assignment 5

Question 1. A is an invertible matrix.

1. Prove that all the eigenvalues of A are non-zero.

Suppose A has an eigenvalue of zero.

$\Rightarrow \quad A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$\Leftrightarrow \quad (A - \lambda I)\underset{\sim}{x} = 0$

$\Leftrightarrow \quad (A - 0.I)\underset{\sim}{x} = 0 \qquad (\text{assume } \lambda = 0)$

$\Leftrightarrow \quad A\underset{\sim}{x} = 0$

$\Leftrightarrow \quad A^{-1}A\underset{\sim}{x} = 0 \qquad (\text{left-multiply by } A^{-1})$

$\Leftrightarrow \quad \underset{\sim}{x} = 0$

However, from the definition of eigenvalue, the corresponding eigenvector is assumed to be non-zero.

$\Rightarrow$ λ cannot be 0.

By contradiction, we can conclude that all the eigenvalues of A are non-zero.

2. Prove that for any eigenvalue λ of A, λ^{-1} is an eigenvalue of A^{-1}.

$$A \underset{\sim}{x} = \lambda \underset{\sim}{x}$$

$$\Leftrightarrow \quad A^{-1} A \underset{\sim}{x} = A^{-1} \lambda \underset{\sim}{x}$$

$$\Leftrightarrow \quad \underset{\sim}{x} = A^{-1} \lambda \underset{\sim}{x}$$

$$\Leftrightarrow \quad \frac{1}{\lambda} \underset{\sim}{x} = A^{-1} \underset{\sim}{x}$$

$$\Leftrightarrow \quad \lambda^{-1} \underset{\sim}{x} = A^{-1} \underset{\sim}{x}$$

$\therefore$ λ^{-1} is an eigenvalue of A^{-1}.

3. Prove that $\det(A^{-1}) = \dfrac{1}{\det(A)}$

We have $\det(A) = \prod_{i=1}^{n} \lambda_i$

using the fact that if λ is an eigenvalue of A with algebraic multiplicity m, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with algebraic multiplicity m.

$\Rightarrow$ We can say if A has a set of eigenvalue S, then the set of eigenvalues in A^{-1} is the reciprocal of each element in S.

$$\det(A^{-1}) = \prod_{i=1}^{n} \frac{1}{\lambda_i} \quad \text{(where } \lambda_i \text{ are the elements of } S\text{)}$$

$$= \frac{1}{\prod_{i=1}^{n} \lambda_i} = \frac{1}{\det(A)}.$$

Question 2. 1. B a square matrix. λ an eigenvalue of B.
Prove that for all integers $n \geqslant 1$, λ^n is an eigenvalue of B^n.

We will use induction to prove this.

Let $P(n)$ be the proposition that λ is an eigenvalue of B, i.e. $B\underset{\sim}{x} = \lambda \underset{\sim}{x}$.

Base case: With $n = 1$, $B^1 = B$ and $\lambda^1 = \lambda$

$\Rightarrow B^1 \underset{\sim}{x} = B\underset{\sim}{x} = \lambda^1 \underset{\sim}{x} = \lambda x$. (True)

Inductive step.

We assume that $P(k)$ holds for any arbitrary integer $k > 1$.

i.e. $B^k \underset{\sim}{x} = \lambda^k \underset{\sim}{x}$

under this assumption, it must be shown that $P(k+1)$ is true

i.e. $B^{k+1} \underset{\sim}{x} = \lambda^{k+1} \underset{\sim}{x}$

$\Rightarrow$ we left-multiply both sides of $(\star)$ by B

$\Leftrightarrow B . B^k \underset{\sim}{x} = B . \lambda^k \underset{\sim}{x}$

$\Leftrightarrow B^{k+1} \underset{\sim}{x} = \lambda^k B \underset{\sim}{x}$

$\Leftrightarrow B^{k+1} \underset{\sim}{x} = \lambda^k \lambda \underset{\sim}{x}$ (since $B\underset{\sim}{x} = \lambda \underset{\sim}{x}$)

$\Leftrightarrow B^{k+1} \underset{\sim}{x} = \lambda^{k+1} \underset{\sim}{x}$

This shows that $P(k+1)$ holds under the assumption that $P(k)$ is true.

$\therefore$ We conclude that λ^n is an eigenvalue for B^n, $\forall n \geqslant 1$.

2. B is a square matrix.
Prove that B and B^T have the same set of eigenvalues.

Since the eigenvalues of a matrix are the roots of its characteristic polynomial, we'll prove that $p_B(\lambda) = p_{B^T}(\lambda)$

$$p_{B^T}(\lambda) = \det(B^T - \lambda I)$$

$$= \det(B^T - \lambda I^T) \quad (\text{since } I = I^T)$$

$$= \det[(B - \lambda I)^T]$$

$$= \det(B - \lambda I) \quad (\text{since } \det(A) = \det(A^T))$$

$$= p_B(\lambda).$$

Question 3. 1. Prove that the determinant of a square upper triangular matrix U is equal to the product of the diagonal elements.

We will prove by induction.

Base case : $n = 1$, $U : 1 \times 1$ matrix

$\Rightarrow \det(U) = \det(u_{11}) = u_{11}$: True

Inductive step :

We assume that $P(k)$ is true for any arbitrary integer $k > 1$.

i.e. for any $k \times k$ upper triangular matrix , $\det(U) = \prod_{i=1}^{k} u_{ii}$

$\Rightarrow$ Let $P(k+1)$ a $(k+1) \times (k+1)$ upper triangular matrix

$$P(k+1) = \begin{bmatrix} u_{11} & \cdots & u_{1,j} & \cdots & u_{1,k+1} \\ \vdots & \ddots & & & \vdots \\ 0 & & u_{jj} & & u_{j,k+1} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & u_{k+1,k+1} \end{bmatrix}$$

Expand along the new $k+1$ row,

$$\det(P(k+1)) = 0 + \cdots + 0 + (-1)^{k+1+k+1} u_{k+1,k+1} \det(U)$$

(where U is the k, k minor of $P(k+1)$)

$$= (-1)^{2k+2} u_{k+1,k+1} \prod_{i=1}^{k} u_{ii} \quad \text{(from the } P(k) \text{ assumption)}$$

$$= u_{k+1,k+1} \prod_{i=1}^{k} u_{ii} \quad \left(2k+2 \text{ is an even number so } (-1)^{2k+2} = 1 \right)$$

$$= \prod_{i=1}^{k+1} u_{ii}$$

$\therefore$ By induction, we can conclude for any integer $n \geq 1$, the determinant of an $n \times n$ upper triangular matrix is the product of its diagonal element.

2. Prove that the determinant of a square lower triangular matrix U is equal to the product of the diagonal elements.

Using the same settings as above, with $P(k+1)$ a $(k+1) \times (k+1)$ lower triangular matrix

$$P(k+1) = \begin{bmatrix} u_{11} & & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ u_{j,1} & & u_{j,j} & & 0 \\ \vdots & & & \ddots & \vdots \\ u_{k+1,1} & \cdots & u_{k+1,j} & \cdots & u_{k+1,k+1} \end{bmatrix}$$

This time we'll perform cofactor expansion along the last column

$$\Rightarrow \det\left(P(k+1)\right) = 0 + \dots + 0 + (-1)^{k+1+k+1} u_{k+1,k+1} \det(U)$$

$$= (-1)^{2k+2} u_{k+1,k+1} \prod_{i=1}^{k} u_{ii}$$

$$= u_{k+1,k+1} \prod_{i=1}^{k} u_{ii}$$

$$= \prod_{i=1}^{k+1} u_{ii} .$$

$\therefore \det(U) = \prod_{i=1}^{n} u_{ii}$ as claimed.

Question 4. Prove that $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$ are orthogonal.

We have

$$\lambda_1 \underset{\sim}{v_1}^T \underset{\sim}{v_2} = A\underset{\sim}{v_1}^T \underset{\sim}{v_2} = A\underset{\sim}{v_2}^T \underset{\sim}{v_1} = \lambda_2 \underset{\sim}{v_2}^T \underset{\sim}{v_1} = \lambda_2 \underset{\sim}{v_1}^T \underset{\sim}{v_2}$$

(since $A\underset{\sim}{v_1} = \lambda_1 \underset{\sim}{v_1}$) (since $A\underset{\sim}{v_2} = \lambda_2 \underset{\sim}{v_2}$)

$$\Rightarrow \quad \lambda_1 \underset{\sim}{v_1}^T \underset{\sim}{v_2} = \lambda_2 \underset{\sim}{v_1}^T \underset{\sim}{v_2}$$

$$\Leftrightarrow \quad \lambda_1 \underset{\sim}{v_1}^T \underset{\sim}{v_2} - \lambda_2 \underset{\sim}{v_1}^T \underset{\sim}{v_2} = 0$$

$$\Leftrightarrow \quad (\lambda_1 - \lambda_2)\, \underset{\sim}{v_1}^T \underset{\sim}{v_2} = 0$$

since $\lambda_1 \neq \lambda_2 \Rightarrow \lambda_1 - \lambda_2 \neq 0$

$$\Rightarrow \quad \underset{\sim}{v_1}^T \underset{\sim}{v_2} = 0$$

$\therefore$ $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$ are orthogonal. $\square$

Question 5. 1. Prove that B is similar to A

We have A is similar to B

i.e. there exists an invertible matrix P s.t.

$$
\begin{aligned}
& B &&= P^{-1}AP \\
\Leftrightarrow\ & PB &&= PP^{-1}AP \\
\Leftrightarrow\ & PB &&= AP \\
\Leftrightarrow\ & PBP^{-1} &&= APP^{-1} \\
\Leftrightarrow\ & PBP^{-1} &&= A
\end{aligned}
$$

if we let $Q = P^{-1}$ and so $Q^{-1} = P$

then $A = Q^{-1}BQ$

$\therefore$ B is similar to A.

2. Prove that A and B share the same characteristic polynomial.

As we have A is similar to B $\Rightarrow B = P^{-1}AP$ or $A = PBP^{-1}$

The characteristic polynomial of B :

$$
\begin{aligned}
p_B(\lambda) &= \det(P^{-1}AP - \lambda I) \\
&= \det(P^{-1}AP - \lambda P^{-1}P) \\
&= \det[P^{-1}(AP - \lambda P)] && \text{(left distributive law)} \\
&= \det[P^{-1}(A - \lambda I)P] && \text{(right distributive law)} \\
&= \det(P^{-1})\det(A - \lambda I)\det(P)
\end{aligned}
$$

$$= \frac{1}{\det(P)} \det(A - \lambda I) \det(P)$$

$$= \det(A - \lambda I) = p_A(\lambda).$$

Question 6.

1. Compute the eigenvalues of A

We have $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 5 \\ 3 & 4-\lambda \end{vmatrix}$

$$= (2-\lambda)(4-\lambda) - 15$$
$$= \lambda^2 - 6\lambda - 7$$
$$= (\lambda+1)(\lambda-7) \qquad \left[\begin{array}{l} \lambda = -1 \\ \lambda = 7. \end{array} \right.$$

$\therefore$ A has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 7$.

2. Find the eigenspaces

$$E_{-1} = \left(\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \underset{\sim}{x} = \underset{\sim}{0}$$

$$\Leftrightarrow \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

$$\Leftrightarrow \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

with x_2 the free variable, we have the system of equations

$$\begin{cases} 3x_1 + 5x_2 = 0 \\ x_2 = t \end{cases} \Leftrightarrow \begin{cases} 3x_1 = -5t \\ x_2 = t \end{cases}$$

$$\Rightarrow \underset{\sim}{x} = \left\{ \begin{bmatrix} -5/3 \\ 1 \end{bmatrix} t \ , \ t \in \mathbb{R} \right\}$$

$$E_7 = \left(\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right) \underset{\sim}{x} = \underset{\sim}{0}$$

$$\Leftrightarrow \begin{bmatrix} -5 & 5 \\ 3 & -3 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

$$\Leftrightarrow \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

$$\Leftrightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

with x_2 the free variable, we have the system of equations

$$\Leftrightarrow \begin{cases} -x_1 + x_2 = 0 \\ x_2 = t \end{cases} \Leftrightarrow \begin{cases} x_1 = t \\ x_2 = t \end{cases}$$

$$\Leftrightarrow \underset{\sim}{x} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} t \ , \ t \in \mathbb{R} \right\}$$

3. Verify the set of all eigenvectors of A span $\mathbb{R}^2$.

Take $\underset{\sim}{p}_1 = \begin{bmatrix} -5/3 \\ 1 \end{bmatrix}$ and $\underset{\sim}{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{cc|c} -5/3 & 1 & x \\ 1 & 1 & y \end{array} \right]$$

$$-\frac{3}{5} R_1 \to R_1 \quad \left[\begin{array}{cc|c} 1 & -3/5 & -3x/5 \\ 1 & 1 & y \end{array} \right]$$

$$R_2 - R_1 \to R_2 \quad \left[\begin{array}{cc|c} 1 & -3/5 & -3x/5 \\ 0 & 8/5 & y - \frac{3x}{5} \end{array} \right]$$

$$\frac{5}{8}R_2 \rightarrow R_2 \qquad \left[\begin{array}{cc|c} 1 & -3/5 & -3x/5 \\ 0 & 1 & -\frac{3}{8}x + \frac{5}{8}y \end{array}\right]$$

$$R_1 + \frac{3}{5}R_2 \rightarrow R_1 \qquad \left[\begin{array}{cc|c} 1 & 0 & \frac{-3(x-y)}{8} \\ 0 & 1 & \frac{3x+5y}{8} \end{array}\right]$$

$\Rightarrow$ For any choice of a and b, we have

$$\left(\frac{-3(x-y)}{8}\right)\begin{bmatrix} -5 \\ 3 \end{bmatrix} + \left(\frac{3x+5y}{8}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, we have shown that the equation $\begin{bmatrix} -5 \\ 3 \end{bmatrix} a + \begin{bmatrix} 1 \\ 1 \end{bmatrix} b = \begin{bmatrix} x \\ y \end{bmatrix}$

can be solved for a and b.

$$\Rightarrow \mathbb{R}^2 = \text{span}\left(\begin{bmatrix} -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and since any other eigenvector of A is a multiple of either p_1 or p_2.

$\therefore$ The set of all eigenvectors of A spans $\mathbb{R}^2$.

4. Diagonalisable

We choose eigenvectors $\begin{bmatrix} -5/3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues -1 and 7 respectively.

So $A = PDP^{-1}$ with $D = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}$ and $P = \begin{bmatrix} -5/3 & 1 \\ 1 & 1 \end{bmatrix}$

Indeed,

$$PDP^{-1} = \begin{bmatrix} -5/3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix} \left(\begin{bmatrix} -5/3 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1}$$

P is invertible

$$= \begin{bmatrix} -5/3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix} \frac{1}{8} \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

5. Find a closed form formula for A^n for any integer $n \geq 0$.

$$A^n = (PDP^{-1})^n = PD^nP^{-1}$$

$$= \begin{bmatrix} -5/3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 7^n \end{bmatrix} \frac{1}{8} \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -\frac{5}{3}(-1)^n & 7^n \\ (-1)^n & 7^n \end{bmatrix} \begin{bmatrix} -3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 5(-1)^n + 3.7^n & -5(-1)^n + 5.7^n \\ (-3)(-1)^n + 3.7^n & 3(-1)^n + 5.7^n \end{bmatrix}$$