Multi objective optimization
and
Game Theory

Example
2 stakeholders

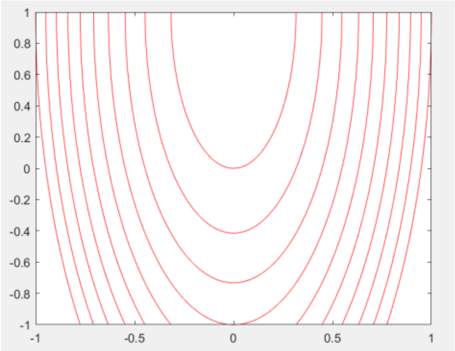
- $X=(xa,xb)$: overall control
- $Ca(X)=(X-a)^T\, Qa\, (X-a)$: cost for stakeholder Sa
- $Cb(X)=(X-b)^T\, Qb\, (X-b)$: cost for stakeholder Sb

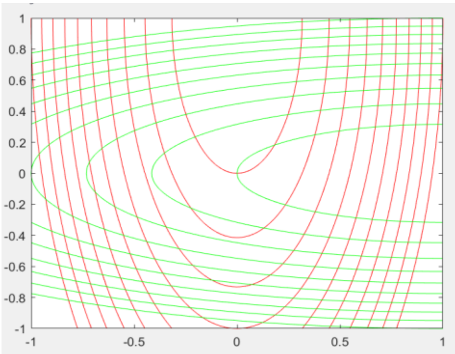
- $C(X)=Ca(X)+Cb(X)$: overall cost

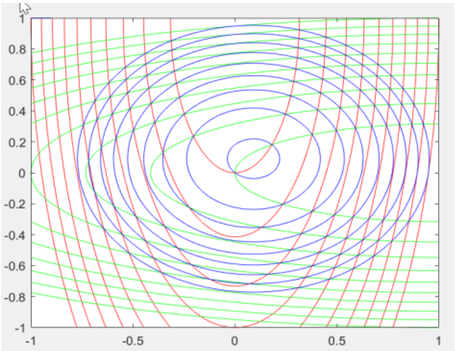
- If Sa decides, a should be chosen (optimal for Sa)
- If Sb decides, b should be chosen (optimal for Sb)

Example
2 stakeholders

- $a=[0\ 1]^T$
- $b=[1\ 0]^T$
- $Q_a=[1\ 0;0\ 0.1]$
- $Q_b=[0.1\ 0;0\ 1]$



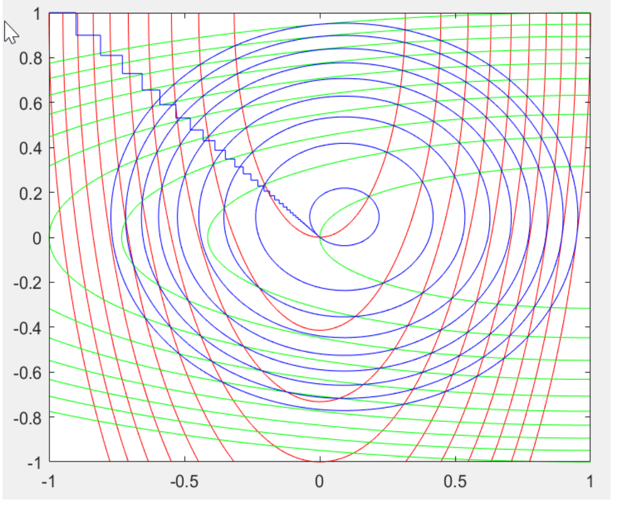




Example
2 stakeholders

- Alternating actions between stakeholders
- Sa: xa = xa - c*dQa/dxa
- Sb: xb = xb - c*dQb/dxb

- dQa/dxa = 0 => xa = a1 = 0
- dQb/dxb = 0 => xb = b2 = 0
- Solution $X=[0\ 0]^T$



Example
2 stakeholders

- Stakeholders are not cooperating, so they reach a “bad” solution.

- Why ??

- Optimal solution for aggregated cost
- $C= (X-a)^T Qa (X-a) + (X-b)^T Qb (X-b)$
- $dC/dx = (X-a)^T Qa + (X-b)^T Qb =$

  $X^T (Qa +Qb) – (a^T Qa + b^T Qb) = 0 =>$

  $Xag = (Qa +Qb)\backslash (Qa\ a + Qb\ b)$

Ca(Xag)=0.091

Cb(Xag)=0.091

$Ca([0\ 0]^T)=0.1$!!

$Cb([0\ 0]^T)=0.1$!!

Multi Objective Optimization Games

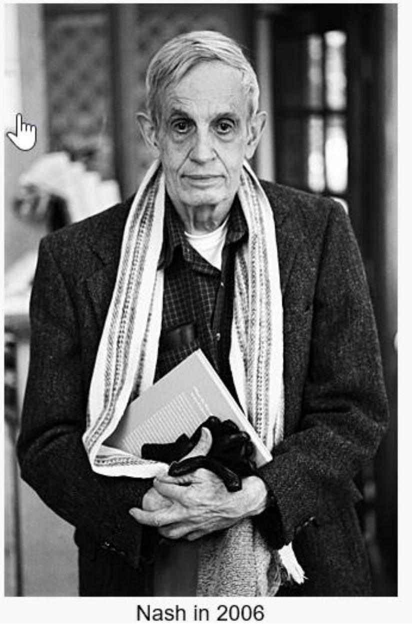
- Where a large collection of subsystems meet a number of different objectives are defined.
- Some are contradictive, i.e.

Optimization variable: x
Objectives: $f_1(x)$, $f_2(x)$,.., $f_N(x)$
$\{\min_x f_i(x)\}$

or

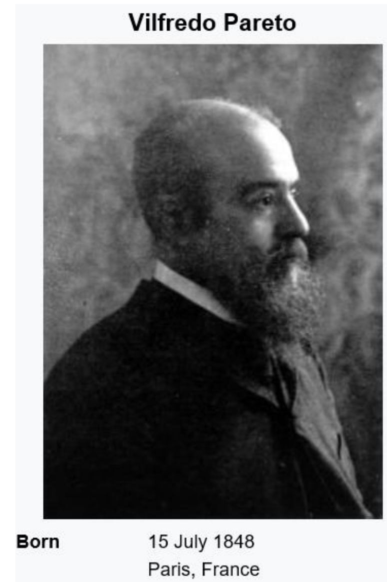
$x=(x_1,..,x_M)$
$\{\min_{xi} f_i(x)\}$ (Nash equilibrium, game theory)

John Forbes Nash Jr.



Nash in 2006

Multi Objective Optimization
Pareto



Optimization variable: x

Objectives: $f_1(x), f_2(x),.., f_N(x)$

$x=(x_1,..,x_M)$

Assume x' such that if

$f_i(x) < f_i(x')$ then for some j: $f_j(x) > f_j(x')$

x' is called a *Pareto* equilibrium

Pareto equilibria and convex combinations

- Consider positive convex combinations
- $f_a(x) = a_1 f_1(x) + a_2 f_2(x) + .. + a_N f_N(x)$, $a_i > 0$

- i) Let x’ be a Pareto equilibrium.
  Then $x' = \arg\min_x f_a(x)$ for some a

- ii) If $x' = \arg\min_x f_a(x)$ for some a
  Then x’ is a Pareto equilibrium.

Proof for N=2

ii) $x' = \arg\min_x f_a(x)$ => for all x

$f_a(x) >= f_a(x')$

Assume $f_1(x) < f_1(x')$

=>

$f_a(x) – a_2f_2(x) < f_a(x') – a_2f_2(x')$

=>

$0 <= f_a(x)-f_a(x') < a_2f_2(x) - a_2f_2(x')$

=>

$f_2(x) > f_2(x')$!!

Proof for N=2

i) Assume x' is not a minimum for any

$f_a(x)/a_1 = f_1(x) + a_2/a_1\, f_2(x) = f_1(x) + c\, f_2(x) = f_c(x)$

So an x exists such that

$f_c(x) < f_c(x')$ for all $0 < c < \inf$

=>

$f_1(x) + c\, f_2(x) < f_1(x') + c\, f_2(x')$ for all $0 < c < \inf$

=>

$f_1(x) - f_1(x') < c(f_2(x') - f_2(x))$ for all $0 < c < \inf$

=>

$f_1(x) - f_1(x') < 0 \Rightarrow f_2(x') - f_2(x) > 0 \Leftrightarrow f_1(x) < f_1(x') \Rightarrow f_2(x) < f_2(x')$

If x' is a Pareto equilibrium then for all x

$f_1(x) < f_1(x') \Rightarrow f_2(x) > f_2(x')$ Contradiction !!!

Pareto frontier

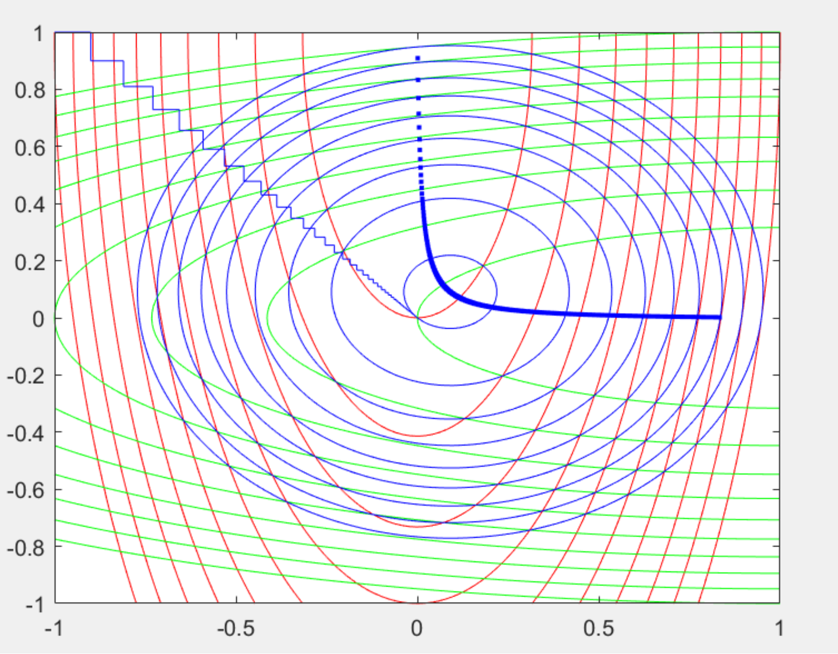
- Pareto equilibria are the minima of

$$f_a(x) = a_1 f_1(x) + a_2 f_2(x) + .. + a_N f_N(x) \quad \text{or}$$
$$f_a(x)/a_1 = f_1(x) + a_2/a_1\, f_2(x) + .. + a_N/a_1\, f_N(x) =$$
$$f_c(x) = f_1(x) + c_1\, f_2(x) + .. + c_{N-1}\, f_N(x)$$

- Thus they are parametrized by an N-1 dim vector c and are (under nice conditions) N-1 dimensional sets
- I.e. they are curves in plane and surfaces in R^3

Example

- $Ca(X)=(X-a)^T Qa (X-a)$
- $Cb(X)=(X-b)^T Qb (X-b)$
- $Cc(X) = Ca(X) + c\, Cb(X)$



Minimum

$Xc = (Qa + c\, Qb) \backslash (Qa\, a + c\, Qb\, b)$

Non Cooperative Game Setup

DEFINITION 3.1 *A non-cooperative game in* strategic (or normal) form *is a triplet* $G = (\mathcal{N}, (\mathcal{S}_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}})$, *where:*

- $\mathcal{N}$ *is a finite set of players, i.e.,* $\mathcal{N} = \{1, \dots, N\}$.
- $\mathcal{S}_i$ *is the set of available strategies for player* i.
- $u_i : \mathcal{S} \to \mathbb{R}$ *is the utility (payoff) function for player* i, *with* $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_i \times \cdots \times \mathcal{S}_N$ *(Cartesian product of the strategy sets).*

Non Cooperative Game Setup

Optimization variable: s

Objectives: $f_1(s)$, $f_2(s)$,.., $f_N(s)$

$s=(s_1,..,s_N)$

$\{\min_{s_i} f_i(s)\}$ (or max)

Example
Dining Philosophers

- 2 philosophers are dining
- They share a set of knife and fork
- Both tools are mandatory to eating
- When both tools are aquired eating is completed in 1 min.
- In order to get to eat they both individially try to maximize the number of tools at hand
- We get the following representation (strategic form)

P2 \ P1	0	1	2
0	(0,0)	(1,0)	(2,0)
1	(0,1)	(1,1)	
2	(0,2)		

Example
Weapons policy

- 2 people (in a population of 2) should choose to carry a firearm or not
- They each try to minimize some percieved probability of getting shot
- The following game is defined

P2\P1	No gun	Gun
No gun	(0,0)	(0,1)
Gun	(1,0)	(1/2,1/2)

Example
rock, paper, scissors
(zero sum)

P2\P1	R	P	S
R	(0,0)	(1,-1)	(-1,1)
P	(-1,1)	(0,0)	(1,-1)
S	(1,-1)	(-1,1)	(0,0)

Evolution of a game

- How will the game evolve ?
- Depends on drawing discipline:
  - Simultaneous
  - Sequential
  - Repeated (simultaneous)
  - Repeated (sequential)

One player reasoning

- Simultaneous: you do not know anything about the other players move (min-max):

  $s_i^* = \arg\max_{s_i} \{ \min\{f_i(s_i,s_{-i})\} \}$
- Sequential (you are first)

  Select the NE that gives you the best outcome and await other player
- Repeated (simultaneous): ??
- Repeated (sequential): ends up in some NE

Dominating/dominated strategies Reducing Games

- If

  $f_i\ (s_i, s_{-i}\ ) > f_i\ (s_i',s_{-i})$ forall s_{-i}

  we say that strategy s_i *strictly dominates* s_i'
- A strictly dominated strategy should/would never be chosen
- I.e. it may be removed from the game
- Let $D_i(A)$ be the player *i* strategies which are not strictly dominated within a set of strategies A
- Let $D(A)=D_1(A) \times D_2(A) \times .. \times D_N(A)$
- $D(S) \subseteq S$ may indeed include dominated strategies, so we try again
- $D(D(S)) \subseteq S$ is the next step and

Dominating/dominated strategies

- Let $D(A)=D_1(A) \times D_2(A) .. \times D_N(A)$
- $D(S) \subseteq S$ may indeed include dominated strategies, so we try again
- $D^2(S) = D(D(S)) \subseteq D(S)$ is the next step
- Now $\{D^n(S)\}_n$ is decreasing, so it has to converge to $D^* = \cap_n D^n(S)$
- $D(D^*) = D^*$, so D^* has no dominated strategies.
- D^* is reached in finite steps for finite games and it includes at least one feasible point – maybe more.

Nash Equilibrium (NE)

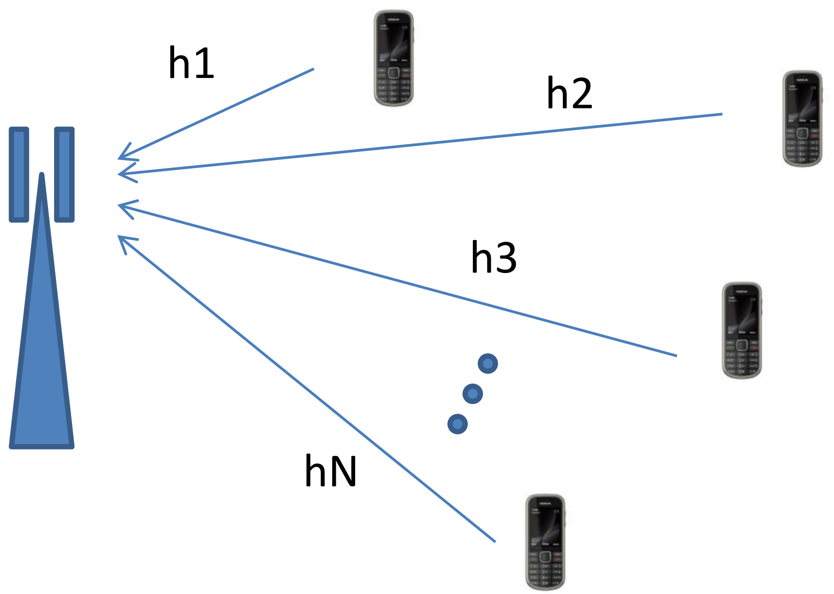
- Consider a strategy profile:

  $s^*=(s^*_1,..,s^*_N)$
- Then s^* is said to be a Nash Equilibrium iff

  $f_i(s^*_i, s^*_{-i}) \geq f_i(s_i , s^*_{-i}) \;\; \forall s_i \in S_i$
- In an NE no player can unilaterally gain by moving.
- $\{NE\} \subseteq D^*(S)$
- An NE is a likely outcome of a game, since no player has an incentive to move.
- An NE needs not be desirable (previous examples)

Example
Uplink power control for CDMA

- In CDMA systems all users transmit simultaneously on the same frequency
- Transmissions are distinguished in code space
- Transmissions interfere with each other



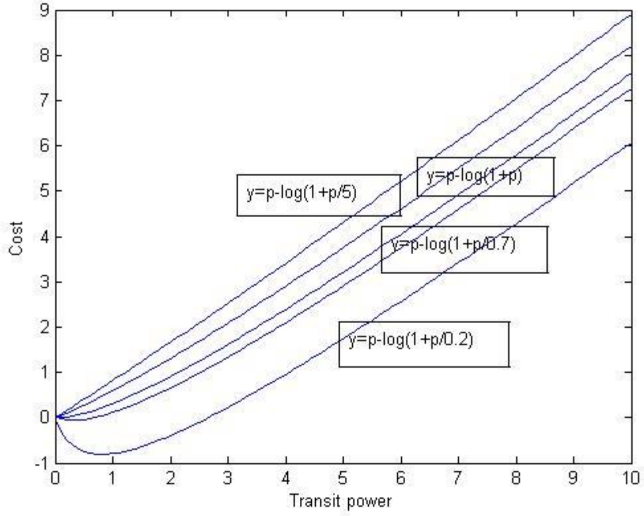
Example
Uplink power control for CDMA

- Consider a single user cost

$$f_i\,(p_i\,, p_{-i}) = \lambda p_i - \alpha \log\,(1+\gamma_i\,),\quad p_i \geq 0, \forall i$$

- *Where*

$$\gamma_i = L\, h_i\, p_i \;/\, (\textstyle\sum_{j \neq i} h_j\, p_j + \sigma^2)$$



Example
Uplink power control for CDMA

- Minimum cost is found at

  $p_i = \max\{0, \alpha/\lambda - Y_i/L/h_i\}$

  $Y_i = \sum_{j \neq i} h_j \, p_j + \sigma^2$

- Any NE must be

  $p_1>0,..,p_M>0,\ p_{M+1}=0,..,P_N=0$

  (in some ordering)

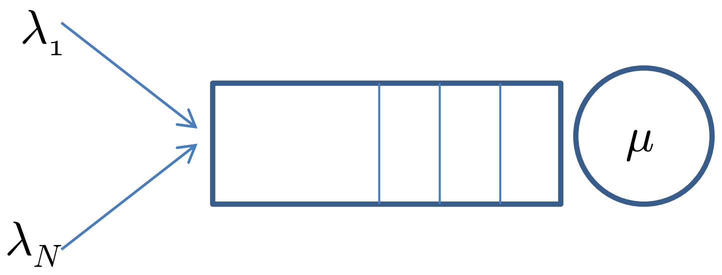
Example
Uplink power control for CDMA

- $p_1>0,..,p_M>0$, $p_{M+1}=0,..,P_N=0$
  (in some ordering)
- Assume for $k \leq M < i$ that $h_k < h_i$
- Then

  $0 < p_k = \alpha/\lambda - Y_k/L/h_k$
  $= \alpha/\lambda - (\sum_{j\neq k}^{M} h_j\, p_j + \sigma^2)/L/h_k$
  $\leq \alpha/\lambda - (\sum^{M} h_j\, p_j + \sigma^2)/L/h_i$!!!
  $= \alpha/\lambda - Y_i/L/h_i = p_i$
- Contradiction

Example
Agregating queues

- Flows $\{\lambda_i\}$
- Aggregated flow $\lambda = \sum_i \lambda_i$
- Service rate μ
- Average delay $T = 1/(\mu - \lambda)$
- Payoff: $f_i = \alpha \lambda_i - T$



Example
Agregating queues

- Payoff: $f_i = \alpha\,\lambda_i - T$

  $= \alpha\,\lambda_i - 1/(\mu - \lambda)$

  $= \alpha\,\lambda_i - 1/(\mu - \sum_j \lambda_j)$
- NE at $\lambda_i = (\mu - (1/\alpha)^{1/2})/N$
- $T = 1/(\mu - (\mu - (1/\alpha)^{1/2})) = 1/((1/\alpha)^{1/2}) = \alpha^{1/2}$
- $\rho = N\lambda_i / \mu = (\mu - (1/\alpha)^{1/2})/\mu$

  $= 1 - (1/\alpha)^{1/2}/\mu$

Power Grid Example
Demand elasticity (none)

- 2 players wish to have 1 unit of electrical energy a time (discrete) $n = 0$
- The benefit of the energy is lowered by late or early delivery

  $$B(n)=K - |n| I_{|n|<K} - K\, I_{|n|=>K}$$

- Cost is constant

  $$C(n) = C$$

- Objective function is $E_i(n_i) = B_i(n_i) - C_i(n_i)$
- Nash equilibrium is $n_1 = n_2 = 0$

Power Grid Example
Demand elasticity

- 2 players wish to have 1 unit of electrical energy a time (discrete) $n = 0$
- The benefit of the energy is lowered by late or early delivery

$$B(n)=5 - 0.02|n| I_{|n|<5}$$

- Cost:

$$C(n) = 1 + 0.1 \sum_n I_{n_1=n \text{ and } n_2=n}$$

Power Grid Example
Demand elasticity

E1 $\rightarrow$

3.8600	3.9800	4.0000	3.9800	3.9600
3.9600	3.8800	4.0000	3.9800	3.9600
3.9600	3.9800	3.9000	3.9800	3.9600
3.9600	3.9800	4.0000	3.8800	3.9600
3.9600	3.9800	4.0000	3.9800	3.8600

E2
$\Downarrow$

3.8600	3.9600	3.9600	3.9600	3.9600
3.9800	3.8800	3.9800	3.9800	3.9800
4.0000	4.0000	3.9000	4.0000	4.0000
3.9800	3.9800	3.9800	3.8800	3.9800
3.9600	3.9600	3.9600	3.9600	3.8600

Mixed Strategies

- We have the original game setup
- A mixed strategy σ_i is a probability distribution over the original *pure* strategies.
- The payoff/cost is computed as an expected value, i.e. :

$$u_i(\sigma)=\sum_{s1} .. \sum_{sN} f_i(s_1,..,s_N)\ \Pi_{j=1}^{N} \sigma_j (s_j)$$

- When player i adopts a mixed strategy σ_i he chooses pure strategy s_i with probability $\sigma_i (s_i)$ **independent** of other players

Pure best responses

- The set of pure best responses $r_i(\sigma)$ for some mixed strategy σ is the set of optimal **pure** player i strategies, i.e.

$$r_i(\sigma) = \{\arg\max_{s_i} u_i(s_i, \sigma_{-i})\}$$

$$r(\sigma) = r_1(\sigma) \times .. \times r_N(\sigma)$$

Best responses

- The set of best responses $mr_i(\sigma)$ for some mixed strategy σ is the set of optimal **mixed** player i strategies, i.e.

$$mr_i\ (\sigma) = \{\arg\max_{\sigma_i} u_i\ (\sigma_i\ ,\sigma_{-i})\}$$

$$mr(\sigma) = mr_1\ (\sigma) \times .. \times mr_N\ (\sigma)$$

Mixed Nash Equilibrium

- A mixed NE σ^* is a mixed strategy st.

  $u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i}) \ \forall \sigma_i$

- Let σ^* be a mixed NE, then

  $\text{supp}\ \sigma^*_i \subseteq r_i(\sigma^*)$

- Let MNE be the set of mixed NE's, then

  $MNE = \{\sigma^* \mid \sigma^* \in mr(\sigma^*)\}$

  (the fixed points of mr)

Existence of Mixed NE’s

- Pure NE’s do not allways exist in a game (*rock, paper, scissors)*
- How about mixed NE’s
- The mixed NE’s are the fixed points (FP’s) of mr
- mr is (hemi)-continuous
- mr maps a compact convex set to the same
- Kakutani: mr has at least one FP
- Every **finite** game has at least one mixed NE.

Mixed NE of *rock, paper, scissors*

- σ_i = (1/3,1/3,1/3) is a mixed NE of *rock, paper, scissors*

  $u_1(\sigma)=\sum_{s1} \sum_{s2} f_1(s_1,s_2)\ \Pi_{j=1}^{2}\ \sigma_j\ (s_j)$
  $= 1/3 \sum_{s1} \sum_{s2} f_1(s_1,s_2)\ \sigma_1\ (s_1)$
  $= 1/3\ (\sigma_1\ (r)\ (f_1(r,r) + f_1(r,p) + f_1(r,s)) +$
  $\sigma_1\ (p)\ (f_1(p,r) + f_1(p,p) + f_1(p,s)) +$
  $\sigma_1\ (s)\ (f_1(s,r) + f_1(s,p) + f_1(s,s))) = 0$ (id)

  $u_2(\sigma) = 0$ (id)

  $mr_1(\sigma) = mr_2(\sigma) = \Sigma$ (all mixed strategies)
  $\sigma \in mr_1(\sigma) \times mr_2(\sigma) = \Sigma \times \Sigma$

Quality (effifiency of NE’s)

- Consider a strategy profile:

  $s^*=(s^*_1,..,s^*_N)$
- Then s^* is said to be *Pareto optimal* iff no other strategy profile s exists st.

  $f_i(s^*) > f_i(s) \quad \forall i$
- You can not improve on one payoff without worsening some other
- NE’s are not allways Pareto optimal (weapons)
- It is desirable that an NE is Pareto optimal

Prize of anarchy/stability

- Minimum aggregated payoff for NE's

  $u_{NE} = \min_{s \in \mathcal{NE}} \sum_i f_i(s)$
- Maximum achievable payoff

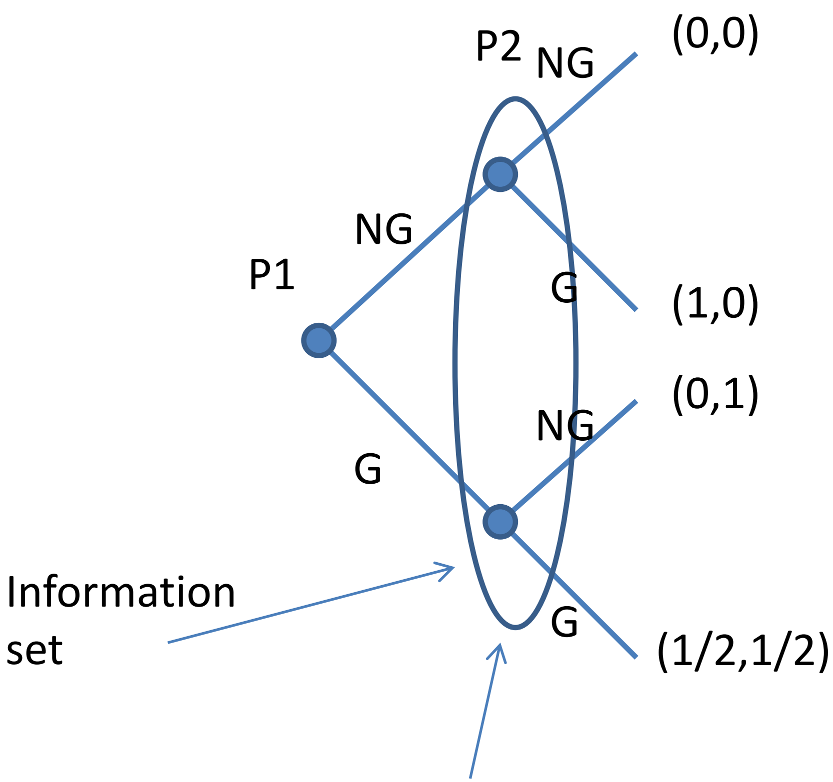
  $u_{MAX} = \max_s \sum_i f_i(s)$
- Prize of anarchy

  $\eta = u_{MAX} / u_{NE}$
- Maximum aggregated payoff for NE's

  $v_{NE} = \max_{s \in \mathcal{NE}} \sum_i f_i(s)$
- Prize of stability

  $\zeta = u_{MAX} / v_{NE}$

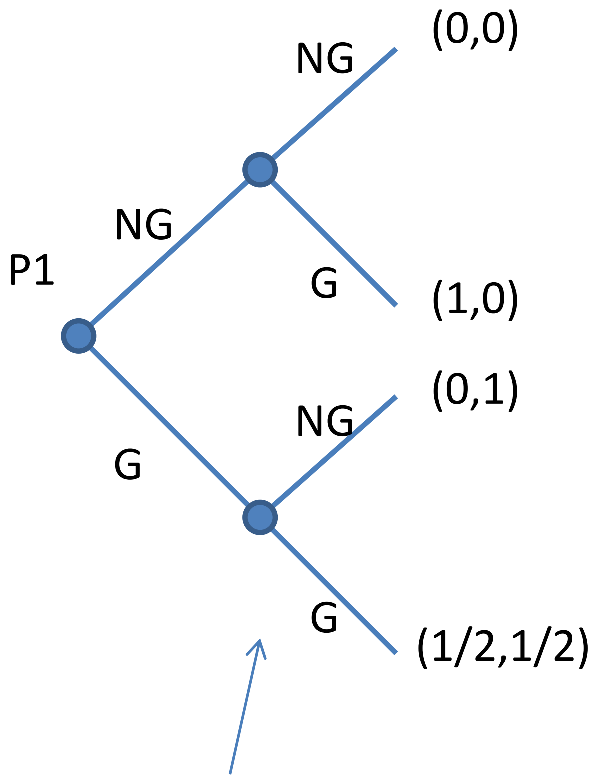
Extended forms



P2\P1	No gun	Gun
No gun	(0,0)	(0,1)
Gun	(1,0)	(1/2,1/2)

P2 does **not** know the choice of P1

Extended forms



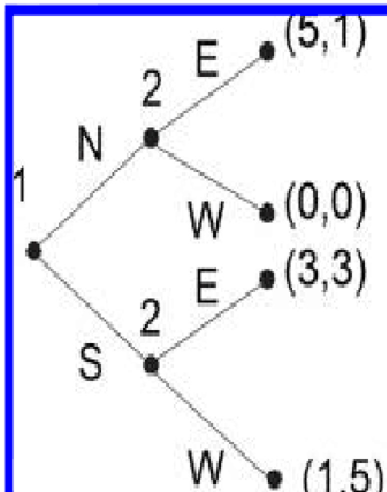
P2 **does** know the choice of P1

- Pure P2 strategies are now functions of s_1
- 4 pure P2 strategies exist

1: s_2(G)=G, s_2(NG)=G
2: s_2(G)=NG, s_2(NG)=G
3: s_2(G)=G, s_2(NG)=NG
4: s_2(G)=NG, s_2(NG)=NG

P2/P1	G	NG
1:	(1/2,1/2)	(1,0)
2:	(0,1)	(1,0)
3:	(1/2,1/2)	(0,0)
4:	(0,1)	(0,0)

Cheap talk



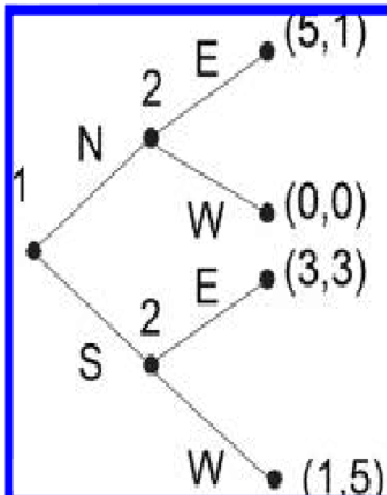
	EE	EW	WE	WW
N	5,1	5,1	0,0	0,0
S	3,3	1,5	3,3	1,5

FIGURE 4.2: Extensive form game example, with actions taken sequentially

From: Game Theory for Wireless Engineers, Allen B. MacKenzie and Luiz A. DaSilva

- (N,EE), (N,EW) and (S,WW) are all NE’s
- However a rational player P2 would never move west after P1 moves north, so the WW strategy would never be chosen
- P2s theath moving west after P1 moving north is called cheap talk

Subgame perfect NE

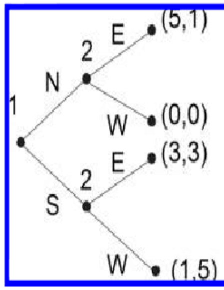


	EE	EW	WE	WW
N	5,1	5,1	0,0	0,0
S	3,3	1,5	3,3	1,5

FIGURE 4.2: Extensive form game example, with actions taken sequentially

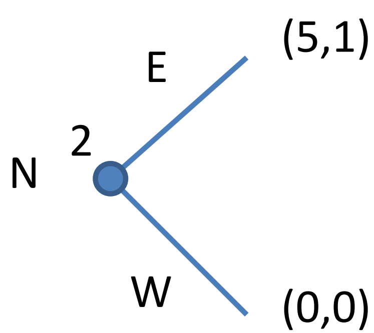
- A strategy profile is a ***subgame perfect eqilibrium*** if it is an NE for every subgame
- In the above example we have 3 different subgames
- Which NE's are subgame perfect ?

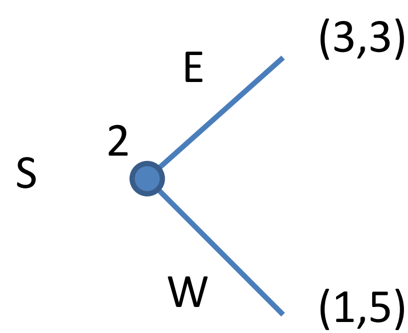
Subgames



	EE	EW	WE	WW
N	5,1	5,1	0,0	0,0
S	3,3	1,5	3,3	1,5

FIGURE 4.2: Extensive form game example, with actions taken sequentially

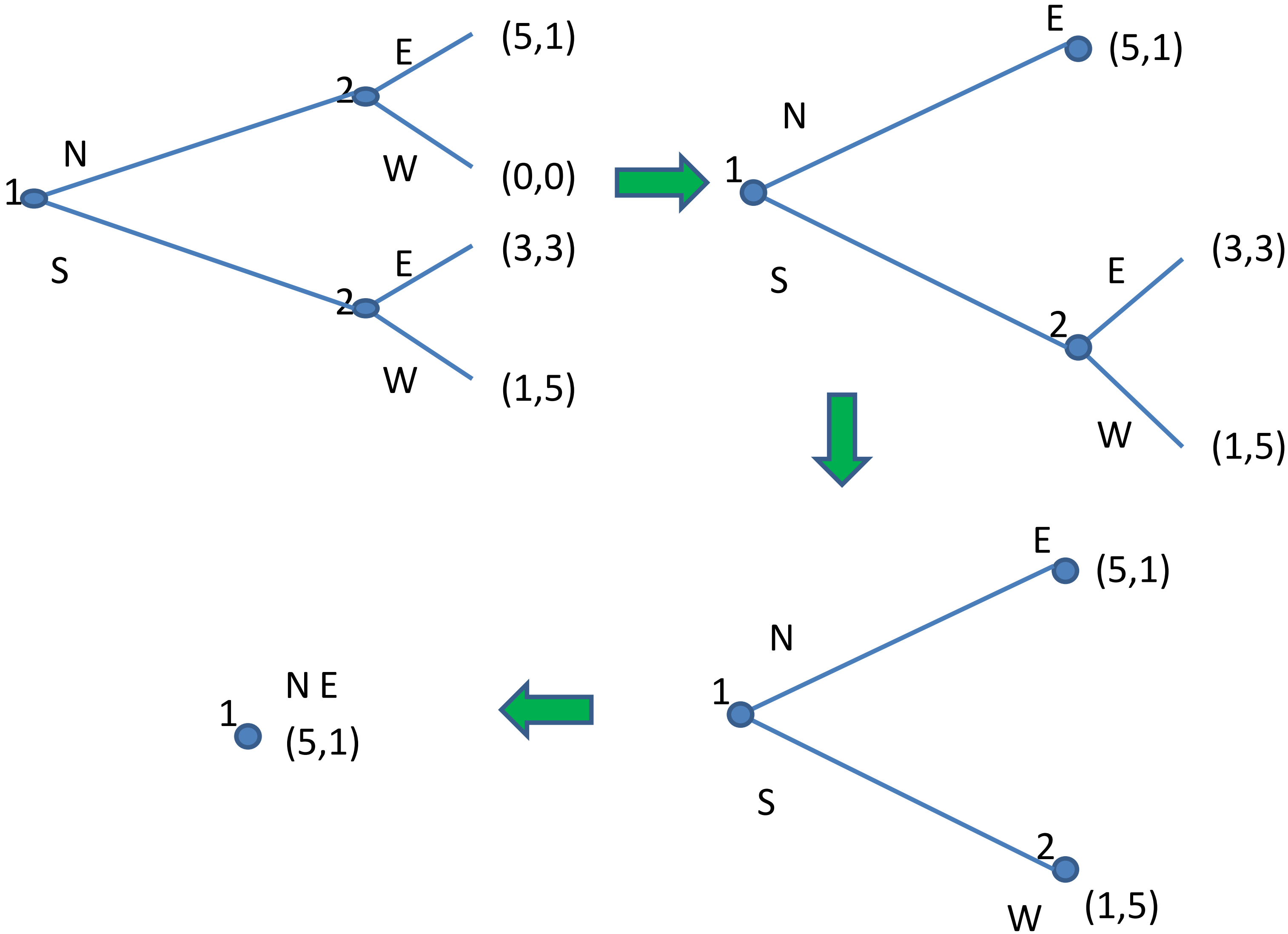




- EE and EW are NE's
- WW is not

- EW and WW are NE's
- EE is not

Backward Induction



Existence of Subgame Perfect NE’s

- Every finite repeated game with perfect information has one
- It may be found by backward induction (BI)
- It is likely to happen even though the game order is opposite to BI
- Earlier players assume rationality of later players
- Therefore they maximize their utility based on the expectation that later players will do the same
- In case of imperfect information an NE has to be found for every subgame under that information set

Applications

- Formation in mobile robotics
- Independent actors in smart grids
- Multi hop ad hoc networks
- Decentralized control of turbines in wind farms
- Autonomous highway platooning

Negotiation exercise
Manufacturer (seller)

- You manufacture items to be sold to retaillers
- You set the prize: P
- The retailler sets the quantity: Q
- Your earnings are $W_M = \min\{20, PQ\}$

(Since you are a start-up, you recieve a substantial tax reduction, which is retained if you earn more than 20)

- For every quantity requested by the retailler, you should optimize your earnings.
- To maintain a good customer relationship you do not want to charge more than nesessary.

Negotiation exercise Retailler (buyer)

- You buy items from the retailler to be sold
- You set the quantity: Q
- The manufacturer sets the prize: P
- You can sell $Q_s = \min\{Q,(1-P)100\}$

(Market shrinks with increased prize)

- Your profit (revenue - cost) is:

$$W_R = Q_s\ P\ 0.1$$

- For every prize offered by the manufacturer, you should optimize your profit.
- You do not want to buy more items than you can sell.

Negotiation Analysis

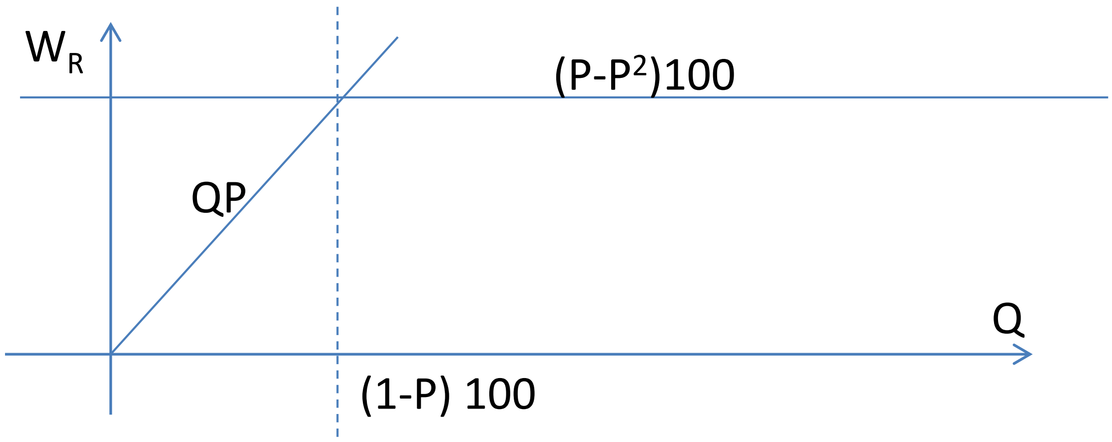
- Manufacturer prize profile:

$$QP = 20 \quad \Rightarrow \quad P = 20/Q$$

- Retailler quantity response:

$$W_R = \min\{Q,(1-P)100\}\, P\, 0.1 =$$

$$\min\{QP,(P-P^2)100\}\, 0.1$$



Negotiation Analysis

- Manufacturer prize profile:

  $QP = 20 \quad \Rightarrow \quad P = 20/Q$

- Retailler quantity response:

  $Q = (1-P)\,100$

- Iteration

  $P_{n+1} = 20/Q_n$

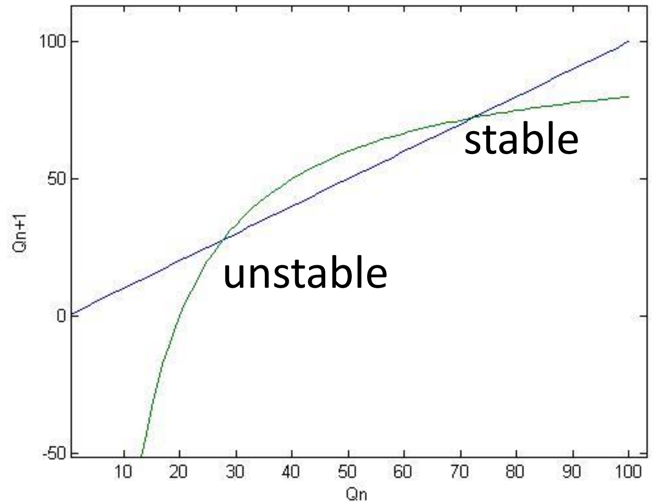
  $Q_{n+1} = (1-P_{n+1})\,100 = (1- 20/Q_n)\,100$

- Equilibria:

  $Q = (1- 20/Q)\,100 \Rightarrow Q^2 - 100\,Q + 2000 = 0$

  $Q=22.5,\ Q=72.5$

Negotiation Analysis



- Returnmap:

  $Q_{n+1} = (1 - 20/Q_n)\ 100$
- Equilibria:

  Q=22.5, Q=72.5

  P=20/22.5, P=20/72.5
- For both equilibria (P,Q) no part can single handedly improve their earnings/profit.
- $W = W_M + W_R = 20 + \min\{QP, (P-P^2)100\}\ 0.1$

  $= 20 + \min\{20, (P-P^2)100\}\ 0.1 = 42$ (P=20/72.5)

  $= 20 + \min\{20, (P-P^2)100\}\ 0.1 = 41$ (P=20/22.5)

Maximizing Collected earnings

- $W = W_M + W_R = 20 + \min\{QP, (P-P^2)100\}\ 0.1$
- $(P-P^2)$ is maximized to 0.25 for $P=0.5$
- Disregarding $QP=20$ allows a collected earning of 45