Quiz 2 Semester 1, 2020

The University of Sydney

Faculties of Arts, Economics, Education,
Engineering and Science

MATH2021: Vector Calculus and Differential Equations

Lecturer: Zhou Zhang

Time allowed: Take-Home Quiz

This booklet contains **6** pages.

SID: ....

Total Marks	*(out of 30)*

- Open book.
- Material in Weeks 6-11 lectures and practice classes (Tutorial Sheets 6-11).
- **Show all necessary work, which will be marked.**
- There is no need to print out the question paper.
- Write by hand on paper, scan and use turnitin for submission.
- Question sheet available at 3:30PM Wednesday Week 13.
- Turnitin link available at 5PM Wednesday Week 13, 27 May.
- Due at 5PM Friday Week 13, 29 May.
- All times for Sydney, Australia.

1. Consider the upper half unit sphere $S_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 4, z \geq 0\}$ and the vector field $F(x, y, z) = (-y, x, z)$.

(a) Show that $n(x, y, z) = \frac{1}{2}(x, y, z)$ is a unit normal vector field of S_2.

(b) Calculate the flux of F through S_2 in the direction of n as in (a),

$$\iint_{S_2} F \cdot n \, dS.$$

(c) For the same n as in (a), applying Stokes' Theorem, calculate

$$\iint_{S_2} (\nabla \times F) \cdot n \, dS.$$

2. Find the solution for the following ODE questions:

(a) $y' - y = -1, \quad y(0) = 0.$

(b) $y' + 2xy = -x, \quad y'(0) = 0.$

(c) $y'' + y = 0, \quad y(1) = y'(1) = 1.$

(d) $y'' + y' - 2y = 0, \quad y(0) = y'(0) = 1.$

3. (a) Use the method of *variation of parameters* to find the general solution for:

$$y'' + y = \cos x - \sin x$$

(b) Use the method involving *Fourier series* to find the general solution for:

$$y'' - y = |x|$$

over the x-interval $[-1, 1]$.

(c) Use *Laplace Transform* to find the particular solution for:

$$y'' - y = 1, \quad y(0) = y'(0) = 2.$$

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Formula Sheet

Most of the formulas and theorems provided are stated without the conditions under which they apply.

Function $f(t), g(t)$	**Laplace Transform** $F(s) = \int_0^\infty e^{-st} f(t)\, dt, G(s)$
e^{at}	$\frac{1}{s-a}$ $\quad (s > a)$
1	$\frac{1}{s}$ $\quad (s > 0)$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$ $\quad (s > 0)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$ $\quad (s > 0)$
$t^n, \quad n = 0, 1, 2, \cdots$	$\frac{n!}{s^{n+1}}$ $\quad (s > 0)$
$a\, f(t) + b\, g(t)$	$a\, F(s) + b\, G(s)$ (linear)
$e^{at} f(t)$	$F(s-a)$ (s-shifting)
$(-t)^n f(t)$	$\frac{d^n}{ds^n} F(s)$ (s-derivatives)
$f'(t)$	$s\, F(s) - f(0)$ (t-derivative)
$f''(t)$	$s^2\, F(s) - s\, f(0) - f'(0)$ (2nd t-derivative)

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Line Integral

$$\int_{\mathcal{C}} \phi(x,y,z)\,ds = \int_a^b \phi(x(t),y(t),z(t))\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt.$$
$$\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{\mathcal{C}} F_1\,dx + F_2\,dy + F_3\,dz = \text{Work done by } \boldsymbol{F} \text{ along } \mathcal{C}.$$

Grad

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\,\boldsymbol{i} + \frac{\partial\phi}{\partial y}\,\boldsymbol{j} + \frac{\partial\phi}{\partial z}\,\boldsymbol{k}.$$

$\nabla\phi$ is normal to the tangent plane of the level surface $\{\phi(x,y,z)=k\}$ for constant k. If $\boldsymbol{F} = \nabla\phi$ for smooth function ϕ, then $\boldsymbol{F}$ is a conservative field, ϕ is a potential function of $\boldsymbol{F}$, and $\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r}$ is path-independent.

Curl

$$\text{curl}\,\boldsymbol{F} = \nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If the domain of $\boldsymbol{F}$ is simply connected (no need to worry about this) and $\nabla\times\boldsymbol{F} = \boldsymbol{0}$ then $\boldsymbol{F}$ is conservative.

Double Integral

$$\text{Area of } R = \iint_R dxdy.$$
$$\text{Volume under the surface } z = f(x,y) \text{ over } R = \iint_R f(x,y)\,dxdy.$$
$$\text{In polar coordinates: } \iint_R f(x,y)\,dxdy = \iint_R f(r\cos\theta, r\sin\theta)r\,d\theta dr.$$

Green's Formula (Theorem)

$$\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_{\mathcal{C}} F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\,dxdy.$$

Using "curl $= \nabla\times$", $\oint_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R (\nabla\times\boldsymbol{F})\cdot\boldsymbol{k}\,dxdy.$

Divergence Theorem in 2D

$$\int_S \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}s = \iint_R \text{div}\boldsymbol{F}\,dxdy$$

Surface Integral

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S is the graph of $z = f(x, y)$ for (x, y) in R. Then

$$\iint_S \phi(x, y, z)\, dS = \iint_R \phi(x, y, f(x, y))\sqrt{1 + f_x^2 + f_y^2}\; dxdy$$

Flux across $S = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$.

Triple Integral

In *cylindrical coordinates*:

$$\iiint f(x, y, z)\; dxdydz = \iiint f(r\cos\theta, r\sin\theta, z)\, r\, drd\theta dz.$$

In *spherical coordinates*:

$$\iiint f(x, y, z)\, dxdydz = \iiint f\big(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi\big)\, r^2 \sin\varphi\, drd\theta d\varphi$$

Gauss' Theorem: Divergence Theorem in 3D

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iiint_V \nabla \cdot \boldsymbol{F}\, dV$$

Stokes' theorem

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$$

Fourier Series

For a $2L$-periodic function $f : \mathbb{R} \to \mathbb{R}$, the *Fourier series of* f:

$$\mathcal{F}(f)(x) := b_0 + \sum_{n=1}^{\infty} \left\{ a_n \sin\left(\tfrac{n\pi}{L}x\right) + b_n \cos\left(\tfrac{n\pi}{L}x\right) \right\}$$

where the *Fourier coefficients* b_0, $(b_n)_{n\geq 1}$ and $(a_n)_{n\geq 1}$ are given by

$$b_0 = \tfrac{1}{2L}\int_{-L}^{L} f(x)$$

$$b_n = \tfrac{1}{L}\int_{-L}^{L} f(x)\, \cos\left(\tfrac{n\pi}{L}x\right)\, dx \qquad \text{for every } n \geq 1$$

$$a_n = \tfrac{1}{L}\int_{-L}^{L} f(x)\, \sin\left(\tfrac{n\pi}{L}x\right)\, dx \qquad \text{for every } n \geq 1$$

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Table of Standard Integrals

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

2. $\int \frac{dx}{x} = \ln|x| + C$

3. $\int e^x\,dx = e^x + C$

4. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$

5. $\int \sin x\,dx = -\cos x + C$

6. $\int \cos x\,dx = \sin x + C$

7. $\int \sec^2 x\,dx = \tan x + C$

8. $\int \sinh x\,dx = \cosh x + C$

9. $\int \cosh x\,dx = \sinh x + C$

10. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

11. $\int x\,\sin x\,dx = -x\cos x + \sin x + C$

12. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

13. $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C'$

14.
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C \quad (x > a)$$
$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C' \quad (x > a \text{ or } x < -a)$$

This is the end of the quiz paper.