Assignment 2

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Question 2 :

a, Let $P(n)$ be the claim that if n is a positive integer, and $X, A_1, A_2, \dots, A_n$ are sets, then

$$X \setminus (A_1 \cup A_2 \cup \dots \cup A_n) = (X \setminus A_1) \cap (X \setminus A_2) \cap \dots \cap (X \setminus A_n)$$

Base case:

We have $P(1)$ holds : $X \setminus (A_1) = X \setminus A_1 \Rightarrow$ true.

Inductive step:

- For inductive hypothesis, we assume that the claim holds for any arbitrary positive integer k, that is, if $X, A_1, A_2, \dots, A_k$ are sets then $X \setminus (A_1 \cup A_2 \cup \dots \cup A_k)$

$$= (X \setminus A_1) \cap (X \setminus A_2) \cap \dots \cap (X \setminus A_k)$$

- It must be shown that under this assumption, $P(k+1)$ also holds

i.e. $X \setminus (A_1 \cup A_2 \cup \dots \cup A_k \cup A_{k+1})$ (*)

$$= (X \setminus A_1) \cap (X \setminus A_2) \cap \dots \cap (X \setminus A_k) \cap (X \setminus A_{k+1})$$

- Let H be a set such that $H = A_1 \cup A_2 \cup \dots \cup A_k$

then (*) will be

$$X \setminus (H \cup A_{k+1})$$

$$= (X \setminus H) \cap (X \setminus A_{k+1}) \quad \text{(using De Morgan's law)}$$

$= [X \setminus (A_1 \cup \dots \cup A_k)] \cap (X \setminus A_{k+1})$

$= (X \setminus A_1) \cap (X \setminus A_2) \cap \dots \cap (X \setminus A_k) \cap (X \setminus A_{k+1})$

This shows that $P(k+1)$ also holds.

$\therefore$ We conclude by induction that the claim is true $\forall n \geq 1$.

b) Let $P(n)$ be the claim that $\forall n \in \mathbb{N}$, 13 divides $4^{2n+1} + 3^{n+2}$.

Base case : We have $P(1)$ holds:

$4^{2.1+1} + 3^{1+2} = 4^3 + 3^3 = 64 + 27 = 91$ is divisible by 3.

Inductive step:

- For inductive hypothesis, we assume that $P(k)$ holds for any arbitrary $k \in \mathbb{N}$
  i.e. 13 divides $4^{2k+1} + 3^{k+2}$
  or $4^{2k+1} + 3^{k+2} = 13d$, $d \in \mathbb{Z}$
- Under this assumption, we want to show that $P(k+1)$ also holds
  i.e. 13 divides $4^{2k+3} + 3^{k+3}$

Note that $4^{2k+3} + 3^{k+3}$

$= 4^2 . 4^{2k+1} + 3 . 3^{k+2}$

$= 4^2 . 4^{2k+1} + 3(13d - 4^{2k+1})$ (inductive hypothesis)

$= 4^2 . 4^{2k+1} + 3.13d - 3.4^{2k+1}$

$= 13 . 4^{2k+1} + 3.13d$

$= 13(4^{2k+1} + 3d) \rightarrow$ divisible by 13

This shows that $P(k+1)$ also holds.

$\therefore$ We conclude by induction that the claim is true $\forall n \in \mathbb{N}$.

Question 1:

a) Let P and Q be sets, we have $p(x): x \in P$
and $q(x): x \in Q$.

- Suppose that $(\forall x)(p(x) \Rightarrow q(x))$ is true, this tells that "for all x, if $x \in P$ then $x \in Q$".
- By the definition of subsets, which follows that "Set A is a subset of B iff every element of A is an element of B", we can conclude that the above compound proposition tells us that $P \subseteq Q$.

- If $p(x)$ and $q(x)$ are logically equivalent, i.e. $(\forall x)((p(x) \Leftrightarrow q(x)$ is true$)$ can also be written as $(\forall x)[(p(x) \Rightarrow q(x)) \wedge (q(x) \Rightarrow p(x))]$.

  Based on the explanation above, this compound proposition tells us that $P \subseteq Q$ and $Q \subseteq P$.

  By the definition of two equal sets, if $P \subseteq Q$ and $Q \subseteq P$, then $P = Q$.

b) Let $a(x)$ be the statement $x \in A$,
$b(x): x \in B$ and $c(x): x \in C$.

(i) $S_1 = (A \setminus B) \setminus C$

$\Rightarrow \quad [a(x) \wedge \sim b(x)] \wedge \sim c(x)$

(ii) $S_2 = A \setminus (B \setminus C)$

$\Rightarrow \quad a(x) \wedge \sim[\, b(x) \wedge \sim c(x)\,]$

(iii) $S_3 = (A \setminus B) \cup (A \cap C)$

$\Rightarrow \quad [\, a(x) \wedge \sim b(x)\,] \vee [\, a(x) \wedge c(x)\,]$

(iv) $S_4 = (A \setminus B) \cap (A \setminus C)$

$\Rightarrow \quad [\, a(x) \wedge \sim b(x)\,] \wedge [\, a(x) \wedge \sim c(x)\,]$

(v) $S_5 = A \setminus (B \cup C)$

$\Rightarrow \quad a(x) \wedge \sim[\, b(x) \vee c(x)\,]$

c) . Since S_2 gives the compound proposition

$$a(x) \wedge \sim[\, b(x) \wedge \sim c(x)\,]$$

$$= a(x) \wedge [\, \sim b(x) \vee c(x)\,] \quad \text{(De Morgan's law)} \quad (1)$$

while S_3 gives

$$[\, a(x) \wedge \sim b(x)\,] \vee [\, a(x) \wedge c(x)\,]$$

$$= a(x) \wedge [\, \sim b(x) \wedge c(x)\,] \quad \text{(distributive law)} \quad (2)$$

Since (1) = (2), we conclude $S_2 = S_3$.

- We have S_5 gives the compound proposition

$$a(x) \wedge \sim[\, b(x) \vee c(x)\,]$$

$$= a(x) \wedge [\, \sim b(x) \wedge \sim c(x)\,] \quad \text{(De Morgan's law)} \quad (3)$$

while S_4 gives

$[a(x) \wedge \sim b(x)] \wedge \sim c(x)$

$\equiv a(x) \wedge [\sim b(x) \wedge \sim c(x)]$ (associative law) (4)

Since (3) $\equiv$ (4), we conclude that $S_1 \equiv S_5$.

Question 3:

a) Suppose that $a = d.k + b$, where $a, b, d, k \in \mathbb{Z}$.

Prove that b is divisible by d iff a is divisible by d.

Let p be the statement "b is divisible by d"

and q be the statement "a is divisible by d".

We will break this proof into two proofs to prove logical equivalence:

- $p \Rightarrow q$:

Assume that we have integers a, b, d, k such that $a = d.k + b$

Suppose that a is divisible by d, i.e. $a = dn$, $n \in \mathbb{Z}$

then $a = dk + b$

$dn = dk + b$

$b = dn - dk = d(n-k)$

Thus, b is divisible by d, by definition.

- $q \Rightarrow p$:

Suppose that b is divisible by d, i.e. $b = dm$, $m \in \mathbb{Z}$

then $a = dk + b$

$= dk + dm = d(k+m)$

This shows that a is divisible by d.

$\therefore$ We can conclude that b is divisible by d if and only if a is divisible by d. $\square$

b) Let $x = abc$ be a three-digit number with digits a, b, c $(a, b, c \in \{0, 1, \ldots, 9\})$. Prove that x is divisible by 3 if and only if $a + b + c$ is divisible by 3.

We will use the same approach as the question above.
Let p be the statement that "x is divisible by 3"
and q be the statement "$a + b + c$ is divisible by 3"

- $p \Rightarrow q$:

Suppose we have a three-digit number as describe above and that $a + b + c$ is divisible by 3

i.e $a + b + c = 3m$, for some $m \in \mathbb{Z}$

Then
$$
\begin{aligned}
x &= 10^2 a + 10b + c \\
&= (99+1)a + (9+1)b + c \\
&= 99a + 9b + \underbrace{a + b + c}_{3m} \\
&= 99a + 9b + 3m \\
&= 3(33a + 3b + m)
\end{aligned}
$$

Thus x is divisible by 3.

- $q \Rightarrow p$

Now we assume that x is divisible by 3

that is $10^2 a + 10b + c = 3n$, for some $n \in \mathbb{Z}$

Then
$$
\begin{aligned}
10^2 a + 10b + c &= 3n \\
(99+1)a + (9+1)b + c &= 3n \\
99a + 99b + a + b + c &= 3n \\
a + b + c &= 3n - 99a - 9b
\end{aligned}
$$

$= 3(n - 33a - 3b)$

Thus, $a + b + c$ is divisible by 3.

$\therefore$ We can now conclude that x is divisible by 3 if and only if $a + b + c$ is divisible by 3. $\square$