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QBUS1040: Foundations of Business Analytics
Homework 2

Semester 2, 2019

This homework consists of seven problems that require you to submit a written response and a coding component. The problems that require a written response are described in this paper. You need to print it and write your answers directly in this paper. You should use scratch paper (which you will not turn in) to do your rough work. You should submit a scanned copy of your written solution as a PDF via Canvas.

For all problems where you are asked for a free-form answer, it must be written in the box below the problem. We won't read anything outside the boxes.

The coding components are described in a separate Jupyter Notebook file. You should also download the Jupyter Notebook file for Homework 2 and enter your code in the space provided. You should submit your code as a Jupyter notebook file via Canvas.

The homework is due by 5pm on Friday, the 30th of August. **Late homework will not be accepted.**

You must provide your SID in the space below.

The problems have unequal weight. Some are easy. Others, not so much.

Tutorial time: Friday 4-6pm

Tutor's name: Mr Kam Fung (Henry) Cheung

Your SID: 4 8 0 0 4 8 6 9 1 1

(For QBUS1040 staff only)

Question:	1	2	3	4	5	6	7	Total
Points:	10	10	10	12	10	8	10	70
Score:								

1. (10 points) *Linear combinations of cash flows.* We consider cash flow vectors over T time periods, with a positive entry meaning a payment received, and negative meaning a payment made. A (unit) single period loan, at time period t, is the T-vector l_t that corresponds to a payment received of \$1 in period t and a payment made of $\$(1+r)$ in period $t+1$, with all other payments zero. Here $r > 0$ is the interest rate (over one period). Let c be a \$1 $T-1$ period loan, starting at period 1. This means that \$1 is received in period 1, $\$(1+r)^{T-1}$ is paid in period T, and all other payments (i.e., $c_2, \dots, c_{T-1}$) are zero. Express c as a linear combination of single period loans.

- We have l_1 a \$1 loan from period 1 to period 2 with r interest:
  $\Rightarrow l_1 = (1, -(1+r), 0)$
- l_2 a \$1 ~~period~~ loan from period 2 to period 3 with r interest
  $\Rightarrow l_2 = (0, 1, -(1+r))$
- Let T = 2, c_3 represent a \$1 2 period loan paid in period 3
  $\Rightarrow c_3 = l_1 + (1+r)\, l_2$
  $= (1, -(1+r), 0) + (1+r)(0, 1, -(1+r)) = (1, 0, -(1+r)^2)$
- Thus any \$1 T − 1 period loan can be expressed as
  $c = l_1 + (1+r) l_2 + \dots + (1+r)^{T-1} l_T$

2. (10 points) *Reverse triangle inequality.* Suppose a and b are vectors of the same size. The triangle inequality states that $\|a+b\| \leq \|a\| + \|b\|$. Show that we also have $\|a+b\| \geq \|a\| - \|b\|$. *Hints.* Draw a picture to get the idea. To show the inequality, apply the triangle inequality to $(a+b)+(-b)$.

Given a and b are vectors of the same size:

Then $\|a\| = \|a+b-b\| = \|a+b+(-b)\|$

Using the triangle inequality stated above, we have

$$\|a+b+(-b)\| \leq \|a+b\| + \|-b\|$$

$$\|a\| \leq \|a+b\| + \|-b\|$$

$$\|a\| \leq \|a+b\| + \|b\|$$

subtract $\|b\|$ from both sides:

$$\|a\| - \|b\| \leq \|a+b\|$$

$$\Rightarrow \|a+b\| \geq \|a\| - \|b\|$$

Thus $\|a+b\| \geq \|a\| - \|b\|$ as desired.

3. (10 points) *Norm identities.* Verify that the following identities hold for any two vectors a and b of the same size.
   (a) $(a+b)^T(a-b) = \|a\|^2 - \|b\|^2$.

~~RHS~~: LHS:

$$
\begin{aligned}
(a+b)^T(a-b) &= a^Ta - a^Tb + b^Ta - b^Tb \text{ (distributivity)} \\
&= a^Ta - a^Tb + a^Tb - b^Tb \text{ (commutavity)} \\
&= a^Ta - b^Tb \\
&= \|a\|^2 - \|b\|^2 \text{ (definition of } \cancel{\text{Euclad}} \text{ Euclidean norm)} \\
&= \text{RHS}.
\end{aligned}
$$

   (b) $\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$. This is called the *parallelogram law.*

LHS:

$$
\begin{aligned}
\|a+b\|^2 + \|a-b\|^2 &= (a+b)^T(a+b) + (a-b)^T(a-b) \\
&= a^Ta + a^Tb + b^Ta + b^Tb + (a^Ta - a^Tb - b^Ta + b^Tb) \\
&= a^Ta + b^Tb + a^Ta + b^Tb \\
&= \|a\|^2 + \|b\|^2 + \|a\|^2 + \|b\|^2 \\
&= 2\|a\|^2 + 2\|b\|^2 = 2(\|a\|^2 + \|b\|^2) \\
&= \text{RHS}.
\end{aligned}
$$

4. (12 points) *k-means with nonnegative, proportions, or Boolean vectors.* Suppose that the vectors $x_1, \ldots, x_N$ are clustered using k-means, with group representatives $z_1, \ldots, z_k$.

(a) Suppose the original vectors x_i are nonnegative, i.e., their entries are nonnegative. Explain why the representatives z_j are also nonnegative.

Representatives can be computed using the formula $z_j = \frac{1}{|G_j|} \sum_{i \in G_j} x_i$

Assume that the group G_j has at least one element, its cardinality will thus be a positive integer.

Given that the original vectors are non-negative, then their sum will also be a non-negative vector.

A non-negative vector with its elements divided by a positive integer will result in z_j is also non-negative.

(b) Suppose the original vectors x_i represent proportions, i.e., their entries are nonnegative and sum to one. (This is the case when x_i are word count histograms, for example.) Explain why the representatives z_j also represent proportions, i.e., their entries are nonnegative and sum to one.

Given the original vectors are proportions,

i.e. $x_1 + x_2 + \ldots + x_n = 1$ and $x_i \in [0, 1]$

Thus the sum of x_i by each group will also be a proportion.

This proportion will then be divided by a positive integer (ie. the cardinality of the group), hence resulting in a smaller proportion, which ~~are~~ is the individual representative.

(c) Suppose the original vectors x_i are Boolean, i.e., their entries are either 0 or 1. Give an interpretation of $(z_j)_i$, the ith entry of the j group representative.

$(z_j)_i$ can be interpreted as

the proportion of 1's of the characteristic i in group j,

such that $(z_j)_i \in [0, 1]$

5. (10 points) *Weighted norm.* On page 51 of the textbook we discuss the importance of choosing the units or scaling for the individual entries of vectors, when they represent heterogeneous quantities. Another approach is to use a weighted norm of a vector x, defined as

$$\|x\|_w = \sqrt{w_1 x_1^2 + \cdots + w_n x_n^2},$$

where $w_1, \ldots, w_n$ are given positive weights, used to assign more or less importance to the different elements of the n-vector x. If all the weights are one, the weighted norm reduces to the usual ('unweighted') norm. It can be shown that the weighted norm is a general norm, i.e., it satisfies the four norm properties listed on page 46. Following the discussion on page 51, one common rule of thumb is to choose the weight w_i as the inverse of the typical value of x_i^2 in the application. A version of the Cauchy-Schwarz inequality holds for weighted norms: For any n-vector x and y, we have

$$|w_1 x_1 y_1 + \cdots + w_n x_n y_n| \leq \|x\|_w \|y\|_w.$$

(The expression inside the absolute value on the left-hand side is sometimes called the weighted inner product of x and y.) Show that this inequality holds. *Hint.* Consider the vectors $\tilde{x} = (x_1\sqrt{w_1}, \ldots, x_n\sqrt{w_n})$ and $\tilde{y} = (y_1\sqrt{w_1}, \ldots, y_n\sqrt{w_n})$, and use the (standard) Cauchy-Schwarz inequality.

Consider $\tilde{x} = (x_1\sqrt{w_1}, \ldots, x_n\sqrt{w_n})$

$$\|\tilde{x}\| = \sqrt{x_1^2 w_1 + x_2^2 w_2 + \ldots + x_n^2 w_n} = \|x\|_w \quad (1)$$

$$\tilde{y} = (y_1\sqrt{w_1}, \ldots, y_n\sqrt{w_n})$$

$$\|\tilde{y}\| = \sqrt{y_1^2 w_1 + y_2^2 w_2 + \ldots + y_n^2 w_n} = \|y\|_w \quad (2)$$

$$|\tilde{x}^T\tilde{y}| = |x_1\sqrt{w_1} \cdot y_1\sqrt{w_1} + x_2\sqrt{w_2} \cdot y_2\sqrt{w_2} + \ldots + x_n\sqrt{w_n} \cdot y_n\sqrt{w_n}|$$

$$= |x_1 y_1 w_1 + x_2 y_2 w_2 + \ldots + x_n y_n w_n|$$

$$\Rightarrow |x_1 y_1 w_1 + \ldots + x_n y_n w_n| = |\tilde{x}^T\tilde{y}|$$

$$\leq \|\tilde{x}\| \|\tilde{y}\| \text{ (Cauchy-Schwarz inequality)}$$

$$= \|x\|_w \|y\|_w \text{ (from (1) \& (2))}.$$

6. (8 points) **Please refer to the Jupyter Notebook file for this problem.**

7. (10 points) **Please refer to the Jupyter Notebook file for this problem.**