Systems of systems

The Hamilton-Jacobi-Bellman Equation and Pontryagin's Maximum Principle

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Example

We consider the state (or control) equation

$$\dot{x}(t) = f(x(t), u(t), t), \;\; x(t_0) = x_0, \;\; t \in [t_0, T]$$

with

$$\begin{aligned} u &= u(t) \in \mathbb{R}^m, \quad \text{control variable (or trajectory)} \\ x &= x(t) = x(t; t_0, x_0, u) \in \mathbb{R}^n, \quad \text{state variable (or trajectory)} \\ f &: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n \quad \text{continuous differentiable (vector field).} \end{aligned}$$

The objective function (or cost functional)

$$J_{t_0}(u) = \int_{t_0}^{T} F(x(t), u(t), t)dt + S(x(T), T)$$

with continuous differentiable functions

$$F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$$
$$S : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

The optimal control problem in Bolza form

$$\max_u J_{t_0}(u)$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \;\; x(t_0) = x_0, \;\; t \in [t_0, T]$$
$$(x(T), T) \in B$$

with $B \subset \mathbb{R}^n \times \mathbb{R}$ a closed (target) set.

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$$\max_u J_{t_0}(u)$$

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$$(x(T), T) \in B$$

with $B \subset \mathbb{R}^n \times \mathbb{R}$ a closed (target) set.

- Assume that we are only allowed to use control values which lie in the (compact) set $\Omega \subset \mathbb{R}^m$. A control u is called admissible if

  u is piecewise continuous, $u(t) \in \Omega$ for all $t \in [t_0, T]$ and $(x(T; t_0, x_0, u), T) \in B$.

  Assume that any (x_0, t_0) can be transferred to B by an admissible control

The optimal control problem in Bolza form

$$\max_{u} J_{t_0}(u)$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \;\; x(t_0) = x_0, \;\; t \in [t_0, T]$$

$$(x(T), T) \in B$$

with $B \subset \mathbb{R}^n \times \mathbb{R}$ a closed (target) set.

- $$\max_{u} = \max_{u=u(t) \text{ admissible}}$$

The optimal control problem in Bolza form

$$\max_u J_{t_0}(u)$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \;\; x(t_0) = x_0, \;\; t \in [t_0, T]$$
$$(x(T), T) \in B$$

with $B \subset \mathbb{R}^n \times \mathbb{R}$ a closed (target) set.

- A solution $u^* = u^*(t)$ to the optimal control problem is called optimal

The optimal control problem in Bolza form

$$\max_u J_{t_0}(u)$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \; x(t_0) = x_0, \; t \in [t_0, T]$$

$$(x(T), T) \in B$$

with $B \subset \mathbb{R}^n \times \mathbb{R}$ a closed (target) set.

- Notation: when B is not explicit specified we assume; T fixed and $x(T)$ free (equivalently, B is implicit set to $\mathbb{R}^n \times \{T\}$)

Example

The optimal control problem in Bolza form

$$\max_u J_0(u) = \int_0^T F(x(t), u(t), t)dt + S(x(T), T),$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \;\; x(0) = x_0 \;\; t \in [0, T].$$

- The linear-quadratic problem:

$$\max \rightsquigarrow \min$$

$$F(x, u, t) = F(x, u) = x'Qx + u'Ru, \quad Q \geq 0, \;\; R > 0,$$

$$S(x, t) = S(x) = x'Gx, \quad G \geq 0,$$

$$f(x, u, t) = f(x, u) = Ax + Bu$$

Example

The optimal control problem in Bolza form

$$\max_u J_0(u) = \int_0^T F(x(t), u(t), t)dt + S(x(T), T),$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t), \; x(0) = x_0 \; t \in [0, T].$$

- Application:

$$\begin{aligned} F(x,u,t) &= \text{instantaneous profit}, \\ S(x,t) &= \text{operation cost } (\leq), \\ f(x,u,t) &= \text{dynamics of a power plant fuel system}, \\ & \quad u \text{ fuel flow reference (coal, gas and oil)}, \\ & \quad x \text{ actual fuel flow}. \end{aligned}$$

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(B) Bolza form

$$\max_u J_0(u), \qquad J_0(u) = \int_0^T F(x,u,t)dt + S(x(T),T),$$

subject to

$$\dot{x}(t) = f(x(t),u(t),t), \;\; x(0) = x_0, \;\; t \in [0,T].$$

(L) Lagrange form if $S = 0$.

(M) Mayer form if $F = 0$.

They are all equivalent. However, it may be preferable to use one particular form or the other.

Clearly $(L),(M) \subset (B)$

(B) Bolza form

$$\max_u J_0(u), \qquad J_0(u) = \int_0^T F(x,u,t)dt + S(x(T),T),$$

subject to

$$\dot{x}(t) = f(x(t),u(t),t),\ x(0) = x_0,\ t \in [0,T].$$

(L) Lagrange form if $S = 0$.

(M) Mayer form if $F = 0$.

To prove $(B) \subset (M)$ we introduce an additional state $\bar{x} \in \mathbb{R}$ by

$$\dot{\bar{x}}(t) = F(x(t),u(t),t),\ \bar{x}(0) = 0,\ t \in [0,T].$$

Then with the new state $z = (\bar{x}, x)$ we obtain the Mayer form

$$\max_u J_0(u),$$

subject to

$$\dot{z}(t) = \bar{F}(x(t),u(t),t),\ z(0) = z_0,\ t \in [0,T].$$

with

$$J_0(u) = \int_0^T F(x,u,t)dt + S(x(T),T) = \bar{x}(T) + S(x(T),T) = \bar{S}(z(T),T),$$

$$\bar{F} = (F,f), \quad z_0 = (0,x_0).$$

The property of being optimal is a “global” property (it applies to the “whole” curve). However, it is also a local property:

> *Principle of optimality: An optimal policy has the property that, whatever the initial state and decision are, the remaining decision must constitute an optimal policy with regards to the outcome resulting from the first decision.*

We embed the optimal control problem in a family of optimal control problems parametrized by initial data, by means of the value function $V = V(x,t)$

$$
\begin{aligned}
V(x,t) = \max_u J_t(u) \\
&\text{subject to} \\
&\quad \dot{x}(s) = f(x(s), u(s), s), \;\; x(t) = x, \;\; s \in [t, T] \\
&\quad (x(T), T) \in B
\end{aligned}
$$

Note that $V(x,T) = S(x(T),T) = S(x,T)$ for all $(x,T) \in B$ since

$$
J_t(u) = \int_t^T F(x(s), u(s), s)ds + S(x(T), T),
$$

We will derive (heuristically) an equation whose solution is the value function $V = V(x, t)$.

Let u^* be optimal on $[0, T]$, $t \in [0, T]$ and $v \in \Omega$. We will construct a control as follows: Apply u^* on $[0, t]$ and then

$$\tilde{u}(s) = \begin{cases} v & t < s \leq t + dt, \\ \hat{u}(s) & t + dt < s \leq T \end{cases}$$

where $\hat{u}(s)$ is optimal from $(t + dt, \tilde{x}(t + dt))$.

$$\begin{aligned} V(x^*(t), t) &= J_t(u^*) \geq J_t(\tilde{u}) \\ &= \int_t^{t+dt} F(\tilde{x}(s), v, s)ds + V(\tilde{x}(t + dt), t + dt) \\ &\approx F(x^*(t), v, t)dt + V(x^*(t), t) + (V_x(x^*(t), t) \cdot \dot{x}(t) + V_t(x^*(t), t))dt + o(dt), \end{aligned}$$

hence

$$\begin{aligned} 0 \geq& F(x^*(t), v, t)dt + (V_x(x^*(t), t) \cdot f(x^*(t), v, t) + V_t(x^*(t), t))dt + o(dt), \\ 0 \geq& F(x^*(t), v, t) + V_x(x^*(t), t) \cdot f(x^*(t), v, t) + V_t(x^*(t), t)), \\ 0 \geq& F(x, v, t) + V_x(x, t) \cdot f(x, v, t) + V_t(x, t). \end{aligned}$$

From above we conclude that for $(x, t) \in \mathbb{R}^n \times [0, T]$

$$0 \geq F(x, u, t) + V_x(x, t) \cdot f(x, u, t) + V_t(x, t), \quad \text{for all } u \in \Omega,$$

with equality if we had used u^* on all of $[0, T]$.

Definition

The Hamilton-Jacobi-Bellman equation (HJBe) is

$$0 = \max_u H(x, u, V_x(x, t), t) + V_t(x, t)$$

or

$$-V_t(x, t) = \max_u H(x, u, V_x(x, t), t)$$

with the Hamiltonian H given by

$$\begin{aligned} H :& \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}; \\ & (x, u, \lambda, t) \mapsto H(x, u, \lambda, t) = F(x, u, t) + \lambda \cdot f(x, u, t) \end{aligned}$$

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$$H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R};$$
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- The HJBe is a (strongly) nonlinear partial differential equation

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- The variable $\lambda \in \mathbb{R}^n$ is called the adjoint vector (also called a costate vector or (time-varying) Lagrange multiplier vector)

Theorem

Assume that the value function $V = V(x,t)$ is continuously differentiable on the compliment of B.

- *The value function $V = V(x,t)$ satisfy the HJBe*

$$0 = \max_u H(x, u, V_x(x,t), t) + V_t(x,t)$$

- *There exist an optimal $u^*(t)$ iff*

$$u^*(t) \in \arg\max_u H(x^*(t), u, V_x(x^*(t), t), t)$$

or equivalently

$$\begin{aligned} &H(x^*(t), u^*(t), V_x(x^*(t), t), t) + V_t(x^*(t), t) \\ &\qquad = \max_u H(x^*(t), u, V_x(x^*(t), t), t) + V_t(x^*(t), t) = 0 \end{aligned}$$

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- One may hope to recover the value function V as the unique solution to the HJBe with boundary condition $V = S$ on B

How to apply the HJBe

- Solve the HJBe.
- Use V to verify a guess of optimal u.

Or

- Guess an “optimal” u^*.
- Set $V^*(x,t) = J_t(u^*)$.
- Check if V^* satisfy the HJBe.

Or

use the classical synthesis procedure

Cost

$$J_0(u) = \int_0^T \Big(x(t)' Q x(t) + u(t)' R u(t) \Big) \, dt + x(T)' G x(T).$$

Bolza problem

$$\min_u J_0(u) \qquad ('' = '' \max_u -J_0(u)),$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \; x(0) = x_0, \; t \in [0, T].$$

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$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0,\ t \in [0, T].$$

We guess that $V(x,t) = -x'P(t)x$ for some symmetric $P = P(t)$ with $P(T) = G$. Then

$$\begin{aligned} 0 &= \max_u \{H(x, u, V_x(x,t), t)\} + V_t(x,t) \\ &= \max_u \{-x'Qx - u'Ru - 2x'P(Ax + Bu)\} - x'\dot{P}x \\ &= -x'Qx - x'PBR^{-1}B'Px - 2x'P(Ax - BR^{-1}B'Px) - x'\dot{P}x \end{aligned}$$

Hence if P satisfy the matrix ODE

$$\dot{P} = -Q + PBR^{-1}B'P - PA - A'P, \quad P(T) = G$$

then V is the value function and

$$u^*(t) = p(x^*(t), t) = -R^{-1}B'P(t)x^*(t)$$

is an optimal (feedback) solution.

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then V is the value function and

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is an optimal (feedback) solution. ARE YOU SURE ???

we need a verification theorem

The construction of the optimal control by means of the value function in the LQ case can be generalized in terms of the following (classical) synthesis procedure for a continuously differentiable value function.

Define the set-valued map

$$U^*(x,t) = \arg\max_u H(x,u,V_x(x,t),t)$$

choose admissible[1] feedback control policy

$$p(x,t) \in U^*(x,t)$$

and let $u^*(t) = p(x^*(t),t)$ with $x^*(t)$ the solution to

$$\dot{x}^*(t) = f(x^*(t), p(x^*(t),t), t) \tag{1}$$

It follows that $u^*(t)$ is optimal.

[1] meaning (1) has a unique solution $x(t)$ and $u(t) = p(x(t),t)$ is admissible

In general it is a hard problem to establish the existence of an admissible feedback control policy[2]. In particular, we mention the following difficulties

- The value function was assumed continuously differentiable
- In general a selection theorem is need in order to choose an admissible feedback control policy (e.g examples show that $p(x, t)$ is not continuous)

[2] see e.g a theorem of Boltyanskii'

Instead of using the HJBe one may use the (Pontryagin) maximum principle. It gives necessary conditions and consists of

- The Hamiltonian maximizing condition.
- The adjoint equation

Since the maximum of the HJBe

$$0 = \max_u \{H(x, u, V_x(x,t), t)\} + V_t(x,t), \;\; V(x,T) = S(x,T),$$

is attained by an optimal solution $(u^*(t), x^*(t))$, we have

$$\begin{aligned} &H(x^*(t), u^*(t), V_x(x^*(t), t), t) + V_t(x^*(t), t) \\ &\qquad \geq H(x^*(t), u, V_x(x^*(t), t), t) + V_t(x^*(t), t), \quad \text{for all } u \in \Omega, \end{aligned}$$

or equivalently (the Hamiltonian maximizing condition)

$$\begin{aligned} &H(x^*(t), u^*(t), \lambda(t), t) \geq H(x^*(t), u, \lambda(t), t), \quad \text{for all } u \in \Omega, \\ &H(x^*(t), u^*(t), \lambda(t), t) = \max_u H(x^*(t), u, \lambda(t), t), \end{aligned}$$

with $\lambda(t) = V_x(x^*(t), t)$.
This gives an implicit pointwise determination of $u^*(t)$. However, it is not of much use if we do not know a simplere expression for

$$\lambda(t) = V_x(x^*(t), t)$$

Can we write the adjoint variable as a solution of an ODE?

$$\dot{\lambda} = \dot{V}_x = V_{xx} \cdot \dot{x} + V_{xt} = V_{xx} \cdot f + V_{tx}.$$

Moreover, by the HJBe we conclude that

$$\begin{aligned} 0 &= H_x(x^*(t), u^*(t), V_x(x^*(t), t), t) + V_{tx}(x^*(t), t) \\ &= F_x + V_{xx} \cdot f + V_x \cdot f_x + V_{tx}. \end{aligned}$$

Hence

$$\begin{aligned} \dot{V}_x(x^*(t), t) &= -F_x(x^*(t), u^*(t), t) - V_x(x^*(t), t) \cdot f_x(x^*(t), u^*(t), t), \\ \dot{\lambda}(t) &= -F_x(x^*(t), u^*(t), t) - \lambda(t) \cdot f_x(x^*(t), u^*(t), t). \end{aligned}$$

Or using the Hamiltonian we obtain (the adjoint equation)

$$\dot{\lambda}(t) = -H_x(x^*(t), u^*(t), \lambda(t), t), \;\; \lambda(T) = S_x(x^*(T), T).$$

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Summarizing we have.

The maximum principle: Let u^* be optimal for the Bolza problem

$$\max_u J_0(u), \qquad J_0(u) = \int_0^T F(x,u,t)dt + S(x(T),T),$$

subject to

$$\dot{x}(t) = f(x(t),u(t),t), \;\; x(0) = x_0, \;\; t \in [0,T].$$

Then

- The Hamiltonian maximizing condition.

$$H(x^*(t),u^*(t),\lambda(t),t) = \max_u H(x^*(t),u,\lambda(t),t),t),$$

- The adjoint equation

$$\dot{\lambda}(t) = -H_x(x^*(t),u^*(t),\lambda(t),t), \;\; \lambda(T) = S_x(x^*(T),T).$$

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- The adjoint equation

$$\dot{\lambda}(t) = -H_x(x^*(t),u^*(t),\lambda(t),t), \;\; \lambda(T) = S_x(x^*(T),T).$$

Two-point boundary value problem

$$\dot{\lambda}(t) = -H_x(x^*(t),u^*(t),\lambda(t),t), \;\; \lambda(T) = S_x(x^*(T),T)$$
$$\dot{x}^*(t) = H_\lambda(x^*(t),u^*(t),\lambda(t),t), \;\; x(0) = x_0.$$

The maximum principle only give necessary conditions, as shown by

$$
\begin{aligned}
&\max_u x_2(T) \\
&\text{subject to} \\
&(\dot{x}_1, \dot{x}_2) = (u, x_1^2) \; x(0) = 0 \\
&u \in \Omega = [-1, 1]
\end{aligned}
$$

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With $u^*(t) = 0$ we find $x^*(t) = 0$ for $t \in [0, T]$. Moreover, the adjoint equation is

$$(\dot{\lambda}_1, \dot{\lambda}_2) = (-2\lambda_2 x_1, 0) \; \lambda(T) = (0, 1)$$

giving $\lambda(t) = (0, 1)$. These data satisfy the maximum principle.

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$$(\dot{\lambda}_1, \dot{\lambda}_2) = (-2\lambda_2 x_1, 0) \; \lambda(T) = (0, 1)$$

giving $\lambda(t) = (0, 1)$. These data satisfy the maximum principle. However, any control $u \neq u^*$ give

$$x_2(T) = \int_0^T \left(\int_0^s u(\tau) d\tau \right)^2 ds > x_2^*(T)$$

Cost

$$J_0(u) = \int_0^T (x(t)'Qx(t) + u(t)'Ru(t))dt + x(T)'Gx(T).$$

Bolza problem

$$\min_u J_0(u) \qquad ('' = '' \max_u -J_0(u)),$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0,\ t \in [0, T].$$

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Hamiltonian

$$H(x, u, \lambda, t) = -x'Qx - u'Ru + \lambda'(Ax + Bu).$$

The Hamiltonian maximizing condition yields

$$\max_u\{-u'Ru + \lambda'Bu\} \ \Rightarrow \ u^*(t) = 1/2R^{-1}B'\lambda(t)$$

We can now solve the dynamical system

$$\dot{\lambda}(t) = 2Qx^*(t) - A'\lambda(t),\ \lambda(T) = -2Gx^*(T),$$

$$\dot{x}^*(t) = Ax^*(t) + 1/2BR^{-1}B'\lambda(t),\ x(0) = x_0,$$

via the guess

$$\lambda(t) = -2P(t)x^*(t),$$

where $P = P(t)$ is the symmetric solution to the matrix Riccati equation

$$\dot{P} = -Q - PA - A'P + PBR^{-1}B'P,\ P(T) = G.$$

Hence the (candidate) optimal controller is given in feedback form by

$$u^*(t) = -R^{-1}B'P(t)x^*(t).$$

For $T = 1$ solve

$$\max_u \int_0^T -x(t)dt,$$

subject to

$$\dot{x}(t) = u(t),\ x(0) = 1,\ t \in [0, T],$$
$$u \in \Omega = [-1, 1]$$

Under certain assumptions the conditions of the maximum principle are also sufficient for optimality. To obtain this define the derived Hamiltonian H^0 as

$$H^0(x, \lambda, t) = \max_u H(x, u, \lambda, t), \tag{2}$$

and assume that (2) implicit defines a unique $u = u^0(x, \lambda, t)$, that is $H^0(x, \lambda, t) = H(x, u^0, \lambda, t)$.

Theorem

Let (u^*, x^*, λ) satisfy the necessary conditions of the maximum principle. If $x \mapsto H^0(x, \lambda(t), t)$ and $x \mapsto S(x, T)$ are concave, then u^* is optimal.

Theorems like the one above giving sufficient conditions for optimality via the maximum principle are sometimes referred to as Mangasarian type theorems.

For $T = 1$ and $T = 2$ solve

$$\max_u \int_0^T -\frac{1}{2}x(t)^2 dt,$$

subject to

$$\dot{x}(t) = u(t),\ x(0) = 1,\ t \in [0, T],$$
$$u \in \Omega = [-1, 1]$$