

# Compositional Stability Analysis Based on Dual Decomposition

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**Abstract**—We propose a compositional stability analysis framework for verifying properties of systems that are interconnections of multiple subsystems. The proposed method assembles stability certificates for the interconnected system based on the certificates for the input-output properties of the subsystems. The decomposition in the analysis is achieved by utilizing dual decomposition ideas from optimization. Decoupled subproblems establish subsystem level input-output properties whereas the “master” problem imposes and updates the conditions on the subproblems toward ensuring interconnected system level stability properties. Both global stability analysis and region-of-attraction analysis are discussed.

## I. INTRODUCTION

We propose a compositional analysis framework for verifying stability properties of systems that are formed as interconnection of multiple subsystems. The method constructs certificates of input-output properties of subsystems in isolation from other subsystems and assembles stability certificates for the interconnected system based on these subsystem certificates. The assembly of system level certificates from subsystem certificates, of course, has to account for the fact that the output of a subsystem is the input of another subsystem (i.e., for the interconnection structure). In particular, consider a signal  $w$  which is the output of subsystem  $A$  and at the same time the input of subsystem  $B$ . Then, in the analysis of subsystem  $A$  the “output” signal  $w$  has to satisfy all properties that are assumed for the “input” signal  $w$  in the analysis of subsystem  $B$ . Such matching conditions at the interfaces between subsystems are introduced as coupling constraints. Then, decomposition of analysis is achieved by utilizing the dual decomposition techniques where violation of these coupling constraints is allowed but iteratively reduced by adapting the subsystem level analysis questions through a subgradient type optimization scheme.

In addition to dual decomposition from optimization, the proposed scheme utilizes ideas from multiple domains. First of all the subsystem level analysis builds on the dissipation inequalities and storage functions [1] to characterize input-output properties. However, local versions of these concepts, where dissipation inequalities hold over bounded subsets of the state space and verify input-output properties for certain levels of inputs norms but not necessarily for signals with larger norm, are emphasized as discussed in [2], [3]. Moreover, especially the compositional analysis framework

for region-of-attraction estimation (introduced in section III) is inspired by the assume-guarantee type compositional techniques that have been proposed as a partial remedy for the “state explosion” problem in software verification [4], [5]. The updates of the dual variables by the subgradient algorithm can be interpreted as automated adjustments to the assumptions in assume-guarantee schemes.

The motivation for the current work stems from the computational complexity of optimization-based analysis of nonlinear dynamical systems and in particular our earlier work on sum-of-squares (SOS) optimization [6] based quantitative local analysis of systems governed by polynomial ordinary differential equations [7], [8], [9], [10]. The growth of the “problem size” in SOS optimization based analysis with the state dimension and the degree of the polynomial certificates (e.g. Lyapunov functions) is so fast that even the complexity of problems for systems of modest state dimension exceeds the capabilities of currently available computational resources [11] (see Table I). Moreover, local (quantitative) analysis leads to bilinear, non-convex optimization problems adding to the computational complexity [9]. Therefore, the use of optimization-based techniques strongly depend on the improvements in the scalability of the algorithms.

Compositional analysis and design have a long history in controls and we here give a very limited list of references mostly as it ties to the current paper. The survey [12] and the volume [13] provide an exposition to the early work. Reference [14] used the primal decomposition to decouple large-scale linear matrix inequalities that appear in the distributed analysis of systems composed of different sub-units, interconnected over an arbitrary graph and [15] employed the dual decomposition in distributed optimal control.

The next section discusses the compositional analysis for global stability and section III presents extensions for the region-of-attraction analysis. These sections are followed by a simple example and concluding remarks. We emphasize that the current paper aims at a simplified exposition of and a proof-of-concept for the proposed methodology. Detailed analysis and demonstrations are subject to current study.

## II. GLOBAL STABILITY ANALYSIS

This section presents the compositional analysis for the case where the subsystems satisfy input-output properties globally. Global input-output properties are relations between inputs and outputs of the system that hold independent of the “level” of the input. For example, for the input  $w$  and the output  $z$  of a system, the global  $\mathcal{L}_2$ -gain relation holds if there exists a constant  $\gamma > 0$  such that  $\|z\|_2 \leq \gamma \|w\|_2$  for all

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n	2d							
	4		6		8		10	
2	6	6	10	27	15	75	21	165
5	21	105	56	1134	126	6714	<b>252</b>	<b>2e4</b>
9	55	825	<b>220</b>	<b>1e4</b>	<b>715</b>	<b>2e5</b>	*	*
14	120	4200	<b>680</b>	*	*	*	*	*
16	<b>153</b>	<b>6936</b>	*	*	*	*	*	*

TABLE I. The number  $N_{\text{decision}}$  of decision variables (right column in each block) and the size  $N_{\text{SDP}}$  of the matrix (left column in each block) in the semidefinite program for checking the existence of a SOS decomposition for a degree  $2d$  polynomial in  $n$  variables.

values of  $\|w\|_2$ , where  $\|\cdot\|_2$  denotes  $\mathcal{L}_2$  signal norm.<sup>1</sup> Note that global  $\mathcal{L}_2$ -gain relations hold for stable linear systems whereas nonlinear systems may or may not satisfy such relations. That is, for the input  $w$  and the output  $z$  of a nonlinear system, there may exist a constant  $\gamma > 0$  such that  $\|z\|_2 \leq \gamma\|w\|_2$  for all  $w$  with  $\|w\|_2 \leq R$  for some  $R$  but this gain relation may not hold for larger values of  $R$ .

For ease of presentation, we will focus the discussion on the interconnection of three subsystems as shown in Figure 1. Extensions of the framework to more general system interconnections is straightforward. In Figure 1,  $w_1, w_2, w_3$ , and  $w_4$  denote the internal signals. Let  $x_1 \in \mathcal{R}^{n_1}$ ,  $x_2 \in \mathcal{R}^{n_2}$ , and  $x_3 \in \mathcal{R}^{n_3}$  be the states of the subsystems whose evolution is governed by the following differential equations with the output maps  $h_1 : \mathcal{R}^{n_1} \rightarrow \mathcal{R}^{m_1}$ ,  $h_2 : \mathcal{R}^{n_2} \rightarrow \mathcal{R}^{m_2+m_3}$ , and  $h_3 : \mathcal{R}^{n_3} \rightarrow \mathcal{R}^{m_4}$

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_1, w_2, w_3) \\
 w_1 &= h_1(x_1) \\
 \dot{x}_2 &= f_2(x_2, w_1) \\
 (w_2, w_3) &= h_2(x_2) \\
 \dot{x}_3 &= f_3(x_3, w_3) \\
 w_4 &= h_3(x_3)
 \end{aligned} \tag{1}$$

that satisfy  $f_1(0, 0, 0) = 0$ ,  $h_1(0) = 0$ ,  $f_2(0, 0) = 0$ ,  $h_2(0) = 0$ ,  $f_3(0, 0) = 0$ , and  $h_3(0) = 0$ . That is  $(x_1, x_2, x_3) = (0, 0, 0)$ ,  $x_1 = 0$ ,  $x_2 = 0$ , and  $x_3 = 0$  are equilibrium points of the interconnected system and respective subsystems.

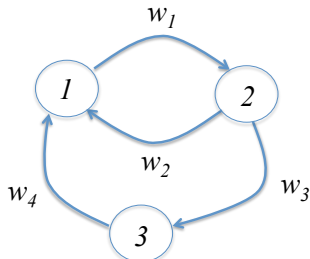


Fig. 1. Interconnection of 3 systems.

The compositional stability analysis builds on the following Lyapunov-type result which provides a sufficient condition for the stability of the interconnected system around the equilibrium point at the origin.

<sup>1</sup>With the extra assumption that the system starts from rest where the output of the system vanishes.

**Theorem 1:** If there exists a positive definite  $\mathcal{C}^1$  function  $V : \mathcal{R}^{n_1+n_2+n_3} \rightarrow \mathcal{R}$  such that  $V(0) = 0$  and  $\dot{V}(x_1, x_2, x_3) < 0$  for all nonzero  $(x_1, x_2, x_3) \in \mathcal{R}^{n_1+n_2+n_3}$ , then the system in Figure 1 is internally asymptotically stable around  $(x_1, x_2, x_3) = (0, 0, 0)$ .  $\triangleleft$

In theorem 1 and hereafter, for a map  $x \rightarrow V(x)$ ,  $\dot{V}(x)$  denotes  $\nabla V(x)\dot{x}$  and  $\mathcal{C}^1$  denotes the set of scalar valued, continuously differential functions on  $\mathcal{R}^n$ .

The goal is to construct a Lyapunov function (i.e., a function that satisfies the conditions in Theorem 1) to verify the internal stability of the interconnected system but through isolated analysis of the input-output properties of subsystems 1, 2, and 3. To this end, we resort to **Willems' dissipation inequalities theory [1]** and use the following proposition to obtain sufficient conditions for asymptotic stability.

**Proposition 1:** If there exist positive definite  $\mathcal{C}^1$  functions  $V_1 : \mathcal{R}^{n_1} \rightarrow \mathcal{R}$ ,  $V_2 : \mathcal{R}^{n_2} \rightarrow \mathcal{R}$ , and  $V_3 : \mathcal{R}^{n_3} \rightarrow \mathcal{R}$  such that  $V_1(0) = 0$ ,  $V_2(0) = 0$ , and  $V_3(0) = 0$  and positive real numbers  $\gamma_{11}, \gamma_{12}, \gamma_{14}, \gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{33}$ , and  $\gamma_{34}$  such that

$$\begin{aligned}
 \dot{V}_1(x_1, w_2, w_4) &< -\gamma_{11}w_1^T w_1 + \gamma_{12}w_2^T w_2 + \gamma_{14}w_4^T w_4 \\
 &= -\gamma_{11}h_1(x_1)^T h_1(x_1) + \gamma_{12}w_2^T w_2 + \gamma_{14}w_4^T w_4 \\
 &\quad \text{for all } w_2, w_4, \text{ and nonzero } x_1
 \end{aligned} \tag{2}$$

$$\begin{aligned}
 \dot{V}_2(x_2, w_1) &< -\gamma_{22}w_2^T w_2 - \gamma_{23}w_3^T w_3 + \gamma_{21}w_1^T w_1 \\
 &= -h_2(x_2)^T \begin{bmatrix} \gamma_{22} & 0 \\ 0 & \gamma_{23} \end{bmatrix} h_2(x_2) + \gamma_{21}w_1^T w_1 \\
 &\quad \text{for all } w_1, \text{ and nonzero } x_2
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 \dot{V}_3(x_3, w_3) &< -\gamma_{34}w_4^T w_4 + \gamma_{33}w_3^T w_3 \\
 &= -\gamma_{34}h_3(x_3)^T h_3(x_3) + \gamma_{33}w_3^T w_3 \\
 &\quad \text{for all } w_3 \text{ and nonzero } x_3
 \end{aligned} \tag{4}$$

and

$$\begin{aligned}
 -\gamma_{11} + \gamma_{21} &\leq 0, & -\gamma_{22} + \gamma_{12} &\leq 0, \\
 -\gamma_{23} + \gamma_{33} &\leq 0, & -\gamma_{34} + \gamma_{14} &\leq 0,
 \end{aligned} \tag{5}$$

then  $(x_1, x_2, x_3) = (0, 0, 0)$  is an (internally) asymptotically stable equilibrium point of the system in Figure 1.  $\triangleleft$

**Remark 1:** Note that the conditions in (2)-(4) and (5) are homogenous in the decision variables in  $V_i$ 's and  $\gamma_{ij}$ 's. This can be avoided by setting some of the  $\gamma_{ij}$ 's (for example  $\gamma_{11}, \gamma_{21}$ , and  $\gamma_{34}$ ) to 1. By such normalization, the number constraints in (5) can be reduced. However, we don't employ this normalization in order to keep the conditions in (2)-(4) and (5) notationally symmetric. Instead, we avoid this homogeneity by properly normalizing one of the decision variables in each  $V_i$ .  $\triangleleft$

**Proof:** (of Proposition 1) Let  $V : \mathcal{R}^{n_1+n_2+n_3} \rightarrow \mathcal{R}$  be defined through  $V(x_1, x_2, x_3) := V_1(x_1) + V_2(x_2) + V_3(x_3)$  and note that  $V(x_1, x_2, x_3) > 0$  for all nonzero  $(x_1, x_2, x_3)$  and  $V(0, 0, 0) = 0$ . Then,  $\dot{V}(x_1, x_2, x_3) < (-\gamma_{11} + \gamma_{21})w_1^T w_1 + (-\gamma_{22} + \gamma_{12})w_2^T w_2 + (-\gamma_{23} + \gamma_{33})w_3^T w_3 + (-\gamma_{34} + \gamma_{14})w_4^T w_4 \leq 0$  for all  $x_1, x_2, x_3, w_1, w_2, w_3$ , and  $w_4$  where the first inequality follows from (2)-(4) and the second from (5). Consequently,  $V$  satisfies the conditions in Theorem 1 and the closed-loop system is asymptotically stable around the origin.  $\blacksquare$

The inequalities in (2)-(4) are dissipation inequalities with the quadratic supply rates  $-\gamma_{11}w_1^T w_1 + \gamma_{12}w_2^T w_2 + \gamma_{14}w_4^T w_4$ ,  $-\gamma_{22}w_2^T w_2 - \gamma_{23}w_3^T w_3 + \gamma_{21}w_1^T w_1$ , and  $-\gamma_{34}w_4^T w_4 + \gamma_{33}w_3^T w_3$ , respectively [1]. In fact, conditions in (2)-(4) are slight generalizations of the dissipation inequalities corresponding to induced  $\mathcal{L}_2$ -gain relations.  $V_1$ ,  $V_2$ , and  $V_3$  are called the storage functions associated with the respective subsystems and the supply rates. Roughly speaking conditions in (2)-(4) constrain the (weighted)  $\mathcal{L}_2$ -norms of the outputs of the subsystems in terms of the (weighted)  $\mathcal{L}_2$ -norms of the inputs and conditions in (5) ensure the internal stability of the interconnection of these subsystems. Proposition 1 can be considered as a generalization of the small-gain theorem [16], [17] for the interconnection of multiple subsystems (with specified governing equations in (1) and “gain” relations in (2)-(4)).

One would typically use a specific finite (linear) parameterization for  $V_i$ 's (e.g. quadratic or polynomial functions) and search for  $V_i$ 's and  $\gamma_{ij}$ 's satisfying the conditions in (2)-(4) and (5) through numerical optimization (or feasibility search). For example, for linear dynamics and output maps and quadratic parameterizations for the storage functions, conditions in (2)-(4) and (5) lead to standard linear matrix inequality (LMI) constraints [18] and for polynomial dynamics and storage functions this search can be performed through sum-of-squares programming [6]. Note that conditions in (2)-(4) are only coupled through  $\gamma_{ij}$ 's. Typically, the number of decision variables in  $\gamma_{ij}$ 's is a small fraction of the total number of decision variables in the numerical optimization problems for the search of  $V_i$ 's and  $\gamma_{ij}$ 's that satisfy the conditions in Proposition 1.<sup>2</sup>

Toward decoupling the conditions in (2)-(4), consider the following related optimization problem:

$$\max_{\lambda_1, \dots, \lambda_4 \geq 0} \min_{\gamma_{ij} \text{'s}, V_i \text{'s}} \lambda_1(\gamma_{21} - \gamma_{11}) + \lambda_2(\gamma_{12} - \gamma_{22}) + \lambda_3(\gamma_{33} - \gamma_{23}) + \lambda_4(\gamma_{14} - \gamma_{34})$$

$$\text{subject to (2) - (4), } V_1(0) = 0, V_2(0) = 0, V_3(0) = 0, \\ V_1, V_2, V_3 \text{ are } \mathcal{C}^1, \text{ positive definite.}$$

(6)

and re-write the problem in (6) as

$$\max_{\lambda_1, \dots, \lambda_4 \geq 0} \varphi_1(\lambda) + \varphi_2(\lambda) + \varphi_3(\lambda), \quad (7)$$

where

$$\varphi_1(\lambda) := \min_{\gamma_{11}, \gamma_{12}, \gamma_{14}, V_1} -\lambda_1\gamma_{11} + \lambda_2\gamma_{12} + \lambda_4\gamma_{14} \\ \text{subject to (2), } V_1(0) = 0, \quad (8)$$

$$V_1 \text{ is } \mathcal{C}^1, \text{ positive definite, and } \gamma_{11}, \gamma_{12}, \gamma_{14} > 0$$

$$\varphi_2(\lambda) := \min_{\gamma_{22}, \gamma_{23}, \gamma_{21}, V_2} \lambda_1\gamma_{21} - \lambda_2\gamma_{22} - \lambda_3\gamma_{23} \\ \text{subject to (3), } V_2(0) = 0, \quad (9)$$

$$V_2 \text{ is } \mathcal{C}^1, \text{ positive definite, and } \gamma_{22}, \gamma_{23}, \gamma_{21} > 0$$

<sup>2</sup>One of the main difficulties of sum-of-squares programming is that the number of the extra decision variables that are introduced to parameterize the Gram matrices corresponding to the polynomial inequalities rapidly grows with the state dimension.

$$\varphi_3(\lambda) := \min_{\gamma_{33}, \gamma_{34}, V_3} \lambda_3\gamma_{33} - \lambda_4\gamma_{34} \\ \text{subject to (4), } V_3(0) = 0, \quad (10)$$

$$V_3 \text{ is } \mathcal{C}^1, \text{ positive definite, and } \gamma_{33}, \gamma_{34} > 0.$$

For given  $\lambda$ , one can compute  $\varphi_1(\lambda)$ ,  $\varphi_2(\lambda)$ , and  $\varphi_3(\lambda)$  by solving (8)-(10) independently and if the optimizing values of  $\gamma_{ij}$ 's satisfy the inequality constraints in (5), then  $V := V_1 + V_2 + V_3$  satisfies the conditions in Proposition 1 and the internal stability of the interconnected system is certified. If the optimizing values of  $\gamma_{ij}$ 's do not satisfy the inequality constraints in (5), then  $\lambda_1, \dots, \lambda_4$  need to be updated. To this end, we attempt to solve (7) using a subgradient algorithm [19]. Define  $\lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$  and for given  $\lambda$ , let  $\gamma_{ij}^*(\lambda)$  denote the optimal value of  $\gamma_{ij}$  in (8)-(10). Define  $g_1(\lambda) := (-\gamma_{11}^*(\lambda), \gamma_{12}^*(\lambda), 0, \gamma_{14}^*(\lambda))^T$ ,  $g_2(\lambda) := (\gamma_{21}^*(\lambda), -\gamma_{22}^*(\lambda), -\gamma_{23}^*(\lambda), 0)^T$ , and  $g_3(\lambda) := (0, 0, \gamma_{33}^*(\lambda), -\gamma_{34}^*(\lambda))^T$ . Then, by the following inequalities that hold for all  $\mu := (\mu_1, \mu_2, \mu_3, \mu_4)$

$$\varphi_1(\mu) \geq \varphi_1(\lambda) - \gamma_{11}^*(\lambda)(\mu_1 - \lambda_1) + \gamma_{12}^*(\lambda)(\mu_2 - \lambda_2) + \gamma_{14}^*(\lambda)(\mu_4 - \lambda_4)$$

$$\varphi_2(\mu) \geq \varphi_2(\lambda) + \gamma_{21}^*(\lambda)(\mu_1 - \lambda_1) - \gamma_{22}^*(\lambda)(\mu_2 - \lambda_2) - \gamma_{23}^*(\lambda)(\mu_3 - \lambda_3)$$

$$\varphi_3(\mu) \geq \varphi_3(\lambda) + \gamma_{33}^*(\lambda)(\mu_3 - \lambda_3) - \gamma_{34}^*(\lambda)(\mu_4 - \lambda_4),$$

$g_1(\lambda)$ ,  $g_2(\lambda)$ , and  $g_3(\lambda)$  are subgradients of  $\varphi_1$ ,  $\varphi_2$ , and  $\varphi_3$  at  $\lambda$ , respectively [19]. Consequently,  $g_1(\lambda) + g_2(\lambda) + g_3(\lambda)$  is a subgradient of  $\varphi$  at  $\lambda$ . Then, the subgradient method solves the problem in (6) iteratively by updating  $\lambda$  at iteration  $k$  for  $\ell = 1, \dots, 4$  using  $\lambda_\ell^{k+1} = \lambda_\ell^k - \alpha^k \delta_\ell(\lambda^k)_\ell$ , where

$$\delta_1(\lambda^k) = \gamma_{11}^*(\lambda^k) - \gamma_{21}^*(\lambda^k) \quad (11a)$$

$$\delta_2(\lambda^k) = \gamma_{22}^*(\lambda^k) - \gamma_{12}^*(\lambda^k) \quad (11b)$$

$$\delta_3(\lambda^k) = \gamma_{23}^*(\lambda^k) - \gamma_{33}^*(\lambda^k) \quad (11c)$$

$$\delta_4(\lambda^k) = \gamma_{34}^*(\lambda^k) - \gamma_{14}^*(\lambda^k) \quad (11d)$$

and  $\alpha^k > 0$  denotes the step size at iteration  $k$ .

Consider now that at iteration  $k$  one of the constraints in (5), say  $-\gamma_{11}^*(\lambda^k) + \gamma_{21}^*(\lambda^k) \leq 0$ , is violated. Then,

$$\lambda_1^{k+1} = \lambda_1^k - \alpha^k \delta_1(\lambda^k) = \lambda_1^k - \alpha^k (\gamma_{11}^*(\lambda^k) - \gamma_{21}^*(\lambda^k)) > \lambda_1^k.$$

Consequently, if  $-\gamma_{11}^*(\lambda^k) + \gamma_{21}^*(\lambda^k) > 0$ , then the update rule (11) increases  $\lambda_1$  (the dual variable corresponding to the constraint  $-\gamma_{11} + \gamma_{21} \leq 0$ ) and this puts a larger penalty on the violation of the constraint  $-\gamma_{11} + \gamma_{21} \leq 0$ . If multiple constraints in (5) are violated, then the increase in the dual variables corresponding to these violated constraints is proportional to the amount of violation, i.e., the weight of the penalty on the largest violation increases most from iteration  $k$  to iteration  $k+1$ . More generally, the problem in (6) penalizes the violations in the constraints in (5) and then the subgradient algorithms adjusts the level of penalty with an adjustment proportional to the relative size of violation of the “dualized” constraints, i.e., constraints in (5).

Note that  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$  and therefore (for given  $\lambda$ ) the objective of the optimization problem in (8) is to minimize

a linear combination of  $\gamma_{12}$  and  $\gamma_{14}$  with respect to  $\gamma_{11}$ . Referring back to the condition in (2), the subproblem in (8) tries to minimize the gain from a weighted  $\mathcal{L}_2$ -norm of the input signals ( $w_2$  and  $w_4$ ) of subsystem 1 to the  $\mathcal{L}_2$ -norm of the output  $w_1$ . Similar relations can be established between the problems in (9) and (10) and the conditions (3) and (4), respectively. Hence, one reaches at an interpretation of the dual variables parallel to the “price” interpretation (see, for example, [20] for such an interpretation in network analysis): the master problem sets the “prices” for the subsystems for deviating from the system level analysis objectives and, given the “price,” subproblems try to compute an optimal estimate for the subsystem level input-output properties.

*Remark 2:* The procedure that led to the structure with the “master” problem in (7) and the decoupled subproblems in (8)-(10) is called the *dual decomposition* [19], [21] and has been used in many engineering applications including communication networks [20] and more recently in distributed optimal control [15].  $\triangleleft$

*Remark 3:* The dual decomposition based procedure employed in this section is not limited to three subsystem configuration in Figure 1. Its generalization for more than three subsystems is straightforward: the conditions in (2)-(4) are extended to include a dissipation inequality for each subsystem and the conditions in (5) are extended to include all conditions of the form  $-\gamma_{jk} + \gamma_{ik} \leq 0$  whenever  $w_k$  is an input signal to a subsystem  $i$  and an output signal of another subsystem  $j$  (with  $i \neq j$ ).

Extensions are possible by using more general supply rates that those in Proposition 1. For example, general quadratic supply rates (instead of those in (2)-(4)) lead to linear matrix inequalities in the decision variables in the supply rates instead of the linear inequalities in (5).  $\triangleleft$

*Remark 4:* When the dynamics in (1) are linear, time-invariant, problems in (8)-(10) can be solved as (affine) semidefinite programs. For  $f_i$ ’s polynomial in the states and affine in the inputs and  $h_i$ ’s polynomial in the states, there exist well-known sum-of-squares programming based relaxations for (8)-(10) that lead to semidefinite programs.  $\triangleleft$

### III. REGION-OF-ATTRACTION ANALYSIS

This section is devoted to a brief discussion on compositional local stability analysis. Building on the following characterization of invariant subsets of the region-of-attraction around the origin, we propose sufficient conditions that enable the construction of a Lyapunov function using local dissipation inequalities for the subsystems.

*Lemma 1:* Consider a system governed by  $\dot{x} = f(x)$  with  $f(0) = 0$  and  $f$  locally Lipschitz. Let  $R \in \mathcal{R}$  be nonnegative. If there exists a  $\mathcal{C}^1$  function  $V : \mathcal{R}^n \rightarrow \mathcal{R}$  such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \neq 0 \quad (12)$$

$$\Omega_{V,R} := \{x \in \mathcal{R}^n : V(x) \leq R\} \text{ is bounded,} \quad (13)$$

$$\Omega_{V,R} \setminus \{0\} \subset \{x \in \mathcal{R}^n : \nabla V(x)f(x) < 0\}, \quad (14)$$

then  $\Omega_{V,R}$  is an invariant subset of the region-of-attraction around the origin.  $\triangleleft$

*Proposition 2:* If there exist positive definite  $\mathcal{C}^1$  functions  $V_1 : \mathcal{R}^{n_1} \rightarrow \mathcal{R}$ ,  $V_2 : \mathcal{R}^{n_2} \rightarrow \mathcal{R}$ , and  $V_3 : \mathcal{R}^{n_3} \rightarrow \mathcal{R}$  such that  $V_1(0) = 0$ ,  $V_2(0) = 0$ , and  $V_3(0) = 0$ , positive real numbers  $\gamma_{11}, \gamma_{12}, \gamma_{14}, \gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{33}$ , and  $\gamma_{34}$ , and nonnegative real numbers  $R_1, R'_1, R_2, R'_2, R_3$ , and  $R'_3$  such that  $\Omega_{V_1,R_1}$ ,  $\Omega_{V_2,R_2}$ , and  $\Omega_{V_3,R_3}$  are bounded,

$$\begin{aligned} \dot{V}_1(x_1, w_2, w_4) &< -\gamma_{11}w_1^T w_1 + \gamma_{12}w_2^T w_2 + \gamma_{14}w_4^T w_4 \\ &\text{for all } w_2, w_4, \text{ and nonzero } x_1 \text{ s.t.} \\ V_1(x_1) &\leq R_1 \text{ and } \|h_1(x_1)\|_2^2 \leq R'_1 \end{aligned} \quad (15)$$

$$\begin{aligned} \dot{V}_2(x_2, w_1) &< -\gamma_{22}w_2^T w_2 - \gamma_{23}w_3^T w_3 + \gamma_{21}w_1^T w_1 \\ &\text{for all } w_1, \text{ and nonzero } x_2 \text{ s.t.} \\ V_2(x_2) &\leq R_2 \text{ and } \|h_2(x_2)\|_2^2 \leq R'_2 \end{aligned} \quad (16)$$

$$\begin{aligned} \dot{V}_3(x_3, w_3) &< -\gamma_{34}w_4^T w_4 + \gamma_{33}w_3^T w_3 \\ &\text{for all } w_3 \text{ and nonzero } x_3 \text{ s.t.} \\ V_3(x_3) &\leq R_3 \text{ and } \|h_3(x_3)\|_2^2 \leq R'_3 \end{aligned} \quad (17)$$

and

$$\begin{aligned} -\gamma_{11} + \gamma_{21} &\leq 0 \\ -\gamma_{22} + \gamma_{12} &\leq 0 \\ -\gamma_{23} + \gamma_{33} &\leq 0 \\ -\gamma_{34} + \gamma_{14} &\leq 0 \\ R'_2 + R'_3 &\leq R_1 \\ R'_1 &\leq R_2 \\ R'_2 &\leq R_3 \end{aligned} \quad (18)$$

then  $\Omega_{V_1+V_2+V_3, \min\{R_1, R'_1, R_2, R'_2, R_3, R'_3\}}$  is an invariant subset of the region-of-attraction around the origin for the interconnected system shown in Figure 1.  $\triangleleft$

*Proof:* Let  $V := V_1 + V_2 + V_3$  and  $R := \min\{R_1, R'_1, R_2, R'_2, R_3, R'_3\}$ . Then,  $V$  is positive definite and vanishes at the origin. Moreover,  $\Omega_{V,R}$  is bounded and  $\dot{V}(x_1, x_2, x_3) < 0$  for all nonzero  $x \in \Omega_{V,R}$ . Consequently, Proposition 2 follows from Lemma 1.  $\blacksquare$

Similar to the previous section, the dual decomposition procedure can be applied to decouple the constraints in (15)-(17) by penalizing the violations of the constraints in (18). Let us introduce the dual variables  $\lambda_1, \lambda_2, \lambda_3$ , and  $\lambda_4$ , as before, to dualize the constraints on  $\gamma_{ij}$  in (18) and the extra dual variables  $\mu_1, \mu_2$ , and  $\mu_3$  for the remaining constraints in (18). Then, the master problem is written as

$$\max_{\lambda_1, \dots, \lambda_4, \mu_1, \mu_2, \mu_3 \geq 0} \phi_1(\lambda, \mu) + \phi_2(\lambda, \mu) + \phi_3(\lambda, \mu), \quad (19)$$

where  $\phi_1, \phi_2, \phi_3$  are solutions to the following three problems respectively.

$$\begin{aligned} \min_{\gamma_{11}, \gamma_{12}, \gamma_{14}, R_1, R'_1, V_1} \quad & -\lambda_1\gamma_{11} + \lambda_2\gamma_{12} + \lambda_4\gamma_{14} \\ \text{subject to} \quad & -\mu_1R_1 + \mu_2R'_1 \\ & \text{conditions in Prop 2 excluding (18)} \\ \min_{\gamma_{21}, \gamma_{22}, \gamma_{23}, R_2, R'_2, V_2} \quad & \lambda_1\gamma_{21} - \lambda_2\gamma_{22} - \lambda_3\gamma_{23} \\ \text{subject to} \quad & + \mu_1R'_2 - \mu_2R_2 + \mu_3R'_2 \\ & \text{conditions in Prop 2 excluding (18)} \\ \min_{\gamma_{33}, \gamma_{34}, R_3, R'_3, V_3} \quad & \lambda_3\gamma_{33} - \lambda_4\gamma_{34} + \mu_1R'_3 - \mu_2R_3 \\ \text{subject to} \quad & \text{conditions in Prop 2 excluding (18)}. \end{aligned}$$

The rules to update  $\lambda_1, \dots, \lambda_4, \mu_1, \dots, \mu_3$  can be adapted in a similar manner to the development in section II.

Conditions in (15)-(17) are local dissipation inequalities, i.e., they hold in certain subsets ( $\Omega_{V_i, R_i}$ ) of the state space. If the subsystem  $i$  starts from rest, then these conditions ensure that the corresponding weighted  $\mathcal{L}_2$ -gain relations hold between the inputs and outputs if the inputs to subsystem  $i$  have an  $\mathcal{L}_2$ -norm less than or equal to  $\sqrt{R_i}$ . On the other hand, recall that every signal  $w_k$  is an input to subsystem  $i$  and at the same time an output of subsystem  $j$ . Therefore, the constraint  $\|w_k\|_2 = \|h_j(x_j)\|_2 \leq R'_j$  in (15)-(17) coupled with  $R'_j \leq R_i$  in (18) ensures that  $w_k$  does not violate the assumptions made on the input norm at subsystem  $i$ . For example, the  $\mathcal{L}_2$ -norm of  $w_1$  as an output of subsystem 1 cannot exceed that as an input to subsystem 2 (i.e.,  $R'_1 \leq R_2$ ). These conditions are motivated by *assume-guarantee* type compositional analysis ideas [4], [5] and the dual decomposition based hierarchical scheme automates the adjustment of *assumptions* in such schemes.

#### IV. EXAMPLE

Let the dynamics and output maps of the subsystems in Figure 1 be given by

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} x_1 + I_2 \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} \\ w_1 &= 0.5 \begin{bmatrix} 1 & 1 \end{bmatrix} x_1 \\ \dot{x}_2 &= \begin{bmatrix} -8 & 0 \\ 12 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_1 \\ \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} &= 0.5 x_2 \\ \dot{x}_3 &= \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_3 \\ w_4 &= 0.4 \begin{bmatrix} 1 & 1 \end{bmatrix} x_3. \end{aligned}$$

We apply the compositional analysis to verify the global asymptotic stability of the interconnected system. Let the step size in the subgradient algorithm be <sup>3</sup>

$$\alpha^k = \frac{0.01}{10 + k}.$$

In this case, the optimization problems in (8)-(10) are LMIs with the decision variables  $\gamma_{ij}$ 's and those in  $V_i$ 's. The dual variables are initialized at  $\lambda_1^0 = 0$ ,  $\lambda_2^0 = 0$ ,  $\lambda_3^0 = 0$ , and  $\lambda_4^0 = 0$ . Figure 2 shows the violations in the constraints on the  $\gamma_{ij}$ 's (negative value means that the corresponding constraint is not violated). The subgradient iterations are terminated when all constraints are satisfied. Figure 3 shows the values of the dual variables  $\lambda_i$ 's versus the iteration number and Figure 4 shows the value of the objective function in (6) versus the iteration number.

#### V. CONCLUSIONS AND CRITIQUE

We proposed a compositional stability analysis methodology for verifying properties of systems that are interconnections of multiple subsystems. The proposed method assembles stability certificates for the interconnected system

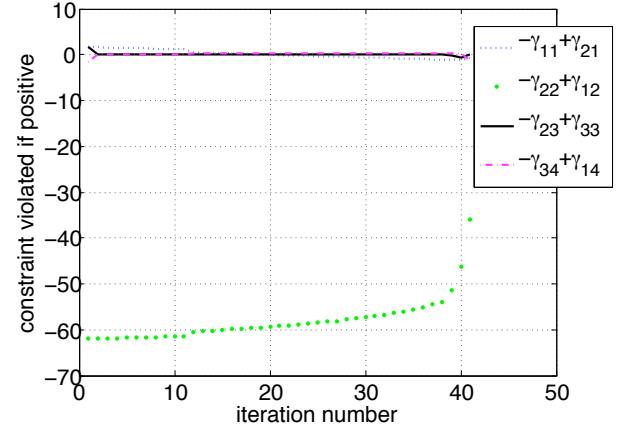


Fig. 2. Violations of the constraints on  $\gamma_{ij}$ 's. Negative value indicates that the corresponding constraint is satisfied.

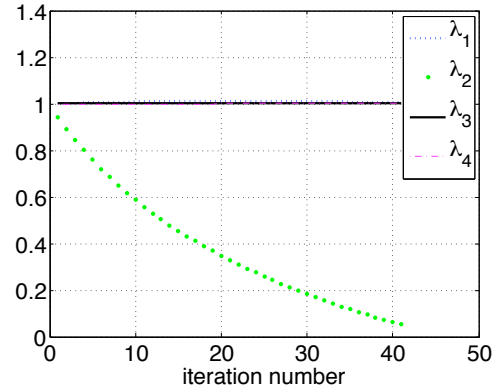


Fig. 3. The dual variable ( $\lambda_i$ 's) versus the iteration number. The correspondence between the dual variables and the constraints is as in (6).

based on the certificates for the input-output properties of the subsystems. The decomposition in the analysis is achieved by utilizing dual decomposition ideas from optimization. Decoupled subproblems establish subsystem level input-output properties whereas the “master” problem imposes and updates the conditions on the subproblems toward ensuring (interconnected) system level stability properties. Both global stability analysis and region-of-attraction analysis were discussed.

There are a series of limitations and also possible extensions of the method proposed here. First of all, the decomposition in analysis is achieved through a fixed decomposition of the sufficient conditions for (local and global) stability based on the specific choice of supply rates, i.e.,  $\mathcal{L}_2$ -gain relations, in (2)-(4) and (15)-(17)) for the subsystems. It may be possible to reduce the conservatism associated with these specific choices by exploring optimization over the choice of supply rates. Also note that the proposed method is inherently more conservative than a “centralized” search (for the overall closed-loop dynamics) for a general Lyapunov function and even a Lyapunov function of the form  $V(x) = \sum_{i=1}^N V_i(x_i)$  (with  $N$  being the number of subsystems).

The convergence of the subgradient based optimization

<sup>3</sup>See [21] for guidelines for choosing the step size  $\alpha$ .



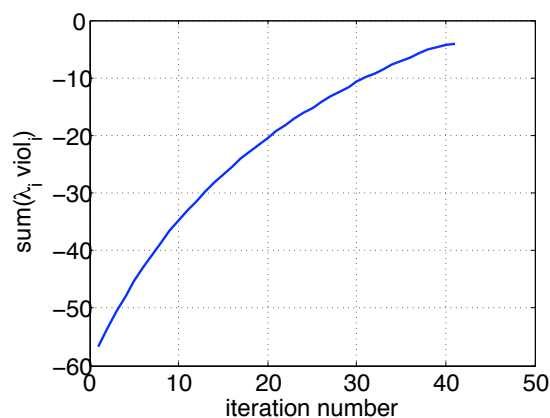


Fig. 4. The value of the objective function in (6) versus the iteration number.

schemes is known to be slow. Therefore, it may be of interest to both theoretically and practically investigate the convergence properties of the proposed scheme. Note that we have presented the compositional analysis framework for a specific interconnection structure. Although the extensions to larger number of subsystems with a general interconnection structure are straightforward, the effect of these extensions on convergence remains to be examined. Nevertheless, we emphasize that one of the main difficulties in solving large-scale semidefinite programs is the memory requirements of the interior-point type algorithms [11]. Therefore, compositional analysis may be the sole option (even if it converges slowly) for certain systems for which solving large-scale semidefinite programming problems corresponding to system level certificates is not practical.

We have only considered stability analysis (i.e., no exogenous signals). It may be possible to extend the framework to identify system level input-output properties (in the presence of exogenous inputs and outputs).

The method proposed here heavily relies on the existence of the notion of a “norm” for the input and output signals (specifically  $\mathcal{L}_2$  signal norms are used here). In fact, the role of these norms is mainly to provide a means (analogous to a “partial order”) for “comparing” inputs of a subsystem to its outputs (i.e., input-output gains) and outputs of a subsystem to the inputs of the subsystems it is connected to. It may be possible to generalize the technique to the cases for which (partial) comparison of signals is possible. An interesting future research direction is to explore such notions for systems with discrete valued inputs and outputs. This may potentially enable compositional analysis for systems where discrete and continuous dynamics interact. For a recent work on related ideas see [22] which discussed small-gain type theorems for hybrid systems.

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