = 2 \iint_{S_1} \vec{V} \cdot \vec{n}\, dS = 2 \cdot \frac{11\pi}{6} = \frac{11\pi}{3}.$

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} (2+2z)r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[2rz + r.z^2 \right]_0^{\sqrt{1-r^2}} dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[2r.\sqrt{1-r^2} + r(1-r^2) \right] dr \, d\theta$$

$$= \int_0^{2\pi} \left(\cancel{2r} \int_0^1 2r\sqrt{1-r^2} \, dr + \int_0^1 r - r^3 \, dr \right) d\theta \qquad (\star)$$

$\star$ Let $u = 1 - r^2 \Rightarrow \frac{du}{dr} = -2r \Rightarrow du = -2r.dr$

$$\Rightarrow \int_0^1 2r\sqrt{1-r^2} \, dr = -\int_0^1 u^{1/2} \, du = -\left[\frac{2}{3} u^{3/2} \right]_0^1$$

$$\Rightarrow -\left[\frac{2}{3}(1-r^2)^{3/2} \right]_0^1 = -\left[\frac{2}{3} \cancel{4}. 0^{3/2} - \frac{2}{3}. 1^{3/2} \right] = \frac{2}{3}$$

$$(\star) = \int_0^{2\pi} \left(\frac{2}{3} + \left[\frac{r^2}{2} - \frac{r^3}{4} \right]_0^1 \right) d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} + \frac{1}{2} - \frac{1}{4} \, d\theta$$

$$= \int_0^{2\pi} \frac{11}{12} \, d\theta = \left[\frac{11}{12}\theta \right]_0^{2\pi} = \frac{11}{12} . 2\pi = \frac{11\pi}{6}.$$

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Question 2.

a) We have the boundary circle C is in the counter-clockwise direction, i.e. positively oriented ⇒ satisfy the conditions of Green's Theorem.

Interpreting the line integral $\oint_C \vec{F}\cdot d\vec{r}$ as the work done to move an object along the closed curve C under the influence of force $\vec{F}$, the z-component of $\vec{F}$ won't affect the total work done since C is in the xy-plane.

$\Rightarrow \vec{F}(x,y) = [\underset{P}{y}, \underset{Q}{-x}]$ with $\frac{\partial P}{\partial y} = 1$ and $\frac{\partial Q}{\partial x} = -1$. ⇒ $\cancel{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} =}$

$\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1-1 = -2.$

Using Green's Theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^{2\pi}\int_0^1 -2r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[-r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} -1 \, d\theta = \left[-\theta\right]_0^{2\pi} = -2\pi.$$

b) We have from Question 1, the outward unit normal of S_1 is $\vec{n} = [x, y, z]$.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & xy\sin(z) \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(xy\sin(z)) + \frac{\partial}{\partial x}(x)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(xy\sin(z)) - \frac{\partial}{\partial z}y\right] + \hat{k}\left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right]$$

$$= \hat{\imath}\left[x\sin(z) + 0\right] - \hat{\jmath}\left[y.\sin(z) - 0\right] + \hat{k}\left[-1-1\right]$$

$$= x.\sin(z)\,\hat{\imath} - y.\sin(z)\,\hat{\jmath} - 2\hat{k}$$

$$(\nabla \times \vec{F}).\vec{n} = [x.\sin(z), -y.\sin(z), -2].[x, y, z]$$
$$= x^2 \sin(z) - y^2.\sin(z) - 2z$$

With only variables x and y:

$$(\nabla \times \vec{F}).\vec{n} = x^2.\sin(z) - y^2.\sin(z) - 2z$$
$$= x^2.\sin(\sqrt{1-x^2-y^2}) - y^2.\sin(\sqrt{1-x^2-y^2}) - 2\sqrt{1-x^2-y^2}$$

c) Stoke's Theorem states that $\int_C \vec{F}.d\vec{r} = \iint_S (\text{curl } \vec{F}.\vec{n})\, dS$ with C the boundary of S.

In our case, we want to compute $\iint_{S_1} (\nabla \times \vec{F}).\vec{n}\, dS$. Note that S_1 is bounded by C

$\Rightarrow$ We can parameterise $C = \vec{r}(t) = [\cos(t), \sin(t)]$, $0 \leq t \leq 2\pi$

$$\Rightarrow \vec{r}'(t) = [-\sin(t), \cos(t)]$$

Once again, C is on the xy-plane and so the z-component ~~of~~ of $\vec{F}$ does not affect the total work done.

$$\Rightarrow \iint_{S_1} (\nabla \times \vec{F}).\vec{n}\, dS = \int_C \vec{F}(\vec{r}(t)) . \vec{r}'(t)\, dt$$

$$= \int_0^{2\pi} [\sin(t), -\cos(t)].[-\sin(t), \cos(t)]\, dt$$

$$= \int_0^{2\pi} -\sin^2(t) - \cos^2(t)\, dt$$

$$= \int_0^{2\pi} -1\, dt = [-t]_0^{2\pi} = -2\pi.$$

Question 3.

a) $y' + y = 0$, $y(0) = 1$

$\Leftrightarrow y' = -y$

$\Leftrightarrow \frac{dy}{dx} = -y$

$\Leftrightarrow \frac{-1}{y} dy = dx$

$\Leftrightarrow \int \frac{-1}{y} dy = \int 1 \, dx$

$\Leftrightarrow -\ln|y| + C_1 = x + C_2$

$\Leftrightarrow -\ln|y| = x + C \quad (C = C_2 - C_1)$

$\Leftrightarrow e^{\ln|y^{-1}|} = e^{x+C}$

$\Leftrightarrow \frac{1}{y} = A.e^{x} \quad (A = e^{C})$

$\Leftrightarrow y(x) = B.e^{-x} \quad \left(B = \frac{1}{A}\right)$

Given the initial condition $y(0) = 1$

$\Rightarrow y(0) = B.e^{-0} = B = 1$

Hence the particular solution for the above ODE is:

$y(x) = e^{-x}$.

b) $y' + y = x$, $y'(0) = 0$

This is a first order ODE with the form $\frac{dy}{dx} + p(x).y = q(x)$

with $p(x) = 1$ and $q(x) = x$.

We have the integrating factor:

$r(x) = \exp\left(\int p(x)\, dx\right) = \exp\left(\int 1 \, dx\right) = e^{x}$.

$\Rightarrow y(x) = \frac{1}{r(x)} \int r(x)\, q(x)\, dx + \frac{C}{r(x)}$

$= e^{-x} \int e^{x}.x \, dx + C.e^{-x}$

$= e^{-x} (x-1).e^{x} + C.e^{-x}$

$= (x-1) + Ce^{-x}$

Hence the particular solution for this ODE is:

$y(x) = x - 1 + e^{-x}$

Given the initial condition $y'(0) = 0$:

$y'(x) = 1 - C.e^{-x}$ $\Rightarrow y'(0) = 1 - C.e^{-0} = 0$

$\Rightarrow C = 1$

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3) $y'' + 2y' + 2y = 0, \quad y(0) = y'(\pi) = 0$

We have the characteristic polynomial

$P(\lambda) = \lambda^2 + 2\lambda + 2 = 0$

$\Rightarrow \lambda_1, \lambda_2 = \dfrac{-2 \pm \sqrt{2^2 - 4.1.2}}{2} = \dfrac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$

$\Rightarrow y(x) = C.e^{(-1+i)x} + D.e^{(-1-i)x}$

$= C.e^{-x}.e^{ix} + D.e^{-x}.e^{-ix}$

$= e^{-x}(Ce^{ix} + D.e^{-ix})$

$= e^{-x}[E.\cos(x) + D.\sin(x)]$ (general solution)

We have $y(0) = e^{-0}[E.\cos(0) + D.\sin(0)] = 0$

$\Leftrightarrow 1.(E.1 + D.0) = 0 \Leftrightarrow E = 0$

Also, $y'(x) = -D.e^{-x}\sin(x) + D.e^{-x}\cos(x)$ (after knowing $E = 0$)

$\Rightarrow y'(\pi) = -D.e^{-\pi}.\sin(\pi) + D.e^{-\pi}.\cos(\pi) = 0$

$\Leftrightarrow -D.e^{-\pi} = 0$

$\Leftrightarrow D = 0$

Hence, the particular solution for this ODE is: $y(x) = 0$.