Assignment 1

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Tutorial: Carslaw 453 Tuesday 12 - 01 pm

1. With $h(x) = \frac{1}{2}$, we have our model: $\frac{dx}{dt} = \frac{1}{2}(1-x)$

a) Find the general solution $x(t)$

$$\frac{1}{1-x}\,dx = \frac{1}{2}\,dt$$

$$\int \frac{1}{1-x}\,dt = \int \frac{1}{2}\,dt$$

$$-\ln|1-x| + C_1 = \frac{1}{2}t + C_2$$

$$-\ln|1-x| = \frac{1}{2}t + C \quad (\text{with } C = C_2 - C_1)$$

$$e^{t/2 + C} = \frac{1}{1-x}$$

$$e^{t/2} \cdot e^{C} = \frac{1}{1-x}$$

$$e^{t/2} \cdot e^{C}(1-x) = 1$$

$$e^{t/2} \cdot e^{C} - e^{t/2} e^{C} x = 1$$

$$-e^{t/2} \cdot e^{C} x = 1 - e^{t/2} \cdot e^{C}$$

$$-x = \frac{1 - e^{t/2} \cdot e^{C}}{e^{t/2} \cdot e^{C}}$$

$$-x = \frac{(e^{t/2} \cdot e^{C})^{-1} - 1}{1}$$

$$x = 1 - e^{-t/2} \cdot e^{-C}$$

$$x = 1 - M \cdot e^{-t/2} \quad (\text{with } M = e^{-C})$$

Thus $x(t) = 1 - M \cdot e^{-t/2}$

b) The initial crowd waiting is $k \in [0,1]$ (i.e. $x(0) = k$)

How full is the venue at t?

From question 1a, we have $x(t) = 1 - M.e^{-t/2}$

knowing that $x(0) = k$, we can solve for M:

$$\Rightarrow \quad 1 - M.e^{-1/2.0} = k$$

$$1 - M = k$$

$$M = 1 - k$$

thus at any given t, the venue is at:

$$x(t) = 1 - M.e^{-t/2}$$

$$x(t) = 1 - (1-k).e^{-t/2}$$

2. With $h(x) = \frac{3}{2}x$, we have our model $\frac{dx}{dt} = \frac{3}{2}x(1-x)$

a) Find the general solution $x(t)$

$$\frac{1}{x(1-x)} dx = \frac{3}{2} dt$$

$$\int \frac{1}{x(1-x)} dx = \int \frac{3}{2} dt$$

Solve LHS using partial fraction:

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} = \frac{A(1-x)}{x(1-x)} + \frac{Bx}{x(1-x)} = \frac{A - Ax + Bx}{x(1-x)} = \frac{A + (B-A)x}{x(1-x)}$$

$$\begin{cases} A = 1 \\ B - A = 0 \end{cases} \quad \begin{cases} A = 1 \\ B = 1 \end{cases} \quad \Rightarrow \int \frac{1}{x(1-x)} dx = \int \frac{1}{x} + \frac{1}{1-x} dx$$

$\Rightarrow$ Back to the equation: $\int \frac{1}{x} + \frac{1}{1-x} dx = \int \frac{3}{2} dt$

$$\ln|x| - \ln|1-x| + C_1 = \frac{3}{2}t + C_2$$

$$\ln\left|\frac{x}{1-x}\right| = \frac{3}{2}t + C \quad (\text{with } C = C_2 - C_1)$$

$$e^{3t/2 + C} = \frac{x}{1-x}$$

$$e^{3t/2} . e^{C} = \frac{1}{1/x - 1}$$

$$e^{3t/2} \cdot e^{c} \left(1/x - 1 \right) = 1$$

$$\frac{e^{3t/2} \cdot e^{c}}{x} - e^{3t/2} \cdot e^{c} = 1$$

$$\frac{1}{x} e^{3t/2} \cdot e^{c} = 1 + e^{3t/2} \cdot e^{c}$$

$$\frac{1}{x} = \frac{1 + e^{3t/2} \cdot e^{c}}{e^{3t/2} \cdot e^{c}}$$

$$x = \frac{e^{3t/2} \cdot e^{c}}{1 + e^{3t/2} \cdot e^{c}}$$

$$x = \frac{1}{(e^{3t/2} \cdot e^{c})^{-1} + 1}$$

$$x = \frac{1}{1 + e^{-3t/2} \cdot e^{-c}}$$

$$x = \frac{1}{1 + N e^{-3t/2}} \quad (\text{with } N = e^{-c})$$

b, What should be the initial crowd $x(0)$ if the band wants to start playing at $t = 2$ hours with 80% capacity ?

At $t = 2$ with 80% capacity can be interproted as: $x(2) = 0.8$

From question 2b, we have: $x(2) = \frac{1}{1 + N \cdot e^{-3/2 \cdot 2}} = 0.8$, we can solve for N:

$$\frac{1}{1 + N \cdot e^{-3}} = 0.8$$

$$0.8 \left(1 + N \cdot e^{-3} \right) = 1$$

$$0.8 + 0.8 N e^{-3} = 1$$

$$N = \frac{1 - 0.8}{0.8 e^{-3}} = \frac{0.2}{0.8 e^{-3}} = 0.25 e^{3} \Rightarrow x(t) = \frac{1}{1 + 0.25 \cdot e^{-3t/2}}$$

Thus at $t = 0$

$$x(0) = \frac{1}{1 + 0.25 e^{3} \cdot e^{-3/2 \cdot 0}} = \frac{1}{1 + 0.25 e^{3}} \approx 0.166$$

Thus, the initial crowd was at 16.6% of the venue's maximum capacity.

3. We have both model starting at 10% full capacity.

From previous questions, we have our models $x_A(t) = 1 - M.e^{-t/2}$ and $x_B(t) = \frac{1}{1+N.e^{-3t/2}}$

Since we know that both models starting at 10% full capacity:

- $x_A(0) = 1 - M.e^{-1/2.0} = 0.1 \Leftrightarrow 1 - M = 0.1 \Leftrightarrow M = 0.9$

$\Rightarrow x_A(t) = 1 - 0.9e^{-t/2}$

- $x_B(0) = \frac{1}{1+N.e^{-3/2.0}} = 0.1 \Leftrightarrow \frac{1}{1+N} = 0.1 \Leftrightarrow 0.1 + 0.1N = 1 \Leftrightarrow 0.1N = 0.9 \Leftrightarrow N = 9$

$\Rightarrow x_B(t) = \frac{1}{1+9.e^{-3t/2}}$

a. Which of the model will first reach 50% full capacity?

$$x_A(t) = 1 - 0.9e^{-t/2} = 0.5$$

$$-0.9e^{-t/2} = -0.5$$

$$e^{-t/2} = \frac{5}{9}$$

$$-\frac{1}{2}t = \ln\left(\frac{5}{9}\right)$$

$$t = \frac{\ln(5/9)}{-1/2} \approx 1.18 \text{ hours}$$

$\Rightarrow$ Model A will reach 50% after approximately 1.18 hours, that is after 1 hour and 11 minutes.

$$x_B(t) = \frac{1}{1+9.e^{-3t/2}} = \frac{1}{1+9.e^{-3t/2}} = 0.5$$

$$(1+9.e^{-3t/2}).0.5 = 1$$

$$1+9.e^{-3t/2} = 2$$

$$e^{-3t/2} = \frac{1}{9}$$

$$-\frac{3}{2}t = \ln\left(\frac{1}{9}\right)$$

$$t = \frac{\ln(1/9)}{-3/2} \approx 1.465 \text{ hours}$$

$\Rightarrow$ Model B will reach 50% after appoximately 1.465 hours, i.e. 1 hour 28 minutes.

Since 1.18 < 1.465, model A will reach 50% of full capacity first.

b. Which of the model will first reach 99% capacity?

$$x_A(t) = 1 - 0.9e^{-t/2} = 0.99$$

$$-0.9e^{-t/2} = -0.01$$

$$e^{-t/2} = \frac{1}{90}$$

$$-\frac{1}{2}t = \ln\left(\frac{1}{90}\right)$$

$$t = \frac{\ln(1/90)}{-1/2} \approx 9 \text{ hours}$$

$\Rightarrow$ Model A will reach 99% after approximately 9 hours.

$$x_B(t) = \frac{1}{1 + 9 \cdot e^{-3t/2}} = \frac{1}{1 + 9 \cdot e^{-3t/2}} = 0.99$$

$$(1 + 9 \cdot e^{-3t/2}) \cdot 0.99 = 1$$

$$1 + 9 \cdot e^{-3t/2} = \frac{100}{99}$$

$$e^{-3t/2} = \frac{1}{891}$$

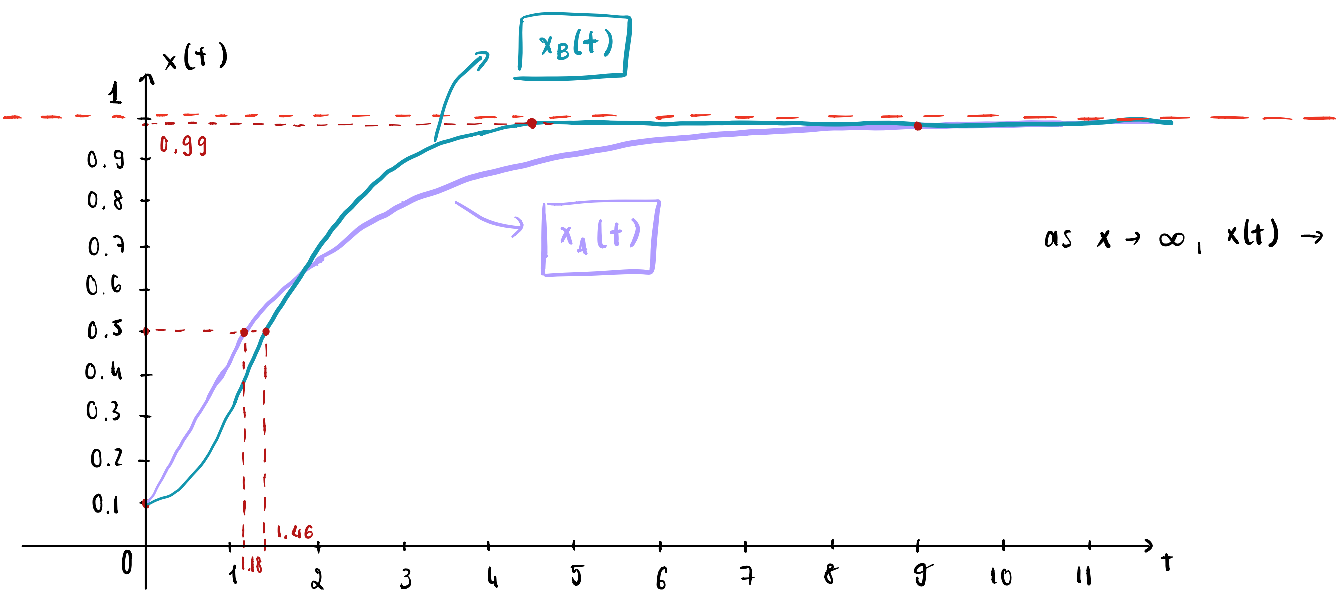
$$-\frac{3}{2}t = \ln\left(\frac{1}{891}\right)$$

$$t = \frac{\ln(1/891)}{-3/2} \approx 4.5 \text{ hours}$$

$\Rightarrow$ Model B will reach 99% after appoximately 4.5 hours.

Since $4.5 < 9$, model B will reach 99% of full capacity first.

c. Plot the curves $x_A(t)$ and $x_B(t)$



as $x \to \infty$, $x(t) \to 1$