Linearization

Subject: Developing a linear model, linearization technique to approximate nonlinear model.

Lecture notes

In this lecture we will present a method by which you can use the nonlinear model to develop a linear model which approximate the nonlinear model in a region near an operating point.

The reasons such a method is relevant are:

- Very often models of physical systems turn out to be nonlinear.
- It can be difficult to analyze nonlinear systems directly.
- Methods to analyze linear systems are well developed.
- Design methods for designing controllers for nonlinear systems are not fully developed and are often difficult to use.
- A large variety methods and software tools (MatLab) for control design and analysis are based on the use of linear models.

Remember that a linear system is a system, where the output (response), $y(t)$ of the model depends linearly of the input (excitation), $u(t)$ of the model. This is the case if the principle of superposition and the principle of proportionality holds. By the principle of superposition if $[u_1(t), y_1(t)]$ and $[u_2(t), y_2(t)]$ are input output pairs, then if the input were $u_1(t) + u_2(t)$ then the response of the system would be $y_1(t) + y_2(t)$. By the proportionality principle if the input were $C_1u_1(t)$ the response would be $C_1y_1(t)$. Superposition and proportionality implies that the response of $C_1u_1(t) + C_2u_2(t)$ will be $C_1y_1(t) + C_2y_2(t)$

Systems which have these properties are:

Amplification (multiplication) with a constant gain $y(t) = Ku(t)$:

$$K(C_1u_1(t) + C_2u_2(t)) = C_1Ku_1(t) + C_2Ku_2(t)$$

Differentiation $y(t) = \frac{du(t)}{dt}$:

$$\frac{d(C_1u_1(t) + C_2u_2(t))}{dt} = C_1\frac{du_1(t)}{dt} + C_2\frac{du_2(t)}{dt}$$

Integration $y(t) = \int_0^t u(t)dt$:

$$\int_0^t (C_1u_1(t) + C_2u_2(t))dt = C_1\int_0^t u_1(t)dt + C_2\int_0^t u_2(t)dt$$

A constant delay $y(t) = D(u(t)) \equiv u(t - t_d)$

$$D(C_1u_1(t) + C_1u_2(t)) = C_1u_1(t - t_d) + C_2u_2(t - t_d)) = C_1D(u_1(t)) + C_2D(u_2(t))$$

A system described by an nth order differential equation with constant coefficients

$$a_n \frac{dy^n(t)}{dt^n} + a_{n-1} \frac{dy^{n-1}(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_n \frac{du^n(t)}{dt^n} + b_{n-1} \frac{du^{n-1}(t)}{dt^{n-1}} + \cdots + b_0 u(t)$$

is linear, which is easily seen by verifying that if $[u_1(t), y_1(t)]$ and $[u_2(t), y_2(t)]$ are input output pairs each satisfying the equation, then the equation will also be satisfied with $u(t) = C_1 u_1(t) + C_2 u_2(t)$ and $y(t) = C_1 y_1(t) + C_2 y_2(t)$.

On the other hand if a constant term c is added on either side of the equation the system is nonlinear.

The concept of linear systems is also used for systems with multiple inputs and outputs, and any linear combination of signals (inputs or outputs or internal signals) differentiated, integrated or delayed with respect to time result in a linear system.

As an example of a linear system consider a rotating system with inertia J, angular velocity ω and angle θ. Assume that the inertia is affected be a input torque $T(t)$ and a friction torque $-b\omega(t)$. Furthermore it is connected to another rotating system with angle $\theta_1(t)$ through a spring giving an additional torque $c(\theta_1(t) - \theta(t)$. This gives us the following equations to describe the system

$$J \frac{d\omega(t)}{dt} = T(t) - b\omega(t) - c\theta(t) + c\theta_1(t)$$

$$\theta(t) = \int_0^t \omega(t) dt$$

The angle of the other system $\theta_1(t)$ may be seen as second input to the system. Note that the second equation may be replaced by

$$\frac{d\theta(t)}{dt} = \omega(t)$$

such that a description containing both terms with integration and terms with differentiation may be replaced by a description with only differentiation. You are also free to choose whether you prefer to use a number of first order differential equations or you want to combine them to one (or generally a smaller number) of higher order differential equations, in this case

$$J \frac{d\theta^2(t)}{dt^2} + b \frac{d\theta(t)}{dt} + c\theta(t) = T(t) + c\theta_1(t)$$

Note also that output of the system at a certain time t is determined by the initial condition of the system (say the condition at $t = 0$) and the profile of the inputs over time from 0 to t.

Any other operation of signals (and their differentiated, integrated or delayed versions) than addition and multiplication with a constant result in nonlinear systems.

As an example of a nonlinear system let us consider a vehicle where the drag from air friction force in a certain region of the velocity v can be assumed to be $F_f = \frac{\rho A c_w}{2} v^2$, where the density of air ρ the area A and the coefficient c_w may be regarded as constants. Let us assume that we can apply a force $F(t)$ which origins from the engine and affects the vehicle through interaction with the ground. Now a simple model of the velocity is

$$M\frac{dv(t)}{dt} = F(t) - F_f = F(t) - \frac{\rho A c_w}{2} v(t)^2$$

Since the velocity appears squared in the equation the system is nonlinear. Other examples of nonlinear functions which often appear in mechanical systems are trigonometric functions which often appear where translational movements and forces are present in systems which also have rotational movement. In thermal and chemical systems products of signals are typical. For instance the product of flow and temperature appears in an expression for thermal power in a stream. A familiar example of nonlinear behavior in electronic systems is the exponential relation between voltage and current through a semiconductor junction.

In the linearization approach a nonlinear element law which can be expressed as an algebraic relation $z = f(x)$ (for instance drag friction force as a function of velocity) is approximated near an operating point using a straight line. We write the variable x as the sum of a constant $\bar{x}$ which is the value it has in the operating point and an incremental value $\hat{x}(t)$

$$x(t) = \bar{x} + \hat{x}(t)$$

Likewise

$$z(t) = \bar{z} + \hat{z}(t)$$

Since the operating points is a point on the curve $z = f(x)$ they are related by

$$\bar{z} = f(\bar{x})$$

Now we want to introduce a linear approximation which relates the incremental variables $\hat{x}(t)$ and $\hat{z}(t)$ like

$$\hat{z}(t) \approx k\hat{x}(t)$$

The value of k may be found using a graphical approach were a tangent to the curve is used to approximate $f(x)$ for x close to $\bar{x}$. The slope of the tangent is

$$k = \left.\frac{df}{dx}\right|_{\bar{x}}$$

where the subscript $\bar{x}$ indicate that the derivative is calculated at $x = \bar{x}$. The function may be approximated using the expression for the tangent:

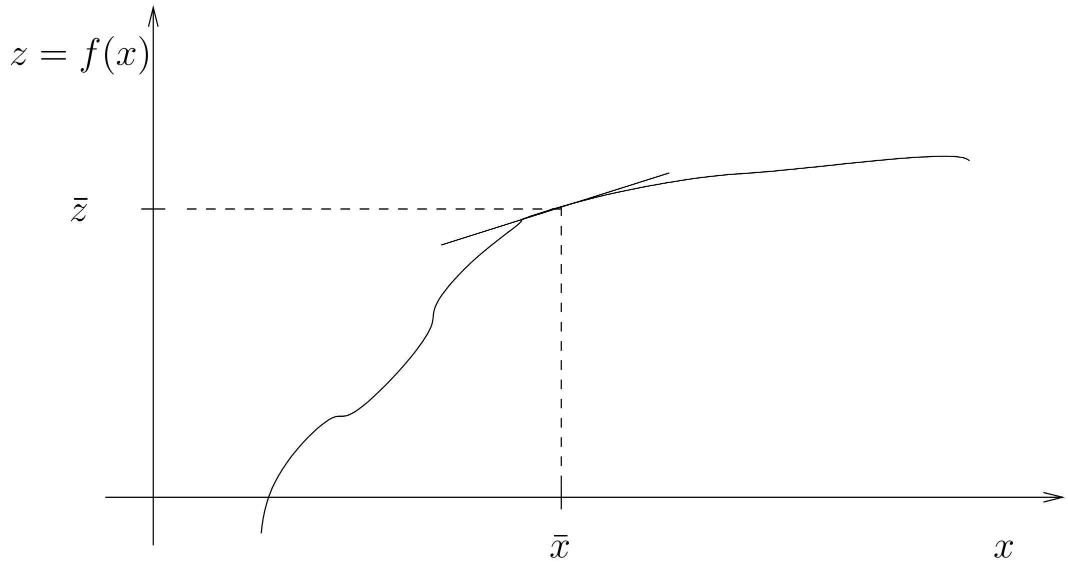
$$f(x) \approx f(\bar{x}) + k(x - \bar{x})$$

Now the function value at $x = \bar{x} + \hat{x}$ is approximated by

$$f(\bar{x} + \hat{x}) \approx f(\bar{x}) + k\hat{x}$$

or

$$\hat{z} = z - \bar{z} = f(x) - f(\bar{x}) \approx k\hat{x}$$



An equivalent way to express the approximation is to introduce a Taylor series expansion of $f(x)$ about the operating point $(\bar{x}, f(\bar{x})$:

$$f(x) = f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} (x - \bar{x}) + \frac{1}{2!} \left.\frac{d^2 f}{dx^2}\right|_{\bar{x}} (x - \bar{x})^2 + \cdots$$

We seek a linear approximation to

$$\hat{z} = f(x) - f(\bar{x})$$

and therefore we truncate the Taylor series after the first order term giving

$$\bar{z} + \hat{z} = f(\bar{x} + \hat{x}) \approx f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} \hat{x}$$

giving the relation we have seen earlier

$$\hat{z} = f(\bar{x} + \hat{x}) - f(\bar{x}) \approx \left.\frac{df}{dx}\right|_{\bar{x}} \hat{x}$$

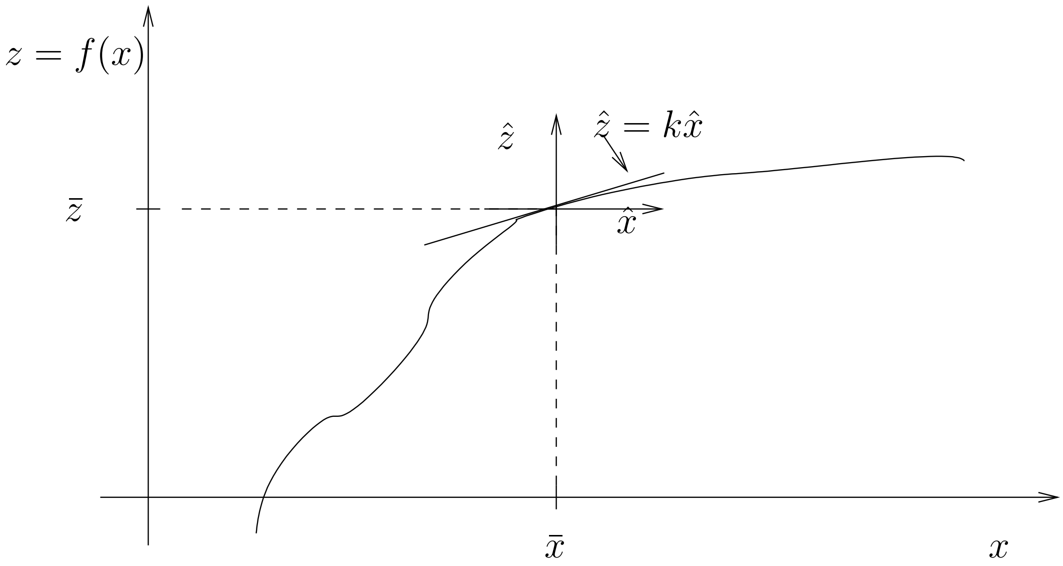
The introduction of incremental variables is a change of coordinates system such that the origin of the new coordinate system is the operating point.

Using these ideas for the air friction force and introducing $v = \bar{v} + \hat{v}$ and $F_f = \bar{F}_f + \hat{F}_f$ the friction force may be approximated like

$$\bar{F}_f + \hat{F}_f \approx F_f(\bar{v}) + \left.\frac{dF_f}{dv}\right|_{\bar{v}} \hat{v} = \frac{\rho A c_w}{2} \bar{v}^2 + \rho A c_w \bar{v} \hat{v}$$

The relation between the incremental variables is

$$\hat{F}_f \approx \rho A c_w \bar{v} \hat{v}$$



If we introduce variables as operating point value plus incremental value in the model for velocity and the above approximation of the friction force in the model for the velocity we obtain

$$M\frac{d(\bar{v}+\hat{v})}{dt} = \bar{F} + \hat{F} - \frac{\rho A c_w}{2}\bar{v}^2 - \rho A c_w \bar{v}\hat{v}$$

We still need to determine the value of the operating point. We will usually require that the system can be in the operating point in steady state, such that the model equation is fulfilled when we put the incremental values to zero. Since

$$\frac{d\bar{v}}{dt} = 0$$

we get the equation

$$0 = \bar{F} - \frac{\rho A c_w}{2}\bar{v}^2$$

This is an algebraic equation which we need to solve to obtain the operating point. The steady operating point equation may further be subtracted from the dynamic equation to obtain

$$M\frac{d\hat{v}}{dt} = \hat{F} - \rho A c_w \bar{v}\hat{v}$$

The nonlinear model will usually consist of one or more differential equations, such that the linearization procedure will involve making linear approximations for every nonlinear term in the equations. If a term is a function of more than one variable the linearization is done using the Taylor expansion for functions of more than one variable. For instance if a term is a function $g(y,z,u)$ of the model variables y and z and the input u, the linear Taylor approximation of this term around an operating point will be

$$g(\bar{y}+\hat{y}, \bar{z}+\hat{z}, \bar{u}+\hat{u}) = g(\bar{y},\bar{z},\bar{u}) + \frac{\partial g}{\partial y}_{\bar{y},\bar{z},\bar{u}}\hat{y} + \frac{\partial g}{\partial z}_{\bar{y},\bar{z},\bar{u}}\hat{z} + \frac{\partial g}{\partial u}_{\bar{y},\bar{z},\bar{u}}\hat{u}$$

Occasionally derivatives of variables appear in nonlinear terms in model differential equations. For instance mechanical equations often involve a nonlinear moment of inertia or mass like

$$M(x)\frac{dv}{dt} = M(x)a$$

This can be approximated like

$$M(\bar{x}+\hat{x})(\bar{a}+\hat{a}) \approx M(\bar{x})\bar{a} + \frac{\partial M}{\partial x}_{\hat{x}}\bar{a}\hat{x} + M(\bar{x})\hat{a} = M(\bar{x})\hat{a}$$

Where the last equality is from the requirement that the operation point is an equilibrium point such that $\bar{a} = 0$

When we want to solve the incremental model equations we need to insert initial values for the incremental variables. These may of course be determined from the corresponding initial values of the original variables by subtracting the nominal values from the operating point.

The steps we need to take to find a linearized model in incremental variables are

1. Determine the operating point of the model by solving the steady state nonlinear algebraic model equations.

2. Rewrite all linear terms in the mathematical model as the sum of their nominal operating point values and incremental variables, noting that the derivatives of constant terms are zero.

3. Replace all nonlinear terms with zero'th an first order terms of their Taylor series expansions. The Taylor series is developed around the operating point, and will include constant terms expressed in operating point variables and linear terms in incremental variables.

4. Use the algebraic equation(s) defining the operating point to cancel the constant terms in the differential equation leaving only the linear terms involving incremental variables.

5. Determine the initial conditions of all incremental variables from the initial conditions of the variables in the nonlinear model.

This method gives the possibility to derive a linear model which approximate the nonlinear model in a region near the operating point.

This also gives the possibility to use the model to design a controller for which the performance can be predicted in the region where the model is valid.

If the operating point value of the control input can be expressed in terms of the reference and the disturbance this give the possibility to use this as a feedforward.

Sometimes it is possible to express control parameters directly in terms of operating point values.

The method is rather straightforward (although sometimes cumbersome) to use. It has some shortcomings

- If the nonlinearities have large curvature the valid region of the model becomes narrow and it may be necessary to use many models to cover the entire operating range of the system.

- Stability can not always be guaranteed if controller parameters depending of operating point change rapidly
- The method does not give answer to cases with discontinuous nonlinearities like Coulomb friction and stiction .