Chapter 12

Differential Games, Distributed Systems, and Impulse Control

In previous chapters, we were mainly concerned with the optimal control problems formulated in Chapters 3 and 4 and their applications to various functional areas of management and to some problems in economics. These problems were described by a single objective function (or a single decision maker) and a set of ordinary differential equations, called the state equations, defined in a deterministic framework.

In this chapter, we deal with generalizations of the (ordinary) deterministic optimal control problems that can be made in one or more of the following directions. See Chapter 13 for stochastic optimal control problems.

There may be more than one decision maker, each having separate objective functions which each is trying to maximize, subject to a set of differential equations. This extension of the optimal control theory is referred to as the *theory of differential games*. Section 12.1 contains a brief introduction to differential games along with an application.

Another extension replaces the system of ordinary differential equations by a set of partial differential equations. These come under the classification of *distributed parameter systems* and are treated in Section 12.2.

Finally in Section 12.3, we treat the theory of *impulse control*. This control is developed to deal with systems which, in addition to conventional controls, allow a controller to make discrete changes in the state variables at selected instants of time in an optimal fashion. This theory is especially useful in dealing with inventory problems such as dynamic lot size problems. It also allow a setup cost associated with the discrete changes in the state variables to be charged to such discrete actions as setting up a machine tool from making one product to making another, opening a warehouse, etc.

Extensions of optimal control problems where uncertainties are present will be discussed in the next chapter.

12.1 Differential Games

The study of differential games was initiated by Isaacs (1965). After the development of Pontryagin's maximum principle, it became clear that there was a connection between differential games and optimal control theory. In fact, differential game problems represent a generalization of optimal control problems in cases where there are more than one controller or player. However, differential games are conceptually far more complex than optimal control problems in the sense that it is no longer obvious what constitutes a solution; see Starr and Ho (1969), Ho (1970), Varaiya (1970), Friedman (1971), Leitmann (1974), Case (1979), Selten (1975), Basar (1986), Mehlmann (1988), Berkovitz (1994), and Dockner, Jørgensen, Long, and Sorger (2000). Indeed, there are a number of different types of solutions such as minimax, Nash, Pareto-optimal, along with possibilities of cooperation and bargaining; see, e.g., Tolwinski (1982) and Haurie, Tolwinski, and Leitmann (1983). We will confine ourselves to minimax solutions for zero-sum differential games and Nash solutions for nonzero-sum games.

12.1.1 Two Person Zero-Sum Differential Games

Consider the state equation

$$\dot{x} = f(x, u, v, t), \quad x(0) = x_0, \tag{12.1}$$

where we may assume all variables to be scalar for the time being. Extension to the vector case simply requires appropriate reinterpretations of each of the variables and the equations. In this equation, we let u and v denote the controls applied by players 1 and 2, respectively. We assume that

$$u(t) \in U, \; v(t) \in V, \; t \in [0, T],$$

where U and V are convex sets in E^1. Consider further the objective function

$$J(u,v) = S[x(T)] + \int_0^T F(x,u,v,t)dt, \tag{12.2}$$

which player 1 wants to maximize and player 2 wants to minimize. Since the gain of player 1 represents a loss to player 2, such games are appropriately termed *zero-sum games*. Clearly, we are looking for admissible control trajectories u^* and v^* such that

$$J(u^*,v) \geq J(u^*,v^*) \geq J(u,v^*). \tag{12.3}$$

The solution (u^*,v^*) is known as the *minimax* solution. Here u^* and v^* stand for $u^*(t)$, $t \in [0,T]$, and $v^*(t)$, $t \in [0,T]$, respectively.

The necessary conditions for u^* and v^* to satisfy (12.3) are given by an extension of the maximum principle. To obtain these conditions, we form the Hamiltonian

$$H = F + \lambda f \tag{12.4}$$

with the adjoint variable λ satisfying the equation

$$\dot{\lambda} = -H_x, \quad \lambda(T) = S_x[x(T)]. \tag{12.5}$$

The necessary condition for trajectories u^* and v^* to be a minimax solution is that for $t \in [0,T]$,

$$H(x^*(t),u^*(t),v^*(t),\lambda^*(t),t) = \min_{v \in V} \max_{u \in U} H(x^*(t),u,v,\lambda^*(t),t), \tag{12.6}$$

which can also be stated, with suppression of (t), as

$$H(x^*,u^*,v,\lambda^*,t) \geq H(x^*,u^*,v^*,\lambda^*,t) \geq H(x^*,u,v^*,\lambda^*,t) \tag{12.7}$$

for $u \in U$ and $v \in V$. Note that (u^*,v^*) is a saddle point of the Hamiltonian function H.

Note that if u and v are unconstrained, i.e., when, $U = V = E^1$, condition (12.6) reduces to the first-order necessary conditions

$$H_u = 0 \text{ and } H_v = 0, \tag{12.8}$$

and the second-order conditions are

$$H_{uu} \leq 0 \text{ and } H_{vv} \geq 0. \tag{12.9}$$

We now turn to the treatment of nonzero-sum differential games.

12.1.2 Nonzero-Sum Differential Games

In this section, let us assume that we have N players where $N \geq 2$. Let $u^i \in U^i$, $i = 1, 2, ..., N$, represent the control variable for the ith player, where U^i is the set of controls from which the ith player can choose. Let the state equation be defined as

$$\dot{x} = f(x, u^1, u^2, ..., u^N, t). \tag{12.10}$$

Let J^i, defined by

$$J^i = S^i[x(T)] + \int_0^T F^i(x, u^1, u^2, ..., u^N, t)dt, \tag{12.11}$$

denote the objective function which the ith player wants to maximize. In this case, a *Nash solution* is defined by a set of N admissible trajectories

$$\{u^{1*}, u^{2*}, ..., u^{N*}\}, \tag{12.12}$$

which have the property that

$$J^i(u^{1*}, u^{2*}, ..., u^{N*}) = \max J^i(u^{1*}, u^{2*}, ..., u^{(i-1)*}, u^i, ..., u^{(i+1)*}, ..., u^{N*}) \tag{12.13}$$

for $i = 1, 2, ..., N$.

To obtain the necessary conditions for a Nash solution for nonzero-sum differential games, we must make a distinction between open-loop and closed-loop controls.

Open-Loop Nash Solution

The open-loop Nash solution is defined when (12.12) is given as functions of time satisfying (12.13). To obtain the maximum principle type conditions for such solutions to be a Nash solution, let us define the Hamiltonian functions

$$H^i = F^i + \lambda^i f \tag{12.14}$$

for $i = 1, 2, ..., N$, with λ^i satisfying

$$\dot{\lambda}^i = -H^i_x, \quad \lambda^i(T) = S^i_x[x(T)]. \tag{12.15}$$

The Nash control u^{i*} for the ith player is obtained by maximizing the ith Hamiltonian H^i with respect to u^i, i.e., u^{i*} must satisfy

$$H^i(x^*, u^{1*}, ..., u^{(i-1)*}, u^{i*}, u^{(i+1)*}, ..., u^{N*}, \lambda^*, t) \geq$$
$$H^i(x^*, u^{1*}, ..., u^{(i-1)*}, u^i, u^{(i+1)*}, ..., u^{N*}, \lambda^*, t), \; t \in [0, T], \tag{12.16}$$

for all $u^i \in U^i$, $i = 1, 2, .., N$.

Deal, Sethi, and Thompson (1979) formulated and solved an advertising game with two players and obtained the open-loop Nash solution by solving a two-point boundary value problem. In Exercise 12.1, you are asked to formulate their problem. See also Deal (1979).

Closed-Loop Nash Solution

A closed-loop Nash solution is defined when (12.12) is defined in terms of the state of the system. To avoid confusion, we let

$$u^{i*}(x,t) = \phi^i(x,t), \quad i = 1, 2, ..., N. \tag{12.17}$$

For these controls to represent a Nash strategy, we must recognize the dependence of the other players' actions on the state variable x. Therefore, we need to replace the adjoint equation (12.15) by

$$\dot{\lambda}^i = -H_x - \sum_{j=1, j\neq i}^{N} H^i_{u^j} \phi^j_x. \tag{12.18}$$

The presence of the summation term in (12.18) makes the necessary condition for the closed-loop solution virtually useless for deriving computational algorithms; see Starr and Ho (1969). It is, however, possible to use a dynamic programming approach for solving extremely simple nonzero-sum games, which require the solution of a partial differential equation. In Exercise 12.2, you are asked to formulate this partial differential equation for $N = 2$.

Note that the troublesome summation term in (12.18) is absent in three important cases: (a) in optimal control problems ($N = 1$) since $H_u u_x = 0$, (b) in two-person zero-sum games because $H^1 = -H^2$ so that $H^1_{u^2} u^2_x = -H^2_{u^2} u^2_x = 0$ and $H^2_{u^1} u^1_x = -H^1_{u^1} u^1_x = 0$, and (c) in open-loop nonzero-sum games because $u^j_x = 0$. It certainly is to be expected, therefore, that the closed-loop and open-loop Nash solutions are going to be different, in general. This can be shown explicitly for the linear-quadratic case.

We conclude this section by providing an interpretation to the adjoint variable λ^i. It is the sensitivity of ith player's profits to a perturbation in the state vector. If the other players are using closed-loop (i.e., feedback) strategies, any perturbation δx in the state vector causes them to revise their controls by the amount $\theta^j_x \delta x$. If the ith Hamiltonian H^i were already extremized with respect to u^j, $j \neq i$, this would not affect the

ith player's profit; but since $\partial H^i/\partial u^j \neq 0$ for $i \neq j$, the reactions of the other players to the perturbation influence the ith player's profit, and the ith player must account for this effect in considering variations of the trajectory.

12.1.3 An Application to the Common-Property Fishery Resources

Consider extending the fishery model of Section 10.1 by assuming that there are two producers having unrestricted rights to exploit the fish stock in competition with each other. This gives rise to a nonzero-sum differential game analyzed by Clark (1976).

Equation (10.2) is modified by

$$\dot{x} = g(x) - q^1u^1x - q^2u^2x, \quad x(0) = x_0, \tag{12.19}$$

where $u^i(t)$ represents the rate of fishing effort and q^iu^ix is the rate of catch for the ith producer, $i = 1, 2$. The control constraints are

$$0 \leq u^i(t) \leq U^i, \quad i = 1, 2, \tag{12.20}$$

the state constraints are

$$x(t) \geq 0, \tag{12.21}$$

and the objective function for the ith producer is the total present value of his profits, namely,

$$J^i = \int_0^\infty (p^iq^ix - c^i)u^ie^{-\rho t}dt, \quad i = 1, 2. \tag{12.22}$$

To find the Nash solution for this model, we let $\bar{x}^1$ denote the turnpike (or optimal biomass) level given by (10.12) on the assumption that the ith producer is the sole-owner of the fishery. Let the bionomic equilibrium x_b^i for producer i be defined by (10.4), i.e.,

$$x_b^i = \frac{c^i}{p^iq^i}. \tag{12.23}$$

As shown in Exercise 10.2, $x_b^i < \bar{x}^i$. If the other producer is not fishing, then producer i can maintain x_b^i by making the fishing effort

$$u_b^i = \frac{g(x_b^i)p^i}{c^i}; \tag{12.24}$$

here we have assumed u_i to be sufficiently large so that $u_b^i \leq U^i$. We also assume that

$$x_b^1 < x_b^2, \tag{12.25}$$

which means that producer 1 is more efficient than producer 2, i.e., producer 1 can make a positive profit at any level in the interval $(x_b^1, x_b^2]$, while producer 2 loses money in the same interval, except at x_b^2, where he breaks even. For $x > x_b^2$, both producers make positive profits.

Since $U^1 \geq u_b^1$ by assumption, producer 1 has the capability of driving the fish stock to a level down to or below x_b^1 which, by (12.25), is less than x_b^2. This implies that producer 2 cannot operate at a sustained level above x_b^2; and at a sustained level below x_b^2, he cannot make a profit. Hence, his optimal policy is bang-bang:

$$u^{2*}(x) = \begin{cases} U^2 & \text{if } x > x_b^2, \\ 0 & \text{if } x \leq x_b^2. \end{cases} \tag{12.26}$$

As far as producer 1 is concerned, he wants to attain his turnpike level $\bar{x}^1$ if $\bar{x}^1 \leq x_b^2$. If $\bar{x}^1 > x_b^2$ and if $x_0 \geq \bar{x}^1$, then from (12.26) producer 2 will fish at his maximum rate until the fish stock is driven to x_b^2. At this level it is optimal for producer 1 to fish at a rate which maintains the fish stock at level x_b^2 in order to keep producer 2 from fishing. Thus, the optimal policy for producer 1 can be stated as

$$u^{1*}(x) = \left\{ \begin{array}{ll} U^1 & \text{if } x > \bar{x}^1 \\ \bar{u}^1 = \frac{g(\bar{x}^1)}{q^1 \bar{x}^1} & \text{if } x = \bar{x}^1 \\ 0 & \text{if } x < \bar{x}^1 \end{array} \right\}, \quad \text{if } \bar{x}^1 < x_b^2, \tag{12.27}$$

$$u^{1*}(x) = \left\{ \begin{array}{ll} U^1 & \text{if } x > x_b^2 \\ \frac{g(x^2)}{q^1 x_b^2} & \text{if } x = x_b^2 \\ 0 & \text{if } x < x_b^2 \end{array} \right\}, \quad \text{if } \bar{x}^1 \geq x_b^2. \tag{12.28}$$

The formal proof that policies (12.26)-(12.28) give a Nash solution requires direct verification using the result of Section 10.1.2. The Nash solution for this case means that for all feasible paths u^1 and u^2,

$$J^1(u^{1*}, u^{2*}) \geq J^1(u^1, u^{2*}), \tag{12.29}$$

and

$$J^2(u^{1*}, u^{2*}) \geq J^2(u^{1*}, u^2). \tag{12.30}$$

The direct verification involves defining a modified growth function

$$g^1(x) = \begin{cases} g(x) - q^2U^2x & \text{if } x > x_b^2, \\ g(x) & \text{if } x \leq x_b^2, \end{cases}$$

and using the Green's theorem results of Section 10.1.2. Since $U^2 \geq u_b^2$ by assumption, we have $g^1(x) \leq 0$ for $x \geq x_b^2$. From (10.12) with g replaced by g^1, it can be shown that the new turnpike level for producer 1 is $\min(\bar{x}^1, x_b^2)$, which defines the optimal policy (12.27)-(12.28) for producer 1. The optimality of (12.26) for producer 2 follows easily.

To interpret the results of the model, suppose that producer 1 originally has sole possession of the fishery, but anticipates a rival entry. Producer 1 will switch from his own optimal sustained yield x^{1*} to a more intensive exploitation policy *prior* to the anticipated entry.

We can now guess the results in situations involving N producers. The fishery will see the progressive elimination of inefficient producers as the stock of fish decreases. Only the most efficient producers will survive. If, ultimately, two or more maximally efficient producers exist, the fishery will converge to a classical bionomic equilibrium, with zero sustained economic rent.

We have now seen that a Nash competitive solution involving $N \geq 2$ producers results in the long-run dissipation of economic rents. This conclusion depends on the assumption that producers face an infinitely elastic supply of all factors of production going into the fishing effort, but typically the methods of licensing entrants to regulated fisheries make some attempt also to control the factors of production such as permitting the licensee to operate only a single vessel of specific size.

In order to develop a model for licensing of fishermen, we let the control variable v^i denote the capital stock of the ith producer and let the concave function $f(v^i)$, with $f(0) = 0$, denote the *fishing mortality function*, for $i = 1, 2, ..., N$. This requires the replacement of q^iu^i in the previous model by $f(v^i)$. The extended model becomes nonlinear in control variables. You are asked in Exercise 12.2 to formulate this new model and develop necessary conditions for a closed-loop Nash solution for this model with N producers. The reader is referred to Clark (1976) for further details.

For other papers on applications of differential games to fishery management, see Hämäläinen, Haurie, and Kaitala (1984, 1985) and Hämäläinen, Ruusunen, and Kaitala (1986, 1990). For applications to problems in environmental management, see the edited volume by Carraro and Filar (1995) on the topic.

Another area in which there have been many applications of differential games is that of marketing in general and optimal advertising in particular. Some references are Bensoussan, Bultez, and Naert (1978), Deal, Sethi, and Thompson (1979), Deal (1979), Jørgensen (1982a), Rao (1984, 1990), Dockner and Jørgensen (1986, 1992), Chintagunta and Vilcassim (1992), Chintagunta and Jain (1994, 1995), and Fruchter (1999). A survey of the literature is done by Jørgensen (1982a) and a monograph is written by Erickson (1991).

For applications of differential games to economics and management science in general, see the book by Dockner, Jørgensen, Long, and Sorger (2000).

12.2 Distributed Parameter Systems

Thus far, our efforts have been directed to the study of the control of systems governed by systems of ordinary differential or difference equations. Such systems are often called *lumped parameter systems*. It is possible to generalize these to systems in which the state and control variables are defined in terms of space as well as time dimensions. These are called *distributed parameter systems* and are described by a set of partial differential or difference equations.

For example, in the lumped parameter advertising models of the type treated in Chapter 7, we need to obtain the optimal advertising expenditures for each instant of time. However, in the analogous distributed parameter advertising model we must obtain the optimal advertising expenditure at every geographic location of interest at each instant of time; see Seidman, Sethi, and Derzko (1987). In other economic problems, the spatial coordinates might be income, quality, age, etc. In Section 12.2.2 we will discuss a cattle-ranching model of Derzko, Sethi, and Thompson (1980), in which the spatial dimension measures the age of a cow.

Let y denote a one dimensional spatial vector, let t denote time, and let $x(t, y)$ be a one dimensional state variable. Let $u(t, y)$ denote a control variable, and let the state equation be

$$\frac{\partial x}{\partial t} = g(t, y, x, \frac{\partial x}{\partial y}, u) \tag{12.31}$$

for $t \in [0, T]$ and $y \in [0, h]$. We denote the region $[0, T] \times [0, h]$ by D, and we let its boundary ∂D be split into two parts Γ_1 and Γ_2 as shown in Figure 12.1. The initial conditions will be stated on the part Γ_1 of the boundary ∂D as

$$x(0, y) = x_0(y) \tag{12.32}$$

and

$$x(t, 0) = v(t). \tag{12.33}$$

In Figure 12.1, (12.32) is the initial condition on the vertical portion of Γ_1, whereas (12.33) is that on the horizontal portion of Γ_1. More specifically, in (12.32) the function $x_0(y)$ gives the starting distribution of x with respect to the spatial coordinate y. The function $v(t)$ in (12.33) is an exogenous *breeding function* at time t of x when $y = 0$. In the cattle ranching example in Section 12.2.2, $v(t)$ measures the number of newly born calves at time t. To be consistent we make the obvious assumption that

$$x(0, 0) = x_0(0) = v(0). \tag{12.34}$$

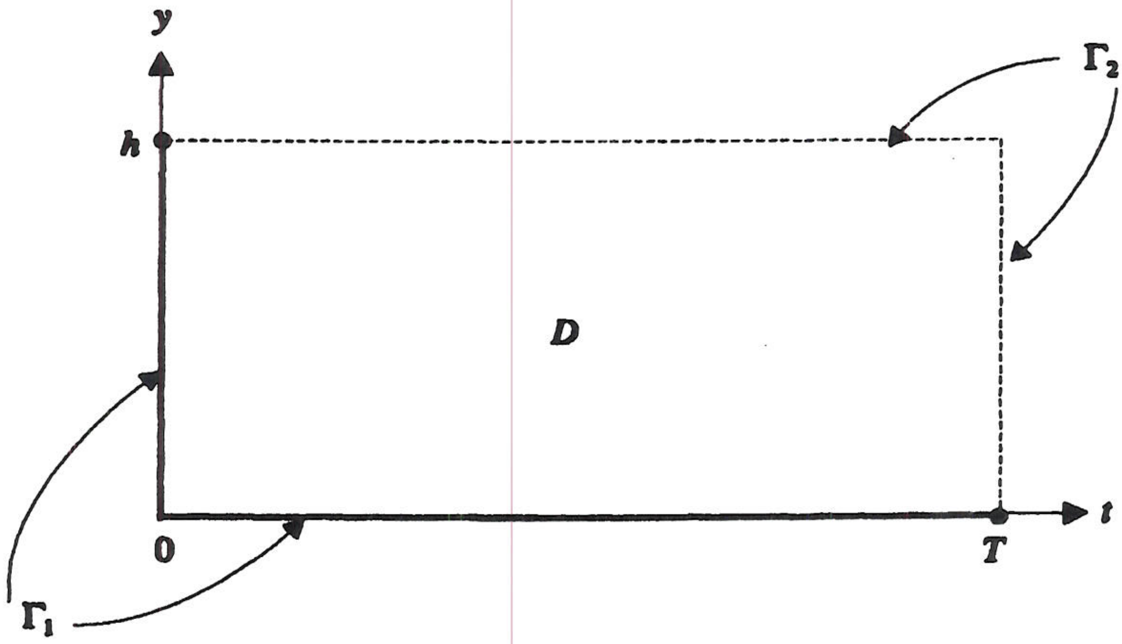

Figure 12.1: Region D with Boundaries Γ_1 and Γ_2

Let $F(t, y, x, u)$ denote the profit rate when $x(t, y) = x$, $u(t, y) = u$ at a point (t, y) in D. Let $Q(t)$ be the value of one unit of $x(t, h)$ at time

t and let $S(y)$ be the value of one unit of $x(T,y)$ at time T. Then the objective function is:

$$\max_{u(t,y)\in\Omega}\left\{J=\int_0^T\int_0^h F(t,y,x(t,y),u(t,y))dydt\right.$$
$$\left.+\int_0^T Q(t)x(t,h)dt+\int_0^h S(y)x(T,y)dy\right\}, \tag{12.35}$$

where Ω is the set of allowable controls.

12.2.1 The Distributed Parameter Maximum Principle

We will formulate, without giving proofs, a procedure for solving the problem in (12.31)-(12.2) by a distributed parameter maximum principle, which is analogous to the ordinary one. A more complete treatment of this maximum principle can be found in Sage (1968, Chapter 7), Butkowskiy (1969), Lions (1971), Derzko, Sethi, and Thompson (1984), and Haurie, Sethi, and Hartl (1984).

In order to obtain necessary conditions for a maximum, we introduce the Hamiltonian

$$H=F+\lambda f, \tag{12.36}$$

where the spatial adjoint function $\lambda(t,y)$ satisfies

$$\frac{\partial\lambda}{\partial t}=-\frac{\partial H}{\partial x}+\frac{\partial}{\partial t}\left[\frac{\partial H}{\partial x_t}\right]+\frac{\partial}{\partial y}\left[\frac{\partial H}{\partial x_y}\right], \tag{12.37}$$

where $x_t=\partial x/\partial t$ and $x_y=\partial x/\partial y$. The boundary conditions on λ are stated for the Γ_2 part of the boundary of D (see Figure 12.1) as follows:

$$\lambda(t,h)=Q(t) \tag{12.38}$$

and

$$\lambda(T,y)=S(y). \tag{12.39}$$

Once again we need a consistency requirement similar to (12.34). It is

$$\lambda(T,h)=P(T)=S(h), \tag{12.40}$$

which gives the consistency requirement in the sense that the price and the salvage value of a unit $x(T,h)$ must agree.

We let $u^*(t,y)$ denote the optimal control function. Then the distributed parameter maximum principle requires that

$$H(t,y,x^*,x_t^*,x_y^*,u^*,\lambda^*) \geq H(t,y,x^*,x_t^*,x_y^*,u,\lambda^*) \tag{12.41}$$

for all $(t,y) \in D$ and all $u \in \Omega$.

We have stated only a simple form of the distributed parameter maximum principle which is sufficient for the cattle ranching example dealt with in the next section. More general forms of the maximum principle are available in the references cited earlier. Among other things, these general forms allow for the function F in (12.2) to contain arguments such as $\partial x/\partial y$, $\partial^2 x/\partial y^2$, etc. It is also possible to consider controls on the boundary. In this case $v(t)$ in (12.33) will become a control variable.

12.2.2 The Cattle Ranching Problem

Let t denote time and y denote the age of an animal. Let $x(t,y)$ denote the number of cattle of age y on the ranch at time t. Let h be the age at maturity at which the cattle are slaughtered. Thus, the set $[0,h]$ is the set of all possible ages of the cattle. Let $u(t,y)$ be the rate at which y-aged cattle are bought at time t, where we agree that a negative value of u denotes a sale.

To develop the dynamics of the process it is easy to see that

$$x(t+\Delta t, y) = x(t, y-\Delta t) + u(t,y)\Delta t. \tag{12.42}$$

Subtracting $x(t,y)$ from both sides of (12.42), dividing by Δt, and taking the limit as $\Delta t \to 0$, yields the state equation

$$\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial y} + u. \tag{12.43}$$

The boundary and consistency conditions for x are given in (12.32)-(12.34). Here $x_0(y)$ denotes the initial distribution of cattle at various ages, and $v(t)$ is an exogenously specified breeding rate.

To develop the objective function for the cattle rancher, we let T denote the horizon time. Let $P(t,y)$ be the purchase or sale price of a y-aged animal at time t. Let $P(t,h) = Q(t)$ be the slaughter value at time t and let $P(T,y) = S(y)$ be the salvage value of a y-aged animal at the horizon time T. The functions Q and S represent the proceeds of the cattle ranching business. To obtain the profit function we must subtract the costs of running the ranch from these proceeds. Let $C(y)$

be the feeding and corralling costs for a y-aged animal per unit of time. Let $\bar{u}(t,y)$ denote the *goal level* purchase rate of y-aged cattle at time t. Any deviation from this goal level is expensive, and the deviation penalty cost is given by $q[u(t,y)-\bar{u}(t,y)]^2$, where q is a constant. Thus, the profit maximizing objective function is

$$\begin{aligned} J &= \int_0^T\int_0^h -\left[q(u(t,y)-\bar{u}(t,y))^2 + C(y)x(t,y) + P(t,y)u(t,y)\right]dydt \\ &\quad + \int_0^T Q(t)x(t,h)dt + \int_0^h S(y)x(T,y)dy. \end{aligned} \tag{12.44}$$

Comparing this with (12.2) we see that

$$F(t,y,x,u) = -[q(u-\bar{u}(t,y))^2 + C(y)x + P(t,y)u].$$

We assume $\Omega = E^1$, which means that the control variable is unconstrained.

To solve the problem we form the Hamiltonian

$$H = -[q(u-\bar{u}(t,y))^2 + C(y)x + P(t,y)u] + \lambda(-\frac{\partial x}{\partial y} + u), \tag{12.45}$$

where the adjoint function $\lambda(t,y)$ satisfies

$$\frac{\partial \lambda}{\partial t} = -\frac{\partial \lambda}{\partial y} + C(y) \tag{12.46}$$

subject to the boundary and consistency conditions (12.38)-(12.40). In order to maximize the Hamiltonian, we differentiate H with respect to u and set it to zero, giving

$$u^*(t,y) = \bar{u}(t,y) + \frac{1}{2q}[\lambda(t,y) - P(t,y)]. \tag{12.47}$$

The form of this optimal control is just like that found in the production-inventory example of Chapter 6. To compute u^*, we must solve for $\lambda(t,y)$. It is easy to verify that the general solution of (12.46) is of the form

$$\lambda(t,y) = -\int_y^k C(\tau)d\tau + g(t-y), \tag{12.48}$$

where g is an arbitrary one-variable function and k is a constant. We will use the boundary conditions to determine g and k.

In order to state the explicit solution for λ (and later for x), we divide the region D of Figure 12.2 into the three regions D_1, D_2, and D_3. The

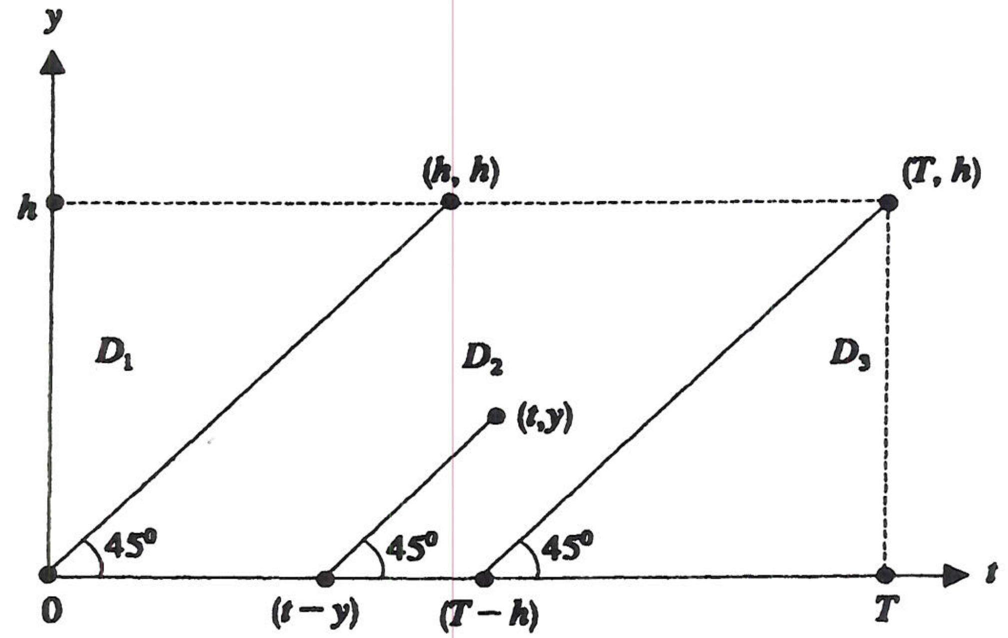

Figure 12.2: A Partition of Region D

45° line from $(0,0)$ to (h,h) belongs to both D_1 and D_2, and the 45° line from $(T-h,0)$ to (T,h) belongs to both D_2 and D_3. Thus, D_1, D_2, and D_3 are closed sets. The reason why these 45° lines are important in the solution for λ comes from the fact that the determination of the arbitrary function $g(t-y)$ involves the term $(t-y)$. We use the condition (12.38) on $\lambda(t,h)$ to obtain $\lambda(t,y)$ in the region $D_1 \cup D_2$. We substitute (12.38) into (12.48) and get

$$\lambda(t,h) = -\int_h^k C(\tau)d\tau + g(t-h) = Q(t).$$

This gives

$$g(t-h) = Q(t) + \int_h^k C(\tau)d\tau,$$

or

$$g(t-y) = Q(t-y+h) + \int_h^k C(\tau)d\tau.$$

Substituting for $g(t-y)$ in (12.48) gives us the solution

$$\lambda(t,y) = Q(t-y+h) - \int_y^k C(\tau)d\tau + \int_h^k C(\tau)d\tau = Q(t-y+h) - \int_y^h C(\tau)d\tau$$

in the region $D_1 \cup D_2$.

For region D_3, we use the condition (12.39) on $\lambda(T,y)$ in (12.48) to obtain

$$\lambda(T,y) = -\int_y^k C(\tau)d\tau + g(T-y) = S(y).$$

This gives

$$g(T-y) = S(y) + \int_y^k C(\tau)d\tau,$$

or

$$g(t-y) = S(T-t+y) + \int_{T-t+y}^k C(\tau)d\tau.$$

Substituting this value of $g(t-y)$ in (12.48) gives the solution

$$\lambda(t,y) = S(T-t+y) - \int_y^{T-t+y} C(\tau)d\tau$$

in region D_3.

We can now write the complete solution for $\lambda(t,y)$ as

$$\lambda(t,y) = \begin{cases} Q(t-y+h) - \int_y^h C(\tau)d\tau & \text{for } (t,y) \in D_1 \cup D_2, \\ S(T-t+y) - \int_y^{(T-t+y)} C(\tau)d\tau & \text{for } (t,y) \in D_3. \end{cases} \tag{12.49}$$

Note that $\lambda(T,h) = P(T,h) = Q(T) = S(h)$. The solution for x is obtained in a similar manner. We substitute (12.47) into (12.43) and use the boundary and consistency conditions (12.32)-(12.34). The complete solution is given as

$$x(t,y) = \begin{cases} x_0(y-t) + \int_0^t u^*(\tau, y-t+\tau)d\tau & \text{for } (t,y) \in D_1, \\ v(t,y) + \int_0^y u^*(t-y+\tau,\tau)d\tau & \text{for } (t,y) \in D_2 \cup D_3. \end{cases} \tag{12.50}$$

We can interpret the solution (12.50) in the region D_1 as the *beginning game*, which is completely characterized by the initial distribution x_0. Also the solution (12.49) in region D_3 is the *ending game*, because in this region the animals do not mature, but must be sold at whatever their age is at the terminal time T. The first expression in (12.49) and the second expression in (12.50) hold in region D_2, which can be interpreted as the *middle game* portion of the solution.

12.2.3 Interpretation of the Adjoint Function

It is instructive to interpret the solution for $\lambda(t, y)$ in (12.49). An animal at age y at time t, where (t, y) is in $D_1 \cup D_2$, will mature at time $t-y+h$. Its slaughter value at that time is $Q(t-y+h)$. However, the total feeding and corralling cost in keeping the animal from its age y until it matures is given by $\int_y^h C(\tau)d\tau$. Thus, $\lambda(t, y)$ represents the net benefit obtained from having an animal at age y at time t. You should give a similar interpretation for λ in region D_3.

Having this interpretation for λ, it is easy to interpret the optimal control u^* in (12.47). Whenever $\lambda(t, y) > P(t, y)$, we buy more than the goal level $\bar{u}(t, y)$, and when $\lambda(t, y) < P(t, y)$, we buy less than the goal level.

Muzicant (1980) considers an extension of the cattle ranching problem to allow the breeding rate $v(t)$ to be controlled. Her model is reproduced in Feichtinger and Hartl (1986). Other applications of the distributed parameter control system model are the following problems: inventory control incorporating product quality deterioration, see Bensoussan, Nissen, and Tapiero (1975); production and inventory systems, see Tzafestas (1982); personnel planning, see Gaimon and Thompson (1984b); social services planning, see Haurie, Sethi, and Hartl (1984); and consumer durables with age/quality structure, see Robson (1985). We believe that many other applications are possible and that they represent fruitful areas for research.

12.3 Impulse Control

In Chapters 3 and 4 we studied the control of systems governed by ordinary differential equations. In these cases the state variable can only change continuously since the control affects only the time derivatives of the state variables. In Chapters 5 and 7, we developed some models in which the state variable did change instantaneously by a finite amount as a result of the application of an impulse control. These situations arose when an optimal control policy was bang-bang, but there was no upper bound on the control variable. However, there are other kinds of situations in management science and related areas, in which a finite change in the value of a state variable is explicitly permitted at some instants of time to be determined, and these do not easily lend themselves to the impulse control methodology given in Chapters 5 and 7. For that reason we shall briefly discuss a more general *theory of impulse control* that is