# Assignment 2

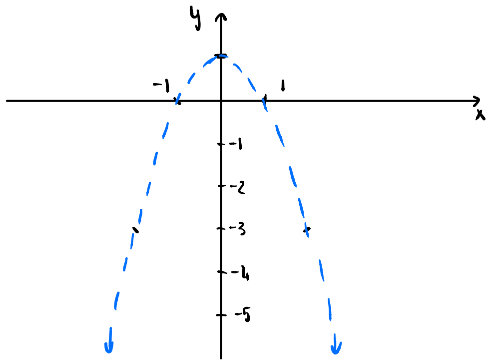
Student ID : 480048691

Tutor's Name : Irina Choubina

Tutorial : Carslaw 453 Tuesday 12pm - 1 pm

let $f(x,y) = \dfrac{x+1}{x^2+y-1}$ be a function $f : \mathbb{R}^2 \to \mathbb{R}$

1. a. Natural domain of $f(x,y)$ : $\mathbb{D} = \{ (x,y) \in \mathbb{R}^2 \mid y \neq -x^2 + 1 \}$




The domain of $f(x,y)$ is every point on the xy-plane except for those lie on the parabola of $y = -x^2 + 1$.

b. The range of $f(x,y)$ is $\mathbb{R}$.

2. a. Compute the partial derivatives $f_x(x,y)$ and $f_y(x,y)$

$$f_x(x,y) = \frac{\partial f}{\partial x}\left(\frac{x+1}{x^2+y-1}\right) = \frac{1(x^2+y-1) - (x+1).2x}{(x^2+y-1)^2}$$

$$= \frac{x^2+y-1-2x^2-2x}{(x^2+y-1)^2} = \frac{-x^2-2x+y-1}{(x^2+y-1)^2}$$

$$f_y(x,y) = \frac{\partial f}{\partial y}\left(\frac{x+1}{x^2+y-1}\right) = \frac{0.(x^2+y-1) - (x+1).1}{(x^2+y-1)^2} = \frac{-x-1}{(x^2+y-1)}$$

b. Evaluate the partial derivatives $f_x$ and $f_y$ at the point $(x,y) = (1,2)$

$$f_x(x,y) = \frac{-x^2-2x+y-1}{(x^2+y-1)^2}$$

$$\Rightarrow f_x(1,2) = \frac{-1^2-2.1+2-1}{(1^2+2-1)^2} = \frac{-2}{4} = \frac{-1}{2}$$

$$f_y(x,y) = \frac{-x-1}{(x^2+y-1)} = \frac{-1-1}{(1^2+2-1)^2} = \frac{-2}{4} = \frac{-1}{2}$$

3. Compute the tangent plane to $f(x,y)$ at the point $(x,y) = (1,2)$.

$\Rightarrow$ We have $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{-1}{2}$ when $(x,y) = (1,2)$ (as shown above).

At the point $(1,2)$, $z = f(x,y) = f(1,2) = \frac{1+1}{1^2+2-1} = \frac{2}{2} = 1$

Therefore, the equation of the tangent plane is

$$z - 1 = \frac{-1}{2}(x-1) - \frac{1}{2}(y-2)$$

which can be arranged to $z = \frac{-1}{2}x - \frac{1}{2}y + \frac{5}{2}$

5. For the points $(x,y) = (a,b)$ given below, compute

$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ or show that the limit does not exist}$$

a. $(a,b) = (0,0)$

Notice that $f(x,y) = \frac{x+1}{x^2+y-1}$ is defined at $(0,0)$

from which it follows that

$$\lim_{(x,y)\to(0,0)} \frac{x+1}{x^2+y-1} = \frac{0+1}{0^2+0-1} = \frac{1}{-1} = -1.$$

Therefore, the limit does exist, $\lim_{(x,y)\to(0,0)} f(x,y) = -1$.

b. $(a,b) = (1,0)$

Since $(1,0)$ is on the parabola $y = -x^2 - 1$, thus $f(x,y)$ is not defined at $(x,y) = (1,0)$.

. Try $x = 1$:

$$\lim_{(x,y)\to(1,0)} \frac{x+1}{x^2+y-1} = \lim_{y\to 0} \frac{2}{y} = \frac{2}{0}$$

By the definition of asymptote, we can conclude that $y = 0$ is a horizontal asymptote of $f(x,y)$ on the $xy$-plane, when fixing $x = 1$.

We can perform a one-sided limit as $y$ approaches $0^-$.

$\Rightarrow \lim_{y \to 0^-} \frac{2}{y} = -\infty$.

Therefore, the limit $\lim_{(x,y)\to(1,0)} \frac{x+1}{x^2+y-1}$ does not exist.

c) $(a,b) = (-1,0)$

Since $(-1,0)$ is also on the parabola $y = -x^2+1$, $f(x,y)$ is not defined at $(x,y) = (-1,0)$.

We will try to approach this point by using different paths:

- Try $y = x+1$, this means that we will plug in $y = x+1$ and then take the limit as $x$ approaches $-1$.

  Note that $f(x,y) = f(x, x+1) = \frac{x+1}{x^2+x+1-1} = \frac{x+1}{x^2+x}$

  $$\Rightarrow \lim_{(x,y)\to(-1,0)} f(x,y) = \lim_{x\to -1} \frac{x+1}{x^2+x} = \lim_{x\to-1} \frac{x+1}{x(x+1)}$$
  $$= \lim_{x\to-1} \frac{1}{x} = \frac{1}{-1} = 1.$$

- Try $y = (x+1)^2$. Note that $f(x,y) = f(x,(x+1)^2) = \frac{x+1}{x^2+(x+1)^2-1} = \frac{x+1}{2x^2+2x}$

  $$\Rightarrow \lim_{(x,y)\to(-1,0)} f(x,y) = \lim_{x\to-1} \frac{x+1}{2x^2+2x} = \lim_{x\to-1} \frac{x+1}{2x(x+1)}$$
  $$= \lim_{x\to-1} \frac{1}{2x} = \frac{1}{2.(-1)} = \frac{-1}{2}$$

Since the limit cannot be both 1 and $\frac{-1}{2}$, it follows that the limit does not exist.

4. a) Draw the level curves of $f(x,y)$ for $c \in \{-2,-1,0,1,2\}$

$\Rightarrow$ Since the range of $f$ is $\mathbb{R}$, $c$ can be of any value.

The level curves are obtained by taking the cross-section at height $c$ has equation

$$f(x,y) = \frac{x+1}{x^2+y-1} = c$$

- When $c = 0$, $f(x,y) = \frac{x+1}{x^2+y-1} = 0$, this gives a straight vertical line $x = -1$.

- With $c = 1$, $f(x,y) = \frac{x+1}{x^2+y-1} = 1 \Leftrightarrow x+1 = x^2+y-1$

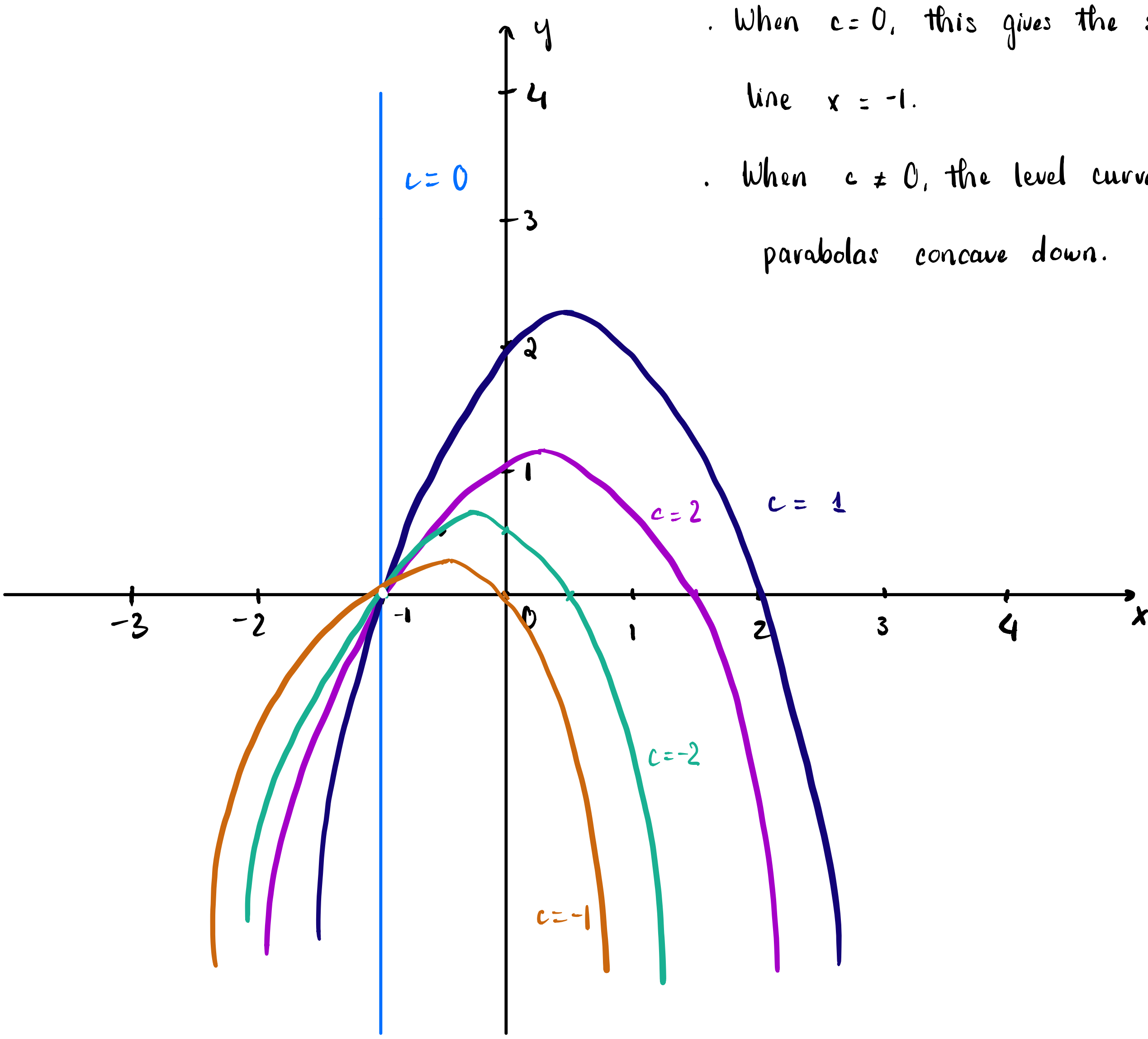
  $\Leftrightarrow y = -x^2+x+2$

- With $c = 2$, $f(x,y) = \frac{x+1}{x^2+y-1} = 2 \Leftrightarrow x+1 = 2(x^2+y-1)$

$\Leftrightarrow y = -x^2 + \frac{1}{2}x + \frac{3}{2}$

- With $c = -1$, $f(x,y) = \frac{x+1}{x^2+y-1} = -1 \Leftrightarrow y = -x^2 - x$

- With $c = -2$, $f(x,y) = \frac{x+1}{x^2+y-1} = -2 \Leftrightarrow y = -x^2 - \frac{1}{2}x + \frac{1}{2}$




- When $c = 0$, this gives the straight line $x = -1$.

- When $c \neq 0$, the level curves are parabolas concave down.

b) Level curves cannot intersect each other. Since $(-1,0)$ is on the parabola $y = -x^2+1$, $f(x,y)$ is not defined at $(x,y) = (-1,0)$. Therefore, the level curves are not intersecting each other at $(-1,0)$.