4.15 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - x_2 - h_2(x_3), \qquad \dot{x}_3 = x_2 - x_3$$

where h_1 and h_2 are locally Lipschitz functions that satisfy $h_i(0) = 0$ and $yh_i(y) > 0$ for all $y \neq 0$.

(a) Show that the system has a unique equilibrium point at the origin.

(b) Show that $V(x) = \int_0^{x_1} h_1(y)\ dy + x_2^2/2 + \int_0^{x_3} h_2(y)\ dy$ is positive definite for all $x \in R^3$.

(c) Show that the origin is asymptotically stable.

(d) Under what conditions on h_1 and h_2, can you show that the origin is globally asymptotically stable?

4.16 Show that the origin of

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 - x_2^3$$

is globally asymptotically stable.

4.17 ([77]) Consider Liénard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\ dy + (1/2)x_2^2$ as a Lyapunov function candidate, find