Boolean Expressions to ILP Constraints

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January 10, 2018

A is the set of atoms. $L = A \cup \{\neg p \mid p \in A\}$ is the set of literals. $C \subset L$ is a clause. For boolean expression B: Transform B to CNF. For each clause $C \in B$, add a constraint

$$\left(\sum_{p \in C \cap A} v_p \right) - \left(\sum_{\neg p \in C \setminus A} v_p \right) \ \geq \ 1 - |C \setminus A|$$

1 Intuition

Binary variable $p \equiv \top$ iff

$$p \geq 1 \tag{1}$$

Negated binary variable $\neg q \equiv \top$ iff

$$(1 - q) \geq 1 \tag{2}$$

Disjunction of variables $p_1 \vee \cdots \vee p_n \equiv \top$ iff

$$\sum_{1 \leq i \leq n} p_i \geq 1 \tag{3}$$

Disjunction of negated variables $\neg q_1 \vee \cdots \vee \neg q_m \equiv \top$ iff

$$(1 - q_1) + \cdots + (1 - q_m) \geq 1 \tag{4}$$

$$m \cdot 1 - \sum_{1 \leq j \leq m} q_j \geq 1 \tag{5}$$

$$- \sum_{1 \leq j \leq m} q_j \geq 1 - m \tag{6}$$

Disjunction of literals $p_1 \vee \cdots \vee p_n \vee \neg q_1 \vee \cdots \vee \neg q_m \equiv \top$ iff

$$\sum_{1 \leq i \leq n} p_i - \sum_{1 \leq j \leq m} q_j \geq 1 - m \tag{7}$$

2 Conjunction and Disjunction Literals over Sets of Atoms

Let X be a set of atoms x. We allow the following "generalized literals" to occur in boolean expressions.

$$\bigwedge_{x \in X} x$$
$$\bigvee_{x \in X} x$$
$$\neg \bigwedge_{x \in X} x$$
$$\neg \bigvee_{x \in X} x$$

In constraints they translate as

$$\begin{array}{rcl}
\bigvee_{x \in X} x & \leadsto & \ldots + \sum_{x \in X} x \; \ldots \geq \ldots \\
\bigwedge_{x \in X} x & \leadsto & \forall x \in X \; : \; \ldots + x \; \ldots \geq \ldots \\
\neg \bigwedge_{x \in X} x \equiv \bigvee_{x \in X} \neg x & \leadsto & \ldots - \sum_{x \in X} x \; \ldots \geq \ldots - |X| \; \ldots \\
\neg \bigvee_{x \in X} x \equiv \bigwedge_{x \in X} \neg x & \leadsto & \forall x \in X \; : \; \ldots - x \; \ldots \geq \ldots - 1 \; \ldots
\end{array}$$