Project 1
Radiative Transfer in a Planetary Atmosphere
Fall 2019

1 Introduction

In this project we will use a Monte Carlo technique to calculate radiative transfer in a planetary atmosphere. The radiative transfer is simulated using a random walk of photons through the atmosphere. Photons are emitted by the planet's surface and random walk their way through the atmosphere until they eventually escape to space or are reabsorbed by the ground.

2 Background

The temperature of a planet is determined by the balance of energy of sunlight absorbed by the planet, which heats the planet, and the energy of thermal Blackbody radiation from the planet to space, which cools the planet.

2.1 Absorbed Sunlight

The solar constant, $S_\odot$, is the amount of solar energy incident per unit area per second at the Earth's distance from the Sun. At Earth's distance of one Astronomical Unit (AU) the incident solar energy is 1367.7 W m^{-2}, so that is the value of $S_\odot$ in the equations below. Note that for *other* distances from the Sun, the incident solar energy would scale inversely with the distance squared.

For a planet of radius R, we can work out the total solar power incident on the planet. It is just the Solar Constant times the cross sectional area of the planet:

$$E_{incident} \;=\; S_\odot \pi R^2$$

Now all of this energy is not absorbed by the planet. A fraction A is reflected, meaning that $(1 - A)$ is the fraction actually absorbed and available to heat up the planet. We call A the *Bond Albedo*. Thus, the amount of absorbed solar energy is:

$$E_{abs} \;=\; (1 - A) S_\odot \pi R^2$$

2.2 Emitted Blackbody Radiation

On the other side of the energy balance, we have the cooling of the planet. It is a pretty good approximation (for our purposes) to treat the planets as blackbodies which allows for a simple and reasonably accurate relationship between the temperature of the planet and the amount of energy radiated into space.

It might be useful to review a couple rules about blackbody radiation at this point.

- A hotter blackbody emits more energy at all wavelengths than a cooler blackbody.
- A hotter blackbody emits a greater fraction of its total energy at a shorter wavelength than a colder blackbody.
- The total amount of energy emitted from the surface of a blackbody is given by $E = \sigma T^4$, where σ is a constant (the Stefan-Boltzmann constant = 5.670 x 10^{-8} W m^{-2} K^{-4}) and T is the temperature of the blackbody.

We can now use the Stefan-Boltzmann law to work out the total blackbody radiation from a planet of radius R at temperature T:

$$E_{rad} = 4\pi R^2 \sigma T^4$$

2.3 Effective Temperature

If the amount of solar energy absorbed by the planet is balanced by the amount of blackbody radiation emitted by the planet, then we can write the following equality:

$$(1 - A) S_{\odot} \pi R^2 \;=\; 4\pi R^2 \sigma T^4$$

It is then straightforward to solve the above for the equilibrium temperature, which we call the *Effective Temperature*:

$$T_{eff} \;=\; \left(\frac{(1 - A) S_{\odot}}{4\sigma} \right)^{\frac{1}{4}}$$

2.4 Effective Temperature and Surface Temperature

Given values for the distance and Bond Albedo of the planets, we may work out the Effective Temperatures, as in Table 1 below. One of the interesting features of the table is that the Effective Temperature is often quite different from the temperature observed at the ground, T_G. The reason for this is the *Greenhouse Effect*, which results from the need to transfer the cooling radiation from the surface through the planet's atmosphere. Note that the difference between surface temperature and effective temperature is correlated with the atmospheric pressure at the ground, P_G, in our table.

Planet	a (AU)	S (W m^{-2})	A	T_{eff} (K)	P_G (atm)	T_G (K)
Mercury	0.39	9127	0.06	441	0	100-700
Venus	0.72	2615	0.71	240	80	700
Earth	1.00	1367	0.33	252	1	288
Moon	1.00	1367	0.07	274	0	100-400
Mars	1.52	589	0.17	215	0.01	220

Table 1: Planetary Data

3 Radiative Transfer as a Random Walk

Energy that is emitted as Blackbody radiation from the surface of a planet is emitted in the infrared part of the electromagnetic spectrum. If the planet has no atmosphere, then this energy can escape directly to space to cool the planet's surface. However, for a planet with an atmosphere, the cooling photons from the surface may be absorbed by the atmosphere before they escape to space. In this case, following the absorption of the photon, its energy will be remitted as a new photon, either towards space or towards the ground. Eventually, some fraction of the photons emitted from the ground escape to space while the rest return to the ground and supply a source of additional heating that raises the ground temperature.

In the case of a thick atmosphere, like Venus, photons might be absorbed and reemited many ten's of times as they work their way through the atmosphere. On the Earth, the atmosphere absorbs and reemits a photon about one time before it finally escapes to cool the Earth. We are going build a model of this process that treats the absorption and emission of photons in the atmosphere as a random walk.

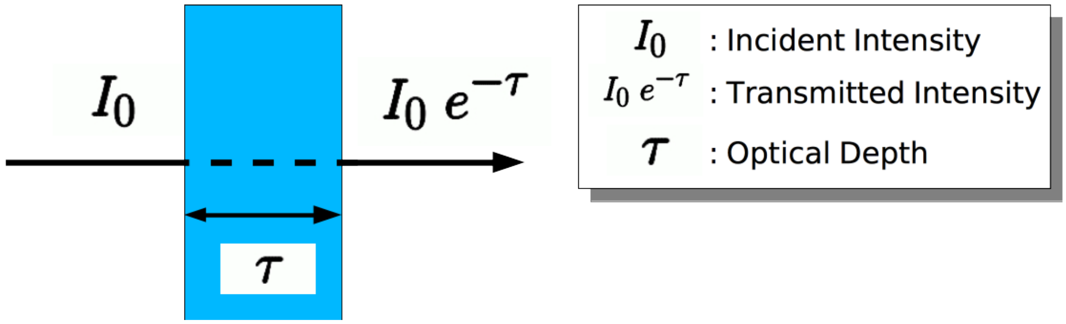

Figure 1: Absorption of Light by Matter

3.1 Absorption of Light in Atmosphere

The law of absorption of light by matter says that the amount of light energy absorbed depends on a property called the *optical depth* if the material. If the optical depth is τ, we find that the incident light intensity, I_0 is decreased by a factor of $e^{-\tau}$ (see Figure 1).

In our model, we will take the optical depth of the entire atmosphere, τ and divide the atmosphere up into N layers of equal optical depth, $d\tau$, where $d\tau = \tau/N$. From the above definition, it is easy to see that the probability that a photon of light is absorbed by a layer $d\tau$ will just be:

$$\text{Probability of Absorption} \;=\; \left(1 - e^{-d\tau}\right)$$

If we have divided the total optical depth, τ, of the atmosphere into a large number of layers, then $d\tau$ will be a small number. In this case, we may expand $e^{-d\tau}$ in a Taylor's series as $e^{-d\tau} = 1 - d\tau + \cdots$, ignoring terms of dt^2 and higher. In this limit, we have:

$$\text{Probability of Absorption} \;=\; d\tau$$

3.2 Random Walk Description

So, to perform a random walk, we can start a photon from the surface and consider each of our N layers in turn. At each layer there is a small probability of $d\tau$ that the layer will absorb our photon. If it does, then the layer will reemit the photon randomly (with 50-50 probability) in the upward or downward direction. If the photon is selected to go UP, then it will move upward through our stack of layers encountering additional layers as it traverses the atmosphere. If it is selected to go DOWN, then it moves towards the surface and encounters layers below the point where it was absobed. In either case, the photon moves layer by layer, with a possible new absorption at each layer, until one of three things happens:

- the photon is absorbed by another layer in the atmosphere, in which case it it reemitted up or down to continue its journey;
- the photon is returned to the surface;
- the photon escapes from the top of the atmosphere.

3.3 Assumption of Optically Thin Layers

Our random walk model for the absorption and emission of layers in the atmosphere is limited by the requirement that the optical thickness of the layers in the atmosphere is very small. First, this assumption allows us to find the probability that the photon is absorbed in a layer directly from its optical depth. That is, the probability is just the optical thickness of the layer $d\tau$. Second, when it comes to reemission, this same assumption means the layer is so transparent that the chance that a photon emitted by the layer will be absorbed by the *same* layer is very small. This requirement means that, afte absorption, I can just move on to the next layers and don't have to consider the possibility that the photon is stuck at the same location. If $d\tau$ is too large, then these conditions are violated and I can't expect to get a good result from my simple random walk simulation.

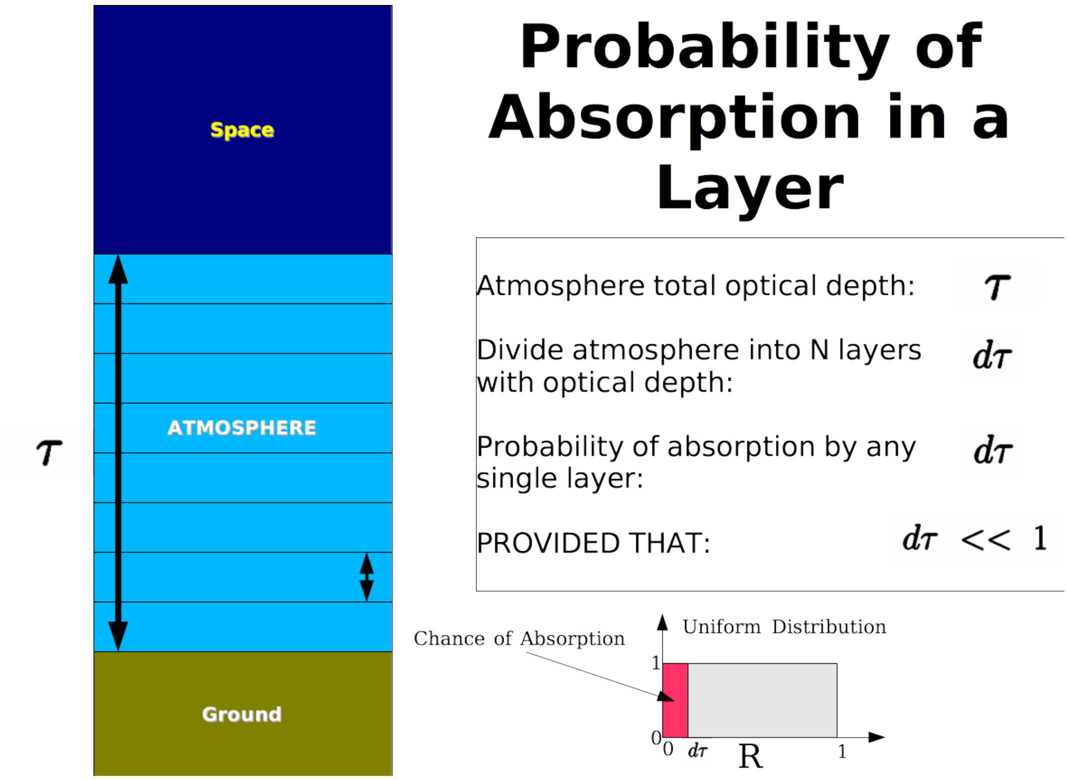

Figure 2: Probability of absorption of Light by a layer in the Atmosphere. Note that we can evaluate whether absorption occurs with the uniform distribution considering only values below the value of $d\tau$.

4 Program Development

The background to this calculation is presented above and was presented in the accompanying lecture. The purpose of these instructions is to suggest a way to develop the program and pose some questions for investigation to be answered in your report.

4.1 Single Random Walk

It is a good idea to begin our development with a single random walk through the atmosphere. We start our photon at the ground moving up through the layers of the atmosphere. As the photon encounters each layer, it has a small chance of being absorbed and reemitted.

How do we set up the layers in the atmosphere, and what is the chance of absorption in each layer? We take as an input to the program the total optical depth of the atmosphere. Then we must divide the atmosphere into a large enough number of layers so that, for each layer, the chance of absorption is small (much less than one). Figure 2 shows a cartoon of the arrangement of the atmosphere into layers.

At each step through the layers in the atmosphere, we can decide whether a photon is absorbed using a uniformly distributed random number between 0 and 1 (i.e. the RandomState.rand() method). The probability of absorption is supposed to be $d\tau$. In a uniform distribution, the chance of drawing a number less than $d\tau$ is also just $d\tau$. So if our random number is less than $d\tau$ then we treat the photon as being absorbed by the layer. Otherwise, the photon is not absorbed and continues on its way.

If our photon is absorbed in a particular layer, then we need to keep track that this has occurred. That's most easilty done by creating an array to hold the count of the number of times each layer absorbs a photon. Then we need to send the photon back on its way, either up or down in the atmosphere. We pick the direction randomly with a 50% probability in each direction (a coin flip).

Obviously, for a given total optical depth, the appoximation that the chance of absorption is small will be better for larger and larger numbers of layers. But, also obviously, the program will take longer to run with larger numbers of layers. So the artistry here is to pick an optical thickness for the layers that is small enough for an accurate calculation, but big enough so that the calculation does not take forever. *In your report you must be prepared to justify your use of a particular value and demonstrate that it gives the correct answer.*

4.2 Simulation with Many Random Walks

Once we have a single random walk function doing the right thing, then we need to carry out this procedure over and over to perform a large number of random walks. Here you will want to keep track of some of the critical data in the calculation:

- Number of photons which escape: P_E.
- Number of photons which are absorbed by the ground: P_G
- For each layer, i in the atmosphere, the number of photons that are absorbed in that layer: $P_L(i)$.

At this point, you will need to decide how many trial photons are required to get accurate answers. In this case, we will define "accurate"to mean that the ground temperature is derived to an accuracy of 0.1K. *In your report, you should be sure to tell the number of trial photons used and justify your selection.*

4.3 Relation of Random Walk Results to Reality

Now that we have the random walk results, we must relate them to the real physical quantities in the atmosphere.

4.3.1 Cooling at Top of Atmosphere

We can relate our number counts in the simulation to a real quantity by considering what comes out of the top of the atmosphere. The count P_E is proportional to the cooling rate of the planet, which in real units averaged over the whole planet is $\sigma T_{eff}^4 4\pi R^2$. So, our value P_E is just proportional to σT_{eff}^4.

4.3.2 Ground Heating

The ground is heated directly by the Sun and it is also heated by emission from the atmosphere downward toward the ground.

The direct heating averaged over the whole planet is $(1-A)S_\odot \pi R^2$. But this is just equal to the same $\sigma T_{eff}^4 4\pi R^2$ we have for the top of the atmosphere. So, in the end, the direct heating of the surface is also just proportional to σT_{eff}^4.

Now the indirect heating of the surface, from the atmosphere, can be derived from the number count P_G which counts the photons that return to the surface from the atmosphere. The ratio $\frac{P_G}{P_E}$ is the additional fraction of σT_{eff}^4 that comes from the atmosphere. So to derive the ground temperature T_G, we just balance the emission from the ground with the direct and indirect heating sources:

$$\sigma T_G^4 \;=\; \sigma T_{eff}^4 \left(1 + \frac{P_G}{P_E}\right)$$

4.3.3 Atmospheric Heating

The absorption of radiation by layers in the atmosphere will heat them. This heating is balanced by cooling of the layer through emission of Blackbody radiation.

We can relate the heating of each layer to σT_{eff}^4 by comparing the number count of photons absorbed in each layer i, $P_L(i)$ to the total number that escaped the atmosphere P_E. So the heating of the layer is:

$$\text{Layer Heating} \;=\; \sigma T_{eff}^4 \frac{P_L(i)}{P_E}$$

For layer i, the emission that cools the layer comes from the top and the bottom of the layer and it depends on the temperature of the layer, T_i and the optical depth, $d\tau$ of the layer as well:

$$\text{Layer Cooling} = 2d\tau\sigma T_i^4$$

Putting these together, we have:

$$2d\tau\sigma T_i^4 = \sigma T_{eff}^4 \frac{P_L(i)}{P_E}$$

So that:

$$\sigma T_i^4 = \frac{1}{2d\tau} \frac{P_L(i)}{P_E} \sigma T_{eff}^4$$

4.3.4 Program Specification

Given the above relations, we can run our simulation and for a given effective temperature, T_{eff}, input by the user, and for a given optical depth, τ, also input by the user, your program should produce:

- The ground temperature T_G of the system.
- The temperature of the top layer of the atmosphere (which will not be the same as T_{eff}!)
- A plot of the temperature of the atmosphere versus optical depth in the atmosphere, where the optical depth defined to be 0 at the top of the atmosphere and increasing with depth into the atmosphere. It is traditional, in atmospheric physics, to plot the independent variable (optical depth) vertically.

5 Project Calculations

5.1 Nominal Calculation (EARTH)

Run your program for the following nominal case and present the results. Please note that, although these values are very similar to the actual situation for the Earth, you should not expect them to exactly reproduce values of atmospheric and ground temperatures found in my table or on Wikipedia.

- Total Optical Depth = 1.1
- Effective Temperature = 253K. (which implies a Bond Albedo of 0.33)

Please document the result of the nominal calculations in your report. Then use your program to address the following questions about the Earth.

- How does the temperature at the top of the atmosphere in the calculation compare to the temperature of the Earth's stratosphere?
- Using the optical depth and effective temperature values given for the nominal calculation, test the sensitivity of the ground temperature to a change in total optical depth. It is thought that the ground temperature of Earth has increased by about 1 degree K over the last 130 years. How big a change in optical depth is required to increase the surface temperature by 1 degree in your model?
- A change in the planet's reflectivity can also change its effective temperature and thereby change the surface temperature. What change in the albedo would be required to increase the Earth's surface temperature by 1 degree K? (Assume nominal value of optical depth of the atmosphere.)

5.2 The Greenhouse Effect on Venus

The effective temperature of Venus is 240K, yet its surface temperature is a whopping 700K! This is due to the greenhouse effect of Venus' atmosphere.

Estimate the optical depth required to achieve this high surface temperature. Is your estimate consistent with what is known about the atmosphere of Venus compared to that of the Earth? (NOTE: remember that Venus is closer to the Sun than the Earth!)

HINT: Be very careful about the number of layers needed for an atmosphere that is thicker than the Earth's!

5.3 Extra Credit

Our calculation has layers of equal optical depth, where the optical depth depends on the amount of material in the layer. In the real atmosphere, the density of air decreases exponentially with height according to:

$$\rho(z) \;=\; \rho(0)e^{-\frac{z}{H}}$$

where z is the altitude, $\rho(z)$ is the density of air at altitude z, and H is called the scale height of the atmosphere. On Earth H is about 8 km. Given this expression, the total mass of air above one square meter of the surface is:

$$M \;=\; \rho(0)\int_0^{\infty} e^{-\frac{z}{H}}\,dz \;=\; \rho(0)H$$

For N layers of equal mass in the atmosphere, the portion of the mass M in each layer will be $dm = \frac{M}{N}$, but because the density changes with altitude, the thickness of the equal mass layers will change (increase) with altitude. Thus, the position of the equal optical depth (equal mass) layers in our calculation will not be a linear function of altitude in the atmosphere. There will be many relatively thin layers close to the surface and fewer, thicker, layers at the top.

Find the positions of the centers of the equal mass layers for your calculation and plot the temperatures as a function of altitude. Compare the temperature profile you derive to the actual temperature profile of the Earth. Specifically, is the temperature gradient in our model steeper or less steep than the Earth's real atmosphere?