1 Introduction

Given an open immersion $U \subset S$ and a smooth algebraic space over U, we can sometimes get informations about the Néron model of this space (existence, nonexistence, explicit construction if it exists) only after base-change to some finite extension S'/S. Examples include smooth curves acquiring nodal reduction over S', Jacobians of smooth curves acquiring nodal reduction, and to some extent abelian varieties acquiring semi-abelian reduction [**?**] *add citations.* Therefore, it is interesting to have tools to turn this into information on the Néron model over S. In [2], one studies the base-change behavior of Néron models of abelian varieties over discrete valuation rings along finite tamely ramified extensions. We are interested in what happens when the base is higher-dimensional. The first complication appearing in this setting is that existence of Néron models is no longer guaranteed. We address this problem by proving

Theorem 1.1 (theorem 2.5)**.** *Let $S' \longrightarrow S$ be a finite, locally free, tamely ramified map between regular schemes and U the complement of a strict normal crossings divisor $D \longrightarrow S$. Suppose $U' := U \times_S S'$ is étale over U.*

Let X_U be a smooth U-algebraic space, such that $X_U \times_U U'$ has a S'-Néron model X'. Then the schematic closure of X_U in $\prod_{S'/S} X'$ is a S-Néron model for X_U (where X_U maps to $\prod_{S'/S} X'$ as in example 1.6).

Moreover, if $S' \longrightarrow S$ is Galois for the right-action of a finite group G, then G acts on X' via its action on S': it follows that G acts on $\prod_{S'/S} X'$ as in remark 1.7.1. In that case, the Néron model of X_U is $(\prod_{S'/S} X')^G$.

Then, when the Néron models N and N' exist, we introduce filtrations of certain strata of N (quite similar to the filtration of the closed fiber described in [2]), and explicit the successive quotients of these filtrations in terms of N'. Namely, after reducing to its hypotheses by working étale-locally on the base (see lemma 2.12), we prove the following result:

Theorem 1.2 (theorem 2.13)**.** *Let $S = \operatorname{Spec} R$ be a regular affine scheme, $f_1, ..., f_r$ regular parameters of R, $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where the n_j are invertible on S, and let $S' = \operatorname{Spec} R'$. Suppose R contains the group μ_{n_j} of n_j-th roots of unity for all j. Let U be the locus in S where all f_j are invertible, and Z the locus where all f_j vanish. Let X_U be a proper smooth U-group algebraic space with a Néron model N' over S'. Then X_U has a Néron model N over S, and we have sub-Z-group spaces $(F^d N_Z)_{d \in \mathbb{N}}$ of N_Z (see definitions 2.7 and 2.10) such that:*

- *For all $d \in \mathbb{N}$, $F^d N_Z \subset F^{d+1} N_Z$.*
- *$F^0 N_Z = N_Z$.*

- *If* $d > \prod_{j=1}^{r}(n_j - 1)$ *then* $F^d N_Z = 0$.

- $F^0 N_Z / F^1 N_Z$ *is the subspace of* N'_Z *invariant under the action of* $G = \prod_{j=1}^{r} \mu_j$, *where* $(\xi_j)_{1 \le j \le r}$ *acts by multiplying* T_j *by* ξ_j.

- *If* $d > 0$, $F^d N_Z / F^{d+1} N_Z$ *is isomorphic (via the isomorphism of proposition 2.8) to the disjoint union* $\coprod_{\mathbf{k}} \mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ *where* $\mathbf{k}$ *ranges through all* r*-uples of integers* $(k_1, ..., k_r)$ *with* $\sum_{j=1}^{r} k_j = d$ *and* $k_j < n_j$ *for all* j, *and* $\mathrm{Lie}_{N'_Z/Z}[\mathbf{k}]$ *is the subspace of* $\mathrm{Lie}_{N'_Z/Z}$ *where all* $(\xi_j)_{1 \le j \le r}$ *in* G *act by multiplication by* $\prod_{j=1}^{r} \xi_j^{k_j}$.

1.1 Néron models

Here we give our definition of Néron models and mention some elementary properties.

Definition 1.3. Let S be a scheme, $U \subset S$ a schematically dense open subscheme and X_U/U an U-algebraic space. We call *S-Néron model of* X_U (or just *Néron model of* X_U if there is no ambiguity) a model N/S of X_U such that for every smooth S-algebraic space Y, the canonical map

$$\mathrm{Hom}_S(Y, N) \longrightarrow \mathrm{Hom}_U(Y_U, X_U)$$

is bijective.

Remark 1.3.1.

- If X/S is a separated model of X_U/U, then for any smooth Y/S, the map $\mathrm{Hom}_S(Y, X) \longrightarrow \mathrm{Hom}_U(Y_U, X_U)$ is automatically injective.

- The universal property implies that a Néron model, if it exists, is unique up to a unique isomorphism.

- One can replace "for every smooth S-algebraic space Y" by "for every smooth S-scheme Y" in the definition above.

- Let $S' \longrightarrow S$ be a smooth morphism of schemes with S locally Noetherian and reduced and let $U \subset S$ be a schematically dense open. It follows $U' := U \times_S S'$ is also schematically dense and open in S'. Consider an algebraic space X/S. If X is the S-Néron model of X_U, then $X_{S'}$ is the S'-Néron model of $X_{U'}$. If $S' \longrightarrow S$ is surjective, the converse also holds.

- In general, given a morphism of schemes $S' \longrightarrow S$ with $U \times_S S'$ schematically dense in S' and a U-algebraic space X_U, if X_U has a S-Néron model N and $X_{U'}$ has a S'-Néron model N', then the universal property just gives a *morphism of base change* $N \times_S S' \longrightarrow N'$.

- X is the Néron model of X_U if and only if for all $s \in S$, $X \times_S \operatorname{Spec}(\mathcal{O}_{S,s}^{sh})$ is the Néron model of its restriction to U, where we note R^{sh} a strict henselization of a ring R.

[?] *add references to make this into an article, delete and refer to previous chapters for the thesis*

1.2 Weil restriction

As in [2], we introduce the Weil restriction functor and give some well-known representability properties.

Definition 1.4. Let S'/S be a morphism of schemes, and X' a contravariant functor from (Sch/S') to (Sets). We call *Weil restriction* of X'/S' to S, and we note $\prod\limits_{S'/S} X'$, the contravariant functor from (Sch/S) to (Sets) sending $T \longrightarrow S$ to $X'(T \times_S S')$. If $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ are affine, we will sometimes write $\prod\limits_{R'/R} X$.

Remark 1.4.1. Given a Grothendieck topology on (Sch/S), inducing one on (Sch/S'), the Weil restriction of a presheaf $X' : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Sets}$ to S is just its pushforward to (Sch/S).

Proposition 1.5. *If $S' \longrightarrow S$ is a flat and proper morphism of schemes, and X' is a quasi-projective S-algebraic space with X'/S factoring through S', then $\prod\limits_{S'/S} X'$ is (representable by) an algebraic space. If in addition X' is a scheme, then $\prod\limits_{S'/S} X'$ is a scheme.*

Proof. Applying the remark above to the étale topology, we reduce to the case where X is a scheme. This is then done in [3], 4.c. □

Example 1.6. Let $S' \longrightarrow S$ be a morphism of schemes and Y/S be an S-algebraic space. Let Y' be the S'-space $Y \times_S S'$. Then for all T/S we have $\mathrm{Hom}_S(T, \prod\limits_{S'/S} Y') = \mathrm{Hom}_{S'}(T \times_S S', Y') = \mathrm{Hom}_S(T \times_S S', Y)$. In particular, there is a natural map $Y \longrightarrow \prod\limits_{S'/S} Y'$.

Proposition 1.7. *If $S' \longrightarrow S$ is a flat and finite morphism of schemes, and X' is a smooth quasi-projective S'-algebraic space, then $\prod\limits_{S'/S} X'$ is smooth over S.*

Proof. The formation of $\prod\limits_{S'/S} X'$ is étale-local on both X' and S, so we can assume $X' = \operatorname{Spec} A'$ and $S = \operatorname{Spec} R$ are affine schemes. Then $S' = \operatorname{Spec} R'$

is affine, with R'/R finite and flat, and $\prod_{S'/S} X'$ is a scheme by proposition 1.5. By hypothesis, $R' \longrightarrow A'$ is formally smooth and locally of finite presentation. But then $\prod_{S'/S} X'/S$ is also formally smooth, and it follows from [4], proposition 8.14.2, that it is locally of finite presentation aswell. $\square$

Remark 1.7.1. Suppose given equivariant right-actions of a group G on a morphism of schemes $S' \longrightarrow S$ and on a morphism from an algebraic space X' to S'. Suppose moreover that G acts trivially on S. Then $\prod_{S'/S} X' \longrightarrow S$ carries a natural G-action, defined as follows: for any S-scheme T, define $T' := T \times_S S'$, every $g \in G$ induces an automorphism $\rho_{X'}(g)$ of X' and an automorphism $\rho_{T'}(g)$ of T' (obtained by extending the automorphism on S' by the identity on T). The action takes $f \in \mathrm{Hom}(T, \prod_{S'/S} X') = \mathrm{Hom}(T', X')$ to

$$f.g = \rho_{X'}(g) \circ f \circ \rho_T(g)^{-1}$$

When $\prod_{S'/S} X'$ is representable, this action is equivariant.

1.3 Fixed points

We will show later that under certain hypotheses, we can construct a Néron model by considering the Weil restriction of a Néron model over a bigger base, and looking at its subspace of fixed points under a Galois action: here we define the functor of fixed points and talk about its representability and eventual smoothness. This is all contained in [2], to which we refer for the proofs unless they are short enough.

Definition 1.8. Let $\pi\colon S' \longrightarrow S$ be a morphism of schemes and G a finite group, acting on the right on S'. We say that π is a *quotient* for this action, or that π is *Galois with group* G, if it is affine, and for every affine open subscheme $\mathrm{Spec}\, A \subset S$ of pullback $\mathrm{Spec}\, A' \subset S'$ by π, A is the subring A'^G of G-invariants of A'.

Definition 1.9. Let S be a scheme and X an algebraic space. Suppose a group G acts equivariantly on $X \longrightarrow S$ with the trivial action on S. We define the *subfunctor of fixed points* $X^G : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Sets}$ by $X^G(T) = (X(T))^G$.

Proposition 1.10. *With notations as above, the formation of X^G commutes with base change. Moreover, X^G is a subspace of X, closed if X/S is separated.*

Proof. Compatibility with base-change is immediate. Each $g \in G$ gives an automorphism $\rho_X(g)$ of X, thus a graph $\Gamma_g : X \longrightarrow X \times_S X$. Γ_g is the composition

$$X \xrightarrow{\Delta} X \times_S X \xrightarrow{(p_1, \rho_X(g) \circ p_2)} X \times_S X$$

where Δ is the diagonal map, and p_1, p_2 are the two projections from $X \times_S X$ onto X. Since $\rho_X(g)$ is an automorphism of X, $(p_1, \rho_X(g) \circ p_2)$ is an automorphism of $X \times_S X$. Moreover, Δ is an immersion, closed if X/S is separated, so by composition Γ_g is an immersion, closed if X/S is separated. Now, call Z the fiber product of all Γ_g: Z is a subscheme of the diagonal since $\Delta = \Gamma_0$. As such, it is a subscheme of X, closed if X/S is separated, and it represents X^G since for any $T \longrightarrow S$, $\mathrm{Hom}(T, Z)$ is precisely the set of maps $f : T \longrightarrow X$ such that for all $g \in G$, $\Gamma_g \circ f = \Gamma_0 \circ f$, i.e. $f = \rho_X(g) \circ f = f.g$. □

Proposition 1.11. *With hypotheses and notations as above, if $f : X \longrightarrow S$ is smooth and $n := \#G$ is invertible on X, then $X^G \longrightarrow S$ is smooth.*

Proof. This is [2], proposition 3.4. □

Corollary 1.12. *Let G be a finite group acting equivariantly on a smooth morphism of algebraic spaces $X \longrightarrow S$. If $\#G$ is invertible on X, then $X^G \longrightarrow S^G$ is smooth.*

1.4 Twisted Lie algebras

We will make use of a slightly broader than usual notion of tangent space and Lie algebra of a group algebraic space over a base scheme, so we present the definition and a few properties here. These objects are studied in much more detail in [1].

Definition 1.13. Let S be a scheme and X a S-group algebraic space. Let $\mathcal{M}$ be a free O_S-module of finite type. We write $\mathrm{T}_{X/S}(\mathcal{M})$ for the functor $(\mathrm{Sch}\,/S)^{op} \longrightarrow \mathrm{Set}$ taking T/S to $\mathrm{Hom}_S(\mathrm{Spec}(\mathcal{O}_S \oplus \mathcal{M}), X)$, where the $\mathcal{O}_S$-module $\mathcal{O}_S \oplus \mathcal{M}$ is endowed with the $\mathcal{O}_S$-algebra structure making $\mathcal{M}$ a square-zero ideal. We write $\mathrm{Lie}_{X/S}(\mathcal{M})$ for the pullback of $\mathrm{T}_{X/S}(\mathcal{M})$ by the unit section $S \longrightarrow X$. In particular, when $\mathcal{M} = \mathcal{O}_S$, they are the usual tangent bundle and Lie algebra of X over S, that we write $\mathrm{T}_{X/S}$ and $\mathrm{Lie}_{X/S}$.

Proposition 1.14. *With hypotheses and notations as above, $\mathrm{T}_{X/S}(\mathcal{M})$ and $\mathrm{Lie}_{X/S}(\mathcal{M})$ are representable by group S-algebraic spaces, and the canonical morphisms*

$$\mathrm{Lie}_{X/S}(\mathcal{M}) \longrightarrow \mathrm{T}_{X/S}(\mathcal{M}) \longrightarrow X$$

are morphisms of S-groups.

Proof. Representability when X is a scheme is [1], exposé 2, proposition 3.3. The case of algebraic spaces is similar. Existence of the group structure, and the fact the canonical maps respect them, is [1], exposé 2, corollaire 3.8.1. □

Proposition 1.15. *Let $\Phi \colon \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{M}$ be an isomorphism, Φ naturally induces isomorphisms $T_{X/S}(\mathcal{M}) = \coprod_{i=1}^{n} T_{X/S}$ and $\mathrm{Lie}_{X/S}(\mathcal{M}) = \coprod_{i=1}^{n} \mathrm{Lie}_{X/S}$*

Proof. The $\mathcal{O}_S$-algebra $\mathcal{O}_S \oplus \mathcal{M}$ (where $\mathcal{M}$ is a square-zero ideal) can be identified via Φ to the tensor product of n copies of $\mathcal{O}_S \oplus \mathcal{I}$, where $\mathcal{I}$ is a square-zero ideal, isomorphic to $\mathcal{O}_S$ as a $\mathcal{O}_S$-module. The rest follows from the definitions. $\square$

Remark 1.15.1. This identification is only canonical up to the choice of a basis for $\mathcal{M}$.

2 The morphism of base-change for tame extensions

2.1 Compatibility with Weil restrictions

In this section, $S' \longrightarrow S$ will be a finite locally free morphism of schemes, X'/S' an algebraic space, U a schematically dense open subscheme of S, and $U' = U \times_S S'$. Note that U' is schematically dense in S' by [4], théorème 11.10.5.

Definition 2.1. Let t be a generic point of $S \backslash U$. t is of codimension 1 in S so $\mathcal{O}_{S,t}$ is a discrete valuation ring. We call *ramification index of S'/S at t* the ppcm of ramification indexes of the discrete valuation ring map $\mathcal{O}_{S,t} \longrightarrow \mathcal{O}_{S',t'}$ for all $t' \in S'$ of image t. Let $\mathbf{t} = (t_1, ..., t_r)$ be a r-uple of generic points of $S \backslash U$, we call *ramification index of S'/S at* $\mathbf{t}$ the r-uple $(n_1, ..., n_r)$, where n_i is the ramification index of S'/S at t_i for all i.

Proposition 2.2 (see [2], proposition 4.1)**.** *Let X' be a smooth quasi-projective S'-algebraic space. Suppose X' is the S'-Néron model of $X'_{U'}$. Then $\prod_{S'/S}(X')$ is the S-Néron model of $\prod_{U'/U}(X'_{U'})$.*

Proof. $\prod_{S'/S}(X')$ is smooth and separated over S by 1.7 and, if Y is a smooth S-algebraic space, then

$$\begin{aligned}
\mathrm{Hom}(Y, \prod_{S'/S}(X')) &= \mathrm{Hom}(Y', X') \\
&= \mathrm{Hom}(Y'_{U'}, X'_{U'}) \\
&= \mathrm{Hom}(Y_U, \prod_{U'/U}(X'_{U'}))
\end{aligned}$$

where Y' denotes the base-change $Y \times_S S'$. $\square$

Proposition 2.3 (Abhyankar's lemma)**.** *Let $S = \operatorname{Spec} R$ be a local scheme and $S' \longrightarrow S$ a finite locally free tamely ramified morphism of schemes, étale over the complement U of a strict normal crossings divisor D of S. Let $(f_1, ..., f_r)$ be part of a regular system of parameters of R such that $D = \mathrm{Div}(f_1 ... f_r)$. Then* *there are integers $n_1, ..., n_r$, prime to the residue characteristic of R, such that if $\tilde{S} = \operatorname{Spec} R[T_1, ..., T_r]/(T_i^{n_i} - f_i)_{1 \leq i \leq r}$, the morphism $S' \times_S \tilde{S} \longrightarrow \tilde{S}$ is étale.*

Proof. This is a consequence of [5], exposé XIII, proposition 5.2. $\square$

Proposition 2.4. *Under the hypotheses of proposition 2.3, suppose S' is connected and S contains all k-th roots of unity for all k prime to the residue characteristic p of S (e.g. S is strictly local). Then there are integers $n_1, ..., n_r$, prime to p, such that $S' = \operatorname{Spec} R[T_1 ..., T_r]/(T_i^{n_i} - f_i)$*

Proof. By proposition 2.3, there are integers $m_1, ..., m_r$ prime to p, such that S' is a quotient of $\tilde{S} = \operatorname{Spec} \tilde{R}$, where

$$\tilde{R} = R[T_1, ..., T_r]/(T_i^{m_i} - f_i)_{1 \leq i \leq r}$$

But the group of S-automorphisms of $\tilde{S}$ is the product $\prod_{i=1}^{r} \mu_{m_i}$ of the groups μ_{m_i} of m_i-th roots of unity of R (where $\xi \in \mu_{m_i}$ acts by sending T_i to ξT_i). Since S' is regular, $\tilde{R}^G$ must be regular (and a fortiori locally factorial), so the subgroup G of S'-automorphisms of $\tilde{S}$ is of the form $< \xi_1, \xi_2, ..., \xi_r >$ with $\xi_i \in \mu_{m_i}$. Therefore, $\prod_{i=1}^{r} \mu_{m_i}/G$ is a product of quotients of the μ_{m_i}, i.e. there are integers $n_1, ..., n_r$ such that n_i divides m_i for all i and S' itself is of the form $\operatorname{Spec} R[T_1, ..., T_r]/(T_i^{n_i} - f_i)_{1 \leq i \leq r}$. $\square$

Remark 2.4.1. The integer n_i is the ramification index of S'/S at the generic point of $\{f_i = 0\}$ in S.

Remark 2.4.2. This proof means that, étale-locally on S, a finite tamely ramified morphism either does nothing more than adding roots to regular parameters, or must break local factoriality of the base. Since in many practical situations, the behavior of Néron models is only well-known over (at least) locally factorial bases, we will only be considering the "adding roots" case. For the same reason, we always take D to be strict: indeed, suppose D is a (non-strict) normal crossings divisor, and suppose there is an étale morphism $\tilde{S} \longrightarrow S$ and an irreducible component D_0 of D which breaks into multiple irreducible components in $\tilde{S}$, then no extension S'/S with ramification index > 1 over the generic point of D_0 can be locally factorial.

Theorem 2.5 (see [2], theorem 4.2)**.** *Let $S' \longrightarrow S$ be a finite, locally free, tamely ramified map between regular schemes and U the complement of a strict normal crossings divisor $D \longrightarrow S$. Suppose $U' := U \times_S S'$ is étale over U.*

Let X_U be a smooth U-algebraic space, such that $X_U \times_U U'$ has a S'-Néron model X'. Then the schematic closure of X_U in $\prod_{S'/S} X'$ is a S-Néron model for X_U (where X_U maps to $\prod_{S'/S} X'$ as in example 1.6).

Moreover, if $S' \longrightarrow S$ is Galois for the right-action of a finite group G, then G acts on X' via its action on S': it follows that G acts on $\prod_{S'/S} X'$ as in remark 1.7.1. In that case, the Néron model of X_U is $(\prod_{S'/S} X')^G$.

Proof. We can assume $S = \operatorname{Spec} R$ is strictly local and S' is connected, in which case proposition 2.4 shows that $S' \longrightarrow S$ is Galois for the action of a finite group G. Call Z the restriction $\prod_{S'/S}(X')$ and N the schematic closure of X_U in Z. We will show that N is the Néron model of X_U, and that $N = Z^G$.

Let Y be a smooth S-algebraic space with a map $f_U : Y_U \longrightarrow X_U$. Call Y' the base-change $Y \times_S S'$. The map $Y'_{U'} \longrightarrow X'_{U'}$ obtained by base-change extends uniquely to a map $Y' \longrightarrow X'$, which induces a map $Y \longrightarrow Z$ extending f_U. By definition of the schematic closure, this factors through a map $f\colon\ Y \longrightarrow N$ extending f_U.

N is separated since Z is. We will now show that $N_U = X_U$, that Z^G is smooth over S, and that $N = Z^G$.

G acts on $Z_U = \prod_{U'/U} X_{U'}$ via its right-action on U', and $U' \longrightarrow U$ is a quotient for the latter. We have a pullback diagram of algebraic spaces

$$\begin{tikzcd} X'_{U'} \arrow[r] \arrow[d] & X_U \arrow[d] \\ U' \arrow[r] & U \end{tikzcd}$$

where both horizontal arrows are quotients for the action of G. We will show $X_U = Z_U^G$, which can be checked Zariski-locally: let V be an affine open subscheme of U, $V' = \operatorname{Spec} A$ its pullback to U', X_0 an affine open of X_U with image contained in V and $X'_0 = \operatorname{Spec} B$ its pullback to $X'_{U'}$. We have $V = \operatorname{Spec}(A^G)$ and $X_0 = \operatorname{Spec}(B^G)$, and there is a pushout diagram of rings

$$\begin{tikzcd} B & B^G \arrow[l] \\ A \arrow[u] & A^G \arrow[l] \arrow[u] \end{tikzcd}$$

Let W be the Weil restriction of X'_0 to V. Then we see that $W = \operatorname{Spec} R$ is affine, and for any A^G-algebra C, we have $\operatorname{Hom}_{A^G}(R, C) = \operatorname{Hom}_A(B, C \otimes_{A^G} A)$. But the G-invariant A-maps from B to $C \otimes_{A^G} A$ are precisely those lying in the image of $\operatorname{Hom}_{A^G}(B^G, C)$. Therefore $W^G = \operatorname{Spec}(B^G)$, and it follows from the sheaf property that $(Z_U)^G = X_U$. Hence, $X_U \longrightarrow Z_U$ is a closed immersion and $N_U = X_U$.

As seen in proposition 2.4, $\#G$ is prime to the residue characteristic of R, so by proposition 1.11, Z^G is S-smooth.

The only point remaining to prove is $N = Z^G$, which follows from observing that $Z^G \longrightarrow Z$ is a closed immersion through which X_U factors, so $N \longrightarrow Z$ factors through a closed immersion $N \longrightarrow Z^G$. But Z^G is S-smooth, hence S-flat, so $Z^G_U = X_U$ is schematically dense in Z^G by [4], théorème 11.10.5, which means N is schematically dense in Z^G and $N \longrightarrow Z^G$ is an isomorphism. □

2.2 A filtration of the Néron model over the canonical stratification

By remark 1.3.1, Néron models can always be described over an étale covering of the base. Therefore, in this section, unless mentioned otherwise, we will work assuming that $S = \operatorname{Spec} R$ is an affine regular connected scheme, that R contains all roots of unity of order invertible on S, that D is a strict normal crossings divisor on S cut out by regular parameters $f_1, ..., f_r$ of R, and that $S' = \operatorname{Spec} R'$ with $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)_{1 \leq j \leq r}$. Note that in our previous setting (where $S' \longrightarrow S$ was a finite, locally free and tamely ramified morphism between regular schemes, étale over the complement of a strict normal crossings divisor), all these assumptions hold in an étale neighbourhood of any given point of S.

We put $A = R/(f_j)_{1 \leq j \leq r}$ and $A' = A \otimes_R R' = A[T_1, ..., T_r]/(T_j^{n_j})_{1 \leq j \leq r}$. The closed subscheme $Z = \operatorname{Spec} A$ of S is the closed stratum of D. We let X_U be a proper and smooth U-group space with a Néron model N' over S'. It follows from theorem 2.5 that X_U has a S-Néron model N, and that N is the subspace of G-invariants of the Weil restriction of N' to S, where the action of $G = \prod_{j=1}^{r} \mu_{n_j}$ on S' is given by multiplying T_j by the j-th coordinate of an element of G.

In [2], section 5, when S is a discrete valuation ring, one computes the successive quotients of a filtration of the closed fiber of N. We adapt this construction to our context to get a filtration of N_Z and express its successive quotients in terms of N'.

For all $d \in \mathbb{N}^*$, we write Λ_d the set of monomials of the form $\prod_{j=1}^{r} T_j^{k_j}$ with $\sum_{j=1}^{r} k_j = d$ and $k_j < n_j$ for all j. The set $A'_d \subset A'$ of homogenous polynomials of degree d in the T_j is a finite free A-module with basis Λ_d.

Definition 2.6. For $d \in \mathbb{N}^*$, we define a sheaf $\operatorname{Res}^d N'_Z \colon \operatorname{Sch}/Z^{op} \longrightarrow \operatorname{Set}$ as follows: for any A-algebra C, we put $\operatorname{Res}^d N'_Z(C) = N'(C \otimes_A A'/(\Lambda_d))$.

Remark 2.6.1. The functor $\operatorname{Res}^d N'_Z$ is (representable by) the Z-algebraic space $\prod_{(A'/(\Lambda_d))/A} N'_{A'/(\Lambda_d)}$. We have $\operatorname{Res}^1 N'_Z = N'_Z$, and for any $d > \prod_{j=1}^{r} (n_j - 1)$, we have $\operatorname{Res}^d N'_Z = \left(\prod_{S'/S} N'\right) \times_S Z$ since Λ_d is empty. There are natural maps $\operatorname{Res}^{d+1} N'_Z \longrightarrow \operatorname{Res}^d N'_Z$.

Definition 2.7. For $d \in \mathbb{N}^*$, we define $F^d N'_Z$ as the kernel of the canonical morphism $\left(\prod_{S'/S} N'\right) \times_S Z \longrightarrow \operatorname{Res}^d N'_Z$ of Z-group spaces. We also put $F^0 N'_Z = \left(\prod_{S'/S} N'\right) \times_S Z$. The $F^d N'_Z$ form a descending filtration of $\left(\prod_{S'/S} N'\right) \times_S Z$ by Z-subgroup spaces, stationary at 0 starting from $d = 1 + \prod_{j=1}^{r} (n_j - 1)$. We call $\operatorname{Gr}^d N'_Z$ the quotient $F^d N'_Z / F^{d+1} N'_Z$.

Proposition 2.8. *We have* $\operatorname{Gr}^0 N'_Z = N'_Z$*, and for any* $d \geq 1$, $\operatorname{Gr}^d N'_Z$ *is canonically isomorphic to* $\operatorname{Lie}_{N'_Z/Z}(A'_d) = \coprod_{P \in \Lambda_d} \operatorname{Lie}_{N'_Z/Z}(PA)$.

Proof. The proof of [2], 5.1. carries over without much change: let $d \geq 1$, and let C be a A-algebra. Let $\lambda_1, ..., \lambda_k$ be parameters for the formal group of N' over R'. An element $a \in F^d N'_Z(C)$ corresponds to a ring map

$$\phi \colon R'[[\lambda_1, ..., \lambda_k]] \longrightarrow C[T_1, ..., T_r]/(T_j^{n_j})_{1 \leq j \leq n}$$

such that for all $1 \leq i \leq k$, $\phi(\lambda_i)$ is in the ideal generated by Λ_d, i.e. is of the form $\sum_P a_{i,P} P$ where the $a_{i,P}$ are in C and P runs over all nonzero monomials $\prod_{j=1}^{r} T^{k_j}$ with $\sum_j k_j \geq d$. Thus, we can associate to a an element of $\operatorname{Lie}_{N'_Z/Z}(A_d)(C)$ by truncature, sending λ_i to $\sum_{P \in \Lambda_d} a_{i,P} P$. This gives a surjective morphism of Z-groups $F^d N'_Z \longrightarrow \operatorname{Lie}_{N'_Z/Z}(A_d)$, with kernel $F^{d+1} N'_Z$, so we are done. $\square$

Proposition 2.9. *If we make* G *act on* $\operatorname{Lie}_{N'_Z/Z}(A_d) = \coprod_{P \in \Lambda_d} \operatorname{Lie}_{N'_Z/Z}(PA)$ *via the isomorphism of proposition 2.8, all the* $\operatorname{Lie}_{N'_Z/Z}(PA)$ *are* G*-stable, and for any* $P = \prod_{j=1}^{r} T_j^{k_j}$ *in* Λ_d*, the bijection* $\operatorname{Lie}_{N'_Z/Z}(PA) = \operatorname{Lie}_{N'_Z/Z}$ *induced by* $P \mapsto 1$ *identifies the subspace* $\operatorname{Lie}_{N'_Z/Z}(PA)^G$ *with the subspace of* $\operatorname{Lie}_{N'_Z/Z}$ *where all* $\xi = (\xi_j)_{1 \leq j \leq r}$ *in* G *act by multiplication by* $\prod_{j=1}^{r} \xi_j^{k_j}$*. We will write this subspace* $\operatorname{Lie}_{N'_Z/Z}[k_1, ..., k_r]$ *or* $\operatorname{Lie}_{N'_Z/Z}[P]$.

Proof. For any A-algebra C, the action of ξ on $\operatorname{Hom}(R'[[\lambda_1, ..., \lambda_k]], C \otimes_A A')$

makes the following diagram commute:

$$\begin{tikzcd}
R'[[\lambda_1, ..., \lambda_k]] \arrow[r, "\xi.\psi"] \arrow[d, "\xi"] & C \otimes_A A' \\
R'[[\lambda_1, ..., \lambda_k]] \arrow[r, "\psi"] & C \otimes_A A' \arrow[u, "T_j \mapsto \xi_j^{-1} T_j"']
\end{tikzcd}$$

where the map $R'[[\lambda_1, ..., \lambda_k]] \xrightarrow{\xi} R'[[\lambda_1, ..., \lambda_k]]$ is given by the G-action on N'. Therefore, the $\mathrm{Lie}_{N'_Z/Z}(PA)$ are G-stable and the action on $\mathrm{Lie}_{N'_Z/Z}(PA)$ for $P = \prod_{j=1}^{r} T_j^{k_j}$ is given by

$$\begin{tikzcd}
R'[[t_1, ..., t_d]] \arrow[r, "\xi.\psi"] \arrow[d, "\xi"] & C \oplus PC \\
R'[[t_1, ..., t_d]] \arrow[r, "\psi"] & C \oplus PC \arrow[u, "P \mapsto \prod_j \xi_j^{-k_j} P"']
\end{tikzcd}$$

from which the proposition follows. □

Definition 2.10. For any integer $d \in \mathbb{N}$, we define $F^d N_Z = (F^d N'_Z)^G$ and $G^d N_Z = F^d N_Z / F^{d+1} N_Z$.

Remark 2.10.1. The $F^d N_Z$ form a descending filtration of sub-Z-group spaces of N_Z, with $F^0 N_Z = N_Z$ and $F^d N_Z = 0$ when $d > \prod_{j=1}^{r} (n_j - 1)$.

Proposition 2.11. *For all $d \in \mathbb{N}$, we have $G^d N_Z = (G^d N'_Z)^G$. In particular, $G^0 N_Z = (N'_Z)^G$, and for all $d \geq 1$, $G^d N_Z = \coprod_{P \in \Lambda_d} \mathrm{Lie}_{N'_Z/Z}[P]$ (see proposition 2.9) via the isomorphism given by the basis Λ_d of A_d.*

Proof. (see [2], 5.2.) The Z-group spaces $F^d N'_Z$ are unipotent for $d \geq 1$ and the order of G is invertible on Z, so the exact sequence

$$0 \longrightarrow F^d N'_Z \longrightarrow F^{d+1} N'_Z \longrightarrow G^d N'_Z \longrightarrow 0$$

remains exact after taking the G-invariants. □

We summarize all this into theorem 2.13 below, and justify its hypotheses by lemma 2.12 and remark 1.3.1.

Lemma 2.12. *Let $S' \longrightarrow S$ be a finite, locally free, morphism between regular connected schemes. Let D be a strict normal crossings divisor of S and $U = S \backslash D$. Suppose $S' \longrightarrow S$ is étale over U. Let s be a point of S, and $D_1, ..., D_r$ the irreducible components of D containing s. Then there is an affine étale neighbourhood $V = \operatorname{Spec} R \longrightarrow S$ of s in S such that:*

- *For all $1 \leq j \leq r$, $D_j|_V$ is cut out by a regular parameter f_j of R.*

- *There is an isomorphism $V \times_S S' = \operatorname{Spec} R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where n_j is the ramification index of $S' \longrightarrow S$ at the generic point of D_j (in particular, if $S' \longrightarrow S$ is tamely ramified, then n_j is invertible on R).*

- *R contains all n_j-th roots of unity for all j.*

Proof. Immediate from proposition 2.3. □

Theorem 2.13. *Let $S = \operatorname{Spec} R$ be a regular affine scheme, $f_1, ..., f_r$ regular parameters of R, $R' = R[T_1, ..., T_r]/(T_j^{n_j} - f_j)$, where the n_j are invertible on S, and let $S' = \operatorname{Spec} R'$. Suppose R contains the group μ_{n_j} of n_j-th roots of unity for all j. Let U be the locus in S where all f_j are invertible, and Z the locus where all f_j vanish. Let X_U be a proper smooth U-group algebraic space with a Néron model N' over S'. Then X_U has a Néron model N over S, and we have sub-Z-group spaces $(F^d N_Z)_{d \in \mathbb{N}}$ of N_Z (see definitions 2.7 and 2.10) such that:*

- *For all $d \in \mathbb{N}$, $F^d N_Z \subset F^{d+1} N_Z$.*

- *$F^0 N_Z = N_Z$.*

- *If $d > \prod_{j=1}^{r} (n_j - 1)$ then $F^d N_Z = 0$.*

- *$F^0 N_Z / F^1 N_Z$ is the subspace of N'_Z invariant under the action of $G = \prod_{j=1}^{r} \mu_j$, where $(\xi_j)_{1 \leq j \leq r}$ acts by multiplying T_j by ξ_j.*

- *If $d > 0$, $F^d N_Z / F^{d+1} N_Z$ is isomorphic (via the isomorphism of proposition 2.8) to the disjoint union $\coprod_{\mathbf{k}} \operatorname{Lie}_{N'_Z/Z}[\mathbf{k}]$ where $\mathbf{k}$ ranges through all r-uples of integers $(k_1, ..., k_r)$ with $\sum_{j=1}^{r} k_j = d$ and $k_j < n_j$ for all j, and $\operatorname{Lie}_{N'_Z/Z}[\mathbf{k}]$ is the subspace of $\operatorname{Lie}_{N'_Z/Z}$ where all $(\xi_j)_{1 \leq j \leq r}$ in G act by multiplication by $\prod_{j=1}^{r} \xi_j^{k_j}$.*

Remark 2.13.1. Our choice of quotienting by all monomials of the same degree in definition 2.6 is somewhat arbitrary, other choices could perharps lead to interesting things aswell.

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