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Research School of Computer Science
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Assignment 5 Theory Questions

COMP3670: Introduction to Machine Learning

Question 1 **Properties of Eigenvalues** (3+5+4=12 credits)

Let $\mathbf{A}$ be an invertible matrix.

1. Prove that all the eigenvalues of $\mathbf{A}$ are non-zero.
2. Prove that for any eigenvalue λ of $\mathbf{A}$, λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.
3. Hence, or otherwise, prove that
$$\det(\mathbf{A}^{-1}) = \frac{1}{\det \mathbf{A}}$$
You may not use the property $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ for this question without proving it.[1]
You may use the following stronger version of Q1.2 without proof:
"If λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity m, then $1/\lambda$ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity m".[2]

Question 2 **Properties of Eigenvalues II** (5+10=15 credits)

1. Let $\mathbf{B}$ be a square matrix. Let λ be an eigenvalue of $\mathbf{B}$.
Prove that for all integers $n \geq 1$, λ^n is an eigenvalue of $\mathbf{B}^n$.
2. Let $\mathbf{B}$ be a square matrix. Prove that $\mathbf{B}$ and $\mathbf{B}^T$ have the same set of eigenvalues.

Question 3 **Properties of Determinants** (15+5=20 credits)

1. Let $\mathbf{U}$ be an square $n \times n$ **upper** triangular matrix. Prove that the determinant of $\mathbf{U}$ is equal to the product of the diagonal elements of $\mathbf{U}$.
2. Let $\mathbf{U}$ be an square $n \times n$ **lower** triangular matrix. Prove that the determinant of $\mathbf{U}$ is equal to the product of the diagonal elements of $\mathbf{U}$.
(Hint: Use the previous exercise to help you.)

Question 4 **Eigenvalues of symmetric matrices** (15 credits)

1. Let $\mathbf{A}$ be a symmetric matrix. Let $\mathbf{v}_1$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ_1, and let $\mathbf{v}_2$ be an eigenvector of $\mathbf{A}$ with eigenvalue λ_2. Assume that $\lambda_1 \neq \lambda_2$. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.
(Hint: Try proving $\lambda_1 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2$. Recall the identity $\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a}$.)

Question 5 **Similar Matrices** (3+15=18 credits)

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices. Assume that $\mathbf{A}$ is similar to $\mathbf{B}$.

[1] The question is trivial with this property, and can be proven without this property.
[2] See `https://piazza.com/class/kcodj1w3jcd6hb?cid=689`

1. Prove that $\mathbf{B}$ is similar to $\mathbf{A}$.
2. Prove that $\mathbf{A}$ and $\mathbf{B}$ share the same characteristic polynomial. (Hint: Note that $\mathbf{I} = \mathbf{P}\mathbf{P}^{-1}$). You may use that property that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

Question 6 **Computations with Eigenvalues** (3+5+4+4+4=20 credits)

Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$.

1. Compute the eigenvalues of $\mathbf{A}$.
2. Find the eigenspace E_λ for each eigenvalue λ.
3. Verify the set of all eigenvectors[3] of $\mathbf{A}$ spans $\mathbb{R}^2$.
4. Hence, find an invertable matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.
5. Hence, or otherwise, find a closed form formula for $\mathbf{A}^n$ for any integer $n \geq 0$.

[3] This used to say "eigenspectra". This was a typo.