5.4 Coprime Factorization over $\mathcal{RH}_\infty$

Recall that two polynomials $m(s)$ and $n(s)$, with, for example, real coefficients, are said to be *coprime* if their greatest common divisor is 1 (equivalent, they have no common zeros). It follows from Euclid's algorithm[1] that two polynomials m and n are coprime iff there exist polynomials $x(s)$ and $y(s)$ such that $xm + yn = 1$; such an equation is called a Bezout identity. Similarly, two transfer functions $m(s)$ and $n(s)$ in $\mathcal{RH}_\infty$ are said to be *coprime over* $\mathcal{RH}_\infty$ if there exists $x, y \in \mathcal{RH}_\infty$ such that

$$xm + yn = 1.$$

The more primitive, but equivalent, definition is that m and n are coprime if every common divisor of m and n is invertible in $\mathcal{RH}_\infty$, i.e.,

$$h, mh^{-1}, nh^{-1} \in \mathcal{RH}_\infty \Longrightarrow h^{-1} \in \mathcal{RH}_\infty.$$

More generally, we have

Definition 5.3 Two matrices M and N in $\mathcal{RH}_\infty$ are *right coprime over* $\mathcal{RH}_\infty$ if they have the same number of columns and if there exist matrices X_r and Y_r in $\mathcal{RH}_\infty$ such that

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = X_r M + Y_r N = I.$$

Similarly, two matrices $\tilde{M}$ and $\tilde{N}$ in $\mathcal{RH}_\infty$ are *left coprime over* $\mathcal{RH}_\infty$ if they have the same number of rows and if there exist matrices X_l and Y_l in $\mathcal{RH}_\infty$ such that

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} X_l \\ Y_l \end{bmatrix} = \tilde{M} X_l + \tilde{N} Y_l = I.$$

Note that these definitions are equivalent to saying that the matrix $\begin{bmatrix} M \\ N \end{bmatrix}$ is left-invertible in $\mathcal{RH}_\infty$ and the matrix $\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}$ is right-invertible in $\mathcal{RH}_\infty$. These two equations are often called Bezout identities.

Now let P be a proper real-rational matrix. A *right-coprime factorization (rcf)* of P is a factorization $P = NM^{-1}$ where N and M are right-coprime over $\mathcal{RH}_\infty$. Similarly, a *left-coprime factorization (lcf)* has the form $P = \tilde{M}^{-1}\tilde{N}$ where $\tilde{N}$ and $\tilde{M}$ are left-coprime over $\mathcal{RH}_\infty$. A matrix $P(s) \in \mathcal{R}_p(s)$ is said to have *double coprime factorization* if there exist a right coprime factorization $P = NM^{-1}$, a left coprime factorization $P = \tilde{M}^{-1}\tilde{N}$, and $X_r, Y_r, X_l, Y_l \in \mathcal{RH}_\infty$ such that

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I. \tag{5.14}$$

Of course implicit in these definitions is the requirement that both M and $\tilde{M}$ be square and nonsingular.

[1] See, e.g., [Kailath, 1980, pp. 140-141].

Theorem 5.9 *Suppose $P(s)$ is a proper real-rational matrix and*

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is a stabilizable and detectable realization. Let F and L be such that $A+BF$ and $A+LC$ are both stable, and define

$$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right] \tag{5.15}$$

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]. \tag{5.16}$$

Then $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ are rcf *and* lcf*, respectively, and, furthermore, (5.14) is satisfied.*

Proof. The theorem follows by verifying the equation (5.14). □

Remark 5.2 Note that if P is stable, then we can take $X_r = X_l = I$, $Y_r = Y_l = 0$, $N = \tilde{N} = P$, $M = \tilde{M} = I$. ♡

Remark 5.3 The coprime factorization of a transfer matrix can be given a feedback control interpretation. For example, right coprime factorization comes out naturally from changing the control variable by a state feedback. Consider the state space equations for a plant P:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

Next, introduce a state feedback and change the variable

$$v := u - Fx$$

where F is such that $A + BF$ is stable. Then we get

$$\begin{aligned} \dot{x} &= (A+BF)x + Bv \\ u &= Fx + v \\ y &= (C+DF)x + Dv. \end{aligned}$$

Evidently from these equations, the transfer matrix from v to u is

$$M(s) = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array}\right],$$

and that from v to y is

$$N(s) = \left[\begin{array}{c|c} A+BF & B \\ \hline C+DF & D \end{array}\right].$$

Therefore

$$u = Mv, \quad y = Nv$$

so that $y = NM^{-1}u$, i.e., $P = NM^{-1}$. ♡

We shall now see how coprime factorizations can be used to obtain alternative characterizations of internal stability conditions. Consider again the standard stability analysis diagram in Figure 5.2. We begin with any rcf's and lcf's of P and $\hat{K}$:

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N} \tag{5.17}$$

$$\hat{K} = UV^{-1} = \tilde{V}^{-1}\tilde{U}. \tag{5.18}$$

Lemma 5.10 *Consider the system in Figure 5.2. The following conditions are equivalent:*

1. *The feedback system is internally stable.*
2. $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$ *is invertible in* $\mathcal{RH}_\infty$.
3. $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ *is invertible in* $\mathcal{RH}_\infty$.
4. $\tilde{M}V - \tilde{N}U$ *is invertible in* $\mathcal{RH}_\infty$.
5. $\tilde{V}M - \tilde{U}N$ *is invertible in* $\mathcal{RH}_\infty$.

Proof. As we saw in Lemma 5.3, internal stability is equivalent to

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$$

or, equivalently,

$$\begin{bmatrix} I & \hat{K} \\ P & I \end{bmatrix}^{-1} \in \mathcal{RH}_\infty. \tag{5.19}$$

Now

$$\begin{bmatrix} I & \hat{K} \\ P & I \end{bmatrix} = \begin{bmatrix} I & UV^{-1} \\ NM^{-1} & I \end{bmatrix} = \begin{bmatrix} M & U \\ N & V \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix}$$

so that

$$\begin{bmatrix} I & \hat{K} \\ P & I \end{bmatrix}^{-1} = \begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1}.$$

Since the matrices

$$\begin{bmatrix} M & 0 \\ 0 & V \end{bmatrix}, \begin{bmatrix} M & U \\ N & V \end{bmatrix}$$

are right-coprime (this fact is left as an exercise for the reader), (5.19) holds iff

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1} \in \mathcal{RH}_\infty.$$

This proves the equivalence of conditions 1 and 2. The equivalence of 1 and 3 is proved similarly.

The conditions 4 and 5 are implied by 2 and 3 from the following equation:

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} \tilde{V}M - \tilde{U}N & 0 \\ 0 & \tilde{M}V - \tilde{N}U \end{bmatrix}.$$

Since the left hand side of the above equation is invertible in $\mathcal{RH}_\infty$, so is the right hand side. Hence, conditions 4 and 5 are satisfied. We only need to show that either condition 4 or condition 5 implies condition 1. Let us show condition 5 → 1; this is obvious since

$$\begin{aligned} \begin{bmatrix} I & \hat{K} \\ P & I \end{bmatrix}^{-1} &= \begin{bmatrix} I & \tilde{V}^{-1}\tilde{U} \\ NM^{-1} & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{V}M & \tilde{U} \\ N & I \end{bmatrix}^{-1} \begin{bmatrix} \tilde{V} & 0 \\ 0 & I \end{bmatrix} \in \mathcal{RH}_\infty \end{aligned}$$

if $\begin{bmatrix} \tilde{V}M & \tilde{U} \\ N & I \end{bmatrix}^{-1} \in \mathcal{RH}_\infty$ or if condition 5 is satisfied. □

Combining Lemma 5.10 and Theorem 5.9, we have the following corollary.

Corollary 5.11 *Let P be a proper real-rational matrix and $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be corresponding rcf and lcf over $\mathcal{RH}_\infty$. Then there exists a controller*

$$\hat{K}_0 = U_0V_0^{-1} = \tilde{V}_0^{-1}\tilde{U}_0$$

with $U_0, V_0, \tilde{U}_0$, and $\tilde{V}_0$ in $\mathcal{RH}_\infty$ such that

$$\begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U_0 \\ N & V_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5.20}$$

Furthermore, let F and L be such that $A+BF$ and $A+LC$ are stable. Then a particular set of state space realizations for these matrices can be given by

$$\begin{bmatrix} M & U_0 \\ N & V_0 \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right] \tag{5.21}$$

$$\begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right]. \tag{5.22}$$

Proof. The idea behind the choice of these matrices is as follows. Using the observer theory, find a controller $\hat{K}_0$ achieving internal stability; for example

$$\hat{K}_0 := \left[\begin{array}{c|c} A + BF + LC + LDF & -L \\ \hline F & 0 \end{array}\right]. \tag{5.23}$$

Perform factorizations

$$\hat{K}_0 = U_0 V_0^{-1} = \tilde{V}_0^{-1}\tilde{U}_0$$

which are analogous to the ones performed on P. Then Lemma 5.10 implies that each of the two left-hand side block matrices of (5.20) must be invertible in $\mathcal{RH}_\infty$. In fact, (5.20) is satisfied by comparing it with the equation (5.14). □

5.5 Feedback Properties

In this section, we discuss the properties of a feedback system. In particular, we consider the benefit of the feedback structure and the concept of design tradeoffs for conflicting objectives – namely, how to achieve the benefits of feedback in the face of uncertainties.

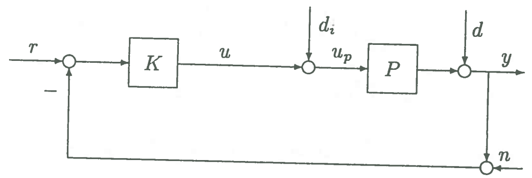


Figure 5.3: Standard Feedback Configuration

Consider again the feedback system shown in Figure 5.1. For convenience, the system diagram is shown again in Figure 5.3. For further discussion, it is convenient to define the *input loop transfer matrix*, L_i, and *output loop transfer matrix*, L_o, as

$$L_i = KP, \quad L_o = PK,$$

respectively, where L_i is obtained from breaking the loop at the input (u) of the plant while L_o is obtained from breaking the loop at the output (y) of the plant. The *input sensitivity* matrix is defined as the transfer matrix from d_i to u_p:

$$S_i = (I + L_i)^{-1}, \quad u_p = S_i d_i.$$

And the *output sensitivity* matrix is defined as the transfer matrix from d to y:

$$S_o = (I + L_o)^{-1}, \quad y = S_o d.$$

The *input* and *output complementary sensitivity* matrices are defined as

$$T_i = I - S_i = L_i(I + L_i)^{-1}$$

$$T_o = I - S_o = L_o(I + L_o)^{-1},$$

respectively. (The word *complementary* is used to signify the fact that T is the complement of S, $T = I - S$.) The matrix $I + L_i$ is called *input return difference matrix* and $I + L_o$ is called *output return difference matrix.*

It is easy to see that the closed-loop system, if it is internally stable, satisfies the following equations:

$$\begin{aligned} y &= T_o(r - n) + S_o P d_i + S_o d && (5.24)\\ r - y &= S_o(r - d) + T_o n - S_o P d_i && (5.25)\\ u &= K S_o(r - n) - K S_o d - T_i d_i && (5.26)\\ u_p &= K S_o(r - n) - K S_o d + S_i d_i. && (5.27) \end{aligned}$$

These four equations show the fundamental benefits and design objectives inherent in feedback loops. For example, equation (5.24) shows that the effects of disturbance d on the plant output can be made "small" by making the output sensitivity function S_o small. Similarly, equation (5.27) shows that the effects of disturbance d_i on the plant input can be made small by making the input sensitivity function S_i small. The notion of smallness for a transfer matrix in a certain range of frequencies can be made explicit using frequency dependent singular values, for example, $\overline{\sigma}(S_o) < 1$ over a frequency range would mean that the effects of disturbance d at the plant output are effectively desensitized over that frequency range.

Hence, good disturbance rejection at the plant output (y) would require that

$$\begin{aligned} \overline{\sigma}(S_o) &= \overline{\sigma}\left((I + PK)^{-1}\right) = \frac{1}{\underline{\sigma}(I + PK)}, && \text{(for disturbance at plant output, } d)\\ \overline{\sigma}(S_o P) &= \overline{\sigma}\left((I + PK)^{-1}P\right) = \overline{\sigma}(P S_i), && \text{(for disturbance at plant input, } d_i) \end{aligned}$$

be made small and good disturbance rejection at the plant input (u_p) would require that

$$\begin{aligned} \overline{\sigma}(S_i) &= \overline{\sigma}\left((I + KP)^{-1}\right) = \frac{1}{\underline{\sigma}(I + KP)}, && \text{(for disturbance at plant input, } d_i)\\ \overline{\sigma}(S_i K) &= \overline{\sigma}\left(K(I + PK)^{-1}\right) = \overline{\sigma}(K S_o), && \text{(for disturbance at plant output, } d) \end{aligned}$$

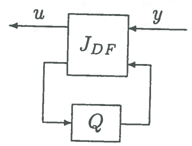
be made small, particularly in the low frequency range where d and d_i are usually significant.

Note that

$$\begin{aligned} \underline{\sigma}(PK) - 1 &\le \underline{\sigma}(I + PK) \le \underline{\sigma}(PK) + 1\\ \underline{\sigma}(KP) - 1 &\le \underline{\sigma}(I + KP) \le \underline{\sigma}(KP) + 1 \end{aligned}$$

The $DF(OE)$ problem can also be considered as the special case of OF by simply setting $D_{21} = I(D_{12} = I)$ and $D_{22} = 0$.

Corollary 12.14 *Consider the DF problem, and assume that (C_2, A, B_2) is stabilizable and detectable. Let F and L be such that $A + LC_2$ and $A + B_2F$ are stable, and then all controllers that internally stabilize G can be parameterized as $\mathcal{F}_l(J_{DF}, Q)$ for some $Q \in \mathcal{RH}_\infty$, i.e. the transfer function from y to u is shown as below*



$$J_{DF} = \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

Remark 12.8 It would be interesting to compare this result with Lemma 12.11. It can be seen that Lemma 12.11 is a special case of this corollary. The condition that $A - B_1C_2$ is stable, which is required in Lemma 12.11, provides the natural injection matrix $L = -B_1$ which satisfies a partial condition in this corollary. ♡

12.4 Structure of Controller Parameterization

Let us recap what we have done. We begin with a stabilizable and detectable realization of G_{22}

$$G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right].$$

We choose F and L so that $A + B_2F$ and $A + LC_2$ are stable. Define J by the formula in Theorem 12.8. Then the proper K's achieving internal stability are precisely those representable in Figure 12.3 and $K = \mathcal{F}_\ell(J, Q)$ where $Q \in \mathcal{RH}_\infty$ and $I + D_{22}Q(\infty)$ is invertible.

It is interesting to note that the system in the dashed box is an observer-based stabilizing controller for G (or G_{22}). Furthermore, it is easy to show that the transfer function between (y, y_1) and (u, u_1) is J, i.e.,

$$\begin{bmatrix} u \\ u_1 \end{bmatrix} = J \begin{bmatrix} y \\ y_1 \end{bmatrix}.$$

It is also easy to show that the transfer matrix from y_1 to u_1 is zero.

This diagram of the parameterization of all stabilizing controllers also suggests an interesting interpretation: every internal stabilization amounts to adding stable dynamics to the plant and then stabilizing the extended plant by means of an observer. The precise statement is as follows: for simplicity of the formulas, only the cases of strictly proper G_{22} and K are treated.

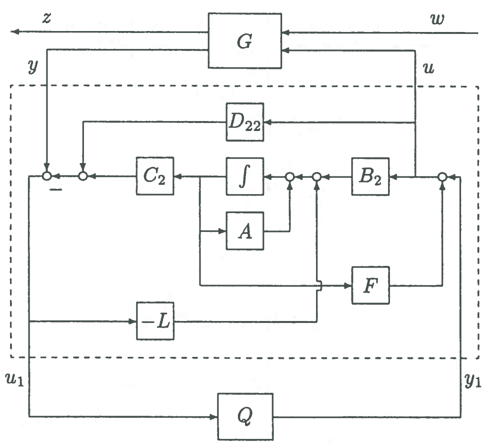


Figure 12.3: Structure of Stabilizing Controllers

Theorem 12.15 *Assume that G_{22} and K are strictly proper and the system is Figure 12.1 is internally stable. Then G_{22} can be embedded in a system*

$$\left[\begin{array}{c|c} A_e & B_e \\ \hline C_e & 0 \end{array}\right]$$

where

$$A_e = \begin{bmatrix} A & 0 \\ 0 & A_a \end{bmatrix}, \; B_e = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \; C_e = \begin{bmatrix} C_2 & 0 \end{bmatrix} \tag{12.3}$$

and where A_a is stable, such that K has the form

$$K = \left[\begin{array}{c|c} A_e + B_eF_e + L_eC_e & -L_e \\ \hline F_e & 0 \end{array}\right] \tag{12.4}$$

where $A_e + B_eF_e$ and $A_e + L_eC_e$ are stable.

Proof. K is representable as in Figure 12.3 for some Q in $\mathcal{RH}_\infty$. For K to be strictly proper, Q must be strictly proper. Take a minimal realization of Q:

$$Q = \left[\begin{array}{c|c} A_a & B_a \\ \hline C_a & 0 \end{array}\right].$$

Since $Q \in \mathcal{RH}_\infty$, A_a is stable. Let x and x_a denote state vectors for J and Q, respectively, and write the equations for the system in Figure 12.3:

$$\begin{aligned} \dot{x} &= (A + B_2F + LC_2)x - Ly + B_2y_1 \\ u &= Fx + y_1 \\ u_1 &= -C_2x + y \\ \dot{x}_a &= A_ax_a + B_au_1 \\ y_1 &= C_ax_a \end{aligned}$$

These equations yield

$$\begin{aligned} \dot{x}_e &= (A_e + B_eF_e + L_eC_e)x_e - L_ey \\ u &= F_ex_e \end{aligned}$$

where

$$x_e := \begin{bmatrix} x \\ x_a \end{bmatrix}, \; F_e := \begin{bmatrix} F & C_a \end{bmatrix}, \; L_e := \begin{bmatrix} L \\ -B_a \end{bmatrix}$$

and where A_e, B_e, C_e are as in (12.3). □

12.5 Closed-Loop Transfer Matrix

Recall that the closed-loop transfer matrix from w to z is a linear fractional transformation $\mathcal{F}_\ell(G, K)$ and that K stabilizes G if and only if K stabilizes G_{22}. Elimination of the signals u and y in Figure 12.3 leads to Figure 12.4 for a suitable transfer matrix T. Thus all closed-loop transfer matrices are representable as in Figure 12.4.

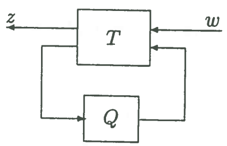


Figure 12.4: Closed loop system

$$z = \mathcal{F}_\ell(G, K)w = \mathcal{F}_\ell(G, \mathcal{F}_\ell(J, Q))w = \mathcal{F}_\ell(T, Q)w. \tag{12.5}$$

It remains to give a realization of T.

Theorem 12.16 *Let F and L be such that $A + BF$ and $A + LC$ are stable. Then the set of all closed-loop transfer matrices from w to z achievable by an internally stabilizing proper controller is equal to*

$$\mathcal{F}_\ell(T, Q) = \{T_{11} + T_{12}QT_{21} : \; Q \in \mathcal{RH}_\infty, \; I + D_{22}Q(\infty) \text{ invertible}\}$$

where T is given by

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} A + B_2F & -B_2F & B_1 & B_2 \\ 0 & A + LC_2 & B_1 + LD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right].$$

Proof. This is straightforward by using the state space star product formula and follows from some tedious algebra, which the interested reader may easily verify. □

An important point to note is that the closed-loop transfer matrix is simply an affine function of the controller parameter matrix Q since $T_{22} = 0$.

12.6 Youla Parameterization via Coprime Factorization*

In this section, all stabilizing controller parameterization will be derived using the conventional coprime factorization approach. Readers should be familiar with the results presented in Section 5.4 of Chapter 5 before proceeding further.

Theorem 12.17 *Let $G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be the rcf and lcf of G_{22} over $\mathcal{RH}_\infty$, respectively. Then the set of all proper controllers achieving internal stability is parameterized either by*

$$K = (U_0 + MQ_r)(V_0 + NQ_r)^{-1}, \quad \det(I + V_0^{-1}NQ_r)(\infty) \neq 0 \tag{12.6}$$

for $Q_r \in \mathcal{RH}_\infty$ or by

$$K = (\tilde{V}_0 + Q_l\tilde{N})^{-1}(\tilde{U}_0 + Q_l\tilde{M}), \quad \det(I + Q_l\tilde{N}\tilde{V}_0^{-1})(\infty) \neq 0 \tag{12.7}$$

for $Q_l \in \mathcal{RH}_\infty$ where $U_0, V_0, \tilde{U}_0, \tilde{V}_0 \in \mathcal{RH}_\infty$ satisfy the Bezout identities:

$$\tilde{V}_0M - \tilde{U}_0N = I, \quad \tilde{M}V_0 - \tilde{N}U_0 = I.$$

Moreover, if $U_0, V_0, \tilde{U}_0$, and $\tilde{V}_0$ are chosen such that $U_0V_0^{-1} = \tilde{V}_0^{-1}\tilde{U}_0$, i.e.,

$$\begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U_0 \\ N & V_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Then

$$\begin{aligned} K &= (U_0 + MQ_y)(V_0 + NQ_y)^{-1} \\ &= (\tilde{V}_0 + Q_y\tilde{N})^{-1}(\tilde{U}_0 + Q_y\tilde{M}) \\ &= \mathcal{F}_\ell(J_y, Q_y) \end{aligned} \tag{12.8}$$

where

$$J_y := \begin{bmatrix} U_0V_0^{-1} & \tilde{V}_0^{-1} \\ V_0^{-1} & -V_0^{-1}N \end{bmatrix} \tag{12.9}$$

and where Q_y ranges over $\mathcal{RH}_\infty$ such that $(I + V_0^{-1}NQ_y)(\infty)$ is invertible

Proof. We shall prove the parameterization given in (12.6) first. Assume that K has the form indicated, and define

$$U := U_0 + MQ_r,\ V := V_0 + NQ_r.$$

Then

$$\tilde{M}V - \tilde{N}U = \tilde{M}(V_0 + NQ_r) - \tilde{N}(U_0 + MQ_r) = \tilde{M}V_0 - \tilde{N}U_0 + (\tilde{M}N - \tilde{N}M)Q_r = I.$$

Thus K achieves internal stability by Lemma 5.10.

Conversely, suppose K is proper and achieves internal stability. Introduce an rcf of K over $\mathcal{RH}_\infty$ as $K = UV^{-1}$. Then by Lemma 5.10, $Z := \tilde{M}V - \tilde{N}U$ is invertible in $\mathcal{RH}_\infty$. Define Q_r by the equation

$$U_0 + MQ_r = UZ^{-1}, \tag{12.10}$$

so

$$Q_r = M^{-1}(UZ^{-1} - U_0).$$

Then using the Bezout identity, we have

$$\begin{aligned} V_0 + NQ_r &= V_0 + NM^{-1}(UZ^{-1} - U_0) \\ &= V_0 + \tilde{M}^{-1}\tilde{N}(UZ^{-1} - U_0) \\ &= \tilde{M}^{-1}(\tilde{M}V_0 - \tilde{N}U_0 + \tilde{N}UZ^{-1}) \\ &= \tilde{M}^{-1}(I + \tilde{N}UZ^{-1}) \\ &= \tilde{M}^{-1}(Z + \tilde{N}U)Z^{-1} \\ &= \tilde{M}^{-1}\tilde{M}VZ^{-1} \\ &= VZ^{-1}. \end{aligned} \tag{12.11}$$

Thus,

$$\begin{aligned} K &= UV^{-1} \\ &= (U_0 + MQ_r)(V_0 + NQ_r)^{-1}. \end{aligned}$$

To see that Q_r belongs to $\mathcal{RH}_\infty$, observe first from (12.10) and then from (12.11) that both MQ_r and NQ_r belong to $\mathcal{RH}_\infty$. Then

$$Q_r = (\tilde{V}_0M - \tilde{U}_0N)Q_r = \tilde{V}_0(MQ_r) - \tilde{U}_0(NQ_r) \in \mathcal{RH}_\infty.$$

Finally, since V and Z evaluated at $s = \infty$ are both invertible, so is $V_0 + NQ_r$ from (12.11), hence so is $I + V_0^{-1}NQ_r$.

Similarly, the parameterization given in (12.7) can be obtained.

To show that the controller can be written in the form of equation (12.8), note that

$$(U_0 + MQ_y)(V_0 + NQ_y)^{-1} = U_0V_0^{-1} + (M - U_0V_0^{-1}N)Q_y(I + V_0^{-1}NQ_y)^{-1}V_0^{-1}$$

and that $U_0V_0^{-1} = \tilde{V}_0^{-1}\tilde{U}_0$. We have

$$(M - U_0V_0^{-1}N) = (M - \tilde{V}_0^{-1}\tilde{U}_0N) = \tilde{V}_0^{-1}(\tilde{V}_0M - \tilde{U}_0N) = \tilde{V}_0^{-1}$$

and

$$K = U_0V_0^{-1} + \tilde{V}_0^{-1}Q_y(I + V_0^{-1}NQ_y)^{-1}V_0^{-1}. \tag{12.12}$$

□

Corollary 12.18 *Given an admissible controller K with coprime factorizations $K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$, the free parameter $Q_y \in \mathcal{RH}_\infty$ in Youla parameterization is given by*

$$Q_y = M^{-1}(UZ^{-1} - U_0)$$

where

$$Z := \tilde{M}V - \tilde{N}U.$$

Next, we shall establish the precise relationship between the above all stabilizing controller parameterization and the parameterization obtained in the previous sections via LFT framework.

Theorem 12.19 *Let the doubly coprime factorizations of G_{22} be chosen as*

$$\begin{bmatrix} M & U_0 \\ N & V_0 \end{bmatrix} = \left[\begin{array}{c|cc} A + B_2F & B_2 & -L \\ \hline F & I & 0 \\ C_2 + D_{22}F & D_{22} & I \end{array}\right]$$

$$\begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A + LC_2 & -(B_2 + LD_{22}) & L \\ \hline F & I & 0 \\ C_2 & -D_{22} & I \end{array}\right]$$

where F and L are chosen such that $A + B_2F$ and $A + LC_2$ are both stable.

Then J_y can be computed as

$$J_y = \left[\begin{array}{c|cc} A + B_2F + LC_2 + LD_{22}F & -L & B_2 + LD_{22} \\ \hline F & 0 & I \\ -(C_2 + D_{22}F) & I & -D_{22} \end{array}\right].$$

Proof. This follows from some tedious algebra. □

Remark 12.9 Note that J_y is exactly the same as the J in Theorem 12.8 and that $K_0 := U_0 V_0^{-1}$ is an observer-based stabilizing controller with

$$K_0 := \left[\begin{array}{c|c} A + B_2F + LC_2 + LD_{22}F & -L \\ \hline F & 0 \end{array}\right].$$

♡

12.7 Notes and References

The special problems FI, DF, FC, and OE were first introduced in Doyle, Glover, Khargonekar, and Francis [1989] for solving the $\mathcal{H}_\infty$ problem, and they have been since used in many other papers for different problems. The new derivation of all stabilizing controllers was reported in Lu, Zhou, and Doyle [1991]. The paper by Moore *et al* [1990] contains some other related interesting results. The conventional Youla parameterization can be found in Youla *et al* [1976], Desoer *et al* [1980], Doyle [1984], Vidyasagar [1985], and Francis [1987]. The parameterization of all two-degree-of-freedom stabilizing controllers is given in Youla and Bongiorno [1985] and Vidyasagar [1985].

13

Algebraic Riccati Equations

We have studied the Lyapunov equation in Chapter 3 and have seen the roles it played in some applications. A more general equation than the Lyapunov equation in control theory is the so-called *Algebraic Riccati Equation* or ARE for short. Roughly speaking, Lyapunov equations are most useful in system analysis while AREs are most useful in control system synthesis; particularly, they play the central roles in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control.

Let A, Q, and R be real $n \times n$ matrices with Q and R symmetric. Then an algebraic Riccati equation is the following matrix equation:

$$A^*X + XA + XRX + Q = 0. \tag{13.1}$$

Associated with this Riccati equation is a $2n \times 2n$ matrix:

$$H := \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}. \tag{13.2}$$

A matrix of this form is called a *Hamiltonian matrix*. The matrix H in (13.2) will be used to obtain the solutions to the equation (13.1). It is useful to note that $\sigma(H)$ (the spectrum of H) is symmetric about the imaginary axis. To see that, introduce the $2n \times 2n$ matrix:

$$J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

having the property $J^2 = -I$. Then

$$J^{-1}HJ = -JHJ = -H^*$$