Systems of systems
Differential Games

Lecturer:

- John Leth

In this lecture I will discuss generalizations of the optimal control problem introduced previously. More precisely, we will discuss two different generalizations; more than one player (control) connected via the same cost but with different objectives (min and max), and more than one player each trying to optimize individual costs depending on all players. These go under the headline differential games.

Headlines:

- Differential Games:

  Zero-sum differential games.

  Nonzero-sum differential games.

Literature:

- Ch. 12.1, 12.2 in [ST00].

Exercises:

1. Consider the nonzero-sum, 2-player differential game

$$\max_{u_i \in \mathbb{R}} J_i = -\frac{1}{2}x(T)^2 - \frac{1}{2}\int_0^T r_i(u_1^2 + u_2^2)dt, \quad i = 1, 2, \tag{1a}$$

   subject to

$$\dot{x} = ax + b_1u_1 + b_2u_2,\ x(0) = x_0, \tag{1b}$$

   where $x_0, a, b_i \in \mathbb{R}$ and $r_i, T > 0$ for $i = 1, 2$ are *given* constants.

   (a) Write-up the Hamiltonian $H^i,\ i = 1, 2$, for the ith player.

   (b) Find the[1] (open-loop) Nash solution $\{\bar{u}_1, \bar{u}_2\}$ for (1) (the $\bar{u}_i$'s will depend on the adjoint variables).

   (c) Write-up the adjoint equation's.

   (d) Solve the adjoint equation's (it depend on $x(T)$) and use the obtained solutions to write-up the Nash solutions, found in exercise (1b), as functions of time.

   (e) (Extra) Use the expression for $\bar{u}_i,\ i = 1, 2$, to solve (1b).

2. (Extra) Consider the nonzero-sum, N-player differential game

$$\max_{u_i \in \mathbb{R}^m} J_i = \frac{1}{2}\int_0^T (x^t Q_i x + \sum_{j=1}^N u_j^t R_{ij} u_j)dt, \tag{2a}$$

   subject to

$$\dot{x} = Ax + \sum_{j=1}^N B_j u_j,\ x(0) = x_0, \tag{2b}$$

   with Q_i and R_{ij} symmetric, and $R_{ii} < 0$. Find the[1] (closed-loop) Nash solution $\{\bar{u}_1, \ldots, \bar{u}_m\}$ for (2) and derive a Riccati equation when it is assumed that $\lambda_i(t) = S_i(t)\bar{x}(t)$ with S_i a unknown symmetric matrix which is to be determined via a Riccati like equation.

3. Let $E_1(t)$ and $E_2(t)$ be two entities in a smart grid with $E_1(t)$ capable of producing $E_2(t) \geq 0$ energy and $E_2(t)$ capable of consuming $E_1(t) \leq 0$ energy. Assume that the price $P(t)$ of energy is given by

[1] In the course you where only given necessary conditions for the existence of a Nash solution. However, for this particular problem these conditions are also sufficient.

$P(t) = -(E_1(t) + E_2(t))$, that the total cost $C_i(t)$ of entity $E_i(t)$ is $(-1)^{i+1}P(t)E_i(t)$, and that the entities can be control via

$$\dot{E}_1(t) = E_1(t) + u_1(t),$$
$$\dot{E}_2(t) = E_2(t) + u_2(t)$$

The task is now for E_1 to maximize and for E_2 to minimize the following total cost

$$J = \int_0^T C(t) - u_1(t)^2 + u_2(t)^2 \; dt$$

with $C(t)$ the sum of the total cost of each entity.

a. Write $C(t)$ as a function of $E_1(t)$ and $E_2(t)$.

b. Use the zero-sum maximum principle to write down the state- and costate equations.

c. For $T = 12$ use Matlab to find the optimal control input when $(E_1(0), E_2(0)) = (100, -50)$. Hint: discretize (with $\Delta t = 1$) the state- and costate equations, and then start by finding $\lambda(0)$.

d. Repeat item c. with $\Delta t = 0.1$.

e. (Extra) Use Matlab to find the optimal control input when $(E_1(0), E_2(0)) = (100, -50)$ without discretizing