Chapter 2

The Maximum Principle: Continuous Time

The main purpose of this chapter is to introduce the maximum principle as a necessary condition that must be satisfied by any optimal control for the basic problem specified in Section 2.1. Although vector notation is used, the reader can think of the problem as having only a single state variable and a single control variable on the first reading. In Section 2.2, the method of dynamic programming is used to derive the maximum principle. We use this method because of the simplicity and familiarity of the dynamic programming concept. The derivation also yields significant economic interpretations. (In Appendix C, the maximum principle is also derived by using a more general method similar to that of Pontryagin et al. (1962), but with certain simplifications.) In Section 2.4, the maximum principle is shown to be sufficient for optimal control under an appropriate concavity condition, which holds in many management science applications.

2.1 Statement of the Problem

Optimal control theory deals with the problem of optimization of dynamic systems. The problem must be well posed before any solution can be attempted. This requires a clear mathematical description of the system to be optimized, the constraints imposed on the system, and the objective function to be maximized (or minimized).

2.1.1 The Mathematical Model

An important part of any control problem is the process of modeling the dynamic system, physical, business, or otherwise, under consideration. The aim is to arrive at a mathematical description which is simple enough to deal with, and realistic enough to be able to predict the response of the system to any given input. The model of the system for our purpose is restricted to systems that can be characterized by a system of ordinary differential equations (or, ordinary difference equations in the discrete-time case treated in Chapter 8). Thus, given the initial state x_0 of the system and control history $u(t), t \in [0,T]$, of the process, the evolution of the system may be described by the first-order differential equation, known also as the *state equation*,

$$\dot{x}(t) = f(x(t), u(t), t), \; x(0) = x_0, \tag{2.1}$$

where the vector of *state variables*, $x(t) \in E^n$, the vector of *control variables*, $u(t) \in E^m$, and $f : E^n \times E^m \times E^1 \to E^n$. Furthermore, the function f is assumed to be continuously differentiable. Here we assume x to be a column vector and f to be a column vector of functions. The path $x(t), t \in [0,T]$, is called a *state trajectory* and $u(t)$, $t \in [0,T]$, is called a *control trajectory* or simply, a *control*. The terms, vector of state variables, state vector, and state will be used interchangeably; similarly for the terms, vector of control variables, control vector, and control. As mentioned earlier, when no confusion arises, we will usually suppress the time notation (t); thus, e.g., $x(t)$ will be written simply as x. Furthermore, whether x denotes the state at time t or the entire state trajectory should be inferred from the context. A similar statement holds for u.

2.1.2 Constraints

In this chapter, we are concerned with problems which do not have state constraints of types (1.4) and (1.5). Such constraints are considered in Chapters 3 and 4, as indicated in Section 1.1. We do impose constraints of type (1.3) on the control variables. We define an *admissible control* to be a control trajectory $u(t)$, $t \in [0,T]$, which is piecewise continuous and satisfies, in addition,

$$u(t) \in \Omega(t) \subset E^m, \; t \in [0,T]. \tag{2.2}$$

Usually the set $\Omega(t)$ is determined by physical or economic constraints on the values of the control variables at time t.

2.1.3 The Objective Function

An objective function is a quantitative measure of the performance of the system over time. An *optimal control* is defined to be an admissible control which maximizes the objective function. In business or economic problems, a typical objective function gives some appropriate measure of quantities such as profits, sales, or negative of costs. Mathematically, we let

$$J = \int_0^T F(x(t), u(t), t)dt + S[x(T), T] \tag{2.3}$$

denote the objective function, where the functions $F : E^n \times E^m \times E^1 \rightarrow E^1$ and $S : E^n \times E^1 \rightarrow E^1$ are assumed for our purposes to be continuously differentiable. In a typical business application, $F(x, u, t)$ could be the *instantaneous profit rate* and $S[x, T]$ could be the *salvage value* of having x as the system state at the *terminal time* T.

2.1.4 The Optimal Control Problem

Given the preceding definitions we can state the optimal control problem with which we will be concerned in this chapter. The problem is to find an admissible control u^*, which maximizes the objective function (2.3) subject to the state equation (2.1) and the control constraints (2.2). We now restate the optimal control problem as:

$$\begin{cases} \max\limits_{u(t)\in\Omega(t)} \left\{ J = \displaystyle\int_0^T F(x, u, t)dt + S[x(T), T] \right\} \\ \text{subject to} \\ \dot{x} = f(x, u, t), \ x(0) = x_0. \end{cases} \tag{2.4}$$

The control u^* is called an *optimal control* and x^*, determined by means of the state equation with $u = u^*$, is called the *optimal trajectory* or an *optimal path*. The optimal value of the objective function will be denoted as $J(u^*)$ or J^*.

The optimal control problem (2.4) specified above is said to be in *Bolza form* because of the form of the objective function in (2.3). It is said to be in *Lagrange form* when $S \equiv 0$. We say the problem is in *Mayer form* when $F \equiv 0$. Furthermore, it is in *linear Mayer form* when $F \equiv 0$

and S is linear, i.e.,

$$\begin{cases} \max\limits_{u(t)\in\Omega(t)} \{J = cx(T)\} \\ \text{subject to} \\ \dot{x} = f(x,u,t),\ x(0) = x_0, \end{cases} \tag{2.5}$$

where $c = (c_1, c_2, \cdots, c_n)$ is an n-dimensional row vector of given constants. In the next paragraph and in Exercise 2.3, it will be demonstrated that all of these forms can be converted into the linear Mayer form.

To show that the Bolza form can be reduced to the linear Mayer form, we define a new state vector $y = (y_1, y_2, \ldots, y_{n+1})$, having $n+1$ components defined as follows: $y_i = x_i$ for $i = 1, \ldots, n$ and

$$\dot{y}_{n+1} = F(x,u,t) + \frac{\partial S(x,t)}{\partial x} f(x,u,t) + \frac{\partial S(x,t)}{\partial t},\ y_{n+1}(0) = 0. \tag{2.6}$$

We also put $c = (0, \cdots, 0, 1)$, where c has $n+1$ components, so that the objective function is $J = cy(T) = y_{n+1}(T)$. If we now integrate (2.6) from 0 to T, we have

$$J = cy(T) = y_{n+1}(T) = \int_0^T F(x,u,t)dt + S[x(T),T], \tag{2.7}$$

which is the same as the objective function J in (2.4). Of course, the price paid for going from Bolza to linear Mayer form is the addition of one state variable and its associated differential equation (2.6).

Exercise 2.3 poses the question of showing in a similar way that the Lagrange and Mayer forms can also be reduced to the linear Mayer form.

In Section 2.2, we derive necessary conditions for optimal control in the form of the maximum principle, and in Section 2.4 we derive sufficient conditions. In any particular application, the existence of a solution will be demonstrated by actually finding a solution that satisfies both the necessary and the sufficient conditions for optimality. We thus avoid the necessity of having to prove general existence theorems, which require advanced and difficult mathematics. Nevertheless, interested readers can consult Hartl, Sethi, and Vickson (1995) for a brief discussion of existence results and references therein including Cesari (1983) for further details.

2.2 Dynamic Programming and the Maximum Principle

We shall now derive the maximum principle by using a dynamic programming approach. The proof is intuitive in nature and is not intended to be mathematically rigorous. For more rigorous derivations, we refer the reader to Appendix C, Pontryagin et al. (1962), Berkovitz (1961), Halkin (1967), Boltyanskii (1971), Hartberger (1973), Bryant and Mayne (1974), Leitmann (1981), and Seierstad and Sydsæter (1987). Additional references can be found in the survey by Hartl, Sethi, and Vickson (1995). For maximum principles for more general optimal control problems including those with nondifferentiable functions, see Clarke (1996, 1983, 1989).

2.2.1 The Hamilton-Jacobi-Bellman Equation

Suppose $V(x,t) : E^n \times E^1 \to E^1$ is a function whose value is the maximum value of the objective function of the control problem for the system, given that we start it at time t in state x. That is,

$$V(x,t) = \max_{u(s)\in\Omega(s)} \int_t^T F(x(s),u(s),s)ds + S(x(T),T), \tag{2.8}$$

where for $s \geq t$,

$$\frac{dx}{ds} = f(x(s),u(s),s), \;\; x(t) = x$$

We initially assume that the *value function* $V(x,t)$ exists for all x and t in the relevant ranges. Later we will make additional assumptions about the function $V(x,t)$.

Richard Bellman (1957) in his book on dynamic programming states the *principle of optimality* as follows:

> An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the outcome resulting from the first decision.

Intuitively this principle is obvious, for if we were to start in state x at time t and did not follow an optimal path from there on, then there would exist (by assumption) a better path from t to T, hence we could

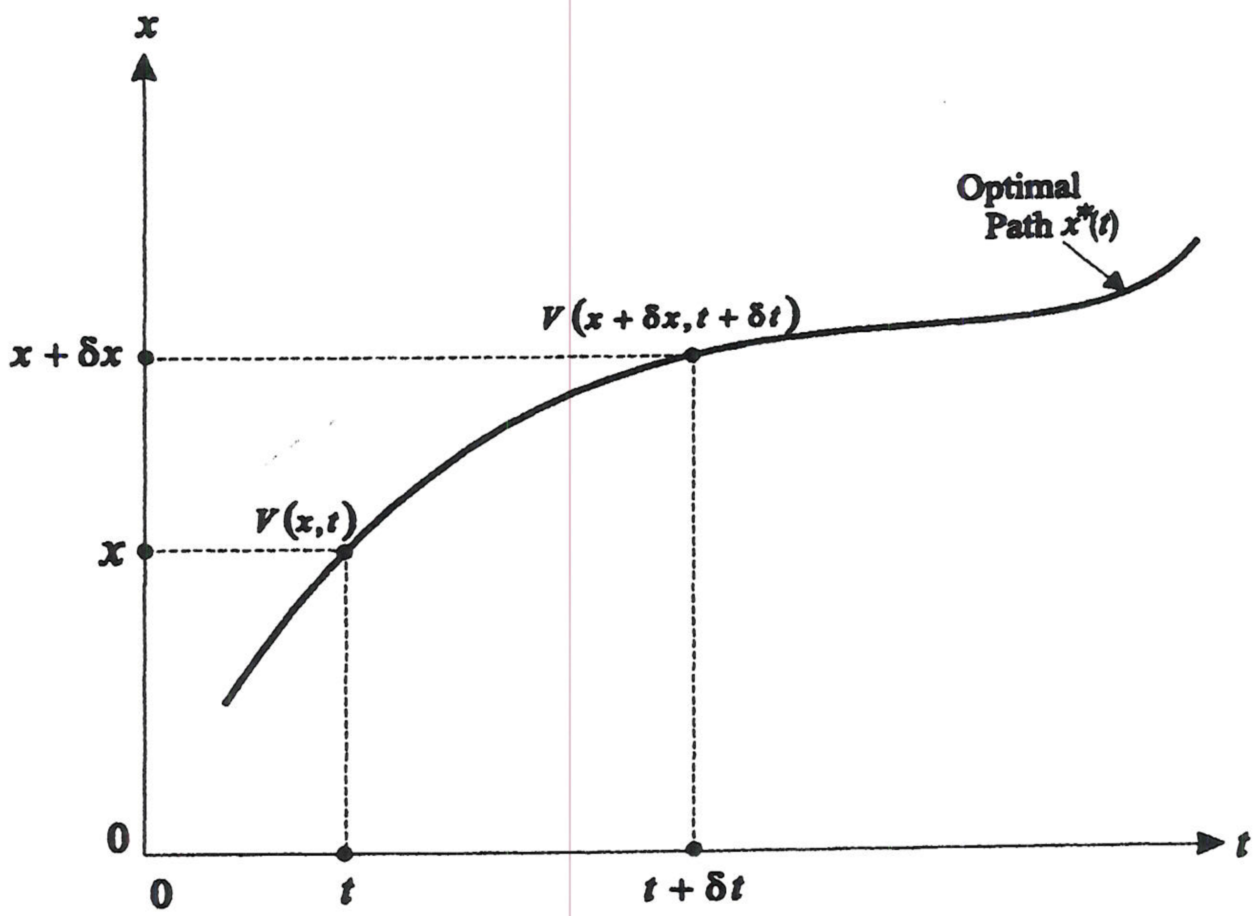

Figure 2.1: An Optimal Path in the State-Time Space

improve the proposed solution by following the better path from time t on. We will use the principle of optimality to derive conditions on the value function $V(x, t)$.

Figure 2.1 is a schematic picture of the optimal path $x^*(t)$ in the state-time space, and two nearby points (x, t) and $(x + \delta x, t + \delta t)$, where δt is a small increment of time and $x + \delta x = x(t + \delta t)$. The value function changes from $V(x, t)$ to $V(x + \delta x, t + \delta t)$ between these two points. By the principle of optimality, the change in the objective function is made up of two parts: first, the incremental change in J from t to $t + \delta t$, which is given by the integral of $F(x, u, t)$ from t to $t + \delta t$; second, the value function $V(x + \delta x, t + \delta t)$ at time $t + \delta t$. The control actions $u(\tau)$ should be chosen to lie in $\Omega(\tau)$, $\tau \in [t, t + \delta t]$, and to maximize the sum of these two terms. In equation form this is

$$V(x,t) = \max_{\substack{u(\tau)\in\Omega(\tau) \\ \tau\in[t,t+\delta t]}} \left\{ \int_t^{t+\delta t} F[x(\tau), u(\tau), \tau]d\tau + V[x(t+\delta t), t+\delta t] \right\}, \tag{2.9}$$

where δt represents a small increment in t. It is instructive to compare this equation to definition (2.8).

Since F is a continuous function, the integral in (2.9) is approximately $F(x, u, t)\delta t$ so that we can rewrite (2.9) as

$$V(x,t) = \max_{u \in \Omega(t)} \{F(x,u,t)\delta t + V[x(t+\delta t), t+\delta t]\} + o(\delta t), \tag{2.10}$$

where $o(\delta t)$ denotes a collection of higher-order terms in δt. (By definition given in Section 1.4, $o(\delta t)$ is a function such that $\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0$).

We now make an assumption of which we will talk more later. We assume that the value function V is a continuously differentiable function of its arguments. This allows us to use the Taylor series expansion of V with respect to δt and obtain

$$V[x(t+\delta t), t+\delta t] = V(x,t) + [V_x(x,t)\dot{x} + V_t(x,t)]\delta t + o(\delta t), \tag{2.11}$$

where V_x and V_t are partial derivatives of $V(x,t)$ with respect to x and t, respectively.

Substituting for $\dot{x}$ from (2.1) in the above equation and then using it in (2.10), we obtain

$$\begin{aligned} V(x,t) &= \max_{u \in \Omega(t)} \{F(x,u,t)\delta t + V(x,t) + V_x(x,t)f(x,u,t)\delta t \\ &\quad + V_t(x,t)\delta t\} + o(\delta t). \end{aligned} \tag{2.12}$$

Canceling $V(x,t)$ on both sides and then dividing by δt we get

$$0 = \max_{u \in \Omega(t)} \{F(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t)\} + \frac{o(\delta t)}{\delta t}. \tag{2.13}$$

Now we let $\delta t \to 0$ and obtain the following equation

$$0 = \max_{u \in \Omega(t)} \{F(x,u,t) + V_x(x,t)f(x,u,t) + V_t(x,t)\}, \tag{2.14}$$

with the boundary condition

$$V(x,T) = S(x,T). \tag{2.15}$$

That this boundary condition must hold follows from the fact that at $t = T$, the value function is simply the salvage value function.

Note that the components of the vector $V_x(x,t)$ can be interpreted as the marginal contributions of the state variables x to the objective function being maximized. We denote the marginal return vector (along the optimal path $x^*(t)$) by the *adjoint* (row) vector $\lambda(t) \in E^n$, i.e.,

$$\lambda(t) = V_x(x^*(t), t) := V_x(x, t) \mid_{x=x^*(t)} . \tag{2.16}$$

From the preceding remark, we can also interpret $\lambda(t)$ as the per unit change in the objective function for a small change in $x^*(t)$ at time t; see Section 2.2.4. Next we introduce the so-called *Hamiltonian*

$$H[x, u, V_x, t] = F(x, u, t) + V_x(x, t) f(x, u, t) \tag{2.17}$$

or, simply,

$$H(x, u, \lambda, t) = F(x, u, t) + \lambda f(x, u, t). \tag{2.18}$$

We can rewrite equation (2.14) as the following equation,

$$0 = \max_{u \in \Omega(t)} [H(x, u, V_x, t) + V_t], \tag{2.19}$$

called the *Hamilton-Jacobi-Bellman equation* or, simply, the HJB equation.

Note that it is possible to take V_t out of the maximizing operation since it does not depend on u.

The Hamiltonian maximizing condition of the maximum principle can be obtained from (2.19) and (2.16) by observing that, if $x^*(t)$ and $u^*(t)$ are optimal values of the state and control variables and $\lambda(t)$ is the corresponding value of the adjoint variable at time t, then the optimal control $u^*(t)$ must satisfy (2.19), i.e., for all $u \in \Omega(t)$,

$$\begin{aligned} H[x^*(t), u^*(t), \lambda(t), t] + V_t(x^*(t), t) \geq\ & H[x^*(t), u, \lambda(t), t] \\ & + V_t(x^*(t), t). \end{aligned} \tag{2.20}$$

Canceling the term V_t on both sides, we obtain

$$H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t] \tag{2.21}$$

for all $u \in \Omega(t)$.

In order to complete the statement of the maximum principle, we must still obtain the adjoint equation.

2.2.2 Derivation of the Adjoint Equation

The derivation of the adjoint equation proceeds from the HJB equation (2.19), and is similar to those in Fel'dbaum (1965) and Kirk (1970). Note that, given the optimal path x^*, the optimal control u^* maximizes the right-hand side of (2.19), and its maximum value is zero. We now consider small perturbations of the values of the state variables in a neighborhood of the optimal path x^*. Thus, let

$$x(t) = x^*(t) + \delta x(t), \tag{2.22}$$

where $\delta x(t)$, $\| \delta x(t) \| < \varepsilon$ for a small positive ε, be such a perturbation.

We now consider a 'fixed' time instant t. We can then write (2.19) as

$$\begin{aligned} &H[x^*(t), u^*(t), V_x(x^*(t), t), t] + V_t(x^*(t), t) \\ &\quad \geq H[x(t), u^*(t), V_x(x(t), t), t] + V_t(x(t), t). \end{aligned} \tag{2.23}$$

To explain, we note from (2.19) that the left-hand side of (2.23) equals zero. The right-hand side can attain the value zero only if $u^*(t)$ is also an optimal control for $x(t)$. In general for $x(t) \neq x^*(t)$, this will not be so. From this observation, it follows that the expression on the right-hand side of (2.23) attains its maximum (of zero) at $x(t) = x^*(t)$. Furthermore, $x(t)$ is not explicitly constrained. In other words, $x^*(t)$ is an unconstrained local maximum of the right-hand side of (2.23), so that the derivative of this expression with respect to x must vanish at $x^*(t)$, i.e.,

$$H_x[x(t), u^*(t), V_x(x(t), t), t] + V_{tx}(x(t), t) = 0. \tag{2.24}$$

In order to take the derivative as we did in (2.24), we must further assume that V is a twice continuously differentiable function of its arguments. Using the definition of the Hamiltonian in (2.17), the identity (1.15), and the fact that $V_{xx} = (V_{xx})^T$, we obtain

$$F_x + V_x f_x + f^T V_{xx} + V_{tx} = F_x + V_x f_x + (V_{xx} f)^T + V_{tx} = 0, \tag{2.25}$$

where the superscript ^{T} denotes the transpose operation. See (1.16) or Exercise 1.9 for further explanation.

The derivation of the necessary condition (2.25) is the crux of the reasoning in the derivation of the adjoint equation. It is easy to obtain the so-called adjoint equation from it. We begin by taking the time derivative of $V_x(x,t)$. Thus,

$$\begin{aligned}
\frac{dV_x}{dt} &= \left(\frac{dV_{x_1}}{dt}, \frac{dV_{x_2}}{dt}, \cdots, \frac{dV_{x_n}}{dt}\right) \\
&= (V_{x_1 x}\dot{x} + V_{x_1 t}, V_{x_2 x}\dot{x} + V_{x_2 t}, \cdots, V_{x_n x}\dot{x} + V_{x_n t}) \\
&= \left(\textstyle\sum_{i=1}^n V_{x_1 x_i}\dot{x}_i, \sum_{i=1}^n V_{x_2 x_i}\dot{x}_i, \cdots, \sum_{i=1}^n V_{x_n x_i}\dot{x}_i\right) + (V_x)_t \\
&= (V_{xx}\dot{x})^T + V_{xt} \\
&= (V_{xx}f)^T + V_{tx}.
\end{aligned} \tag{2.26}$$

Since the terms on the right-hand side of (2.26) are the same as the last two terms in (2.25), we see that (2.26) becomes

$$\frac{dV_x}{dt} = -F_x - V_x f_x. \tag{2.27}$$

Because λ was defined in (2.16) to be V_x, we can rewrite (2.27) as

$$\dot{\lambda} = -F_x - \lambda f_x.$$

To see that the right-hand side of this equation can be written simply as $-H_x$, we need to go back to the definition of H in (2.18) and recognize that when taking the partial derivative of H with respect to x, the adjoint variables λ are considered to be independent of x. We note further that along the optimal path, λ is a function of t only. Thus,

$$\dot{\lambda} = -H_x. \tag{2.28}$$

Also, from the definition of λ in (2.16) and the boundary condition (2.15), we have the *terminal boundary condition*, which is also called the *transversality condition*:

$$\lambda(T) = \frac{\partial S(x,T)}{\partial x}\Big|_{x=x(T)} = S_x[x(T),T]. \tag{2.29}$$

The *adjoint equation* (2.28) together with its boundary condition (2.29) determine the adjoint variables.

From the definition of the Hamiltonian in (2.18), it is also obvious that the state equation can be written as

$$\dot{x} = f = H_\lambda, \tag{2.30}$$

using an argument similar to the one used in the derivation of (2.28). The system of equations (2.28) and (2.30) along with their respective boundary conditions can be collected together and expressed as the system

$$\begin{cases} \dot{x} = H_\lambda, & x(0) = x_0, \\ \dot{\lambda} = -H_x, & \lambda(T) = S_x[x(T), T], \end{cases} \tag{2.31}$$

called a *canonical system of equations* or *canonical adjoints*. Hence, the name *adjoint vector* for λ.

This completes our derivation of the maximum principle using dynamic programming. We can now summarize the main results in the following section.

2.2.3 The Maximum Principle

The necessary conditions for u^* to be an optimal control are:

$$\begin{cases} \dot{x}^* = f(x^*, u^*, t), x^*(0) = x_0, \\ \dot{\lambda} = -H_x[x^*, u^*, \lambda, t], \quad \lambda(T) = S_x[x^*(T), T], \\ H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t], \end{cases} \tag{2.32}$$

for all $u \in \Omega(t)$, $t \in [0, T]$.

It should be emphasized that the state and the adjoint arguments of the Hamiltonian are $x^*(t)$ and $\lambda(t)$ on both sides of the Hamiltonian maximizing condition in (2.32), respectively,. Furthermore, $u^*(t)$ must provide a *global* maximum of the Hamiltonian $H[x^*(t), u, \lambda(t), t]$ over $u \in \Omega(t)$. For this reason the necessary conditions in (2.32) are called the *maximum principle.*

Note that in order to apply the maximum principle, we must simultaneously solve two sets of differential equations with u^* obtained from the Hamiltonian maximizing condition in (2.32). With the control variable u^* so obtained, the state equation for x^* is given with the initial value x_0, and the adjoint equation for λ is specified with a condition on the terminal value $\lambda(T)$. Such a system of equations, where initial values of some variables and final values of other variables are specified, is called a *two-point boundary value problem* (TPBVP). The general solution of such problems can be very difficult; see Bryson and Ho (1969), Roberts and Shipman (1972), and Feichtinger and Hartl (1986). However, there are certain special cases which are easy. One such is the case in which the adjoint equation is independent of the state and the control variables; here we can solve the adjoint equation first, then get the optimal control u^*, and then solve for x^*.

In subsequent chapters we will solve many two-point boundary value problems of varying degrees of difficulty.

Note also that if we can solve the Hamiltonian maximizing condition for an optimal control function in closed form as

$$u^*(t) = u[x^*(t), \lambda(t), t],$$

then we can substitute these into the state and adjoint equations to get a two-point boundary value problem just in terms of a set of differential equations, i.e.,

$$\begin{cases} \dot{x}^* = f(x^*, u(x^*, \lambda, t), t), & x^*(0) = x_0, \\ \dot{\lambda} = -H_x(x^*, u(x^*, \lambda, t), t), & \lambda(T) = S_x[x^*(T), T]. \end{cases}$$

In Exercise 2.17, you are asked to derive such a two-point boundary value problem.

One final remark should be made. Because an integral is unaffected by values of the integrand at a finite set of points, some of the arguments made in this chapter may not hold at a finite set of points. This does not affect the validity of the results.

In the next section, we give economic interpretations of the maximum principle, and in Section 2.3, we solve five simple examples by using the maximum principle.

2.2.4 Economic Interpretations of the Maximum Principle

Recall from Section 2.1.3 that the objective function (2.3) is

$$J = \int_0^T F(x, u, t)dt + S[x(T), T],$$

where F is considered to be the instantaneous profit rate measured in dollars per unit of time, and $S[x, T]$ is the salvage value, in dollars, of the system at time T when the terminal state is x. For purposes of discussion it will be convenient to consider the system as a firm and the state $x(t)$ as the stock of capital at time t.

In (2.16), we interpreted $\lambda(t)$ to be the per unit change in the value function $V(x,t)$ for small changes in capital stock x. In other words, $\lambda(t)$ is the marginal value per unit of capital at time t, and it is also referred to as the *price* or *shadow price* of a unit of capital at time t. In particular, the value of $\lambda(0)$ is the marginal rate of change of the maximum value of J (the objective function) with respect to the change in the initial capital stock, x_0.

The interpretation of the Hamiltonian function in (2.18) can now be obtained. Multiplying (2.18) formally by dt and using the state equation (2.1) gives

$$Hdt = Fdt + \lambda f dt = Fdt + \lambda \dot{x} dt = Fdt + \lambda dx.$$

The first term $F(x,u,t)dt$ represents the *direct contribution* to J in dollars from time t to $t+dt$, if the firm is in state x (i.e., it has a capital stock of x), and we apply control u in the interval $[t, t+dt]$. The differential $dx = f(x,u,t)dt$ represents the change in capital stock from time t to $t+dt$, when the firm is in state x and control u is applied. Therefore, the second term λdx represents the value in dollars of the incremental capital stock, dx, and hence can be considered as the *indirect contribution* to J in dollars. Thus, Hdt can be interpreted as the *total contribution* to J from time t to $t + dt$ when $x(t) = x$ and $u(t) = u$ in the interval $[t, t + dt]$.

With this interpretation of the Hamiltonian, it is easy to see why the Hamiltonian must be maximized at each instant of time t. If we were just to maximize F at each instant t, we would not be maximizing J, because we would ignore the effect of control in changing the capital stock, which gives rise to indirect contributions to J. The maximum principle derives the adjoint variable $\lambda(t)$, the price of capital at time t, in such way that $\lambda(t)dx$ is the correct valuation of the indirect contribution to J from time t to $t+dt$. As a consequence, the Hamiltonian maximizing problem can be treated as a static problem at each instant t. In other words, the maximum principle *decouples* the dynamic maximization problem (2.4) in the interval $[0,T]$ into a set of static maximization problems associated with instants t in $[0,T]$. Thus, the Hamiltonian can be interpreted as a surrogate profit rate to be maximized at each instant of time t.

The value of λ to be used in the maximum principle is given by (2.28) and (2.29), i.e.,

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\frac{\partial F}{\partial x} - \lambda \frac{\partial f}{\partial x}, \ \lambda(T) = S_x[x(T), T].$$

Rewriting the first equation as

$$-d\lambda = H_x dt = F_x dt + \lambda f_x dt,$$

we can observe that along the optimal path, the decrease $d\lambda$ in the price of capital from t to $t + dt$, which can be considered as the *marginal cost of holding that capital*, equals the *marginal revenue $H_x dt$ of investing the capital.* In turn the marginal revenue, $H_x dt$, consists of the sum of direct marginal contribution, $F_x dt$, and the indirect marginal contribution, $\lambda f_x dt$. Thus, the adjoint equation becomes the equilibrium relation—*marginal cost equals marginal revenue*, which is a familiar concept in the economics literature. See, e.g., Cohen and Cyert (1965, p.189) or Takayama (1974, p.712).

Further insight can be obtained by integrating the above adjoint equation from t to T as follows:

$$\begin{aligned}\lambda(t) &= \lambda(T) + \textstyle\int_t^T H_x(x(\tau), u(\tau), \lambda(\tau), \tau)d\tau \\ &= S_x[x(T), T] + \textstyle\int_t^T H_x d\tau.\end{aligned}$$

Note that the price $\lambda(T)$ of a unit of capital at time T is its marginal salvage value, $S_x[x(T), T]$. The price $\lambda(t)$ of a unit of capital at time t is the sum of its terminal price, $\lambda(T)$, plus the integral of the marginal surrogate profit rate, H_x, from t to T.

The above interpretations show that the adjoint variables behave in much the same way as do the *dual variables* in linear (and nonlinear) programming. The differences being that here the adjoint variables are time dependent and satisfy derived differential equations. These connections will become clearer in Chapter 8, which addresses the discrete maximum principle.

2.3 Elementary Examples

In order to absorb the maximum principle, the reader should study very carefully the examples in this section, all of which are problems having only one state and one control variable. Some or all of the exercises at the end of the chapter should also be worked.

In the following examples and others in this book, we shall omit the superscript $*$ on the optimal value of the state variable when there is no confusion arising in doing so.

Example 2.1 Consider the problem:

$$\max \left\{ J = \int_0^1 -x dt \right\} \tag{2.33}$$

subject to the state equation

$$\dot{x} = u, \ x(0) = 1 \tag{2.34}$$

and the control constraint

$$u \in \Omega = [-1, 1]. \tag{2.35}$$

Note that $T = 1$, $F = -x, S = 0$, and $f = u$. Because $F = -x$, we can interpret the problem as one of minimizing the (signed) area under the curve $x(t)$ for $0 \leq t \leq 1$.

Solution. First, we form the Hamiltonian

$$H = -x + \lambda u \tag{2.36}$$

and note that, because the Hamiltonian is linear in u, the form of the optimal control, i.e., the one that would maximize the Hamiltonian, is

$$u^*(t) = \begin{cases} 1 & \text{if} \quad \lambda(t) > 0, \\ \text{undefined} & \text{if} \quad \lambda(t) = 0, \\ -1 & \text{if} \quad \lambda(t) < 0, \end{cases} \tag{2.37}$$

or referring to the notation in Section 1.4,

$$u^*(t) = \text{bang}[-1, 1; \lambda(t)]. \tag{2.38}$$

To find λ, we write the adjoint equation

$$\dot{\lambda} = -H_x = 1, \ \lambda(1) = S_x[x(T), T] = 0. \tag{2.39}$$

Because this equation does not involve x and u, we can easily solve it as

$$\lambda(t) = t - 1. \tag{2.40}$$

It follows that $\lambda(t) = t - 1 \leq 0$ for all $t \in [0, 1]$ and since we can set $u^*(1) = -1$, which defines u at the single point $t = 1$, we have the optimal control

$$u^*(t) = -1 \text{ for } t \in [0, 1].$$

Substituting this into the state equation (2.34) we have

$$\dot{x} = -1, \; x(0) = 1, \tag{2.41}$$

whose solution is

$$x(t) = 1 - t \text{ for } t \in [0, 1]. \tag{2.42}$$

The graphs of the optimal state and adjoint trajectories appear in Figure 2.2. Note that the optimal value of the objective function is $J^* = -1/2$.

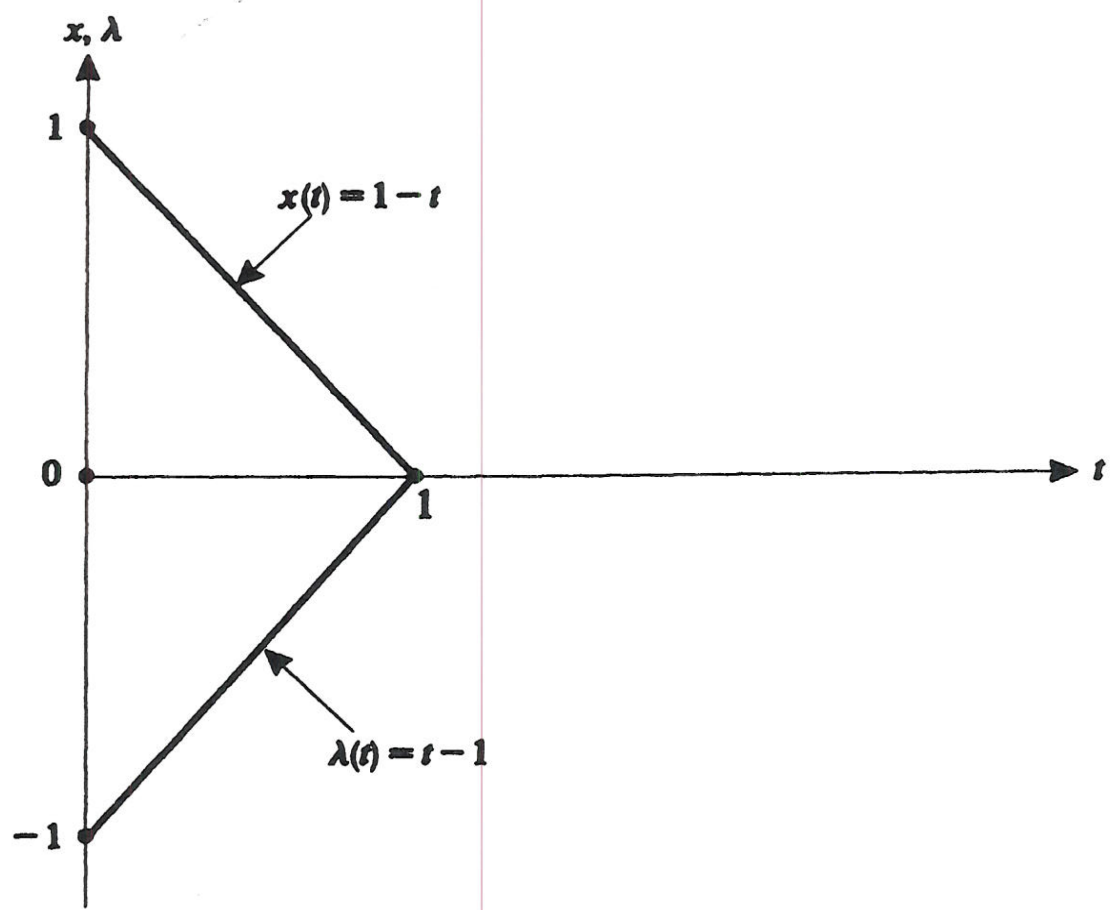

Figure 2.2: Optimal State and Adjoint Trajectories for Example 2.1

Example 2.2 Let us solve the same problem as in Example 2.1 over the interval $[0, 2]$ so that the objective is to

$$\text{maximize} \left\{ J = \int_0^2 -x dt \right\}. \tag{2.43}$$

The dynamics and constraints are (2.34) and (2.35), respectively, as before. Here we want to minimize the *signed* area between the horizontal axis and the trajectory of $x(t)$ for $0 \leq t \leq 2$.

Solution. As before the Hamiltonian is defined by (2.36) and the optimal control is as in (2.38). The adjoint equation

$$\dot{\lambda} = 1,\ \lambda(2) = 0 \tag{2.44}$$

is the same as (2.39) except that now $T = 2$ instead of $T = 1$. The solution of (2.44) is easily found to be

$$\lambda(t) = t - 2,\ t \in [0, 2]. \tag{2.45}$$

The graph of $\lambda(t)$ is shown in Figure 2.3.

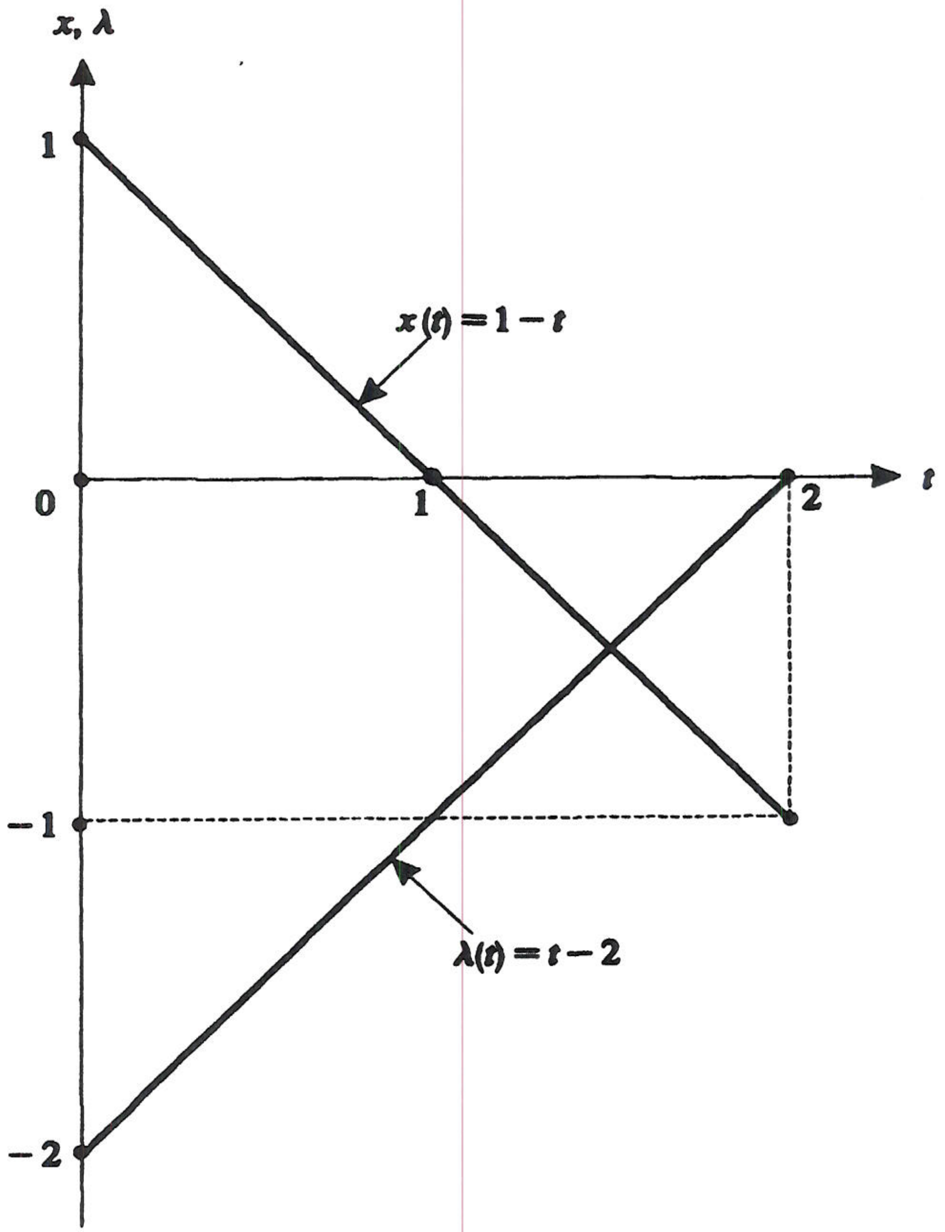

Figure 2.3: Optimal State and Adjoint Trajectories for Example 2.2

With $\lambda(t)$ as in (2.45), we can determine $u^*(t) = -1$ throughout. Thus, the state equation is the same as (2.41). Its solution is given by

(2.42) for $t \in [0, 2]$. The optimal value of the objective function is $J^* = 0$. The graph of $x(t)$ is also sketched in Figure 2.3.

Example 2.3 The next example is:

$$\max \left\{ J = \int_0^1 -\frac{1}{2}x^2 dt \right\} \tag{2.46}$$

subject to the same constraints as in Example 2.1, namely,

$$\dot{x} = u, \; x(0) = 1, \; u \in \Omega = [-1, 1]. \tag{2.47}$$

Here $F = -(1/2)x^2$ so that the interpretation of the objective function (2.46) is that we are trying to find the trajectory $x(t)$ in order that the area under the curve $(1/2)x^2$ is minimized.

Solution. The Hamiltonian is

$$H = -\frac{1}{2}x^2 + \lambda u, \tag{2.48}$$

which is linear in u so that the optimal policy is

$$u^*(t) = \text{bang}\,[-1, 1; \lambda]. \tag{2.49}$$

The adjoint equation is

$$\dot{\lambda} = -H_x = x, \; \lambda(1) = 0. \tag{2.50}$$

Here the adjoint equation involves x so that we cannot solve it directly. Because the state equation (2.47) involves u, which depends on λ, we also cannot integrate it independently without knowing λ.

The way out of this dilemma is to use some intuition. Since we want to minimize the area under $(1/2)x^2$ and since $x(0) = 1$, it is clear that we want x to decrease as quickly as possible. Let us therefore temporarily *assume* that λ is nonpositive in the interval $[0, 1]$ so that from (2.49) we have $u = -1$ throughout the interval. (In Exercise 2.5, you will be asked to show that this assumption is correct.) With this assumption, we can solve (2.47) as

$$x(t) = 1 - t. \tag{2.51}$$

Substituting this into (2.50) gives

$$\dot{\lambda} = 1 - t.$$

Integrating both sides of this equation from t to 1 gives

$$\int_t^1 \dot{\lambda}(\tau)d\tau = \int_t^1 (1-\tau)d\tau,$$

or

$$\lambda(1) - \lambda(t) = (\tau - \frac{1}{2}\tau^2)\mid_t^1,$$

which, using $\lambda(1) = 0$, yields

$$\lambda(t) = -\frac{1}{2}t^2 + t - \frac{1}{2}. \tag{2.52}$$

The reader may now verify that $\lambda(t)$ is nonpositive in the interval $[0,1]$, verifying our original assumption. Hence, (2.51) and (2.52) satisfy the necessary conditions. In Exercise 2.6, you will be asked to show that they satisfy sufficient conditions derived in Section 2.4 as well, so that they are indeed optimal. Figure 2.4 shows the graphs of the optimal trajectories.

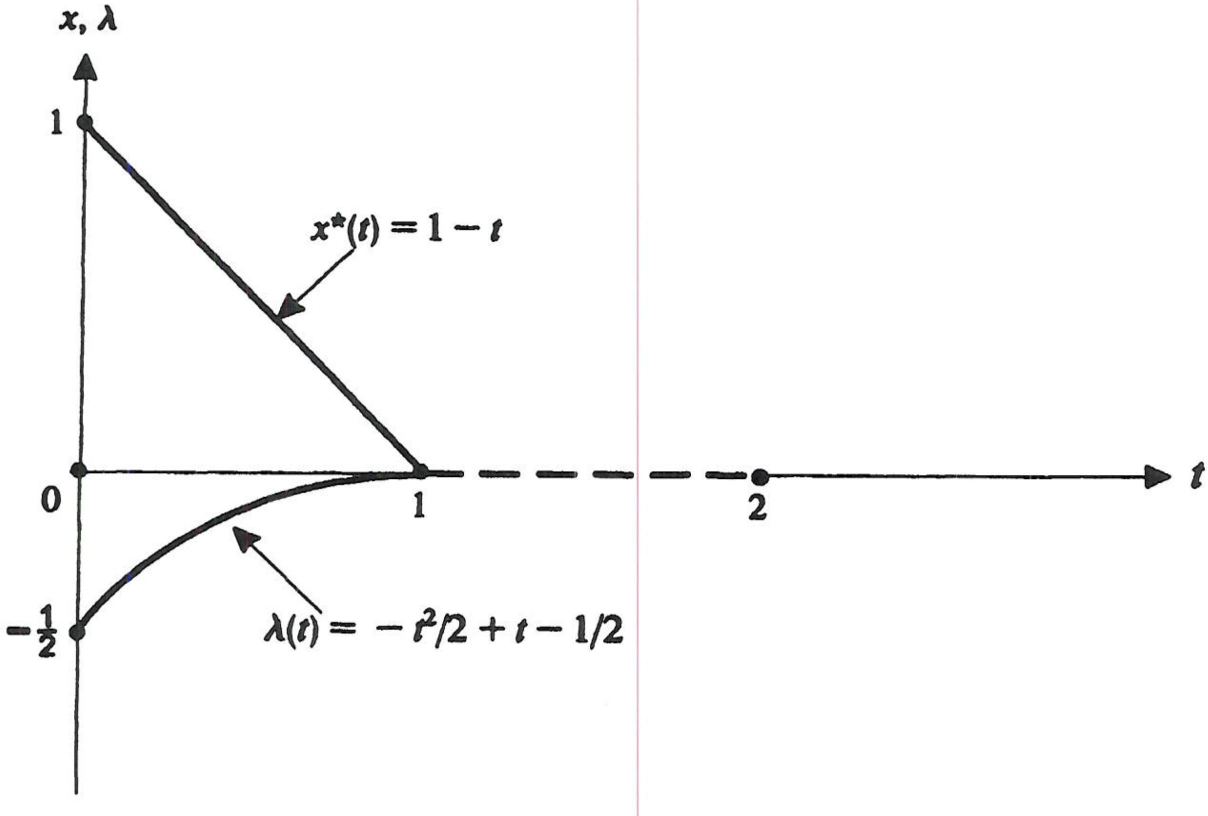

Figure 2.4: Optimal Trajectories for Examples 2.3 and 2.4

Example 2.4 Let us rework Example 2.3 with $T = 2$, i.e., with the objective function:

$$\max\left\{J=\int_0^2 -\frac{1}{2}x^2dt\right\} \tag{2.53}$$

subject to the constraints (2.47).

Solution. The Hamiltonian is still as in (2.48) and the form of the optimal policy remains as in (2.49). The adjoint equation is

$$\dot{\lambda}=x,\ \lambda(2)=0,$$

which is the same as (2.50) except $T=2$ instead of $T=1$. Let us try to extend the solution of the previous example from $T=1$ to $T=2$. Note from (2.52) that $\lambda(1)=0$. If we recall from the definition of the bang function that bang $[-1,1;0]$ is not defined, it allows us to choose u in (2.49) arbitrarily when $\lambda=0$. This is an instance of *singular control*, so let us see if we can *maintain* the singular control by choosing u appropriately. To do this we choose $u=0$ when $\lambda=0$. Since $\lambda(1)=0$ we set $u(1)=0$ so that from (2.47), we have $\dot{x}(1)=0$. Now note that if we set $u(t)=0$ for $t>1$, then by integrating equations (2.47) and (2.50) forward from $t=1$ to $t=2$, we see that $x(t)=0$ and $\lambda(t)=0$ for $1<t<2$; in other words, $u(t)=0$ maintains singular control in the interval. Intuitively, this is the correct answer since once we get $x=0$, we should keep it at 0 in order to maximize the objective function J in (2.53). We will later give further discussion of *singular control* and will state an additional necessary condition in Appendix D.3 for such cases; see also Bell and Jacobson (1975). In Figure 2.4, we can get the singular solution by extending the graphs shown to the right (as shown by thick dotted line), making $x(t)=0$ and $u^*(t)=0$ for $1\le t\le 2$.

Example 2.5 Our last example is slightly more complicated and the optimal control is not bang-bang. The problem is:

$$\max\left\{J=\int_0^2 (2x-3u-u^2)dt\right\} \tag{2.54}$$

subject to

$$\dot{x}=x+u,\ x(0)=5 \tag{2.55}$$

and the control constraint

$$u\in\Omega=[0,2]. \tag{2.56}$$

Solution. Here $T = 2$, $F = 2x - 3u - u^2$, $S = 0$, and $f = x + u$. The Hamiltonian is

$$\begin{aligned} H &= (2x - 3u - u^2) + \lambda(x + u) \\ &= (2 + \lambda)x - (u^2 + 3u - \lambda u). \end{aligned} \tag{2.57}$$

Let us find the optimal control policy by differentiating (2.57) with respect to u. Thus,

$$\frac{\partial H}{\partial u} = -2u - 3 + \lambda = 0,$$

so that the form of the optimal control is

$$u(t) = \frac{\lambda(t) - 3}{2}, \tag{2.58}$$

provided this expression stays within the interval $\Omega = [0, 2]$. Note that the second derivative of H with respect to u is $\partial^2 H/\partial u^2 = -2 < 0$, so that (2.58) satisfies the second-order condition for the maximum of a function.

We next derive the adjoint equation as

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -2 - \lambda,\ \lambda(2) = 0, \tag{2.59}$$

which can be rewritten as

$$\dot{\lambda} + \lambda = -2,\ \lambda(2) = 0.$$

This equation can be solved by the techniques explained in Appendix A. Its solution is

$$\lambda(t) = 2(e^{2-t} - 1).$$

If we substitute this into (2.58) and impose the control constraint (2.56), we see that the optimal control is

$$u^*(t) = \begin{cases} 2 & \text{if} \quad e^{2-t} - 2.5 > 2, \\ e^{2-t} - 2.5 & \text{if} \quad 0 \le e^{2-t} - 2.5 \le 2, \\ 0 & \text{if} \quad e^{2-t} - 2.5 < 0, \end{cases} \tag{2.60}$$

or referring to the notation defined in (1.22),

$$u^*(t) = \text{sat}\ [0, 2; e^{2-t} - 2.5].$$

The graph of $u^*(t)$ appears in Figure 2.5. In the figure, t_1 is the solution of $e^{2-t} - 2.5 = 2$, i.e., $t_1 \approx$0.496, while t_2 solves $e^{2-t} - 2.5 = 0$, which gives $t_2 \approx 1.08$.

In Exercise 2.4 you will be asked to compute the optimal state trajectory $x^*(t)$ corresponding to $u^*(t)$ shown in Figure 2.5 by piecing together the solutions of three separate differential equations obtained from (2.55) and (2.60).

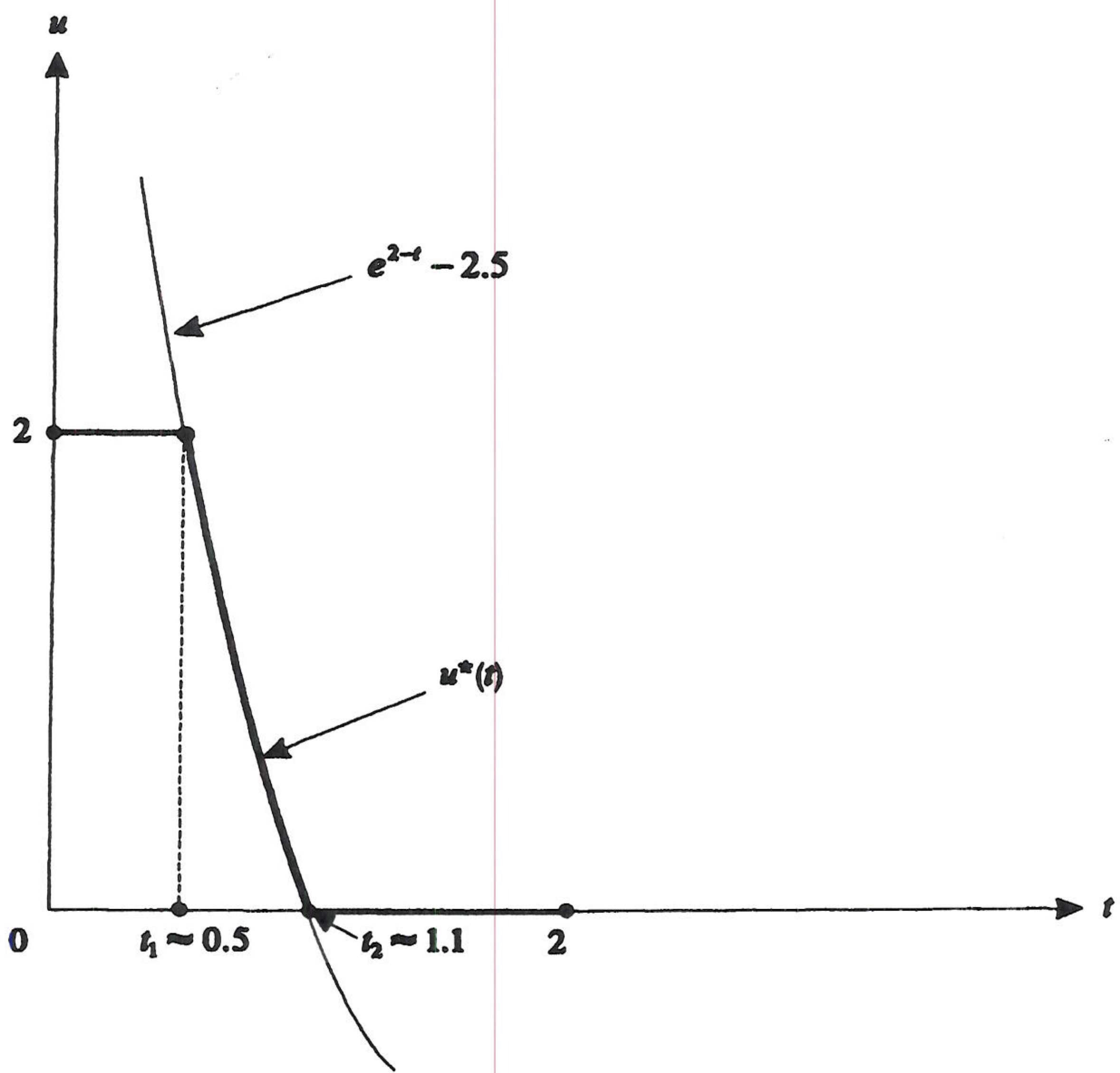

Figure 2.5: Optimal Control for Example 2.5

2.4 Sufficiency Conditions

So far, we have shown the necessity of the maximum principle conditions for optimality. We next prove a theorem that gives qualifications under which the maximum principle conditions are also sufficient for optimality. This theorem is important from our point of view since the models derived from many management science applications will satisfy conditions required for the sufficiency result. As remarked earlier, our technique for proving existence will be to display for any given model, a solution that satisfies both necessary and sufficient conditions. A good reference for sufficiency conditions is Seierstad and Sydsæter (1987).

We first define a function $H^0 : E^n \times E^m \times E^1 \to E^1$ called the *derived Hamiltonian* as follows:

$$H^0(x, \lambda, t) = \max_{u \in \Omega(t)} H(x, u, \lambda, t). \tag{2.61}$$

We assume that by this equation a function $u = u^0(x, \lambda, t)$ is implicitly and uniquely defined. Given these assumptions we have by definition,

$$H^0(x, \lambda, t) = H(x, u^0, \lambda, t). \tag{2.62}$$

It is also possible to show that

$$H^0_x(x, \lambda, t) = H_x(x, u^0, \lambda, t). \tag{2.63}$$

To see this for the case of differentiable u^0, let us differentiate (2.62) with respect to x:

$$H^0_x(x, \lambda, t) = H_x(x, u^0, \lambda, t) + H_u(x, u^0, \lambda, t)\frac{\partial u^0}{\partial x}. \tag{2.64}$$

Let us look at the second term on the right-hand side of (2.64). We must show

$$H_u(x, u^0, \lambda, t)\frac{\partial u^0}{\partial x} = 0 \tag{2.65}$$

for all x. There are two cases to consider: (i) The unconstrained global maximum of H occurs in the interior of $\Omega(t)$. Here $H_u(x, u^0, \lambda, t) = 0$. (ii) The unconstrained global maximum of H occurs outside of $\Omega(t)$. Here $\partial u^0/\partial x = 0$, because changing x does not influence the optimal value of u. Thus (2.65) and, therefore, (2.63) hold. Exercise 2.15 gives a specific instance of this case.

Remark 2.1. We have shown the result in (2.63) for cases where u^0 is a differentiable function of x. It holds more generally provided $\Omega(t)$ is appropriately qualified; see Derzko, Sethi, and Thompson (1984).

Theorem 2.1 *(Sufficiency Conditions). Let $u^*(t)$, and the corresponding $x^*(t)$ and $\lambda(t)$ satisfy the maximum principle necessary condition (2.32) for all $t \in [0,T]$. Then, u^* is an optimal control if $H^0(x, \lambda(t), t)$ is concave in x for each t and $S(x,T)$ is concave in x.*

Proof. The proof is a minor extension of the arguments in Arrow and Kurz (1970) and Mangasarian (1966). By definition

$$H[x(t), u(t), \lambda(t), t] \leq H^0[x(t), \lambda(t), t]. \tag{2.66}$$

Since H^0 is differentiable and concave, we can use the applicable definition of concavity given in Section 1.4 to obtain

$$H^0[x(t), \lambda(t), t] \leq H^0[x^*(t), \lambda(t), t] + H_x^0[x^*(t), \lambda(t), t][x(t) - x^*(t)]. \tag{2.67}$$

Using (2.66), (2.62), and (2.63) in (2.67), we obtain

$$\begin{aligned} H[x(t), u(t), \lambda(t), t] \leq\ & H[x^*(t), u^*(t), \lambda(t), t] \\ & + H_x[x^*(t), u^*(t), \lambda(t), t][x(t) - x^*(t)]. \end{aligned} \tag{2.68}$$

By definition of H in (2.18) and the adjoint equation of (2.32)

$$\begin{aligned} F[x(t), u(t), t] + \lambda(t) f[x(t), u(t), t] \leq\ & F[x^*(t), u^*(t), t] \\ & + \lambda(t) f[x^*(t), u^*(t), t] \\ & - \dot{\lambda}(t)[x(t) - x^*(t)]. \end{aligned} \tag{2.69}$$

Using the state equation in (2.32), transposing, and regrouping,

$$\begin{aligned} F[x^*(t), u^*(t), t] - F[x(t), u(t), t] \geq\ & \dot{\lambda}(t)[x(t) - x^*(t)] \\ & + \lambda(t)[\dot{x}(t) - \dot{x}^*(t)]. \end{aligned} \tag{2.70}$$

Furthermore, since $S[x,T]$ is a concave function in its first argument, we have

$$S[x(T), T] \leq S[x^*(T), T] + S_x[x^*(T), T][x(T) - x^*(T)] \tag{2.71}$$

or,

$$S[x^*(T), T] - S[x(T), T] + S_x[x^*(T), T][x(T) - x^*(T)] \geq 0. \tag{2.72}$$

Integrating both sides of (2.70) from 0 to T and adding (2.72), we have

$$\begin{aligned} & J(u^*) - J(u) + S_x[x^*(T), T][x(T) - x^*(T)] \\ & \geq \lambda(T)[x(T) - x^*(T)] - \lambda(0)[x(0) - x^*(0)], \end{aligned} \tag{2.73}$$

where $J(u)$ is the value of the objective function associated with a control u. Since $x^*(0) = x(0) = x_0$, the initial condition, and since $\lambda(T) = S_x[x^*(T), T]$ from the terminal adjoint condition in (2.32), we have

$$J(u^*) \geq J(u). \tag{2.74}$$

Thus, u^* is an optimal control. This completes the proof. □

Note that if the problem is given in the Lagrange form, i.e., $S(x,T) \equiv 0$, then all we need is the concavity of $H^0(x, \lambda, t)$ in x for each t. Since $\lambda(t)$ is not known *a priori*, it is usual to test H^0 for a stronger assumption, i.e., to check for the concavity of the function $H^0(x, \lambda(t), t)$ in x for any λ and t. Sometimes the stronger condition given in Exercise 2.19 can be used.

Example 2.6 Let us show that the problems in Examples 2.1 and 2.2 satisfy the sufficient conditions. We have from (2.36) and (2.61),

$$H^0 = -x + \lambda u^0,$$

where u^0 is given by (2.37). Since u^0 is a function of λ only, $H^0(x, \lambda, t)$ is certainly concave in x for any t and λ (and in particular for $\lambda(t)$ supplied by the maximum principle). Since $S(x,T) = 0$, the sufficient conditions hold.

Finally, it is important to mention that thus far in this chapter, we have considered problems in which the terminal values of the state variables are not constrained. Such problems are called *free-end-point problems*. The problems at the other extreme, where the terminal values of the state variables are completely specified, are termed *fixed-end-point problems*. Then, there are problems in between these two extremes. While a detailed discussion of terminal conditions on state variables appears in Section 3.4 of the next chapter, it is instructive here to briefly indicate how the maximum principle needs to be modified in the case of fixed-end-point problems. Suppose $x(T)$ is completely specified, i.e., $x(T) = \alpha \in E^n$, where α is a vector of constants. Observe then that the first term on the right-hand side of inequality (2.73) vanishes regardless of the value of $\lambda(T)$, since $x(T) - x^*(T) = \alpha - \alpha = 0$ in this case. This means that the sufficiency result would go through for any value of $\lambda(T)$. Not surprisingly, therefore, the transversality condition (2.29) in the fixed-end-point case changes to

$$\lambda(T) = K, \tag{2.75}$$

where $K \in E^n$ is a vector of constants to be determined. The maximum principle for fixed-end-point problems can be restated as (2.32) with $x(T) = \alpha$ and without $\lambda(T) = S_x[x^*(T), T]$. The resulting two-point boundary value problem has initial and final values on the state variables, whereas both initial and terminal values for the adjoint variables are unspecified, i.e., $\lambda(0)$ and $\lambda(T)$ are constants to be determined.

In Exercises 2.9 and 2.18, you are asked to solve the fixed-end-point problems given there.

2.5 Solving a TPBVP by Using Spreadsheet Software

A number of examples and exercises found in the rest of this book involve finding a numerical solution to a two-point boundary value problem (TPBVP). In this section we shall show how the GOAL SEEK function in the EXCEL spreadsheet software can be used for this purpose. We will solve the following example.

Example 2.7 Consider the problem:

$$\max\left\{J = \int_0^1 -\frac{1}{2}(x^2 + u^2)dt\right\}$$

subject to

$$\dot{x} = -x^3 + u,\ x(0) = 5. \tag{2.76}$$

Solution. We form the Hamiltonian

$$H = -\frac{1}{2}(x^2 + u^2) + \lambda(-x^3 + u),$$

where the adjoint variable λ satisfies the equation

$$\dot{\lambda} = x + 3x^2\lambda,\ \lambda(1) = 0. \tag{2.77}$$

Since u is unconstrained, we set $H_u = 0$ to obtain $u^* = \lambda$. With this, the state equation (2.76) becomes

$$\dot{x} = -x^3 + \lambda,\ x(0) = 5. \tag{2.78}$$

Thus, the TPBVP is given by the system of equations (2.77) and (2.78).

In order to solve these equations we discretize them by replacing dx/dt and $d\lambda/dt$ by

$$\frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t} \text{ and } \frac{\Delta \lambda}{\Delta t} = \frac{\lambda(t+\Delta t) - \lambda(t)}{\Delta t},$$

respectively. Substitution of $\Delta x/\Delta t$ for $\dot{x}$ in (2.78) and $\Delta\lambda/\Delta t$ for $\dot{\lambda}$ in (2.77) gives the discrete version of the TPBVP:

$$x(t+\Delta t) = x(t) + [-x(t)^3 + \lambda(t)]\,\Delta\, t,\ x(0) = 5, \tag{2.79}$$

$$\lambda(t+\Delta t) = \lambda(t) + [x(t) + 3x(t)^2\lambda(t)]\,\Delta\, t,\ \lambda(1) = 0. \tag{2.80}$$

In order to solve these equations, open an empty spreadsheet, choose the unit of time to be $\Delta t = 0.01$, make a guess for the initial value $\lambda(0)$ to be, say -0.2, and make the entries in the cells of the spreadsheet as specified below:

Enter -0.2 in cell A1.
Enter 5 in cell B1.
Enter = A1 + (B1 + 3 * (B1^2) * A1) * 0.01 in cell A2.
Enter = B1 + (-B1^3 + A1) * 0.01 in cell B2.

Note that $\lambda(0) = -0.2$ shown as the entry -0.2 in cell A1 is merely a guess. The correct value will be determined by the use of the GOAL SEEK function.

Next blacken cells A2 and B2 and drag the combination down to row 101 of the spreadsheet. Using EDIT in the menu bar, select FILL DOWN. Thus, EXCEL will solve equations (2.79) and (2.80) numerically from $t = 0$ to $t = 1$ in steps of $\Delta t = 0.01$, and that solution will appear as entries in columns A and B of the spreadsheet. In other words, the guessed solution for $\lambda(t)$ will appear in cells A1 to A101 and the guessed solution for $x(t)$ will appear in cells B1 to B101. In order to find the correct value for $\lambda(0)$, use the GOAL SEEK function under TOOLS in the menu bar and make the following entries:

Set cell: A101.
To value: 0.
By changing cell: A1.

It finds the correct initial value for the adjoint variable as $\lambda(0) = -0.10437$, which should appear in cell A1, and the correct ending value of the state variable as $x(1) = 0.62395$, which should appear in cell B101. You will notice that the entry in cell A101 may not be exactly zero as instructed, although it will be very close to it. In our example, it is -0.0007. By using the CHART function, the graphs of $x(t)$ and $\lambda(t)$ can be printed out by EXCEL as shown in Figure 2.6.

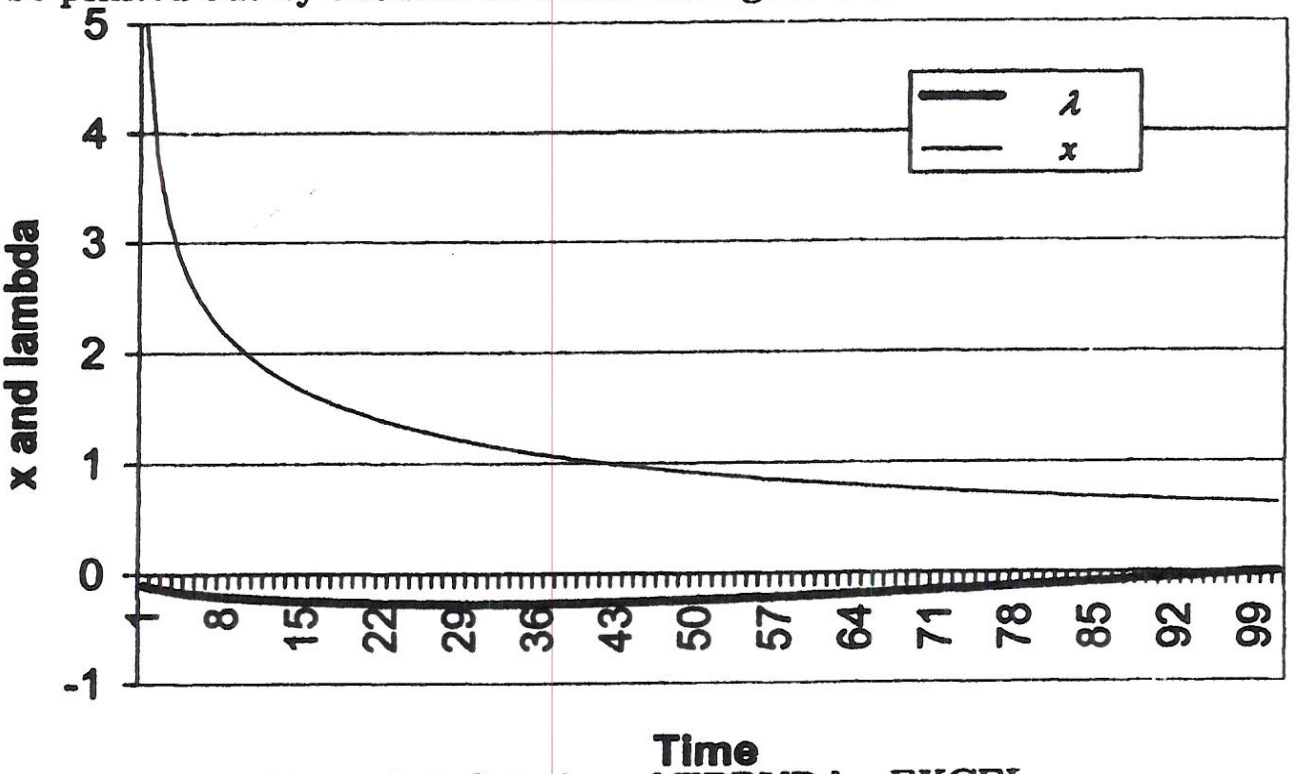

Figure 2.6: Solution of TPBVP by EXCEL

EXERCISES FOR CHAPTER 2

2.1 (a) In Example 2.1, show $J^* = -\frac{1}{2}$.

(b) In Example 2.2, show $J^* = 0$.

(c) In Example 2.3, show $J^* = -\frac{1}{6}$.

(d) In Example 2.4, show $J^* = -\frac{1}{6}$.

2.2 Rework Example 2.5 with $F = 2x - 3u$.

2.3 Show that both the Lagrange and Mayer forms of the optimal control problem can be reduced to the linear Mayer form (2.5).

2.4 Complete Example 2.5 by writing the optimal $x^*(t)$ in the form of integrals over the three intervals $(0, t_1)$, (t_1, t_2), and $(t_2, 2)$ shown in Figure 2.5.
[Hint: It is not necessary to actually carry out the numerical evaluation of these integrals unless you are ambitious.]