Project 3
Orbital Perturbations

Fall 2019

1 Background

The study of the orbits of objects in the solar system has a long tradition, but they would be pretty boring — just an object going round and round the Sun on a fixed path — if not for the fact that there are lots of interesting forces besides gravity that can affect the orbit. In this project I'd like to compute orbits including interesting perturbing forces and study how the orbits evolve.

2 Two Body Orbit Equation

The classic "two body" orbit considers the Sun to be a fixed point in space. This corresponds to the case where the Sun's mass is much much greater than that of the orbiting object so that we can consider the center of mass of the Sun-object system to be the same as that of the center of mass of the Sun. If we think of an asteroid, like the near Earth asteroid 433 Eros, the relative mass is about 3×10^{-15} suggesting that this might be a good approximation for many things in the solar system.

The two body equation of motion is just:

$$m\ddot{\mathbf{r}} = -\frac{GM_\odot m}{r^2}\hat{r}$$

where $\mathbf{r}$ is the position vector, m is the object's mass, $GM_\odot$ is the Universal gravitational constant times the mass of the Sun, and $\hat{r}$ is a unit vector from the Sun to the object. We will use the common convention that $\mathbf{r}$ (bold face) denotes a vector, whereas r denotes the magnitude of the vector ($r = |\mathbf{r}|$). We use the "dot" notation to show a derivative with respect to time.

The mass of the object appears on both sides of our equation, and so it is customary to rewrite the above without the mass:

$$\ddot{\mathbf{r}} = -\frac{GM_\odot}{r^2}\hat{r} \tag{1}$$

From first principles, considering this radial force law, we know that energy and angular momentum of the object are conserved. The energy per mass of the object, E, is just

$$E = \frac{1}{2}\dot{r}^2 - \frac{GM_\odot}{r}$$

The angular momentum per mass of the orbiting body, $\mathbf{H}$ is

$$\mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}}$$

The magnitude of $\mathbf{H}$, H, is $r^2\dot{\theta}$, where θ is the angle of a vector pointing to the object in the plane of the orbit from some reference point.

3 Orbital Elements and Constants of the Motion

For objects which are bound to the Sun, the solution can be shown to be an ellipse with the Sun at one focus of the ellipse. Given a position **r** and a velocity **v** at some initial time, I can solve equation 1 to find the position at any time in the past or in the future. Since I know that the solution is an ellipse, I can also express the solution geometrically, as described below.

Figure 1 shows the definitions of key properties of an elliptical orbit around the Sun. The left panel describes the shape and size of the ellipse using the semimajor axis (a in the figure) and the eccentricity (e in the figure. The body's position (P) along the orbit can be described by the polar angle from perihelion, identified as TA for true anomaly in celestial mechanics. When we describe orbits in the real solar system, we must also account for the orientation of the ellipse in space. This is done using three angles identified in the right hand panel: the inclination (i); the longitude of the ascending node (Ω); and the longitude of perihelion (ω). Two of these, i and Ω give the orientation of the plane of the orbit in space with respect to the x-y plane of the solar system. The third, ω, defines the orientation of the ellipse within the plane. There is also a sixth number needed to fully specify the orbit. That is the time that the object is at perihelion (often called τ).

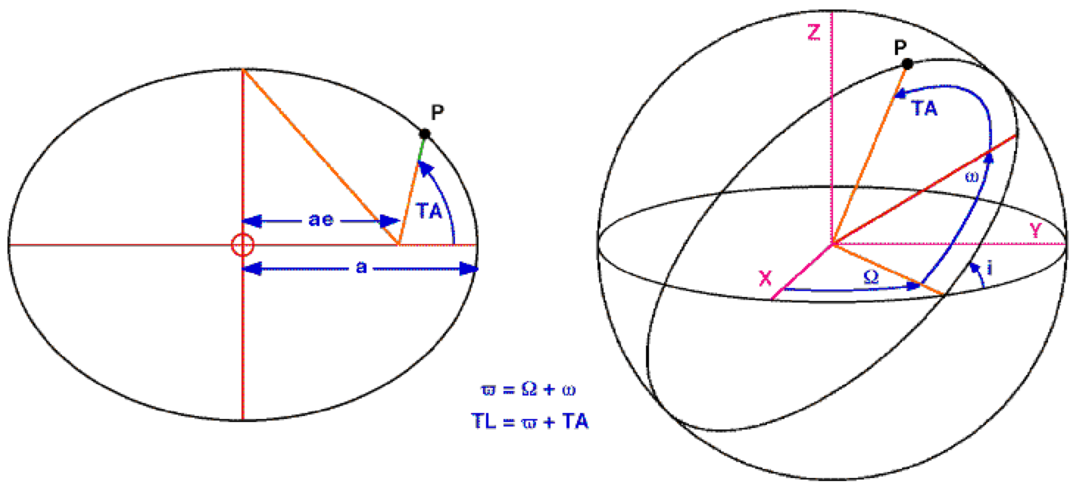

Figure 1: Orbital Element Definitions

The six numbers a, e, i, Ω, ω, and τ are called the orbital elements describing an orbit. They are entirely equivalent to the six numbers contained in the 3D position and velocity vectors, and astronomers like them because we can easily visualize an orbit with this description.

Perhaps it won't surprise you that we can make a connection between the orbital elements and the fundamental constants of the motion, E and $\mathbf{H}$. The energy of the orbit is just related to the semimajor axis, so we can write

$$a \;=\; -\frac{GM_\odot}{2E}$$

The eccentricity of the orbit is a little more complicated and includes a dependence on the magnitude of the angular momentum, H.

$$e \;=\; \big(1 + \frac{2H^2E}{(GM_\odot)^2}\big)^{\frac{1}{2}}$$

Note that these two quantities describe the shape of the orbit in the plane of the orbit. The orbital elements describing the orbit plane's orientation can also be derived from the direction of the angular momentum vector. If $\mathbf{H}$ has components H_x, H_y, and H_z, then we can show:

$$\cos i \;=\; \frac{H_z}{H}$$

$$\tan\Omega = -\frac{H_x}{H_y}$$

So, given $\mathbf{r}$ and $\dot{\mathbf{r}}$ for some instant in time, I can calculate E and $\mathbf{H}$ and easily compute the above orbital elements to study the way that orbits evolve under perturbing forces as they may change the energy and angular momentum of the orbiting object. Of course, if there are no perturbing forces, the all these quantities are constants.

4 Perturbing Forces in Celestial Mechanics

We are going to add three perturbing forces to the force of gravity in our two body equation of motion (Equation 1). At any instant, the force will be considered to have components in: (1) the radial direction of the body from the Sun; (2) the direction perpendicular to the radial direction in the orbit plane; and (3) the direction normal to the orbit plane. Traditionally these three components of the perturbing force are called R, T, and N. We are going to parameterize the forces by considering them to be some fraction of the gravitational force felt by the object. Thus, for example, the value R would be the fraction of the gravity force tugging the object away from the Sun.

I can write down the total force, including gravity, according to this specification:

$$\mathbf{F} = \frac{GM_\odot}{r^2}\big(-\hat{r} + R\hat{r} + T\hat{\theta} + N\hat{n}\big) \tag{2}$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{n}$ are unit vectors in the radial direction, the "theta" direction perpendicular to radial direction in the plane of the orbit, and normal to the orbit plane. If I know position $\mathbf{r}$ and velocity $\mathbf{v}$ at some time in the orbit, then I can find $\hat{r}$, $\hat{\theta}$, and $\hat{n}$ easily:

$$\hat{r} = \frac{\mathbf{r}}{r}$$

$$\hat{n} = \frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|}$$

$$\hat{\theta} = \hat{n}\times\hat{r}$$

So finally I can rewrite the equation of motion, with the perturbing forces, as:

$$\ddot{\mathbf{r}} = \frac{GM_\odot}{r^2}\big(-\hat{r} + R\hat{r} + T\hat{\theta} + N\hat{n}\big) \tag{3}$$

5 Calculation

5.1 Funny Units

We are going to carry out our calculation using units that are traditionally used in solar system studies. Our unit of length will be the astronomical unit (AU), which is the distance between the Earth and the Sun. Our unit of time will be years, which of course corresponds to the period of the Earth's orbit. With this choice of units, the product $GM_\odot$ is just $4\pi^2$.

As an example in these units, a circular orbit at 1 AU in this system will have a period of 1 year. This means that the velocity of the object in AU per year will be the circumference of the orbit, $2\pi\times 1$AU, divided by 1 year or just 2π AU per year.

5.2 Useful Functions

We are going to solve Equation 3 using the 4th order Runge-Kutta method. To help implement this and to interpret the results, it is a good idea to create a few supporting functions. (It is also a good idea to make a single function that can do the whole problem since we will be running this many times with different values of R, T, and N to see what happens.)

One useful function is to compute the position and velocity derivatives for each time step according to the Runge-Kutta method.

To implement the above, it is also a good idea to define a separate function to compute our complicated force in Equation 3 given position, velocity, R, T, and N.

Finally, for visualization and understanding of what is going on, it is useful to provide a function to calculate energy, E, angular momentum, H, and the four orbital elements a, e, i, and Ω given the instantaneous position and velocity at any point during the orbit.

For each calculation we do below, please present plots to illustrate the motion of the object. Also present plots of the constants of the motion and the orbital elements over the duration of your calculation.

5.3 Step 1

The first thing to do is to prove that our orbit program works. So we should reproduce a circular orbit at 1 AU. Make the orbit plane coincident with the x-y plane in our coordinate system. Run your program for 20 years of time to be sure that the object maintains the same distance from the Sun at all times and returns exactly to where you started after 20 years. It is not too hard to repeat the starting position at the level of 10^{-7} or better. Be sure to demonstrate that your program does this and tell me about the choice of time step.

Another thing to calculate is the energy, angular momentum and orbital elements for each point in the calculation. For this case you should find that these quantities are constant to high accuracy over the entire calculation. The values of a, e, and i should be 1, 0, and 0 respectively to high precision. (Ω is a bit tricky to define in the case of an orbit whose plane is the same as the reference x-y plane, so don't worry about that one. Your program will probably produce values jumping between $-\pi$ and π.)

5.4 Step 2

Once the program is working for the simple case of a circular orbit, think of at least one other test you might do to verify that the program is working and that the orbital elements are being computed correctly for the case with no perturbations.

5.5 Step 3 - Perturbing Forces

Let's explore what happens if we turn on the perturbing forces. It is more amusing and instructive to do these calculations using an orbit with a small (say 10 degrees) inclination and a non-zero eccentricity. You can get this started by picking an initial velocity that is not the circular velocity and by starting you object from above or below the x-y plane. Be sure to run a case with no perturbing forces to show what the original orbit looks like before turning on the perturbing force. Run your program for 50 years to see the evolution of the orbit under the influence of each perturbing force. (Be sure to keep the time step size the same as you used to demonstrate that the program worked!)

5.5.1 Radial Perturbing Force

A good example of an R force is the effect of radiation pressure on our object. This force has little effect on large bodies, but can be very important for small dust grains.

- What happens to the orbit for the case R=0.1 (10% of the gravitational force)?
- Are all orbital elements affected by the R force? If some are unaffected, can you explain that result?
- Try some different values of R to find the value that is required for our object to be ejected from the solar system?

5.5.2 Tangential Perturbing Force

Now let's consider the T force. Here there is an important effect on dust grains called the Poynting-Robertson effect. This is also related to radiation pressure and results from the fact that the motion of our particle perpendicular to the direction to the Sun makes the incoming photons of sunlight approach the particle from a direction slightly different from the radial direction. The result is a perturbing force in the -T direction.

- Orbits are very sensitive to T perturbing forces, so even a tiny value can have a big impact. See what happens to your nominal orbit with T = -0.001.

- Are all orbital elements affected by the T force? If some are unaffected, can you explain that result?
- You should see that over the first 10 years or so the a and e values will decrease more or less linearly. Try a smaller value of T to see if the rate of change of a and e over the first 10 years is proportional to T.
- What is the ultimate fate of dust particles in the solar system?

5.5.3 Normal Perturbing Force

Finally, lets look at the N force which is normal to the orbit plane. There is an actual force which is important for big particles, even up to the scale of asteroids, which can have a component in the normal direction. It is called the Yarkovsky effect, and it results from the fact that the temperature of the surface of a rotating body is hottest in the local afternoon. Hotter parts of the asteroid's surface emit more blackbody radiation than colder parts, and so there are more photons coming off the afternoon side of the asteroid than the morning side. Thus we wind up with a small net force.

Because the Yarkovsky depends on the orientation of the rotational axis, this force can be in any direction. So lets see what will happen if things are lined up to give a perturbing force normal to the orbit plane.

- This one is also pretty sensitive compared to the R perturbation, so lets use N=0.01 as our nominal case.
- Which orbital elements are affected by the N perturbing force? Can you explain why?