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Néron models and tame ramification

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1. Introduction

In this article we study the behaviour of Néron models of abelian varieties with respect to tamely ramified extensions of discrete valuation rings. Let D be a discrete valuation ring with field of fractions K and residue field k, and let A be an abelian variety over K. Then among all extensions of A to a scheme over $S = \mathrm{Spec}(D)$ there exists a canonical "best" one, the so-called Néron model $\mathscr{A}/S$, named after its discoverer A. Néron. It is characterized by the property that it is smooth over S and that for every smooth morphism of schemes $T \to S$ the induced map $\mathscr{A}(T) \to A(T_K)$ is bijective. For more information about these models we refer to the book [1] on this subject.

Let K'/K be a finite dimensional separable extension of fields, and let D' be the localization at one of the maximal ideals of the integral closure of D in K'. Then one can ask what the relations are between $\mathscr{A}$ and $\mathscr{A}'$, where $\mathscr{A}'$ is the Néron model over $S' = \mathrm{Spec}(D')$ of $A_{K'}$. By the defining properties of $\mathscr{A}$ and $\mathscr{A}'$ we get a morphism $\mathscr{A}_{S'} \to \mathscr{A}'$. It is this morphism that we want to understand, especially in the case where A does not have semi-stable reduction over D.

Let us assume that D' is Galois over D with group G (by this we mean that G acts on D' and that D is the subring of G-invariants). By the universal property of $\mathscr{A}'$ the right-action of G on $A_{K'}$ extends to a G-equivariant right-action of G on $\mathscr{A}'$ over S'. Let $X = \Pi_{S'/S}(\mathscr{A}'/S')$ denote the Weil restriction of scalars of $\mathscr{A}'$ to S (see Section 2). The group G acts on X in a natural way, and we have a morphism $\mathscr{A} \to X$. The key result of this article is that this morphism $\mathscr{A} \to X$ identifies $\mathscr{A}$ with the closed subscheme X^G of fixed points of X, provided that the extension D'/D is tamely ramified. This makes it possible to study $\mathscr{A}$ in terms of $\mathscr{A}'$ with its G-action. If D'/D is not tame, then $\mathscr{A}$ is obtained from the closure of A in X by what is called the "smoothening process" in [1] (see [1] 7.2/4). We will restrict ourselves to tamely ramified extensions. Section 5 contains a detailed description of $\mathscr{A}_k$ in terms of $\mathscr{A}'_k$ with its G-action, and in Section 6 some criteria for exactness properties of Néron models to be true are given. I want to thank Hendrik Lenstra for suggestions concerning the proof of lemma 3.3, and René Schoof for his computations concerning elliptic curves.

2. Generalities on Weil restriction of scalars

Let $\pi\colon S' \to S$ be a morphism of schemes, and let $X \to S'$ be a scheme over S'. Then the Weil restriction of X to S, à la Grothendieck, is defined as the functor

$$\prod_{S'/S} (X/S')\colon (\mathrm{Sch}/S) \to (\mathrm{Sets}), \quad (T \to S) \mapsto X(T'),$$

where $T' = T \times_S S'$ and $X(T') = \mathrm{Hom}_{S'}(T', X)$.

2.1. REMARK. If we consider $X \to S'$ as a presheaf for some Grothendieck topology on (Sch/S'), then $\Pi_{S'/S}(X/S')$ is just the push forward to (Sch/S). It is proved in [2], exp. 221, 4c, see also Proposition 5.7 and Section 7 of loc. cit., that $\Pi_{S'/S}(X/S')$ is representable by an open subscheme of the Hilbert scheme of X over S, if $\pi\colon S' \to S$ is proper and flat, and $X \to S$ quasi-projective. We will use this result only in the case where π is finite and flat, and $X \to S'$ is quasi-projective. In that case, $\Pi_{S'/S}(X/S') \to S$ is quasi-projective. It is clear from the definition that the formation of $\Pi_{S'/S}(X/S')$ commutes with base change on S: for all $T \to S$, *we have* $\Pi_{T'/T}(X_{T'}/T') = \Pi_{S'/S}(X/S') \times_S T$.

2.2. LEMMA. *Let $\pi\colon S' \to S$ be finite and flat and let $X \to S'$ be quasi-projective and smooth. Then $\Pi_{S'/S}(X/S')$ is smooth over S.*

Proof. By the remark above, $\Pi_{S'/S}(X/S')$ is representable. By definition, see [4], IV, 17.3.1, $X \to S'$ is locally of finite presentation and formally smooth. It follows right from the definitions, see [4], IV, 8.14.2 and 17.3.1, plus the fact that $S' \to S$ is affine, that $\Pi_{S'/S}(X/S')$ is locally of finite presentation and formally smooth over S. □

2.3. CONSTRUCTION. Suppose that X is obtained by base change via $\pi\colon X = Y' = Y_{S'}$, for some $Y \to S$. Then $\Pi_{S'/S}(X/S')(T) = X(T') = Y_T(T') = Y(T')$. We have natural maps $Y(T) \to Y(T')$, giving an S-morphism $Y \to \Pi_{S'/S}(Y'/S')$. If π: $S' \to S$ is faithfully flat, then all the maps $Y(T) \to \Pi_{S'/S}(Y'/S')(T)$ are injections.

2.4. CONSTRUCTION. Suppose that a group G acts, on the right, equivariantly on $\pi\colon S' \to S$ and on $X \to S'$, with the trivial action on S. Then we can define an equivariant right-action by G on $\Pi_{S'/S}(X/S') \to S$ by:

$$P \cdot g = \rho_X(g) \circ P \circ \rho_{T'}(g)^{-1},$$

where T is a scheme over $S, P \in \Pi_{S'/S}(X/S')(T) = \mathrm{Hom}_{S'}(T', X), g \in G$, $\rho_X(g)$ the automorphism of X induced by g, $\rho_{S'}(g)$ the automorphism of S' induced by g, and $\rho_{T'}(g) = \rho_{S'}(g) \times 1_T$.

Note that if $X = Y'$, as in construction 2.3, with G acting equivariantly on $Y \to S$, then the map $Y \to \Pi_{S'/S}(Y'/S')$ of construction 2.3 is G-equivariant.

3. Generalities about fixed points

Let $X \to S$ be a morphism of schemes, and let G be a finite group acting equivariantly on $X \to S$, with the trivial action on S. We define the functor X^G of fixed points by:

$$X^G\colon (\mathrm{Sch}/S) \to (\mathrm{Sets}), \quad (T \to S) \mapsto X(T)^G.$$

3.1. PROPOSITION. *The functor X^G is represented by a subscheme of X. The formation of X^G commutes with base change on S. If $X \to S$ is separated, then X^G is a closed subscheme of X.*

Proof. Let Z be the fibered product, over $X \times_S X$, of the graphs $\Gamma_g\colon X \to X \times_S X$, where $g \in G$. The Γ_g are immersions ([4], I, 5.1.4); hence $Z \to X \times_S X$ is an immersion ([4], I, 4.3.4). Since Z is a subscheme of the diagonal (the graph of the unit element of G), we can consider it as a subscheme of X. As such, it represents X^G. If $X \to S$ is separated, then all the Γ_g are closed immersions. Hence $Z \to X \times_S X$ is a closed immersion, and can be considered as a closed subscheme of X. □

3.2. PROPOSITION. *Let $\mathrm{T}_{X/S}$ and $\mathrm{T}_{X^G/S}$ denote the tangent bundles of X/S and X^G/S (see [4], IV, 16.5.12.) Let $x \in X^G$; then we have*:

$$\mathrm{T}_{X^G/S}(x) = \mathrm{T}_{X/S}(x)^G.$$

Proof. For arbitrary $X \to S$, and $x \in X$ we have [4], IV, 16.5.13.1:

$$\mathrm{T}_{X/S}(x) = \mathrm{Hom}_{k(x)}(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} k(x), k(x)).$$

Let $Y_0 = \mathrm{Spec}(k(x))$, and $Y = \mathrm{Spec}(k(x)[\varepsilon])$, with $\varepsilon^2 = 0$, both considered as schemes over S. Let $u_0\colon Y_0 \to X$ be the canonical morphism with image x, and let $i\colon Y_0 \to Y$ denote the closed immersion of Y_0 into Y. Then [4], IV, 16.5.17 tells us that

$$\mathrm{T}_{X/S}(x) = \{u \in X(Y) | u \circ i = u_0\}.$$

Applying this to the situation mentioned in the proposition gives:

$$\mathrm{T}_{X^G/S}(x) = \{u \in X^G(Y) | u \circ i = u_0\} = \{u \in X(Y) | u \circ i = u_0\}^G = \mathrm{T}_{X/S}(x)^G. \qquad \square$$

3.3. LEMMA. *Let A be a complete local ring with residue field k, d a non-negative integer, $B = A[[T_1, \ldots, T_d]]$. Let G be a finite group acting on the A-algebra B, and let $n = \# G$. Suppose that A is a $\mathbf{Z}[1/n]$-algebra. Then there exist $S_1, \ldots, S_d \in B$, with $B = A[[S_1, \ldots, S_d]]$, such that the A-submodule $M = AS_1 \oplus \cdots \oplus AS_d$ is G-stable. Moreover, M is then a direct sum of $A[G]$-*

modules M_i which are free as A-modules, and have the property that $M_i \otimes_A k$ is an irreducible $k[G]$-module.

Proof. Let m_A and m_B denote the maximal ideals of A and B. Let $\bar{V}$ be a finitely generated $k[G]$-module. We claim that there exists an A-free $A[G]$-module V, unique up to isomorphism, such that $V \otimes_A k$ is isomorphic to $\bar{V}$. To prove this, it suffices to show that for any $m \geqslant 1$, a morphism of groups $G \to \mathrm{GL}_d(A/m_A^m)$ can be lifted to a morphism of groups $G \to \mathrm{GL}_d(A/m_A^{m+1})$, and that such a lift is unique up to an inner automorphism of $\mathrm{GL}_d(A/m_A^{m+1})$. Now let K_{m+1} be the kernel of $\mathrm{GL}_d(A/m_A^{m+1}) \to \mathrm{GL}_d(A/m_A^m)$; then K_{m+1} is the additive group of a k-vector space. Since $n = \# G$ is invertible in k, we have $\mathrm{H}^i(G, H_{m+1}) = 0$, for all $i > 0$. For $i = 2$ this means that the required lift exists, and for $i = 1$ it means that the lift is unique up to inner automorphisms.

Let $\bar{V}$ be the $k[G]$-module $m_B/(Bm_A + m_B^2)$. Note that $\bar{V}$ has a k-basis consisting of the images of the T_i. We can write $\bar{V} = \oplus \bar{V}_i$, where the $\bar{V}_i$ are irreducible $k[G]$-modules. Let V_i be a A-free $A[G]$-module with $V_i \otimes_A k \cong \bar{V}_i$, and let $V = \oplus V_i$. After changing the $(T_1, \ldots, T_d)$ by an element of $\mathrm{GL}_d(A)$, we may assume that each $\bar{V}_i$ has a basis consisting of a subset of $\{T_1, \ldots, T_d\}$. We lift the k-basis of each $\bar{V}_i$ arbitrarily to a A-basis of V_i, and get a basis of V.

Now consider ϕ': $V \to B$, sending the ith vector of the basis to T_i. Then we get a G-equivariant morphism ϕ: $V \to B$ by:

$$\phi := \frac{1}{n} \sum_{g \in G} g \cdot \phi' \cdot g^{-1}.$$

Let $S_i \in B$ be the image, under ϕ, of the ith basis vector. Then S_i and T_i have the same image in $m_B/(Bm_A + m_B^2)$; hence $B = A[\![S_1, \ldots, S_d]\!]$. □

3.4. PROPOSITION. *If the morphism $f: X \to S$ is smooth, and $n := \# G$ is invertible on X, then X^G is smooth over S.*

Proof. Let $x \in X^G$, and let $s = f(x)$. We have to show that f is smooth at x. By an argument of [3] 3, exp. V, 5b, there exist an affine open G-stable neighborhood U of x in X, and an affine open neighborhood V of s, with $fU \subset V$, such that $\mathcal{O}_X(U)$ is a $\mathcal{O}_S(V)$-algebra of finite presentation. We can now replace $X \to S$ by $U \to V$. By [4] IV, 8.8.3 and 17.7.8, we may assume that S is noetherian. It is sufficient to show smoothness at x after the (flat) base change $\mathrm{Spec}(\mathcal{O}_{X,x}) \to S$. Then we have $k(x) = k(s)$. Let $A = \hat{\mathcal{O}}_{S,s}$ and $B = \hat{\mathcal{O}}_{X,x}$. By [4] IV, 17.5.3, B is isomorphic, as an A-algebra, to $A[\![T_1, \ldots, T_d]\!]$, where d is the relative dimension of X over S at x. In order to test the formal smoothness of X^G it is enough to consider only artinian local A-algebras ([4] IV, 17.5.4), which can be done in terms of the G-action on B.

By lemma 3.3, we may assume that the action of G on $A[\![T_1, \ldots, T_d]\!]$ is linearized: the A-submodule generated by the T_i is G-stable. We may also assume that T_i is G-invariant for $i \leqslant e$, and that the T_i with $i > e$ generate a G-stable A-submodule of B that, modulo m_A, contains only non-trivial irreducible representations.

Let I be the ideal in B generated by the $(1-g)b$, where $g \in G$ and $b \in B$. Then for every A-algebra C, the G-invariant morphisms $B \to C$ are precisely those that factor through B/I. Let J be the ideal $\Sigma_{i>e} BT_i$. We want to show that $I = J$. It is immediate that $I \subset J$. For the other inclusion, we consider the morphism of finitely generated B-modules $\phi\colon \bigoplus_{g \in G} J \to J$, which is $1-g$ on the g-component. Since $J/m_B J$ is a direct sum of nontrivial irreducible $k[G]$-modules, $\phi \otimes k$ is surjective. Then ϕ is surjective by Nakayama's lemma, which proves that $J \subset I$. To finish the proof, we note that B/I is a formally smooth A-algebra since it is isomorphic to $A[\![T_1, \ldots, T_e]\!]$. □

3.5. PROPOSITION. *Let G be a finite group, acting equivariantly on a smooth morphism of schemes $f\colon X \to S$. If $\# G$ is invertible on X, then the induced morphism $f\colon X^G \to S^G$ is smooth.*

Proof. Apply the preceding proposition to $X \times_S S^G \to S^G$. □

4. Application to Néron models

Let D be a discrete valuation ring with field of fractions K, and residue field k. Let K' be a finite separable extension of K, and let D' be the integral closure of D in K'. Note that D' is not necessarily a discrete valuation ring, but that the localizations at its finitely many maximal ideals are. Let $S = \mathrm{Spec}(D)$ and $S' = \mathrm{Spec}(D')$. Let A be an abelian variety over K, and let $\mathscr{A}'$ be the Néron model over S' of $A_{K'}$.

4.1. PROPOSITION. *$\Pi_{S'/S}(\mathscr{A}'/S')$ is the Néron model over S of the abelian variety $\Pi_{K'/K}(A_{K'}/K')$ over K.*

Proof. First of all, we have to prove that $\Pi_{K'/K}(A_{K'}/K')$ is an abelian variety over K. This is clear after base change to the separable closure K^{sep} of K, since then we have:

$$\left(\prod_{K'/K} (A_{K'}/K')\right)_{K^{\mathrm{sep}}} = \prod_{\phi: K' \to K^{\mathrm{sep}}} A_{K',\phi}.$$

We have already seen in lemma 2.2 that $\Pi_{S'/S}(\mathscr{A}'/S') \to S$ is smooth. Let $T \to S$ be smooth, then:

$$\prod_{S'/S} (\mathscr{A}'/S')(T) = \mathscr{A}'(T') = A_{K'}(T'_{K'}) = \prod_{K'/K} (A_{K'}/K')(T_K) = \left(\prod_{S'/S} (\mathscr{A}'/S')\right)_K (T_K). \quad \square$$

Let $\mathscr{A}$ be the Néron model over S of A. Construction 2.3 gives us a closed immersion: $A \hookrightarrow (\Pi_{S'/S}(\mathscr{A}'/S'))_K$. Since $\mathscr{A}$ is smooth over S, and $\Pi_{S'/S}(\mathscr{A}'/S')$ has the Néronian property, this closed immersion extends to a morphism $\mathscr{A} \to \Pi_{S'/S}(\mathscr{A}'/S')$.

4.2. THEOREM. *If $S' \to S$ is tamely ramified, then $\mathscr{A} \to \Pi_{S'/S}(\mathscr{A}'/S')$ is a closed immersion. If moreover K' is a Galois extension of K with group G, then this closed immersion induces an isomorphism between $\mathscr{A}$ and the subscheme $(\Pi_{S'/S}(\mathscr{A}'/S'))^G$ of fixed points, where G acts as in* 2.4.

Proof. This can be checked after the base change from S to its strict henselization. Then S' is a finite disjoint union of schemes $S'_i = \mathrm{Spec}(D'_i)$, where the D'_i are discrete valuation rings, tamely ramified over D. By definition, $\Pi_{S'/S}(\mathscr{A}'/S')$ is the fibered product, over S, of the schemes $\Pi_{S'_i/S}(\mathscr{A}'_{S'_i}/S'_i)$. Since all the factors in this fibered product are separated over S (they are even quasi-projective over S), it is sufficient to prove that all the morphisms

$$\mathscr{A} \to \prod_{S'_i/S} (\mathscr{A}'_{S'_i}/S'_i)$$

are closed immersions. This means that we have reduced the proof of the theorem to the case where S is strictly henselian. From now on we assume that S is strictly henselian.

Let X denote $\Pi_{S'/S}(\mathscr{A}'/S')$, and let $\bar{A}$ be the scheme theoretic closure of the image of $A = \mathscr{A}_K$ in X (see [6] 2.1). By definition, the morphism $\mathscr{A} \to X$ factors through $\bar{A}$. We will now show that $\bar{A}$ is a Néron model of A, which proves the theorem. It is clear that $\bar{A}$ has the Néronian property: for all $T \to S$ smooth, all elements of $A(T_K)$ extend uniquely to an element of $X(T)$, which must be in $\bar{A}(T)$ by the definition of scheme theoretic closure. It remains to be seen that $\bar{A}$ is smooth over S. By construction, $\bar{A}$ is a group scheme over S.

Recall that S is strictly henselian, and that S' is tamely ramified. Hence K' is Galois over K with Galois group $G = \mu_n$, where $n = [K' \colon K]$ is prime to the characteristic of k, and μ_n is the group of nth roots of unity of D. We let G act, from the right, on $A_{K'}$ via its right-action on $\mathrm{Spec}(K')$. By the Néron property, this action extends uniquely to a right-action on $\mathscr{A}'$, such that $\mathscr{A}' \to S'$ is equivariant. As in construction 2.4, we get a right G-action, over S, on X. The morphism $\mathscr{A} \to X$ is G-equivariant.

Since the G-action on A is trivial, $\bar{A}$ is a closed subscheme of the scheme of fixed points X^G. It is easy to check that $X^G_K = A$. By proposition 3.4, X^G is smooth over S. Now both $\bar{A}$ and X^G are closed subschemes of X, flat over S, and they have the same generic fibre. It follows that $\bar{A} = X^G$. This proves that $\bar{A} \to S$ is smooth. □

4.3. AN EXAMPLE. We will now give an example where S' is wildly ramified over S, and where $(\Pi_{S'/S}(\mathscr{A}'/S'))^G$ is not smooth over S. Let $S = \mathrm{Spec}(\mathbf{Z})$ and

$S' = \mathrm{Spec}(\mathbf{Z}[i])$. In order to have simple equations, we will study the Néron model over S of a twisted form of the multiplicative group $\mathbf{G}_{m,S}$. To be precise, we will consider only the connected component of the Néron model involved. More information about Néron models of not necessarily proper group schemes can be found in [1] 10.

We write $G = \mathrm{Gal}(\mathbf{Q}(i)/\mathbf{Q}) = \{1, \sigma\}$, and we let $A_{\mathbf{Q}(i)} = \mathbf{G}_{m,\mathbf{Q}(i)} = \mathrm{Spec}(\mathbf{Q}[i][X, X^{-1}])$. We let G act on $A_{\mathbf{Q}(i)}$ by $\sigma^{\#}: i \mapsto -i, X \mapsto X^{-1}$, and we define A to be the corresponding quotient. Then A is a twisted form of $\mathbf{G}_{m,\mathbf{Q}}$. According to [1] 10/5, $\mathscr{A}' = \mathbf{G}_{m,\mathbf{Z}[i]}$ is the connected component of the Néron model of $A_{\mathbf{Q}(i)}$. Now let C be any $\mathbf{Z}$-algebra. Then we have:

$$\begin{aligned}\left(\prod_{S'S} \mathscr{A}'/S'\right)(C) &= \mathrm{Hom}_{\mathbf{Z}[i]}\left(\mathbf{Z}[i][X, X^{-1}], C \otimes_{\mathbf{Z}} \mathbf{Z}[i]\right)\\ &= (C \otimes_{\mathbf{Z}} \mathbf{Z}[i])^{*}\\ &= \{(x+yi, u+vi) \in C \otimes_{\mathbf{Z}} \mathbf{Z}[i] \mid (x+yi)(u+vi) = 1\}.\end{aligned}$$

From this we see that $\Pi_{S'/S}(\mathscr{A}'/S')$ is represented by $\mathrm{Spec}(\mathbf{Z}[X, Y, U, V]/(XU - YV = 1,\ XV + YU = 0))$. Since σ acts by $(x + yi, u + vi) \mapsto (u - vi, x - yi)$, the subscheme of G-invariants is given by the equations $U = X$ and $V = -Y$. It follows that $(\Pi_{S'/S}(\mathscr{A}'/S'))^G$ is isomorphic to $\mathrm{Spec}(\mathbf{Z}[X, Y]/(X^2 + Y^2 = 1))$, which has a non-smooth fibre at 2. To obtain the Néron model of A, one has to apply the smoothening process of [1] 3 to $(\Pi_{S'S}(\mathscr{A}'/S'))^G$. In this case, that amounts to a blow-up in the two points with maximal ideals $(2, X, Y - 1)$ and $(2, X - 1, Y)$.

5. The special fibre in the totally ramified case

Let the notation and hypotheses be as in the preceding section. Let $n = [K' : K]$. In this section we assume moreover that D contains the nth roots of unity, and that D' is a discrete valuation ring with uniformizer π', such that $\pi = \pi'^n$ is a uniformizer of D. Note that one always gets into this situation if D is strictly henselian. Then the group $G = \mu_n$ acts on S' with quotient S. To be precise, for any $\zeta \in G$, we define $\zeta^{\#}(\pi') = \zeta\pi'$. This defines a right-action by G on S'. Let X denote $\Pi_{S'/S}(\mathscr{A}'/S')$; then G acts on $X \to S$ and we have seen in the proof of theorem 4.2 that $\mathscr{A} = X^G$. This implies that $\mathscr{A}_k$ is the closed subscheme of fixed points under G in the restriction of $\mathscr{A}' \otimes_{D'} (D'/\pi D')$ to $k = D/\pi D$. Note that $D'/\pi D' = D[t]/(t^n - \pi, \pi) = k[t]/(t^n)$; hence for any k-algebra C we have:

$$\mathscr{A}_k(C) = X_k^G(C) = X_k(C)^G = \mathscr{A}'(C[t]/(t^n))^G.$$

According to this formula, in order to understand $\mathscr{A}_k$ we must know what

$C[t]/(t^n)$-valued points of $\mathscr{A}'$ are, and how G acts on them. Our aim in this section is to describe $\mathscr{A}_k$ as accurately as we can in terms of $\mathscr{A}'_k$ together with its G-action. The fact that $\mathscr{A}'(C[t]/(t^n))$ depends not only on $\mathscr{A}'_k$, but on its $(n-1)^{\text{st}}$ infinitesimal neighborhood probably makes it impossible to describe $\mathscr{A}_k$ itself. What we will do instead is to consider a natural filtration on $\mathscr{A}_k$, and describe only the successive quotients.

5.1. The filtration on X_k

For any k-algebra C, and for any i with $0 \leqslant i \leqslant n$ we define:

$$(F^i X_k)C = \ker(X_k(C) \xrightarrow{\sim} \mathscr{A}'(C[t]/(t^n)) \to \mathscr{A}'(C[t]/(t^i))).$$

This defines a filtration of X_k by subfunctors: $X_k = F^0 X_k \supset F^1 X_k \supset \cdots \supset F^n X_k = 0$. The functor $C \mapsto \mathscr{A}'(C[t]/(t^i))$ is represented by the scheme $\Pi_{(k[t]/(t^i))/k} \mathscr{A}'_{k[t]/(t^i)}$; hence the functors $F^i X_k$ are represented by closed subgroup schemes of X_k.

We will now investigate the successive quotients of the filtration. For i with $0 \leqslant i \leqslant n-1$ and C any k-algebra we define $(\mathrm{Gr}^i X_k)C = ((F^i X_k)C)/((F^{i+1} X_k)C)$. We have, for any C, that $(\mathrm{Gr}^0 X_k)C = \mathscr{A}'_k(C)$; hence $\mathrm{Gr}^0 X_k = \mathscr{A}'_k$. To determine the $\mathrm{Gr}^i(X_k)$ for $i > 0$, we choose parameters $T_1, \ldots, T_d$ for the formal group of $\mathscr{A}'$ over D'.

Let P be an element of $(F^i X_k)(C)$, with $0 < i < n$; then P corresponds to a morphism of rings $\phi\colon D'[\![T_1, \ldots, T_d]\!] \to C[t]/(t^n)$ such that the $\phi(T_j)$ are 0 modulo t^i. This means that we can write $\phi(T_j) = \Sigma_{l=i}^{n-1} a_{j,l} t^l$, with $a_{j,l} \in C$. Now consider the C-module $\mathrm{T}_{\mathscr{A}'_k,0} \otimes_k C$. Elements of this module correspond to morphisms of rings $\psi\colon D'[\![T_1, \ldots, T_d]\!] \to C[\varepsilon]/(\varepsilon^2)$ which have $\psi(T_j) \in (\varepsilon)$ for all j. It follows that we can associate such a ψ to P by setting $\psi(T_j) = a_{j,i}\varepsilon$. This gives us a map:

$$(F^i X_k)C \to (\mathrm{T}_{\mathscr{A}'_k,0} \otimes_k (m/m^2)^{\otimes i}) \otimes_k C\colon P \mapsto \psi \otimes t^i, \tag{5.1.1}$$

where m is the maximal ideal of D'. Both source and target of this map are groups; the group structure on the right hand side is induced from the k-vector space structure on $\mathrm{T}_{\mathscr{A}'_k,0} \otimes_k (m/m^2)^{\otimes i}$. Since the group law on $D[\![T_1, \ldots, T_d]\!]$ is of the form:

$$(a_1, \ldots, a_d) + (b_1, \ldots, b_d) = (a_1 + b_1, \ldots, a_d + b_d) + \text{higher terms},$$

the map 5.1.1 is actually a morphism of groups. It is clear that it is surjective, and that its kernel is $(F^{i+1} X_k)C$. This means that we have an isomorphism of group schemes over k:

$$\mathrm{Gr}^i X_k \xrightarrow{\sim} \mathrm{T}_{\mathscr{A}'_k,0} \otimes_k (m/m^2)^{\otimes i}. \tag{5.1.2}$$

It is easy to check that this isomorphism does not depend on the choice of the parameters $T_1, \ldots, T_d$ and the uniformizers π and π'.

5.2. Taking the G-invariants

Since $\mathscr{A}_k = X_k^G$, the $\mathrm{F}^i X_k$ induce a filtration $\mathrm{F}^i \mathscr{A}_k = (\mathrm{F}^i X_k)^G$; we denote its successive quotients by $\mathrm{Gr}^i \mathscr{A}_k$. Since the group schemes $\mathrm{F}^i X_k$ are unipotent for $i > 0$, and $\# G$ is invertible in k, the short exact sequences:

$$0 \to \mathrm{F}^{i+1} X_k \to \mathrm{F}^i X_k \to \mathrm{Gr}^i X_k \to 0$$

remain exact after taking the G-invariants. This means that $\mathrm{Gr}^0 \mathscr{A}_k$ is simply $(\mathscr{A}'_k)^G$. For $i > 0$ it follows that $\mathrm{Gr}^i \mathscr{A}_k = (\mathrm{T}_{\mathscr{A}'_k,0} \otimes_k (m/m^2)^{\otimes i})^G$. In order to make the isomorphism 5.1.2 G-equivariant, we have to let G act on the target as follows. The right-action on $\mathrm{T}_{\mathscr{A}'_k,0}$ is induced by the action by G on $\mathscr{A}'_k$ from the right which is usually called the inertia action (as defined in the proof of theorem 4.2). The right-action of $\zeta \in G$ on $(m/m^2)^{\otimes i}$ is by multiplication by ζ^{-i} (note the sign). This means that for i with $0 < i < n$, $\mathrm{Gr}^i \mathscr{A}_k = \mathrm{T}_{\mathscr{A}'_k,0}[i] \otimes_k (m/m^2)^{\otimes i}$, where $\mathrm{T}_{\mathscr{A}'_k,0}[i]$ denotes the k-subspace of $\mathrm{T}_{\mathscr{A}'_k,0}$ on which the action of all $\zeta \in G = \mu_n$ is by multiplication by ζ^i. We summarize these results in the following theorem.

5.3. THEOREM. *Let D be a discrete valuation ring with field of fractions K and residue field k, and let n be a positive integer. We suppose that n is prime to the characteristic of k, and that D contains the group μ_n of nth roots of unity. Let K'/K be a totally ramified Galois extension of degree n, and let D' be the integral closure of D in K'. Let m be the maximal ideal of D', and let $G = \mathrm{Gal}(K'/K)$. We identify G with μ_n via its action on m/m^2: $\zeta \in \mu_n$ acts on m/m^2 by multiplication by ζ. Let A be an abelian variety over K; let $\mathscr{A}$ and $\mathscr{A}'$ be its Néron models over* Spec(D) *and* Spec(D'), *respectively. By functoriality of the Néron model, μ_n acts from the right on the situation $\mathscr{A}'/\mathrm{Spec}(D')$, and induces a right action on $\mathscr{A}'_k$. The constructions above define a filtration by closed subgroup schemes on $\mathscr{A}_k$:*

$$\mathscr{A}_k = \mathrm{F}^0 \mathscr{A}_k \supset \mathrm{F}^1 \mathscr{A}_k \supset \cdots \supset \mathrm{F}^n \mathscr{A}_k = 0.$$

This filtration is functorial with respect to morphism of abelian varieties over K. For $0 \leqslant i < n$ let $\mathrm{Gr}^i \mathscr{A}_k = \mathrm{F}^i \mathscr{A}_k / \mathrm{F}^{i+1} \mathscr{A}_k$. Then $\mathrm{Gr}^0 \mathscr{A}_k = (\mathscr{A}'_k)^{\mu_n}$, and for $i > 0$ there are natural isomorphisms:

$$\mathrm{Gr}^i \mathscr{A}_k \xrightarrow{\sim} \mathrm{T}_{\mathscr{A}'_k,0}[i] \otimes_k (m/m^2)^{\otimes i},$$

where $\mathrm{T}_{\mathscr{A}'_k,0}[i]$ denotes the k-subspace of $\mathrm{T}_{\mathscr{A}'_k,0}$ on which the action of all $\zeta \in \mu_n$ is by multiplication by ζ^i.

5.4. REMARKS. 1. Let Φ and Φ' denote the groups of connected components of $\mathscr{A}_{\bar{k}}$ and $\mathscr{A}'_{\bar{k}}$, respectively. Since $\mathrm{F}^1\mathscr{A}_{\bar{k}}$ is connected, and $\mathrm{Gr}^0\mathscr{A}_k = \mathscr{A}'_k$, Φ is the group of connected components of $(\mathscr{A}'_{\bar{k}})^{\mu_n}$. This induces an exact sequence:

$$0 \to \Phi_0 \to \Phi \to \Phi_1 \to 0,$$

where Φ_0 is the set of connected components of $(\mathscr{A}'^{0}_{\bar{k}})^{\mu_n}$ and where Φ_1 denotes the subgroup of Φ'^{μ_n} consisting of those components that contain at least one fixed point.

2. Suppose that D has residue characteristic $p > 0$. Let $T_1, \ldots, T_d$ be parameters for the formal group of $\mathscr{A}'$. Then multiplication by p is given by d power series of the form $pf_i(T_1, \ldots, T_d) + g_i(T_1^p, \ldots, T_d^p)$. It follows that $\mathrm{F}^1\mathscr{A}_k$ is annihilated by p^a, as soon as $p^a \geqslant n$. In particular, if $\mathscr{A}'$ is semi-abelian, then the unipotent part of $\mathscr{A}_k$ is annihilated by p^a, as soon as $p^a \geqslant n$.

3. Suppose that $k \supset \mathbf{F}_p$, and that $ip > n$. Then the power series for the formal logarithm and exponential map induce an isomorphism:

$$\mathrm{F}^i\mathscr{A}_k \xrightarrow{\sim} \mathrm{F}^i\Bigg(\prod_{(D'\otimes_D k)/k} \mathrm{T}_{\mathscr{A}'_{D'\otimes_D k}}(0)\Bigg)^{\mu_n}$$

If $k \supset \mathbf{Q}$, then this formula is valid for all $i > 0$.

4. It is probably useful to note that *every* extension D' of D, not necessarily tamely ramified, induces a filtration on $\mathscr{A}$. Namely, for every D-algebra C, consider the morphism $\mathscr{A}(C) \to \mathscr{A}'(C \otimes_D D')$, and for $i \geqslant 0$ let

$$\mathrm{F}^i\mathscr{A}(C) = \ker(\mathscr{A}(C) \to \mathscr{A}(C \otimes_D D'/m^i)),$$

where m denotes the maximal ideal of D'. The jumps of this filtration occur exactly at the i for which there exists a $P \in \mathscr{A}(D^{\mathrm{sh}})$ whose image in $\mathscr{A}'(D^{\mathrm{sh}'})$ is 0 mod m^i, but is not 0 mod m^{i+1}. For tamely ramified extensions, this gives the same filtration on $\mathscr{A}_k$ as discussed above. Let $\tilde{D}$ denote the minimal extension of D over which A acquires semi-stable reduction, let $\tilde{n} = [\tilde{D}: D]$, and let $\tilde{\mathrm{F}}$ denote the filtration induced by $\tilde{D}$. Then for every D' containing $\tilde{D}$ we have $\mathrm{F}^i\mathscr{A} = \tilde{\mathrm{F}}^{\lceil i\tilde{n}/n \rceil}$, where for any real number x, $\lceil x \rceil$ denotes the smallest integer j with $j \geqslant x$.

5. If D is strictly henselian and of residue characteristic p, then we have a whole tower of tamely ramified extensions, with Galois group $\varprojlim \mu_n$, where n ranges through the positive integers that are prime to p. This induces a filtration F^i on $\mathscr{A}_k$, with $i \in \mathbf{Z}_{(p)} \cap [0, 1]$, and $\mathrm{F}^1\mathscr{A}_k = 0$. Namely, if $i = a/n$, with n not divisible by p, then $\mathrm{F}^i\mathscr{A}_k$ is the $\mathrm{F}^a\mathscr{A}_k$ induced by the Néron model of A over the extension of degree n of K as above. It follows from our description of $\mathscr{A}_k$ that this does not depend on the choice of a and n. It would be interesting to know

where the jumps in this filtration occur. If A acquires semi-stable reduction over a tamely ramified extension D' of degree n of D, then for $i \in \mathbf{Z}_{(p)} \cap [0, 1]$ we have $F^i\mathscr{A}_k = F^{[ni]}\mathscr{A}_k$ (where the latter filtration is the one induced by D'), and the jumps occur at indices $x \in (1/n)\mathbf{Z}/\mathbf{Z}$.

In general, we do not even know if the jumps occur at rational numbers. If A is a jacobian we can say more. Suppose that $A = \mathrm{Pic}^0_{X_K/K}$, where X_K is a smooth, geometrically irreducible curve over K. Let X be a regular model over S of X_K. We assume in addition that $k = \bar{k}$ and that the morphism $\mathrm{Pic}^0_{X/S} \to \mathscr{A}^0$ is an isomorphism (this happens for example if $X_K(K) \neq \varnothing$, see [1] 9.5/4). After blowing up we may assume that X_k is a divisor with normal crossings (locally for the Zariski topology). Let l be the least common multiple of the multiplicities of the irreducible components of X_k. Then we claim that the filtration with indices in $\mathbf{Z}_{(pl)} \cap [0, 1]$ as above depends only on the following data: the multiplicities and genera of the irreducible components of X_k, and their intersection graph. In particular, it does not depend on p. In order to prove this, one has to show that these data suffice to compute, for n prime to pl and D'/D tamely ramified of degree n, the action of μ_n on $\mathrm{T}_{\mathscr{A}'_k/k}(0)$. The key point here is that one can compute the same combinatorial data for a similar model (i.e., regular, normal crossings) X'/S' of $X_{K'}$. Then one computes the character of the representation of μ_n on the formal difference $\mathrm{H}^0(X'_k, \mathcal{O}_{X'_k}) - \mathrm{H}^1(X'_k, \mathcal{O}_{X'_k})$. For elliptic curves we give the index where the (unique) jump in the filtration occurs in the following table:

Type	I_0	I_ν	II	III	IV	I_0^*	I_ν^*	IV^*	III^*	II^*
i	0	0	1/6	1/4	1/3	1/2	1/2	2/3	3/4	5/6

For the meaning of the entries of the first row (Kodaira symbols) we refer to [7].

6. The map $\mathscr{A}_{D'} \to \mathscr{A}'$ is given as follows. For every D'-algebra C, we have:

$$\mathscr{A}_{D'}(C) = \mathscr{A}(C) = (\mathscr{A}'(C \otimes_D D'))^{\mu_n} \subset \mathscr{A}'(C \otimes_D D') \to \mathscr{A}'(C).$$

In particular, the kernel of the natural map $\mathscr{A}_k \to \mathscr{A}'_k$ is $F^1\mathscr{A}_k$, and the image is $(\mathscr{A}'_k)^{\mu_n}$. Let $\mathscr{K}$ be the kernel of $\mathscr{A}_{D'} \to \mathscr{A}'$, so that we have an exact sequence:

$$0 \to \mathscr{K} \to \mathscr{A}_{D'} \to \mathscr{A}'.$$

Let $\omega_{\mathscr{K}/D'} = 0^*\Omega^1_{\mathscr{K}/D'}$. Then we have an exact sequence

$$0 \to \omega_{\mathscr{A}'/D'} \to \omega_{\mathscr{A}_{D'}/D'} \to \omega_{\mathscr{K}/D'} \to 0.$$

By choosing parameters T_i for the formal group of $\mathscr{A}'$ on which each $\zeta \in \mu_n$ acts by $\zeta^{\#}(T_i) = \zeta^{a_i} T_i$, for some a_i with $0 \leqslant a_i < n$, it can be seen that $\omega_{\mathscr{K}/D'} \cong \bigoplus D'/(\pi')^{a_i} D'$.

6. Exactness and specialization

Let D be a discrete valuation ring of with field of fractions K, and let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian varieties over K. Let e be the absolute ramification index of D: e is the valuation of $p \in D$. Passing to Néron models gives a sequence $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ of group schemes over D. Suppose that $e < p - 1$. Then it is known ([1] 7.5 Thm. 4) that:

1. If $\mathscr{A}$ is semi-abelian, then $\mathscr{A} \to \mathscr{B}$ is a closed immersion.
2. If $\mathscr{B}$ is semi-abelian, then $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C}$ is exact.
3. If $\mathscr{B}$ is abelian, then $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ is an exact sequence of abelian schemes over D.

In this section we will generalize this type of results to the case where A, B and C do not necessarily have semi-stable reduction over D. It turns out that assertions 1 and 2 are still true, provided that p is sufficiently large with respect to e and the dimensions of A, B and C. After proving this result, we give an example where $p = 5$, $e = 1$, A is an elliptic curve, $A \to B$ is a closed immersion, but $\mathscr{A} \to \mathscr{B}$ is not injective.

6.1. THEOREM. *Consider the following statements:*

1. $e < (p-1)/n$, *for all* $n > 0$ *with* $\phi(n) \leqslant 2\dim(A)$.
2. *A acquires semi-stable reduction over a tamely ramified extension D' of D with* $e' < p - 1$.
3. $\mathscr{A} \to \mathscr{B}$ *is a closed immersion.*
4. $e < (p-1)/n$, *for all* $n > 0$ *with* $\phi(n) \leqslant 2\dim(B)$.
5. *B acquires semi-stable reduction over a tamely ramified extension D' of D with* $e' < p - 1$.
6. $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C}$ *is exact, and* $\mathscr{B} \to \mathscr{C}$ *is smooth.*

Then we have the following implications: $1 \Rightarrow 2 \Rightarrow 3$ *and* $4 \Rightarrow 5 \Rightarrow 6$.

Proof. First of all, we may suppose that D is complete, with separably closed residue field. In that case the theory of the preceding section applies. The implications $1 \Rightarrow 2$ and $4 \Rightarrow 5$ follow from the fact that for n the order of an automorphism of a d-dimensional semi-abelian variety one has $\phi(n) \leqslant 2d$. Now assume that 2 holds. Then by the theorem from [1] cited above, $\mathscr{A}' \to \mathscr{B}'$ is a closed immersion. By proposition 7.6/2 of [1], the induced morphism of schemes

$$\prod_{D'/D} \mathscr{A}' \to \prod_{D'/D} \mathscr{B}'$$

is a closed immersion. Since $\mathscr{A}$ and $\mathscr{B}$ are closed subschemes of these, the morphism $\mathscr{A} \to \mathscr{B}$ is a closed immersion too.

Now assume that 5 holds. By the theorem of [1] cited above, the sequence $0 \to \mathscr{A}' \to \mathscr{B}' \to \mathscr{C}'$ is exact. Then the sequence $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C}$ is exact since it is constructed from the sequence above by composing two left-exact functors (namely: push-forward of sheaves, and taking the subsheaf of invariants under a group action). Since passing from $\mathscr{A}$ to $\mathrm{T}_{\mathscr{A}_k/k}(0)$ is a left exact functor (it is taking the kernel of $\mathscr{A}(k[\varepsilon]) \to \mathscr{A}(k)$), we find that

$$0 \to \mathrm{T}_{\mathscr{A}_k/k}(0) \to \mathrm{T}_{\mathscr{B}_k/k}(0) \to \mathrm{T}_{\mathscr{C}_k/k}(0)$$

is exact. By dimension considerations, it follows that $\mathrm{T}_{\mathscr{B}_k/k}(0) \to \mathrm{T}_{\mathscr{C}_k/k}(0)$ is surjective. From [4] IV 17.11.1 we can conclude that $\mathscr{B} \to \mathscr{C}$ is smooth. □

6.2. The connection with finite flat group schemes

Let D, K and e be as above. Suppose that we have a closed immersion $A \to B$ of abelian varieties over K. Then there exists an abelian subvariety C of B such that the induced morphism $A \times C \to B$ is an isogeny, and we get an exact sequence $0 \to G_K \to A \times C \to B \to 0$, with G_K a finite group scheme over K. Let $G_{1,K}$ and $G_{2,K}$ be the images of G under the projections to A and C, respectively. After replacing C by $C/(C \cap G_K)$ we may assume that the projections from $A \times C$ to A and C induce isomorphisms $G_K \to G_{1,K}$ and $G_K \to G_{2,K}$.

Now suppose that B has semi-stable reduction over a tamely ramified extension K'/K of degree n. Then $\mathscr{A}' \times_{D'} \mathscr{C}' \to \mathscr{B}'$ is flat, by the following argument. By the fibre-wise criterion for flatness ([4] IV 11.3.11), it suffices to show that $\mathscr{A}'_k \times_k \mathscr{C}'_k \to \mathscr{B}'_k$ is flat. The morphism $(\mathscr{A}'_k \times_k \mathscr{C}'_k)^0 \to \mathscr{B}'^0_k$ is surjective (consider l-torsion, for some $l \neq p$), hence the open part where it is flat ([4] IV 11.1.1) is not empty. Since we deal with a morphism of group schemes, the result follows.

We have now an exact sequence $0 \to G \to \mathscr{A}' \times_{D'} \mathscr{C}' \to \mathscr{B}'$, where G is the closure of $G_{K'}$. The image of $\mathscr{A}' \times_{D'} \mathscr{C}' \to \mathscr{B}'$, which is an open subscheme of $\mathscr{B}'$, represents $(\mathscr{A}' \times_{D'} \mathscr{C}')/G$. Let G_1 and G_2 denote the closures of $G_{K'}$ in $\mathscr{A}'$ and $\mathscr{C}'$, respectively, and let $\mathscr{K}'$ be the kernel of $\mathscr{A}' \to \mathscr{B}'$. From the diagram:

$$\begin{array}{ccccccc} & & G & \to & G_2 & & \\ & & \downarrow & & \downarrow & & \\ 0 \to \mathscr{A}' & \to & \mathscr{A}' \times_{D'} \mathscr{C}' & \to & \mathscr{C}' & \to & 0 \\ & & \downarrow & & & & \\ & & \mathscr{B}' & & & & \end{array}$$

it follows that $\mathscr{K}' = G \cap \mathscr{A}' = \ker(G \to G_2)$. Since $G_{K'} \to G_{2,K'}$ is an isomorphism, we can replace G by its finite part G^{f}, $G_i (i = 1, 2)$ by the closure of $G^{\mathrm{f}}_{K'}$ in G_i, and still have that $\mathscr{K}' = \ker(G \to G_2)$. Now G, G_1 and G_2 are finite flat groups schemes over D', all three extending $G_{K'}$. In the terminology of Raynaud's article [6], we have that G dominates G_1 and G_2, and that $G \hookrightarrow G_1 \times_{D'} G_2$. Therefore G is the maximum of G_1 and G_2. Let $G_- = G_1 \times_{D'} G_2/G$; then G_- is the minimum of G_1 and G_2, and it is not hard to see that $\mathscr{K}' = \ker(G_1 \to G_-)$.

Anyhow, given G_1 and G_2, and the isomorphism between their generic fibres, we know how to describe $\mathscr{K}'$. Also, note that $\mathscr{K}'$ is p-power torsion. Finally, we get an exact sequence over D:

$$0 \to \mathscr{K} \to \mathscr{A} \to \mathscr{B}, \text{ where } \mathscr{K} = \left(\prod_{D'/D} \mathscr{K}'\right)^{\mu_n}$$

Let us discuss one easy special case: suppose that G^0_1 is of multiplicative type (i.e., it is a direct sum of copies of $\mu_{p,D'}$). This happens if the abelian variety part of $\mathscr{A}'_k$ is ordinary. Then $G^0_1 \to G_-$ is a closed immersion, since G^0_1 is minimal as a group scheme over D'. Hence $\mathscr{K}'^0 = 0$, implying that $\mathscr{K}'$ is unramified over D'. Therefore $\mathscr{K}' \cong \operatorname{Spec}(D') \coprod_i \operatorname{Spec}(D'/\pi'^{n_i} D')$, for some set of $n_i > 0$. Taking the quotient gives an exact sequence:

$$0 \to \mathscr{K}' \to G_1/G^0_1 \to G_-/G^0_1.$$

From this it is not hard to see that for all i: $n_i \leqslant e'/(p-1)$ (one uses that G_1/G^0_1 is étale, and that the statement does not change under base change so that one may assume that G_- is multiplicative). Now, if $\Pi_{D'/D} \mathscr{K}' \neq 0$, then $\mathscr{K}'(D'/\pi'^n D') = (\Pi_{D'/D} \mathscr{K}')(k) \neq 0$ (since $\Pi_{D'/D} \mathscr{K}'$ is unramified over D). But then there exists i such that $n_i \geqslant n = e'/e$, which implies that $e \geqslant p - 1$. So we have the following result.

6.3. PROPOSITION. *If in the situation above, $e < p - 1$ and $\mathscr{A}'_k[p]^0$ is multiplicative, then $\mathscr{A} \to \mathscr{B}$ is a monomorphism (i.e., its kernel is* 0).

In this case, we do not know if $\mathscr{A} \to \mathscr{B}$ is a closed immersion.

6.4. AN EXAMPLE. In this example, we will have $\pi = p = 5$, and $n = 6$. Let D be the ring of Witt vectors of $\bar{\mathbf{F}}_p$, and let $D' = D[\pi']$, with $\pi'^n = \pi$; hence $e = 1$ and $e' = 6$. Let $\mathscr{E}$ be an elliptic curve over D with j-invariant equal to 0. Let μ_6 be the group of 6th roots of unity in D. For every $\zeta \in \mu_6$, let $g(\zeta)$ be the automorphism of $S' = \operatorname{Spec}(D')$ given by: $g(\zeta)^{\#}(\pi') = \zeta\pi'$. Then $\zeta \mapsto g(\zeta)$ gives an isomorphism $\mu_6 \xrightarrow{\sim} \operatorname{Gal}(K'/K)$. We define an action by μ_6 on $\mathscr{E}/D$ as follows: to $\zeta \in \mu_6$ we associate the element $a(\zeta) \in \operatorname{Aut}_D(\mathscr{E})$ which acts by multiplication by ζ on the cotangent space at 0 of $\mathscr{E}$. Let $E = \mathscr{E}_K$.

We let $\zeta \in \mu_6$ act on $E_{K'} = E \times_{\operatorname{Spec}(K)} \operatorname{Spec}(K')$ by the automorphism $(a(\zeta)^{-1}, g(\zeta))$, and we let A be the quotient. Then A is an elliptic curve over K (a twist of E), having reduction type II^*. To define B, we need to know more about $\mathscr{E}[p]$. It is well known (see for example [6] 3.4.7) that $\mathscr{E}[p]$ can be described by the equation $X^{p^2} - pX = 0$ in $D[X]$. In $D'[X]$, this equation can be factored: $X\prod(X^{p-1} - \lambda\pi')$, where λ ranges through μ_6. For each $\lambda \in \mu_6$, we get a subgroup scheme G_λ of rank p of $\mathscr{E}[p]$ described by the equation $X^p - \lambda\pi' X = 0$. We let $E_\lambda = E_{K'}/G_{\lambda,K'}$, and $B_{K'} = \prod E_\lambda$, where the product is the fibered product over K', ranging over the $\lambda \in \mu_6$. Now we want to descend $B_{K'}$ to K, in such a way that the "diagonal" morphism $A_{K'} = E_{K'} \to \prod E_\lambda$ descends too.

Let us first consider the action of the $a(\zeta)$ on the G_λ, where ζ and λ are elements of μ_6. From the theory in [6], it follows that $a(\zeta)^{\#}$ acts on $D[X]/(X^{p^2} - pX)$ by $X \mapsto \zeta^a X$, for some $a \in \mathbf{Z}/6\mathbf{Z}$. Since dX generates the cotangent space at 0 of $\mathscr{E}$, we must have that $a = -1$. It follows that the inverse image $a(\zeta)^{-1}G_\lambda$ is described by the equation $(\zeta X)^p - \pi'\lambda(\zeta X) = 0$. Hence $a(\zeta)^{-1}G_\lambda = G_{\lambda\zeta^2}$, and $a(\zeta)G_\lambda = G_{\lambda\zeta^{-2}}$. This gives us a diagram:

$$\begin{array}{ccccccccc}
0 \to & G_\lambda & \to & E_{K'} & \xrightarrow{\phi_\lambda} & E_\lambda & \to 0 \\
& \downarrow & & \downarrow{\scriptstyle a(\zeta)} & & \downarrow{\scriptstyle a(\zeta)} & \\
0 \to & G_{\lambda\zeta^{-2}} & \to & E_{K'} & \xrightarrow{\phi_{\lambda\zeta^{-2}}} & E_{\lambda\zeta^{-2}} & \to 0
\end{array}$$

where the vertical arrows are isomorphisms. In particular, we have the formula $a(\zeta)\phi_\lambda = \phi_{\lambda\zeta^{-2}}a(\zeta)$.

Then we need to know the action of the $g(\zeta)$ on the G_λ. Since G_λ is described by the equation $X^p - \lambda\pi' X = 0$, $g(\zeta)^{-1}G_\lambda$ is given by $X^p - \lambda\zeta\pi' X = 0$. From this we see that $g(\zeta)G_\lambda = G_{\lambda\zeta^{-1}}$, and we get a diagram as above, and the formula $g(\zeta)\phi_\lambda = \phi_{\lambda\zeta^{-1}}g(\zeta)$.

Finally, we can give the descent data for $B_{K'}$. For $\zeta \in \mu_6$, we let $c(\zeta)$ be the automorphism of $\prod E_\lambda$ that sends the factor E_λ to the factor $E_{\lambda\zeta^{-1}}$ via the automorphism $a(\zeta^{-1})g(\zeta)$. It follows from the formulas above that this defines an action by μ_6 on $B_{K'}$, compatible with its action on $\mathrm{Spec}(K')$, such that the "diagonal" morphism $(\phi_\lambda)_\lambda\colon E_{K'} \to B_{K'}$ descends, say to a closed immersion $\phi\colon A \to B$ over K.

Let $\mathscr{K}$ be the kernel of $\phi\colon \mathscr{A} \to \mathscr{B}$. We have seen that $\mathscr{K} = (\prod_{D'/D}\mathscr{K}')^{\mu_n}$, where $0 \to \mathscr{K}' \to \mathscr{A}' \to \mathscr{B}'$. Now $\mathscr{A}' = \mathscr{E}_{D'}$ and $\mathscr{B}' = \prod \mathscr{E}_{D'}|G_\lambda$, from which it follows that $\mathscr{K}' = \cap\, G_\lambda = \mathrm{Spec}(D'[X]/(X^p, \pi' X))$. The next thing we do is to compute $\prod_{D'/D}\mathscr{K}'$. Let C be any D-algebra; then

$$\Bigl(\prod_{D'/D}\mathscr{K}'\Bigr)(C) = \mathscr{K}'(C \otimes_D D') = \mathrm{Hom}_{D'}(D'[X]/(X^p, \pi' X),\ C \otimes_D D')$$

$$= \{a \in C \otimes_D D' \mid a^p = 0, \pi' a = 0\}.$$

Since $C \otimes_D D' = \bigoplus_0^{n-1} C\pi'^i$, we can write every $a \in C \otimes_D D'$ uniquely as $a = \Sigma_0^{n-1} a_i \pi'^i$. The condition $\pi' a = 0$ is then equivalent to: $a_i = 0$ for $0 \leqslant i < n-1$, and $pa_{n-1} = 0$. The other condition, $a^p = 0$, is then a consequence of $\pi' a = 0$. So we see that $\Pi_{D'/D} \mathscr{K}'$ is represented by $\mathrm{Spec}(D[Y]/(pY))$. Now let $X \mapsto a_5 \pi'^5$ be in $(\Pi_{D'/D} \mathscr{K}')(C)$, and let $\zeta \in \mu_p$. Then, since $a(\zeta^{-1})g(\zeta)$ sends X to $\zeta^{-1}X$ and sends π'^5 to $\zeta^5 \pi'^5$, it follows that the action by μ_p on $\Pi_{D'/D} \mathscr{K}'$ is trivial. The final conclusion is that $\mathscr{K} = \mathrm{Spec}(D[Y]/(pY)) \neq 0$.

6.5. REMARK. In this example, A has dimension 1 and B has dimension 6. With a little bit more work, one can make an example where A has dimension 1 and B has dimension 2. Also, 5 is the largest prime such that an example as above with $e = 1$ exists: by theorem 6.1 and proposition 6.3, $\mathscr{A} \to \mathscr{B}$ will be a closed immersion if $p > 7$, and is injective if $p = 7$.

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