Assignment 1 - COMP3670 - Theory part
Student ID: u7205329

Exercise 1.

a. Augmenting matrix A and b and performing row reduction, we have:

$$\left[\begin{array}{ccc|c} 2 & -2 & -5 & -4 \\ 1 & -1 & 3 & 9 \\ 3 & -3 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 9 \\ 2 & -2 & -5 & -4 \\ 3 & -3 & -2 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & -1 & 3 & 9 \\ 0 & 0 & -11 & -22 \\ 0 & 0 & -11 & -22 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We thus have the system of linear equations:

$$\begin{cases} x_1 - x_2 = 3 \\ x_3 = 2 \\ x_2 = x_2 \end{cases} \Rightarrow \begin{cases} x_1 = 3 + x_2 \\ x_2 = x_2 \\ x_3 = 2 \end{cases}$$

As x_2 is a free variable, letting $x_2 = t$, we get the solution set given by the parametric equation

$$\begin{cases} x_1 = 3 + t \\ x_2 = t \\ x_3 = 2 \end{cases} \quad \therefore \quad \underset{\sim}{x} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ t \in \mathbb{R} \right\}$$

b. Augmenting matrix A and b and performing row reduction, we have:

$$[A \mid b] = \left[\begin{array}{ccc|c} 2 & -3 & -10 & 4 \\ -4 & 2 & 3 & 0 \\ 10 & -3 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -3 & -10 & 4 \\ 0 & -4 & -17 & 8 \\ 0 & 12 & 51 & -19 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -3 & -10 & 4 \\ 0 & -4 & -17 & 8 \\ 0 & 0 & 0 & 5 \end{array}\right] \rightarrow \text{no solution}$$

The last equation represents $0x_1 + 0x_2 + 0x_3 = 5$, which clearly has no solutions. Therefore our solution set is an empty set, $\underset{\sim}{x} = \emptyset$.

Exercise 2. We will row reduce the matrix and find the conditions on λ such that the matrix is invertible.

let $A = \begin{bmatrix} \lambda & 1 & 2 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad R_1 \Leftrightarrow R_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ \lambda & 1 & 2 \end{bmatrix}$

$R_3 - \lambda R_1 \to R_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1-\lambda & 2-\lambda \end{bmatrix}$ $\quad \frac{1}{3} R_2 \to R_2 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/3 \\ 0 & 1-\lambda & 2-\lambda \end{bmatrix}$

$R_3 - (1-\lambda) R_2 \to R_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 5/3 - 2\lambda/3 \end{bmatrix}$

In order for there to be an invertible matrix, there must be a pivot corresponding to the last row. i.e. we require

$$\frac{5}{3} - \frac{2}{3}\lambda \neq 0 \Leftrightarrow \frac{2}{3}\lambda \neq \frac{5}{3} \Leftrightarrow \lambda \neq \frac{5}{2}.$$

Exercise 3.

a. $A = \{(x, y, 1) : x, y \in \mathbb{R}\}$

$\Rightarrow$ A is not a subspace of $\mathbb{R}^3$ as the zero vector $\underset{\sim}{0} = [0\ 0\ 0]^T \notin A$.
Even if we take $x = y = 0$, we get the vector $\underset{\sim}{x} = [0\ 0\ 1]^T$.

b. $B = \{(x, y, z) : x + 4y - 3z = t\}$, where t is some real number.

Let us check the three conditions:

1) The zero vector belongs to B. let $x = y = z = 0$, $0 + 4.0 - 3.0 = 0$.
$\Rightarrow \underset{\sim}{0} \in B$ if $t = 0$.

2) The subset B is closed under vector addition.
let $\underset{\sim}{v_1}, \underset{\sim}{v_2} \in B$
i.e. $\underset{\sim}{v_1} = [x_1\ y_1\ z_1]^T$ such that $x_1 + 4y_1 - 3z_1 = t$, and
$\underset{\sim}{v_2} = [x_2\ y_2\ z_2]^T$ such that $x_2 + 4y_2 - 3z_2 = t$
where t is some fixed constant.

Adding $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$, we have $\underset{\sim}{v_1} + \underset{\sim}{v_2} = [x_1 + x_2\ \ y_1 + y_2\ \ z_1 + z_2]^T$ where

$$
\begin{aligned}
&(x_1 + x_2) + 4(y_1 + y_2) - 3(z_1 + z_2) \\
&= x_1 + x_2 + 4y_1 + 4y_2 - 3z_1 - 3z_2 \\
&= (x_1 + 4y_1 - 3z_1) + (x_2 + 4y_2 - 3z_2) \\
&= t + t = 2t.
\end{aligned}
$$

As t is a fixed constant, for $\underset{\sim}{x} + \underset{\sim}{y} \in B$ we need $t = 0$ so that $2t = t$.

3) The subset B is closed under scalar multiplication.
Let $\underset{\sim}{v} = [x\ y\ z]^T \in B$ that is $x + 4y - 3z = t$.

Let $c \in \mathbb{R} \Rightarrow c\underset{\sim}{v} = [cx\ \ cy\ \ cz]^T$
and so $cx + 4cy - 3cz = c(x + 4y - 3z) = ct$

$c\underset{\sim}{v} \in B$ if we can prove that $ct = t$. Once again, we need $t = 0$ so that a multiple of t can be equal to itself.

Therefore, B is a subspace of $\mathbb{R}^3$ if and only if $t = 0$.

c) $C = \{(x, y, z) : x \geq 0,\ y \geq 0,\ z \geq 0\}$

1) The zero vector is in C (take $x = y = z = 0$), so axiom (1) holds.

2) The set C is closed under vector addition.

Let $\underset{\sim}{u} = [u_1\ u_2\ u_3]^T \in C$ and $\underset{\sim}{v} = [v_1\ v_2\ v_3]^T \in C$

Then $\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$

since the sum of non-negative numbers is non-negative.

$\rightarrow \underset{\sim}{u} + \underset{\sim}{v} \in C$

3) The set C is not closed under scalar multiplication

Let $\underset{\sim}{u} = [u_1\ u_2\ u_3]^T \in C$ and $\lambda = -3$.

$\Rightarrow \lambda \underset{\sim}{u} = -3\underset{\sim}{u} = [-3u_1\ \ -3u_2\ \ -3u_3]^T \notin C$ as a product of a non-negative number and a negative number is negative number.

Hence, C is not a subspace in $\mathbb{R}^3$.

d) $D = \{(x, y, z) : x, y \in \mathbb{R},\ z \in \mathbb{Q}\}$

1) The vector 0 is in D, where $z = \frac{0}{t}$ and $t \neq 0$.

2) The set D is closed under vector addition.

let $\underset{\sim}{u} = [x_1\ y_1\ z_1]^T \in D_1$, $\underset{\sim}{u_2} = [x_2\ y_2\ z_2]^T \in D$

$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \in D$ because the sum of real numbers is a real number and the sum of rational numbers is a rational number.

3) The set D is not closed under scalar multiplication

For example: $\underset{\sim}{v} = [x\ y\ z]^T$ and $c = \pi \in \mathbb{R}$.

$\Rightarrow \pi \underset{\sim}{v} = [\pi x\ \ \pi y\ \ \pi z]^T$, since $\pi z \notin \mathbb{Q}$ $\therefore \pi \underset{\sim}{v} \notin D$.

And so, D is not a subspace in $\mathbb{R}^3$.

Exercise 4. We have $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3 \in \mathbb{R}^2$. letting these vectors be column vectors of a matrix A.

$$\Rightarrow A = [\underset{\sim}{v}_1 \;\; \underset{\sim}{v}_2 \;\; \underset{\sim}{v}_3] = \begin{bmatrix} v_{1a} & v_{2a} & v_{3a} \\ v_{1b} & v_{2b} & v_{3b} \end{bmatrix}$$

By the properties of rank, we have that the rank of matrix A is at most $\min(3,2) = 2$.

But from the rank-nullity theorem, we know that

$$\begin{aligned} \text{nullity}(A) &= n - \text{rank}(A) \\ &= 3 - 2 = 1. \end{aligned}$$

That is, the dimension of the subspace of solutions for $A\underset{\sim}{x} = 0$ is at least 1.

$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$ for some $\underset{\sim}{x} \neq \underset{\sim}{0}$.

Therefore, $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3$ are linearly dependent (by definition).

Exercise 5.

a. $\langle x, y\rangle = y_1(x_1 - x_2) + y_2(x_2 - x_1)$

Let us check the three inner product axioms:

1) Bilinearity mapping holds

. First argument: $\langle \underset{\sim}{x} + \underset{\sim}{y}, \underset{\sim}{z}\rangle = \langle \underset{\sim}{x}, \underset{\sim}{z}\rangle + \langle \underset{\sim}{y}, \underset{\sim}{z}\rangle$

$$\begin{aligned}
\langle \underset{\sim}{x} + \underset{\sim}{y}, \underset{\sim}{z}\rangle &= z_1\left[(x_1 + y_1) - (x_2 + y_2)\right] + z_2\left[(x_2 + y_2) - (x_1 + y_1)\right] \\
&= z_1(x_1 + y_1) - z_1(x_2 + y_2) + z_2(x_2 + y_2) - z_2(x_1 + y_1) \\
&= x_1 z_1 + y_1 z_1 - x_2 z_1 - y_2 z_1 + x_2 z_2 + y_2 z_2 - x_1 z_2 - y_1 z_2 \\
&= x_1 z_1 - x_1 z_2 + x_2 z_2 - x_2 z_1 + y_1 z_1 - y_1 z_2 + y_2 z_2 - y_2 z_1 \\
&= x_1(z_1 - z_2) + x_2(z_2 - z_1) + y_1(z_1 - z_2) + y_2(z_2 - z_1) \\
&= \langle \underset{\sim}{x}, \underset{\sim}{z}\rangle + \langle \underset{\sim}{y}, \underset{\sim}{z}\rangle \quad \checkmark
\end{aligned}$$

. Second argument: $\langle \underset{\sim}{x}, \underset{\sim}{y} + \underset{\sim}{z}\rangle = \langle \underset{\sim}{x}, \underset{\sim}{y}\rangle + \langle \underset{\sim}{x}, \underset{\sim}{z}\rangle$

$$\begin{aligned}
\langle \underset{\sim}{x}, \underset{\sim}{y} + \underset{\sim}{z}\rangle &= (y_1 + z_1)(x_1 - x_2) + (y_2 + z_2)(x_2 - x_1) \\
&= y_1(x_1 - x_2) + z_1(x_1 - x_2) + y_2(x_2 - x_1) + z_2(x_2 - x_1) \\
&= y_1(x_1 - x_2) + y_2(x_2 - x_1) + z_1(x_1 - x_2) + z_2(x_2 - x_1) \\
&= \langle \underset{\sim}{x}, \underset{\sim}{y}\rangle + \langle \underset{\sim}{x}, \underset{\sim}{z}\rangle \quad \checkmark
\end{aligned}$$

2) Symmetry holds   We want to prove $\langle \underset{\sim}{x}, \underset{\sim}{y}\rangle = \langle \underset{\sim}{y}, \underset{\sim}{x}\rangle$

$$\begin{aligned}
\langle \underset{\sim}{x}, \underset{\sim}{y}\rangle &= y_1(x_1 - x_2) + y_2(x_2 - x_1) \\
&= x_1 y_1 - x_2 y_1 + x_2 y_2 - x_1 y_2 \\
&= x_1 y_1 - x_1 y_2 + x_2 y_2 - x_2 y_1 \\
&= x_1(y_1 - y_2) + x_2(y_2 - y_1) = \langle \underset{\sim}{y}, \underset{\sim}{x}\rangle. \quad \checkmark
\end{aligned}$$

3) Positive definite $\langle \underset{\sim}{x}, \underset{\sim}{x}\rangle > 0 \ \forall x \setminus \{\underset{\sim}{0}\}$ and $\langle \underset{\sim}{x}, \underset{\sim}{x}\rangle = 0 \Leftrightarrow \underset{\sim}{x} = \underset{\sim}{0}$

$$\begin{aligned}
\langle \underset{\sim}{x}, \underset{\sim}{x}\rangle &= x_1(x_1 - x_2) + x_2(x_2 - x_1) \\
&= x_1^2 - x_1 x_2 + x_2^2 - x_1 x_2 \\
&= x_1^2 - 2x_1 x_2 + x_2^2 \\
&= (x_1 - x_2)^2 > 0 \quad \text{for } \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ such that } x_1 \neq x_2 \quad \times
\end{aligned}$$

$\Rightarrow$ The function does not satisfy positive definite. For example:

Let $\underset{\sim}{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \langle \underset{\sim}{x}, \underset{\sim}{x}\rangle = (2-2)^2 = 0$ where $\underset{\sim}{x} \neq \underset{\sim}{0}$.

As the function above satisfies only 2 axioms, we can conclude that the function is not an inner product.

b, $\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle := x_1 y_1 - x_2 y_2$

This function is not an inner product as it does not satisfy the positive definite property.

$$\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = x_1 x_1 - x_2 x_2 = x_1^2 - x_2^2$$

Let $\underset{\sim}{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = 3^2 - 4^2 = 9 - 16 = -7 < 0.$

Exercise 6.

a. Prove that if $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are linearly dependent vectors, then $|\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle| = \|\underset{\sim}{x}\| . \|\underset{\sim}{y}\|$

Suppose that $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are linearly dependent, that is $\underset{\sim}{x} = k\underset{\sim}{y}$ for some $k \in \mathbb{R}$.

$$\text{LHS} = |\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle| = |\langle k\underset{\sim}{y}, \underset{\sim}{y} \rangle| = |k \langle \underset{\sim}{y}, \underset{\sim}{y} \rangle| = |k| . \|\underset{\sim}{y}\|^2$$

$$\text{RHS} = \|\underset{\sim}{x}\| . \|\underset{\sim}{y}\| = \|k\underset{\sim}{y}\| . \|\underset{\sim}{y}\| = |k| . \|\underset{\sim}{y}\|^2 = \text{LHS}.$$

b. Show that we can retrieve the inner product from the norm via

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \frac{1}{2}\left(\|\underset{\sim}{x} + \underset{\sim}{y}\|^2 - \|\underset{\sim}{x}\|^2 - \|\underset{\sim}{y}\|^2 \right)$$

$$\begin{aligned}
\text{RHS} &= \frac{1}{2}\left(\|\underset{\sim}{x} + \underset{\sim}{y}\|^2 - \|\underset{\sim}{x}\|^2 - \|\underset{\sim}{y}\|^2 \right) \\
&= \frac{1}{2}\left[\langle \underset{\sim}{x} + \underset{\sim}{y}, \underset{\sim}{x} + \underset{\sim}{y} \rangle - \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle - \langle \underset{\sim}{y}, \underset{\sim}{y} \rangle \right] \\
&= \frac{1}{2}\left[\langle \underset{\sim}{x}, \underset{\sim}{x} + \underset{\sim}{y} \rangle + \langle \underset{\sim}{y}, \underset{\sim}{x} + \underset{\sim}{y} \rangle - \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle - \langle \underset{\sim}{y}, \underset{\sim}{y} \rangle \right] \\
&= \frac{1}{2}\left[\langle \underset{\sim}{x}, \underset{\sim}{x} + \underset{\sim}{y} \rangle - \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle + \langle \underset{\sim}{y}, \underset{\sim}{x} + \underset{\sim}{y} \rangle - \langle \underset{\sim}{y}, \underset{\sim}{y} \rangle \right] \\
&= \frac{1}{2}\left[\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle + \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle \right] \\
&= \frac{1}{2} . 2 . \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \text{LHS}. \qquad \square
\end{aligned}$$

c. Show that norm equivalence is an equivalent relation.

Let us prove the three properties:

1. Reflexive : $\|\cdot\|_a \sim \|\cdot\|_a$
2. Symmetric : If $\|\cdot\|_a \sim \|\cdot\|_b$ then $\|\cdot\|_b \sim \|\cdot\|_a$
3. Transitive : If $\|\cdot\|_a \sim \|\cdot\|_b$ and $\|\cdot\|_b \sim \|\cdot\|_c$ then $\|\cdot\|_b \sim \|\cdot\|_c$

. Reflexive

Suppose $\|\cdot\|_a$ is equivalent to $\|\cdot\|_a$, i.e. $\exists M_1, M_2 > 0$ such that

$$M_1 \|\cdot\|_a \leq \|\cdot\|_a \leq M_2 \|\cdot\|_a$$

We see that this is true with $M_1 = M_2 = 1$.

$\Rightarrow$ Norm equivalence is reflexive.

. Symmetric

Suppose $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$, i.e. $\exists M_1, M_2 > 0$ such that

$$M_1 \|\cdot\|_a \leq \|\cdot\|_b \leq M_2 \|\cdot\|_a \qquad (1)$$

then also exists $N_1, N_2 > 0$ such that

$$N_1 \|\cdot\|_b \leq \|\cdot\|_a \leq N_2 \|\cdot\|_b$$

From (1), we can rewrite the inequality on the left

$$M_1 \|\cdot\|_a \leq \|\cdot\|_b \iff \|\cdot\|_a \leq \frac{1}{M_1} \|\cdot\|_b \qquad (2)$$

and the inequality on the right

$$\|\cdot\|_b \leq M_2 \|\cdot\|_a \iff \frac{1}{M_2} \|\cdot\|_b \leq \|\cdot\|_a \qquad (3)$$

From (2) and (3), we found N_1 and N_2

$$\Rightarrow \frac{1}{M_2} \|\cdot\|_b \leq \|\cdot\|_a \leq \frac{1}{M_1} \|\cdot\|_b$$

$\therefore$ Norm equivalence is symmetric.

. Transitive

Assume that $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ and $\|\cdot\|_b$ equivalent to $\|\cdot\|_c$

that is, $\exists M_1, M_2, N_1, N_2 > 0$ such that

$$M_1 \|\cdot\|_a \leq \|\cdot\|_b \leq M_2 \|\cdot\|_a \text{ and,} \qquad (5)$$

$$N_1 \|\cdot\|_b \leq \|\cdot\|_c \leq N_2 \|\cdot\|_b \qquad (6)$$

From (6), we have $\|\cdot\|_b \leq \frac{1}{N_1} \|\cdot\|_c$ and $\frac{1}{N_2} \|\cdot\|_c \leq \|\cdot\|_b$

in combination with (5)

$$M_1 \|\cdot\|_a \leq \frac{1}{N_1} \|\cdot\|_c \iff M_1 . N_1 \|\cdot\|_a \leq \|\cdot\|_c \text{ and}$$

$$\frac{1}{N_2} \|\cdot\|_c \leq M_2 \|\cdot\|_a \iff \|\cdot\|_c \leq M_2 N_2 \|\cdot\|_a$$

$$\Rightarrow M_1 N_1 \|\cdot\|_a \leq \|\cdot\|_c \leq M_2 N_2 \|\cdot\|_a$$

$\therefore$ Norm equivalence is transitive.

d. Assuming $V = \mathbb{R}^2$, show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.

⇒ We want to prove that there exists constants C_1 and C_2 such that

$$C_1 \|\cdot\|_1 \leq \|\cdot\|_2 \leq C_2 \|\cdot\|_1$$

Firstly, we have $\|\underset{\sim}{x}\|_1 = |x_1| + |x_2|$ and $\|\underset{\sim}{x}\|_2 = \sqrt{x_1^2 + x_2^2}$

$$\Rightarrow \|\underset{\sim}{x}\|_1^2 = (|x_1|^2 + |x_2|^2)$$
$$= |x_1|^2 + 2|x_1||x_2| + |x_2|^2$$
$$= x_1^2 + x_2^2 + 2|x_1||x_2|$$

while $\|\underset{\sim}{x}\|_2^2 = x_1^2 + x_2^2$

and so $\|\underset{\sim}{x}\|_1^2 = \|\underset{\sim}{x}\|_2^2 + 2|x_1||x_2|$

$$\|\underset{\sim}{x}\|_1^2 \geq \|\underset{\sim}{x}\|_2^2$$
$$\Rightarrow \|\underset{\sim}{x}\|_1 \geq \|\underset{\sim}{x}\|_2 \qquad (1)$$

Secondly, we have

$$\|\underset{\sim}{x}\|_1 = |x_1| + |x_2|$$

we can write this summation as a dot product between two vectors $\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\Rightarrow \|\underset{\sim}{x}\|_1 = |x_1| + |x_2|$$
$$= \begin{bmatrix} |x_1| & |x_2| \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\leq \left| \begin{bmatrix} |x_1| & |x_2| \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right| \qquad (a \leq \|a\| \ \forall a \in \mathbb{R})$$
$$\leq \left\| \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2, \quad \forall \underset{\sim}{x} \in \mathbb{R}^2 \quad \text{(Cauchy - Schwartz inequality)}$$
$$= \sqrt{(|x_1|^2 + |x_2|^2)} \cdot \sqrt{1^2 + 1^2}$$
$$= \sqrt{x_1^2 + x_2^2} \cdot \sqrt{2} = \sqrt{2} \|\underset{\sim}{x}\|_2 \qquad (2)$$

From (1) and (2), we have

$$\|\underset{\sim}{x}\|_2 \leq \|\underset{\sim}{x}\|_1 \leq \sqrt{2} \|\underset{\sim}{x}\|_2 \quad \forall \underset{\sim}{x} \in \mathbb{R}^2$$

Thus, we can conclude that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Exercise 7.

a. Show that $\underset{\sim}{x} \notin U$

We want to see if $\underset{\sim}{x}$ a linear combination of the vectors that span U, i.e. we want to find scalars such that

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

whose corresponding augmented matrix is $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{array}\right]$

Performing row reduction:

$$\begin{array}{l} \\ R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] \to \text{no solution}$$

$\therefore \underset{\sim}{x} \notin U$.

b) Determine the orthogonal projection $\pi_U(x)$ onto U.

We can first write the spanning vectors as columns of

matrix $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$$B^T B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 17 \end{bmatrix}$$

$$\Rightarrow (B^T B)^{-1} = \frac{1}{3.17 - 7.7} \begin{bmatrix} 17 & -7 \\ -7 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 17 & -7 \\ -7 & 3 \end{bmatrix}$$

$$\Rightarrow (B^T B)^{-1} B^T = \frac{1}{2} \begin{bmatrix} 17 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 3 & -4 \\ -1 & -1 & 2 \end{bmatrix}$$

We can then find the coordinates of the projection with respect to U:

$$\underset{\sim}{\lambda} = (B^T B)^{-1} B^T \underset{\sim}{x} = \frac{1}{2} \begin{bmatrix} 3 & 3 & -4 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -9 \\ 5 \end{bmatrix}$$

$\Rightarrow \pi_u(\underset{\sim}{x}) = B.\underset{\sim}{\lambda} = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -9 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 3 \end{bmatrix}$

To show that $\pi_u(\underset{\sim}{x})$ is a linear combination of the vectors spanning, we once again want to find scalars

$$c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 3 \end{bmatrix}$$

Performing row reduction on the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 1/2 \\ 1 & 2 & 1/2 \\ 1 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 1 & 5/2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 0 & -9 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

yields the system:

$$\begin{cases} 2c_1 = -9 \\ 2c_2 = 5 \end{cases} \Rightarrow c_1 = \frac{-9}{2} \text{ and } c_2 = \frac{5}{2}.$$

The corresponding linear combination is:

$$\frac{-9}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{5}{2}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 3 \end{bmatrix}$$

c) Determine the distance $d(x, u)$

We have $d(x, u) = \|\underset{\sim}{x} - \pi_u(\underset{\sim}{x})\|$

$$\underset{\sim}{x} - \pi_u(\underset{\sim}{x}) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \|\underset{\sim}{x} - \pi_u(\underset{\sim}{x})\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = \frac{1}{\sqrt{2}}.$$