Consensus in Dynamic Networks

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This lecture note describes consensus of interconnected dynamical systems, i.e., a network of systems connected with some communication given by a graph. The lecture will only address consensus of an interconnection of agents governed by integrator dynamics. The material is based on [1] that gives an overview on control of interconnected dynamical systems, and [3, 5] and Chapters 2 and 3 of [4] that provide more details on consensus problems. The exposition is given in continuous time, but equivalent results exist for the discrete time formulation. Furthermore, the topology of the network is fixed and there are no communication delays. These assumptions can all be relaxed. The interested reader may consult [4] for details.
After reading this note, you must

- have knowledge about graph theory,
- have knowledge about consensus algorithms,
- be able to analyze and design consensus protocols.

The structure of the note is as follows; first, graph theoretic concepts are defined and basic results are presented. Subsequently, consensus problems are addressed on undirected graphs and directed graphs. Finally, the performance of consensus protocols is assessed and small extensions with practical applications are given.

1 Graph Theory

This section introduces basic graph theoretic concepts. We consider both directed graphs [1] and undirected graphs [2]. We start by defining a directed graph, also called a digraph.

Definition 1 (Directed Graph). *A directed graph is a pair $G = (V, E)$, where $V = \{v_1, \dots, v_n\}$ is a nonempty set of nodes (or vertices), and $E \subseteq V \times V$ is a set of edges.*

An example of a digraph is given in the following.

Example 1. *Let $G = (V, E)$ be a directed graph, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_1, v_4), (v_3, v_4)\}$. The graph G is illustrated in Figure 1.*

The set of undirected graphs is a subset of directed graphs, where all edges are bidirectional. To simplify notation, we denote the edge (v_i, v_j) by e_{ij}.

Definition 2 (Undirected Graph). *A graph $G = (V, E)$ is undirected if*

$$e_{ij} \in E \Rightarrow e_{ji} \in E.$$

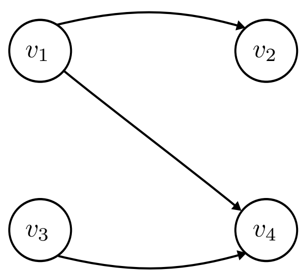

Figure 1: Illustration of directed graph G from Example 1.

An undirected graph is exemplified next.

Example 2. *To make G from Example 1 undirected, we add edges and get $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_2, v_1), (v_1, v_4), (v_4, v_1), (v_3, v_4), (v_4, v_3)\}$.*

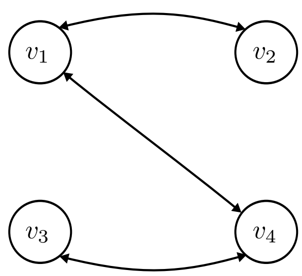

Figure 2: Illustration of the undirected graph G from Example 2.

An important concept in graph theory is a neighbor.

Definition 3 (Neighbors of Node)**.** *Let $G = (V, E)$ be a graph. The set of neighbors of node $v_i \in V$ is*

$$N_i = \{v_j \in V |\ (v_j, v_i) \in E\}.$$

In this context, there is an edge from neighbors of v_i to v_i. Note that the notion of neighbor is not consistently defined in the literature. However, for the subsequent treatment of consensus, this definition is natural.

Definition 4 (Cluster)**.** *Let V be a set of vertices. A cluster is any subset $J \subseteq V$.*

Example 3 (Continuation of Example 1)**.** *The sets of neighbors of the graph in Example 1 are $N_1 = \emptyset$, $N_2 = \{v_1\}$, $N_3 = \emptyset$, and $N_4 = \{v_1, v_3\}$.*

The notion of a path is used to determine if a graph is connected. An undirected path does not take the direction of the edges into account.

Definition 5 (Undirected Path)**.** *Let $(\tilde{V}, \tilde{E})$ be a directed graph whose vertex set may be numbered $\{\tilde{v}_1, \ldots, \tilde{v}_m\}$ and edges may be numbered $\{\tilde{e}_1, \ldots, \tilde{e}_{m-1}\}$*

so that $\tilde{e}_i = (\tilde{v}_i, \tilde{v}_{i+1})$ for every $i \in \{1, \dots, m-1\}$. The graph $(\tilde{V}, \tilde{E})$ is an undirected path of a directed graph (V, E) if there is an injection (one-to-one) $\phi : \tilde{V} \to V$ such that

$$(\phi(\tilde{v}_i), \phi(\tilde{v}_{i+1})) \in E \quad \text{or} \quad (\phi(\tilde{v}_{i+1}), \phi(\tilde{v}_i)) \in E \quad \forall i \in \{1, \dots, m-1\}.$$

A directed path respects the direction of the edges and is defined as follows.

Definition 6 (Directed Path)**.** *Let $(\tilde{V}, \tilde{E})$ be a directed graph whose vertex set may be numbered $\{\tilde{v}_1, \dots, \tilde{v}_m\}$ and edges may be numbered $\{\tilde{e}_1, \dots, \tilde{e}_{m-1}\}$ so that $\tilde{e}_i = (\tilde{v}_i, \tilde{v}_{i+1})$ for every $i \in \{1, \dots, m-1\}$. The graph $(\tilde{V}, \tilde{E})$ is a directed path of a directed graph (V, E) if there is an injection $\phi : \tilde{V} \to V$ such that*

$$(\phi(\tilde{v}_i), \phi(\tilde{v}_{i+1})) \in E \quad \forall i \in \{1, \dots, m-1\}.$$

The distinction between directed and undirected paths only makes sense for digraphs; hence, they are often referred to as paths in an undirected graph.

A digraph is called strongly connected if and only if any two distinct nodes of the graph can be connected via a path that follows the directions of the edges of the digraph.

Definition 7 (Strongly Connected Graph)**.** *A digraph $G = (V, E)$ is strongly connected if for any ordered pair (v_i, v_j) with $v_i \neq v_j \in V$ there exists a directed path $(\tilde{V}, \tilde{E})$ with $v_i = \tilde{v}_1$ and $v_j = \tilde{v}_m$.*

Notation 1. *The cardinality of a set V is denoted by $|V|$. We say that the order of a graph $G = (V, E)$ is $|V|$.*

The graph Laplacian matrix is instrumental for the analysis of interconnected system; hence, it is defined in the following from the adjacency matrix and input degree matrix.

Definition 8 (Adjacency Matrix)**.** *Given a graph $G = (V, E)$, the adjacency matrix $A \in \mathbb{R}^{|V| \times |V|}$ is*

$$A = [a_{ij}], \qquad \text{where } a_{ij} = \begin{cases} 1, & j \in N_i \\ 0, & \text{otherwise.} \end{cases}$$

Definition 9 (Input Degree Matrix)**.** *Given a graph $G = (V, E)$, the input degree matrix $D \in \mathbb{R}^{|V| \times |V|}$ is a diagonal matrix*

$$D = [d_{ij}], \qquad \text{where } d_{ij} = \begin{cases} deg_{\text{in}}(v_i), & i = j \\ 0, & \text{otherwise.} \end{cases}$$

where $deg_{in}(v_i)$ is the number of ingoing edges e_{ki}.

Definition 10 (Graph Laplacian Matrix)**.** *The Laplacian of a graph G is*

$$L = D - A$$

or equivalently

$$L = [l_{ij}], \qquad \text{where } l_{ij} = \begin{cases} -1, & j \in N_i \\ |N_i|, & j = i \\ 0, & \text{otherwise.} \end{cases}$$

The set of edges of a fixed orientation of an undirected graph is denoted by E_o. Thus, E_o contains one and only one of the two edges $e_{ij}, e_{ji} \in E$. We use two projections π_1 and π_2, such that for $e = (a, b)$, $a = \pi_1 e$ and $b = \pi_2 e$.

Definition 11 (Incidence Matrix)**.** *Fix an orientation of an undirected graph and let $E_\mathrm{o} = \{e_1, \ldots, e_m\}$. Define the incidence matrix which is an $n \times m$ matrix as*

$$C = [c_{ij}], \qquad \textit{where } c_{ij} = \begin{cases} 1, & \textit{if } v_i = \pi_2 e_j \\ -1, & \textit{if } v_i = \pi_1 e_j \\ 0, & \textit{otherwise.} \end{cases}$$

The incidence matrix is used to show that the Laplacian of an undirected graph is symmetric and positive semidefinite.

Lemma 1. *Let G be an undirected graph, with incidence matric C, then the Laplacian is given by*

$$L = CC^T. \tag{1}$$

Proof. To prove the equality (1), we set up equations for the elements of CC^T. It is seen that

$$[CC^T]_{ij} = \sum_{k=1}^{m} c_{ik} c_{jk}.$$

We have two cases

1. For $i = j$ we get

$$[CC^T]_{ij} = \sum_{k=1}^{m} c_{ik}^2 = deg_{\mathrm{in}}(v_j) = deg_{\mathrm{out}}(v_j).$$

2. For $i \neq j$ we have two cases

   - if there is no edge between v_i and v_j then

$$[CC^T]_{ij} = 0.$$

   - if there is an edge between v_i and v_j then

$$[CC^T]_{ij} = -1.$$

From this it is seen that $L = CC^T$.

$\square$

From Lemma 1, it is also seen that the Laplacian matrix of an undirected graph is symmetric and positive semidefinite.

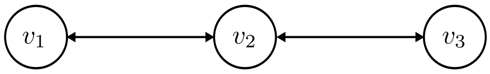

Figure 3: Illustration of undirected graph.

Example 4. *Consider the undirected graph illustrated in Figure 3 given by* $G = (V, E)$*, where* $V = \{v_1, v_2, v_3\}$ *and* $E = \{e_{12}, e_{21}, e_{23}, e_{32}\}$*.*

The set E_o *is given as* $E_o = \{(v_1, v_2), (v_2, v_3)\}$*. Thus, the incidence matrix is*

$$C = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and the Laplacian is

$$L = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

It is seen that each row of the Laplacian graph matrix sums to zero, which means that the right nullspace of L is nontrivial.

Proposition 1. *Any Laplacian matrix* L *has a zero eigenvalue corresponding to the right eigenvector*

$$w_r = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Thus, $rank(L) \leq n - 1$*.*

Proof. Per definition, every row sum of a Laplacian matrix is zero. Therefore, the Laplacian matrix always has a zero eigenvalue corresponding to the right eigenvector $1_{n\times 1}$. The null space of L is therefore at least one-dimensional and $rank(L) \leq n - 1$. □

Directed spanning trees play an essential role in consensus for directed graphs.

Definition 12 (Directed Spanning Tree)**.** *A directed spanning tree for a digraph* $G = (V, E)$ *is a subgraph* $G' = (V, E')$*, where* $E' \subseteq E$ *and* G' *is a tree.*

Note that a directed spanning tree includes every vertex of the original graph. The rank of the Laplacian matrix can be used for determining if a graph has a directed spanning tree.

Theorem 1. *Let* $G = (V, E)$ *be a digraph with Laplacian* L*. The graph* G *has a directed spanning tree if and only if*

$$rank(L) = |V| - 1.$$

Lemma 2. *Let $G = (V, E)$ be a strongly connected digraph. Then G has a directed spanning tree.*

For undirected graphs connectedness is equivalent with the existence of a directed spanning tree. Thus, we have the following result.

Corollary 1. *Let $G = (V, E)$ be an undirected graph with Laplacian L. The graph G is connected if and only if*

$$\mathit{rank}(L) = |V| - 1.$$

The presented graph theory is applied to consensus problems in the next section.

2 Consensus Problems on Graphs

This section presents the consensus problem for a network of agents. The section is based on [3, 5].

After reading this section, you must be able to determine if a dynamic graph, with the studied protocol, reaches consensus by examining only the network graph. In particular, you should be able to determine which of the three graphs in Figure 4 that

- does not reach consensus for all initial conditions,
- reaches consensus for all initial conditions,
- reaches average-consensus.

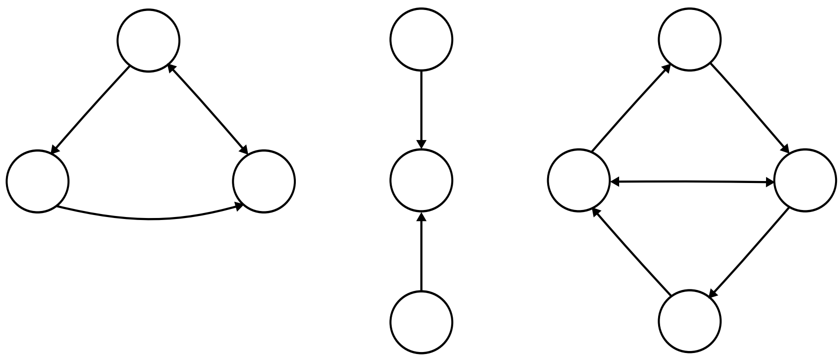

Figure 4: Three graphs with different consensus properties.

Additionally, you should be able to argue how the network graph (algebraic connectivity) affects the performance of a consensus algorithm, and relate consensus problems to practical applications.

In the following, a state variable is associated to each node in the graph G. Let $x_i \in \mathbb{R}^p$ denote the value of node v_i. We refer to $G_x = (G, x)$ with $x = \begin{bmatrix} x_1^T & \cdots & x_n^T \end{bmatrix}^T$ as a network (or algebraic graph) with value $x \in \mathbb{R}^{np}$ and topology G. Let $a \in \mathbb{R}^n$ then $||a|| = \sqrt{a_1^2 + \cdots + a_n^2}$.

We study consensus or agreement, which is defined next.

Definition 13 (Consensus)**.** *We say that the nodes of a network G_x have reached consensus if and only if $x_i = x_j$ for all $i, j \in \{1, \dots, n\}$.*

Whenever the nodes of a network are all in agreement, the common value of all nodes is called the ***group decision value***.

A dynamic graph (or dynamic network) is a dynamical system with a state (G, x) in which the value of x evolves according to the network dynamics

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = f(x, u) = \begin{bmatrix} f_1(x_1, u_1) \\ \vdots \\ f_n(x_n, u_n) \end{bmatrix},$$

where u_i is an input to agent i. Thus the dynamics of agent i are given by $\dot{x}_i = f_i(x_i, u_i)$.

A dynamic graph with topology G is illustrated in Figure 5, where G is a directed graph.

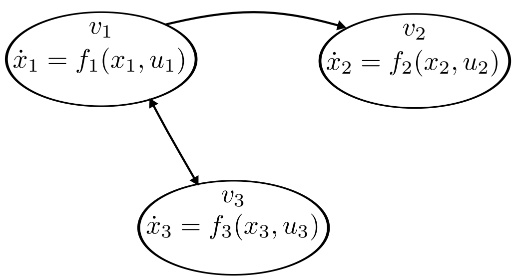

Figure 5: Illustration of a dynamic graph, with topology G.

A network is often required to reach a particular consensus value, which is a function of the initial condition of the system. Thus, we define the χ-consensus problem.

Definition 14 (Consensus Problem)**.** *Let $\chi : \mathbb{R}^{np} \to \mathbb{R}^p$ be a map of variables $x_1, \dots, x_n \in \mathbb{R}^p$ and let $z = x(0)$ denote the initial state of the system. The χ-consensus problem in a dynamic graph is a distributed way to calculate $\chi(z)$ by applying inputs u_i that only depend on the states of node v_i and the states of its neighbors.*

For a network graph to solve a consensus problem, some control must be designed. In this context a control is called a protocol.

Definition 15 (Protocol)**.** *Let $G = (V, E)$ be a graph of order n, and define the cluster $J_i = \{j_{i1}, \dots, j_{im_i}\} \subseteq \{v_i\} \cup N_i$ with $m_i = |J_i|$. We say that a state feedback*

$$u_i = k_i(x_{j_{i1}}, \dots, x_{j_{im_i}})$$

is a protocol with topology G.

If in addition $|J_i| < n$ for all $i \in \{1, \dots, n\}$ then the state feedback is called a **distributed protocol**.

A protocol asymptotically solves the χ-consensus problem if and only if there exists an asymptotically stable equilibrium point x^* of $\dot{x} = f(x, k(x))$ satisfying $x_i^* = \chi(z)$ for all $i \in \{1, \ldots, n\}$. Notice that a particular equilibrium should be reached by every agent. This differs from just reaching consensus - here the equilibrium point is arbitrary. Special consensus problems are:

- Average consensus ($\chi(x) = 1/n \sum_{i=1}^{n} x_i$)
- Max consensus $\chi(x) = \max_i(x_i)$
- Min consensus $\chi(x) = \min_i(x_i)$

The next subsection explains the design of protocols (or controls) for solving consensus problems.

2.1 Consensus Protocols

In the following, we study very simple consensus protocols and agent dynamics. The convergence of a considered protocol is proven using the Laplacian potential. This is the general tool for studying convergence of consensus protocols, see [1] and the references therein for more advanced protocols.

We consider a dynamical network with dynamics for each agent given as

$$\dot{x}_i = u_i$$

and define a consensus protocol (consensus algorithm)

$$u_i = \sum_{j \in N_i} (x_j - x_i). \tag{2}$$

It is seen that the closed-loop dynamics, with protocol (2) can be written as

$$\dot{x}(t) = -(L \otimes I_p)x(t),$$

where $x \in \mathbb{R}^{pn}$ is the state of all agents, $L \in \mathbb{R}^{n \times n}$ is the graph Laplacian matrix, I_p is the $p \times p$ identity matrix, and $\otimes$ is the Kronecker product defined as follows.

Definition 16 (Kronecker Product)**.** *Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$. The Kronecker product is*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

This setup is similar to a distributed control architecture.

Example 5. *Consider the dynamic graph $G_x = (G, x)$, with topology $G = (\{v_1, v_2, v_3, v_4\}, \{e_{14}, e_{21}, e_{24}, e_{31}, e_{34}, e_{43}\})$ illustrated in Figure 6. Let the dynamics of the agents be given by*

$$\dot{x}_i = u_i$$

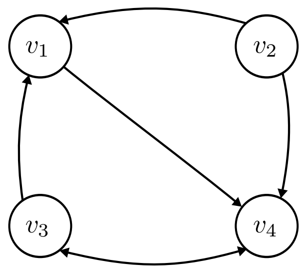

Figure 6: Illustration of the graph G.

with $x_i \in \mathbb{R}^2$ *and*

$$u_i = \sum_{j \in N_i} (x_j - x_i).$$

The closed-loop dynamics are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} (x_2 - x_1) + (x_3 - x_1) \\ 0 \\ (x_4 - x_3) \\ (x_1 - x_4) + (x_2 - x_4) + (x_3 - x_4) \end{bmatrix} = - \begin{bmatrix} 2x_1 - x_2 - x_3 \\ 0 \\ x_3 - x_4 \\ 3x_4 - x_1 - x_2 - x_3 \end{bmatrix}.$$

The neighbors and the Laplacian matrix are

$$N_1 = \{v_2, v_3\},\ N_2 = \emptyset,\ N_3 = \{v_4\},\ N_4 = \{v_1, v_2, v_3\}$$

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

Furthermore, we have

$$L \otimes I_2 = \begin{bmatrix} 2I_2 & -I_2 & -I_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 & -I_2 \\ -I_2 & -I_2 & -I_2 & 3I_2 \end{bmatrix}.$$

Now it is clear that the closed-loop system is

$$\dot{x} = -(L \otimes I_2)x.$$

To simplify presentation, we assume that $p = 1$ in the remainder of the chapter.

To study the convergence of a dynamic graph, we use the Laplacian potential, which is introduced next.

Definition 17 (Laplacian Potential). *We define the Laplacian potential of a graph* G *as*

$$\Psi_G(x) = \frac{1}{2} x^T L x,$$

where L *is the Laplacian matrix of* G.

The Laplacian potential is similar to a Lyapunov function (in particular a quadratic Lyapunov function) used for stability analysis. This becomes apparent later. From Definition 17, it is not clear if $\Psi_G(x)$ is nonnegative, which is required for a Lyapunov function. This property is studied in the following sections.

2.1.1 Undirected Graphs

First, we study dynamic networks with a topology given by an undirected graph.

The Laplacian potential of an undirected graph has the following property. Recall that E_o denotes the set of edges of a fixed orientation of an undirected graph, i.e., E_o contains one and only one of the two edges $e_{ij}, e_{ji} \in E$.

Lemma 3. *The Laplacian potential of an undirected graph is positive semidefinite and satisfies*

$$\Psi_G(x) = x^T L x = \sum_{e_{ij} \in E_\text{o}} (x_i - x_j)^2.$$

Moreover, given a connected graph, $\Psi_G(x) = 0$ if and only if $x_i = x_j$ for all i, j.

Proof. Recall from Lemma 1 that the Laplacian of an undirected graph is given by $L = CC^T$. Thus, the Laplacian potential is

$$\Psi_G(x) = x^T L x = x^T C C^T x = ||C^T x||^2.$$

This implies that $\Psi_G(x)$ is positive semidefinite.

If $\sum_{e_{ij} \in E_\text{o}} (x_i - x_j)^2 = 0$, then for all edges $e_{ij} \in E_\text{o}$, $x_i - x_j = 0$. If the graph is connected, then the value of all nodes must be the same. Finally, it is seen that if all nodes have the same value, then $\Psi_G(x) = 0$. □

The study of consensus can now be analyzed via properties of the Laplacian potential in relation to the system dynamics.

We have established that the Laplacian potential $\Psi_G(x)$ is positive semidefinite for undirected graphs. Thus, if Ψ_G is decreasing along the solution of the system $\dot{x} = f(x)$ then consensus will eventually be reached. This principle is illustrated in Figure 7, where $t_1 < t_2 < t_3$ i.e. the value of Ψ_G is decreasing along the solution of the system. Recall from Theorem 1 that $rank(L) = n - 1$ for a connected graph; hence, the null space of L is one-dimensional. This implies that the Laplacian potential Ψ_G will have a graph similar to the one shown in Figure 8 for $L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. The equilibrium point (group decision) of the system becomes

$$x_\text{ss} = \alpha \otimes I_n$$

with $\alpha \in \mathbb{R}^p$.

We know that a system with a connected undirected graph with the proposed protocol reaches consensus, but we can say more about the consensus value.

Theorem 2. *Consider a dynamic network of integrators $\dot{x}_i = u_i$ where each node applies the protocol*

$$u_i = \sum_{j \in N_i} (x_j - x_i). \tag{3}$$

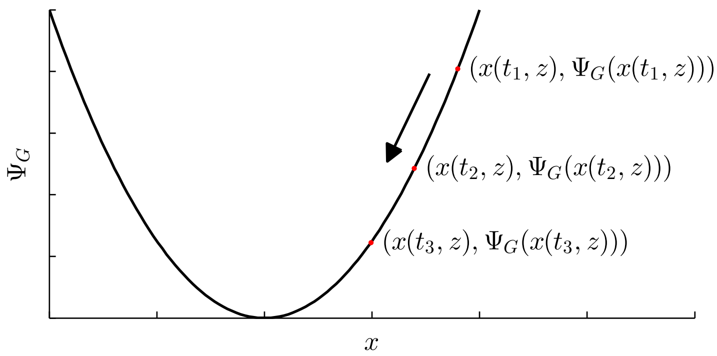

Figure 7: Graph of the Laplacian potential decreasing along the solution of the system, where $x(t, z)$ denotes the solution at time t, initialized at $x(0) = z$.

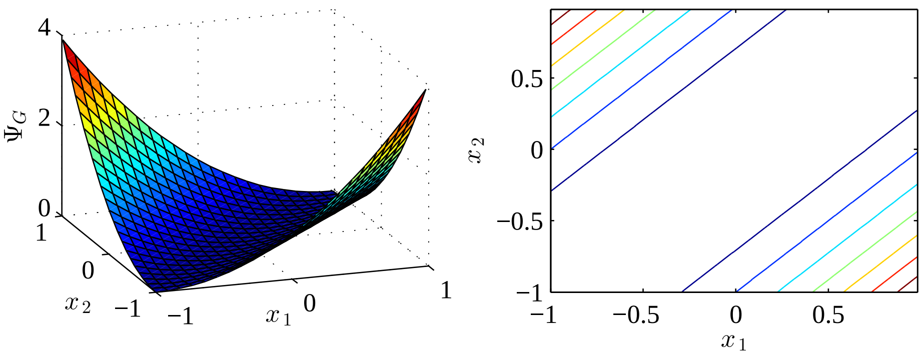

Figure 8: Laplacian potential of a connected undirected graph.

Suppose that G is a connected undirected graph. Then the values of the nodes x is the solution of a gradient system associated with the Laplacian potential $\Psi_G(x)$, i.e.,

$$\dot{x} = -\nabla \Psi_G(x) \qquad x(0) \in \mathbb{R}^n.$$

In addition, all nodes of the graph G globally asymptotically reach average-consensus, i.e. let $x^* = \lim_{t\to\infty} x(t)$, then $x_i^* = x_j^* = Ave(x(0))$ for all i, j.

Proof. Let x^* be an equilibrium point of the system $\dot{x} = -Lx$. Then $Lx^* = 0$ which implies that x^* is the right eigenvector of L associated to $\lambda_1 = 0$, i.e., by Proposition 1, $x^* = \begin{bmatrix} a & \dots & a \end{bmatrix}^T$ with $a \in \mathbb{R}$. Note that since the graph is undirected $\sum_{i=1}^n u_i = 0$; thus, $\sum_{i=1}^n \dot{x}_i = 0$. This implies that $x_i^* = Ave(x(0))$ for all $i \in \{1, \dots, n\}$. □

Example 6. *Consider an undirected graph $G = (V, E)$, with vertices $V = \{v_1, v_2, v_3\}$ and edges $E = \{e_{12}, e_{21}, e_{23}, e_{32}\}$. The graph is illustrated in Figure 9 and is a connected graph. Thus, it reaches average-consensus, which is confirmed by the simulation depicted in Figure 10.*

This finalizes the exposition on consensus of networks with topology given by undirected graphs.

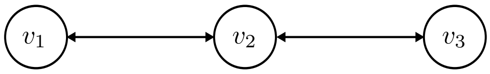

Figure 9: Illustration of an undirected graph.

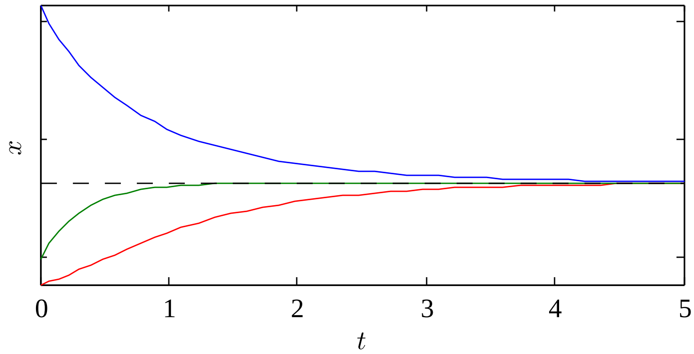

Figure 10: Trajectory of a system with connected undirected graph (solid lines), and average value of $x(0)$ (dashed line).

2.1.2 Directed Graphs

The Laplacian of a directed graph is not necessarily positive semi-definite. A counterexample is provided in the following.

Example 7. *Consider the directed graph $G = (\{v_1, v_2\}, \{e_{21}\})$ illustrated in Figure 11. The graph Laplacian is*

$$L = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Thus, the Laplacian potential of G is indefinite ($\Psi_G(x) = x_1^2 - x_1 x_2$).

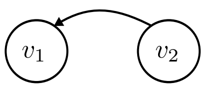

Figure 11: Illustration of directed graph with indefinite Laplacian potential.

The next result is used for the proof of the subsequent theorem.

Theorem 3 (Spectral Location)**.** *Let $G = (V, E)$ be a digraph with Laplacian L. Denote the maximum node out-degree of the digraph G by $d_{max}(G) = \max_i deg_{out}(v_i)$. Then all the eigenvalues of L are located in the following disk*

$$D(G) = \{z \in \mathbb{C} | \; |z - d_{max}(G)| \leq d_{max}(G)\}$$

centered at $z = d_{max}(G) + 0j$ in the complex plane.

We can prove consensus of digraphs with a directed spanning tree.

Theorem 4. *Consider a dynamic network of integrators $\dot{x}_i = u_i$ where each node applies the protocol*

$$u_i = \sum_{j \in N_i} (x_j - x_i). \tag{4}$$

The protocol (4) *globally asymptotically solves a consensus problem if and only if the digraph G has a directed spanning tree.*

Proof. By Theorem 1, the graph G has a directed spanning tree is equivalent with $rank(L) = n-1$. Thus, the eigenvalue at zero is simple, and by Theorem 3 the remaining eigenvalues of $-L$ have negative real part; hence, the system is stable. This implies that $x^* = \alpha 1_{n\times 1}$ with $\alpha \in \mathbb{R}$ are the only possible equilibrium points; hence, the system asymptotically reaches consensus. □

Connected undirected graphs reach average-consensus, but connectedness is not sufficient for a digraph to reach average-consensus.

Example 8 (Counterexample Average Consensus Problem)**.** *Consider the graph $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{e_{12}, e_{23}, e_{31}, e_{13}\}$, and the dynamics of the nodes given by $\dot{x}_i = u_i$. The digraph G is* strongly connected *(there exists a directed path connecting every pair of distinct nodes) and is illustrated in Figure 12. We apply the following consensus protocol*

$$u_i = \sum_{v_j \in N_i} (x_j - x_i).$$

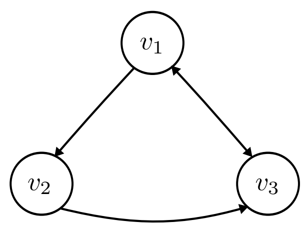

Figure 12: Illustration of the graph G that is strongly connected, but the system does not reach average-consensus.

A sufficient condition for the decision value α of each node to be equal to $Ave(x(0))$ is that $\sum_{i=1}^n u_i = 0$, since this implies that $\sum_{i=1}^n \dot{x}_i = 0$, i.e., $\sum_{i=1}^n x_i(t) = \sum_{i=1}^n x_i(0)$.

For the considered digraph

$$\sum_{i=1}^{n} u_i = \underbrace{x_3 - x_1}_{u_1} + \underbrace{x_1 - x_2}_{u_2} + \underbrace{x_2 - x_3 + x_1 - x_3}_{u_3} = x_3 - x_1.$$

Figure 13 shows a simulated trajectory, with each of the three states with solid lines and the average value of $x(0)$ with a dashed line.

For a digraph to reach average-consensus, the concept of a balanced graph plays an important role.

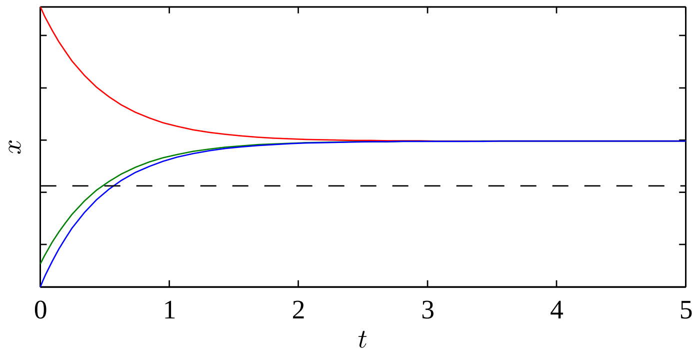

Figure 13: Trajectory of system (solid lines) and average value of $x(0)$ (dashed line).

Definition 18 (Balanced Graph)**.** *We say that the node v_i of a digraph $G = (V, E)$ is balanced if and only if its in-degree and out-degree are the same ($deg_{in}(v_i) = deg_{out}(v_i)$). A graph $G = (V, E)$ is called balanced if and only if all its nodes are balanced.*

Remark that any undirected graph is balanced. The next two theorems lead to the necessary and sufficient condition shown in Theorem 7 for a dynamic network with topology given by digraph to reach average-consensus.

Theorem 5. *Consider a network of integrator agents with a topology $G = (V, E)$ that contains a digraph with a directed spanning tree. The protocol*

$$u_i = \sum_{j \in N_i} (x_j - x_i)$$

globally asymptotically solves the average-consensus problem if and only if $1^T L = 0$.

Note that a balanced digraph with a directed spanning tree is also strongly connected.

Theorem 6. *Let $G = (V, E)$ be a directed graph. Then all the following statements are equivalent:*

1. *G is balanced.*
2. *$w_l = 1$ is the left eigenvector of the Laplacian matrix L of G associated with the zero eigenvalue, i.e., $1^T L = 0$.*
3. *$\sum_{i=1}^{n} u_i = 0$ for all $x \in \mathbb{R}^n$ with $u_i = \sum_{j \in N_i} (x_j - x_i)$.*

Theorem 7. *Consider a dynamic network of integrators $\dot{x}_i = u_i$ where each node applies the protocol*

$$u_i = \sum_{j \in N_i} (x_j - x_i). \tag{5}$$

Let the network have topology G that is a strongly connected digraph. Then the protocol globally asymptotically solves the average-consensus problem if and only if G is balanced.

Example 9. *Consider the digraph $G = (V, E)$, with vertices $V = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{e_{13}, e_{21}, e_{23}, e_{32}, e_{34}, e_{42}\}$. The graph is illustrated in Figure 14 and is a connected graph. From the following graph Laplacian, it is seen that the graph is balanced*

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

From this, we conclude that the system reaches average-consensus, which is confirmed by the simulation depicted in Figure 15.

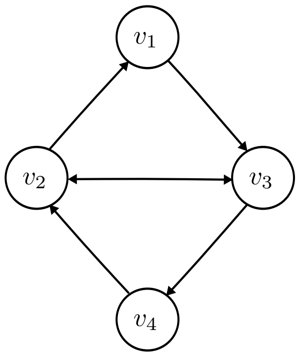

Figure 14: Illustration of a balanced connected directed graph.

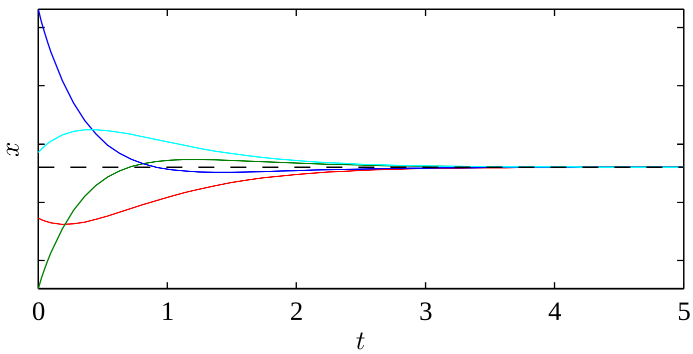

Figure 15: Trajectory of system (solid lines) and average value of $x(0)$ (dashed line).

2.2 Performance of Consensus Protocols

This section gives a relation between the Fiedler eigenvalue of an undirected graph and the performance (minimum decay rate) of its consensus protocol. For directed graphs it is necessary to study an induced undirected graph called a mirror graph. Initially, we find bounds on the eigenvalues of the Laplacian matrix. Such bounds can be derived from Courant-Fischer Theorem.

Theorem 8 (Courant-Fischer Formula)**.** *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. Then*

$$\lambda_1 = \min_{||x||=1} x^T A x = \min_{x \neq 0} \frac{x^T A x}{x^T x},$$

$$\lambda_2 = \min_{\substack{||x||=1 \\ x \perp v_1}} x^T A x = \min_{\substack{x \neq 0 \\ x \perp v_1}} \frac{x^T A x}{x^T x},$$

$$\lambda_n = \lambda_{max} = \max_{||x||=1} x^T A x = \max_{x \neq 0} \frac{x^T A x}{x^T x}.$$

In general, for $1 \leq k \leq n$, let S_k denote the span of $v_1, \ldots, v_k$ (with $S_0 = \{0\}$), and let $S_k^\perp$ denote the orthogonal complement of S_k. Then

$$\lambda_k = \min_{\substack{||x||=1 \\ x \in S_{k-1}^\perp}} x^T A x = \min_{\substack{x \neq 0 \\ x \in S_{k-1}^\perp}} \frac{x^T A x}{x^T x}.$$

We apply Courant-Fischer Theorem directly on the Laplacian matrix and obtain the following bound. The result only applies to undirected graphs, since L must be symmetric.

Lemma 4. *Let G be a connected undirected graph, then*

$$\min_{||x||=1} x^T L x = \min_{\substack{x \neq 0 \\ 1^T x = 0}} \frac{x^T L x}{x^T x} = \lambda_2(L).$$

Proof. Connected undirected graphs have an eigenvalue $\lambda_1 = 0$ of multiplicity 1, which is associated to the eigenvector 1. Thus, the lemma is a special case of Courant-Fischer Theorem with $\lambda_1 = 0$ and $v_1 = 1$. $\square$

The eigenvalue λ_2 is called the Fiedler eigenvalue and is pivotal in the study of dynamic networks, as it gives a bound on the minimum decay rate of the system. The Fiedler eigenvalue depends on the topology of the system - for a sparse graph it is small and for a dense graph it is large. Therefore, λ_2 is called the algebraic connectivity.

In the following, we write the state x, as a function of the average value $\alpha = Ave(x(0)) \in \mathbb{R}$, and the disagreement $\delta \in \mathbb{R}^n$. Thus, we have

$$x = \alpha 1 + \delta.$$

Since $Ave(x(0))$ is an invariant quantity, the system dynamics are given as

$$\dot{\delta} = -L\delta.$$

For studying the algebraic connectivity via Fiedler eigenvalues for directed graphs, it is necessary to introduce the mirror graph.

Definition 19 (Mirror Graph)**.** *Let $G = (V, E)$ be a directed graph. Let $\tilde{E}$ be the set of reverse edges of G, i.e. $\tilde{E} = \{e_{ij} | e_{ji} \in E\}$. The mirror graph of G denoted by $\hat{G}$ is an undirected graph $\hat{G} = (V, \hat{E})$, where $\hat{E} = E \cup \tilde{E}$.*

Lemma 5. *Let $G = (V, E)$ be a digraph with Laplacian matrix L. Then $(L + L^T)/2$ is a Laplacian for the mirror graph $\bar{G}$.*

Finally, the convergence of a dynamic network with a topology given by a digraph can be assessed via the following result.

Theorem 9. *Consider a dynamic network of integrators $\dot{x}_i = u_i$ with topology G that is a strongly connected digraph. Let each node apply the protocol*

$$u_i = \sum_{j \in N_i} (x_j - x_i). \tag{6}$$

Then the following statements hold

1. *The group disagreement δ globally asymptotically vanishes with exponential decay rate $\kappa = \lambda_2(\hat{G})$, i.e.,*

$$||\delta(t)|| \leq ||\delta(0)|| \exp(-\kappa t).$$

2. *The smooth, positive semidefinite, and proper function $V : \mathbb{R}^n \to \mathbb{R}$ given by*

$$V(\delta) = \frac{1}{2}\delta^T \delta$$

is a Lyapunov function for the disagreement dynamics.

Example 10. *We consider the two directed graphs illustrated in Figure 14. For the graph G_1 $\lambda_2 = 0.2929$ and for the graph G_2 $\lambda_2 = 0.6340$. Thus, the convergence of the right graph should be the fastest. This is confirmed by the simulation depicted in Figure 17.*

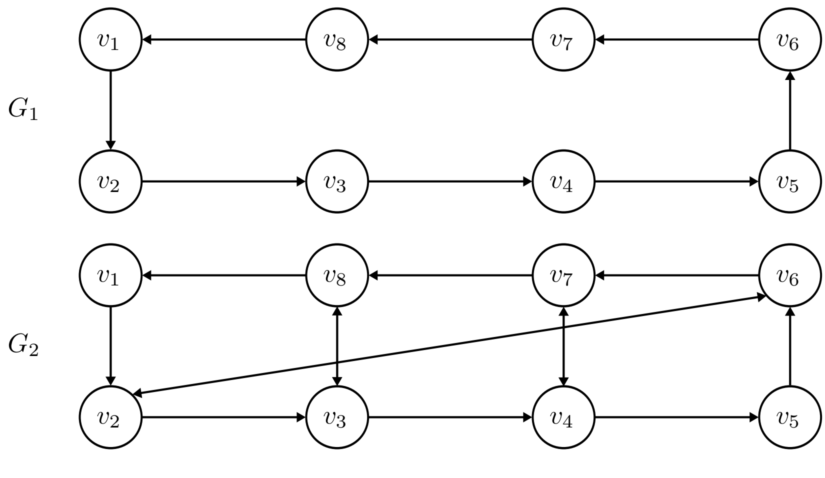

Figure 16: Illustration of two balanced connected directed graphs G_1 and G_2.

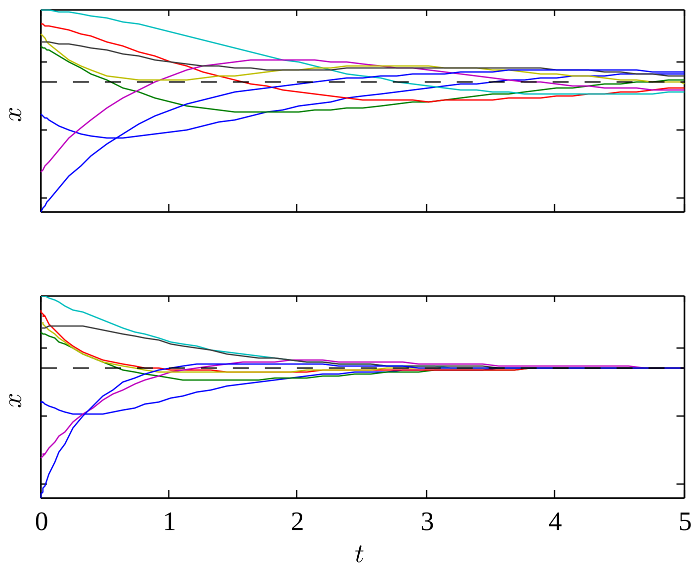

Figure 17: Trajectory of system (solid lines) and average value of $x(0)$ (dashed line). The upper subplot resembles G_1 and the lower subplot resembles G_2.

2.3 Applications of Consensus Problems

This section presents some applications of consensus via small variations of the original problem, the consensus algorithms can be used for something useful [4].

By slightly modifying the consensus protocol (4), the dynamic network can converge towards a relative difference between their states. Specifically, we provide a protocol, which ensures that $x_i - x_j$ converges to a value Δ_{ij}.

Corollary 2. *Consider a dynamic network of integrators $\dot{x}_i = u_i$ with topology G where each node applies the protocol*

$$u_i = \dot{\delta}_i + \sum_{j \in N_i} (x_j - x_i) + (\delta_j - \delta_i), \tag{7}$$

and $\Delta_{ij} \equiv \delta_i - \delta_j$. Then $x_i(t) - x_j(t) \to \Delta_{ij}(t)$ as $t \to \infty$ if and only if G has a directed spanning tree.

Proof. Define a new state $\hat{x}_i \equiv x_i - \delta_i$. Then the dynamics of each agent becomes

$$\dot{\hat{x}}_i = \sum_{j \in N_i} (\hat{x}_j - \hat{x}_i). \tag{8}$$

It is seen that this dynamics is identical to the original consensus problem. Thus, by Theorem 4 it follows that $\hat{x}_i - \hat{x}_j \to 0$ as $t \to \infty$. The result follows from the fact that $\hat{x}_i - \hat{x}_j = 0$ implies that $x_i - x_j = \delta_i - \delta_j$. $\square$

The dynamic network can also track a reference, for simplicity, we assume that the reference is constant.

Corollary 3. *Consider a dynamic network of integrators $\dot{x}_i = u_i$ with topology G. Add an additional vertex v_{n+1} to the graph (and possibly some edges and denote the topology by $\bar{G}$). Let each node apply the protocol*

$$u_i = \sum_{j \in N_i} (x_j - x_i) \tag{9}$$

for $i = 1, \ldots, n$ and let $u_{n+1} = 0$. Then $x_i(t) \to x_{n+1}(t)$ as $t \to \infty$ if and only if $\bar{G}$ has a directed spanning tree.

3 Exercises

Exercise 1. *Let the graphs G_1, G_2, and G_3 be given as shown in Figure 18.*

1. *Derive the graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, and $G_3 = (V_3, E_3)$.*
2. *Derive the Laplacian matrices of the graphs.*
3. *Let G_1, G_2, and G_3 be the topology of a network of agents with integrator dynamics and consensus protocol (2). Will the agents reach consensus, and if so, what type of consensus?*

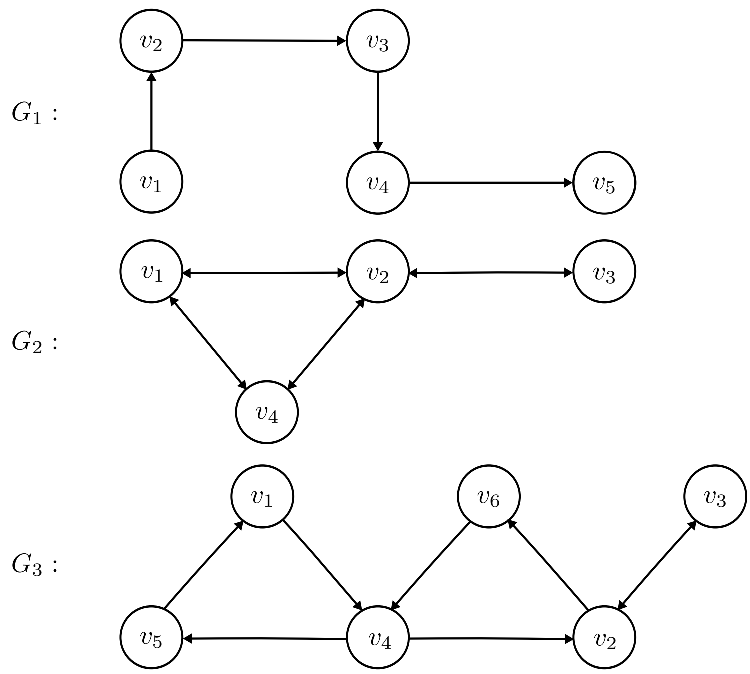

Figure 18: Illustration of a dynamic graph.

Exercise 2. *Consider the two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$,*
$E_1 = \{e_{12}, e_{21}, e_{23}, e_{32}, e_{24}, e_{42}, e_{34}, e_{43}, e_{14}, e_{41}\}$,
$E_2 = \{e_{12}, e_{23}, e_{31}, e_{14}, e_{41}, e_{43}, e_{36}, e_{65}, e_{54}\}$.

1. *Derive the convergence rate of the disagreement of two dynamic networks with topology G_1 and G_2 and agents governed by integrator dynamics and consensus protocol* (2)*.*

2. *Simulate the dynamic networks and compare the simulated results with the bound.*

Exercise 3. *The graph $G = (V, E)$ with vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edges $E = \{e_{12}, e_{23}, e_{34}, e_{45}\}$ reaches consensus with consensus protocol* (2)*. Modify the consensus algorithm such that the network converges to*

$$\begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_4 \\ x_4 - x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \\ 2 \end{bmatrix}.$$

Exercise 4. *Add a virtual agent to the dynamic network in Figure 19 such that the state of every agent follows the constant reference of the virtual agent.*

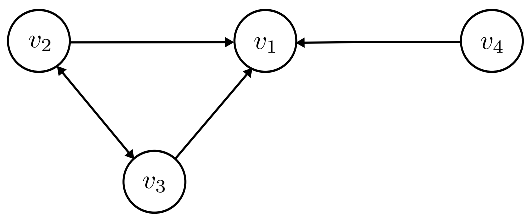

Figure 19: Illustration of a dynamic graph.

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