Néron models of nodal curves and their Jacobians in arbitrary base dimension

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Abstract

We work with a smooth relative curve X_U/U with nodal reduction over a regular and excellent scheme S. We show that blowing up a nodal model of X_U in the ideal sheaf of a section yields a new nodal model, and describe how these models relate to each other. We show that, locally on S, a finite sequence of such blowups always yields a nodal model that remains locally factorial after any smooth base change. We construct Néron models of X_U and of its Jacobian, and provide combinatorial criteria for the separatedness of these Néron models.

Contents

General introduction

Néron models

Given a scheme S and a dense open $U \subset S$, proper and smooth schemes over U often have no proper and smooth model over S. Even so, they may still have a canonical smooth S-model, the *Néron model*, first introduced in [Né64]. The Néron model of X_U/U is defined as a smooth S-model satisfying the *Néron mapping property*: for every smooth S-scheme T and every U-morphism $\phi_U \colon T_U \to X_U$, there exists a unique morphism $\phi \colon T \to N$ extending ϕ_U. Néron models are unique up to a unique isomorphism, and inherit a group structure from X_U when the latter has one.

Néron proved in the original article [Né64] that abelian varieties over the fraction field of an integral Dedekind scheme always have Néron models. Recently, people have taken interest in constructing Néron models in different settings. It was proved in 2013 by Qing Liu and Jilong Tong in [LT13] that smooth and proper curves of positive genus over a Dedekind scheme always have Néron models. This does not apply to genus 0: Proposition 4.12 of [LT13] shows that if S is the spectrum of a discrete valuation ring with field of fractions K, then $\mathbb{P}^1_K$ does not have a Néron model over S.

When the base S is a regular scheme of arbitrary dimension, and X/S a nodal curve[1], smooth over U, David Holmes exhibited in [Hol19b] a combinatorial criterion on X/S called *alignment*, necessary for the Jacobian of X_U to have a separated Néron model, and sufficient when X is regular. In [Ore18], Giulio Orecchia introduces the *toric-additivity* criterion. Consider an abelian scheme A/U with semi-abelian reduction $\mathcal{A}/S$, where S is a regular base and U the complement in S of a strict normal crossings divisor. Toric-additivity is a condition on the Tate module of A. When A is the generic Jacobian of an S-curve with a nodal model, toric-additivity is sufficient for a Néron model of A to exist. It is also necessary up to some restrictions on the base characteristic. For general abelian varieties, it is proven in [Ore19] that toric-additivity is still sufficient when S is of equicharacteristic zero, and a partial converse holds, i.e. existence of a Néron model implies a weaker version of toric-additivity. When S is a

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[1] A proper, flat, finitely presented morphism with geometric fibers of pure dimension 1, and at worst ordinary double point singularities.

toroidal variety and X/S a nodal curve, smooth over the complement U of the boundary divisor, a construction of the Néron model of the Jacobian of X_U is given in [HMOP20], together with a moduli interpretation for it.

Among the concrete applications of the theory of Néron models, we can cite the semi-stable reduction theorem (an abelian variety over the fraction field of a discrete valuation ring acquires semi-abelian reduction after a finite extension of the base field); the Néron-Ogg-Shafarevich criterion for good reduction of abelian varieties; the computation of canonical heights on Jacobians; as well as the linear and quadratic Chabauty methods to determine whether or not a list of rational points on a curve is exhaustive. For a geometric description of the quadratic Chabauty method, see [EL19]. Parallels can also be drawn to some problems in which Néron models do not explicitly intervene, such as extending the double ramification cycle on the moduli stack of smooth curves to the whole moduli stack of stable curves as in [Hol19a]. Here, one is interested in models in which one given section extends, instead of all sections simultaneously, but the two problems are closely related.

Structure of the paper, main results

In this article, we work with a nodal curve X/S, smooth over a dense open $U \subset S$, where S is a regular and excellent scheme. We are interested in constructing Néron models for X_U and for its Jacobian, without any restriction on the discriminant locus of X/S.

In Part I, we describe what can be seen as a higher-dimensional version of the smoothening process of [BLR90]. We call *refinements* of X/S the blowing-ups of X in the ideal sheaves of S-sections. We show that the blow-up $X' \to X$ in a section $\sigma \colon S \to X$ is a nodal curve, and compute the smoothing parameters of the singularities of X'/S in terms of those of X/S. We show that, étale-locally on S around a point s, X' only depends on the *type* of σ at $\sigma(s)$, a combinatorial invariant. The reader familiar with logarithmic geometry can establish a parallel between our refinements and logarithmic modifications of a log curve inducing a given subdivision of its tropicalization (although since we often work with refinements of smooth base changes of X/S, one should be careful with this analogy, see Remark 6.2.2).

In Part II, Section 5, we start with a general discussion about the base change properties of Néron models, and we show:

Corollary 0.1 (Corollary 5.10)**.** *Let S be a scheme, $U \subset S$ a scheme-theoretically dense open subscheme, $N_U \to U$ a smooth, separated U-group algebraic space and $f \colon N \to S$ a smooth S-group model of N_U. Call E the scheme-theoretical closure of the unit section in N. Then, for any smooth S-scheme Y, the sequence of abelian groups*

$$0 \to \mathrm{Hom}(Y, E^{\mathrm{et}}) \to \mathrm{Hom}(Y, N) \to \mathrm{Hom}(Y_U, N_U)$$

is exact. In particular, the quotient space N/E^{et} is a smooth S-group model of N_U with uniqueness in the Néron mapping property.

Here, E^{et} denotes the maximal open subspace of E which is étale over S. Then, in Section 6, we discuss Néron models of Jacobians. We obtain a general existence result:

Theorem 0.2 (Theorem 6.2). *Let S be a regular scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U. Then, the Jacobian J of X_U admits a Néron model N over S. Moreover, there exists an étale cover $V \to S$ such that N_V is of the form $\text{Pic}_{X'/V}^{\text{tot}\,0} / E^{\text{et}}$, where X'/V is a nodal model of $X_U \times_S V$ and E is the closure of the unit section in $\text{Pic}_{X'/V}^{\text{tot}\,0}$.*

Then, we provide a simple way to determine whether or not the Néron model is separated. Following the ideas of [Hol19b], we say X/S is *strictly aligned* when the smoothing parameters of its singularities satisfy a certain combinatorial condition (Definition 6.3); and we prove

Theorem 0.3 (Theorem 6.8). *Let S be a regular and excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U. Call J the Jacobian of X_U/U. Then, the S-Néron model N of J exhibited in Theorem 6.2 is separated if and only if X/S is strictly aligned.*

In Section 7, we aim at constructing a Néron model for X_U/U itself. We characterize when a section of X/S factors through the smooth locus of a given refinement (Subsection 7.1). We glue the smooth loci of several refinements of X/S into a space called the *smooth aggregate* of X/S, and show that it satisfies a universal property close to Néron's:

Proposition 0.4 (Proposition 7.8). *Let S be a regular and excellent scheme and X/S a quasisplit nodal curve. Let N be the smooth aggregate of X/S (Construction 7.3). Then for any smooth S-algebraic space Y and any morphism $f: Y \to X$, f factors uniquely through the canonical map $N \to X$.*

This allows us to find a Néron model for X_U, under some restrictions on the rational curves appearing in the geometric fibers of X/S. We obtain:

Theorem 0.5 (Theorem 7.38). *Let S be a regular excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Suppose X has no rational loops, and suppose no geometric fiber of X contains a rational component meeting the non-exceptional other irreducible components in three points or more. Then X_U/U has a Néron model N/S. If in addition X/S is quasisplit, then N is the smooth aggregate of the stable model X^{stable} of X_U (Definition 7.24).*

Then, we determine when this Néron model is separated as follows.

Theorem 0.6 (Theorem 7.45). *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. If X_U has a separated Néron model over S, then the two following conditions are met:*

- *The singular locus $\text{Sing}(X/S)$ is étale-locally irreducible around every non-exceptional singular geometric point of X/S.*

- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}_{S,s}^{\mathrm{et}}$ have the same radical.*

Conversely, suppose these conditions are met. Suppose in addition that no geometric fiber of X/S contains either a rational cycle or a rational component meeting the non-exceptional other components in at least three points. Then the Néron model of X_U/U exhibited in Theorem 7.38 is separated over S.

Notations

We will adopt the following conventions:

- If $f\colon X \to S$ is a morphism of algebraic spaces locally of finite type, we call *smooth locus* of f, and write $(X/S)^{sm}$ (or X^{sm} if there is no ambiguity) the open subspace of X at which f is smooth.

- If $f\colon X \to S$ is a morphism of schemes, locally of finite presentation, with fibers of pure dimension 1, we call *singular locus* of f, and write $\mathrm{Sing}(X/S)$, the closed subscheme of X cut out by the first Fitting ideal of the sheaf of relative 1-forms of X/S. The set-theoretical complement of $\mathrm{Sing}(X/S)$ in X is precisely $(X/S)^{sm}$.

- Unless specified otherwise, if A is a local ring, we write $\mathfrak{m}_A$ for its maximal ideal, k_A for the its residue field and $\widehat{A}$ for its $\mathfrak{m}_A$-adic completion.

Acknowledgements

Part I

Nodal curves, dual graphs and resolutions

1 Local structure of nodal curves and their dual graphs

1.1 First definitions

The results of this subsection are mostly either well-known facts about nodal curves, or come from [Hol19b]. When the proofs are short enough, we reproduce them for convenience.

Definition 1.1. A *graph* G is a pair of finite sets (V, E), together with a map $f : E \to (V \times V)/\mathcal{S}_2$. We call V the set of vertices of G and E its set of edges. We think of f as the map sending an edge to its endpoints. We call *loop* any edge in the preimage of the diagonal of $(V \times V)/\mathcal{S}_2$. We will often omit f in the notations and write $G = (V, E)$.

Let v, v' be two vertices of G. A *path between v and v' in G* is a finite sequence $(e_1, ..., e_n)$ of edges, such that there are vertices $v_0 = v, v_1, ..., v_n = v'$ satisfying $f(e_i) = (v_{i-1}, v_i)$ for all $1 \le i \le n$. We call n the *length* of the path. A *chain* is a path as above, with positive n, where the only repetition allowed in the vertices $(v_i)_{0 \le i \le n}$ is $v_0 = v_n$. A *cycle* is a chain from a vertex to itself. The cycles of length 1 of G are its loops.

Let M be a semigroup. A *labelled graph over M* (or *labelled graph* if there is no ambiguity) is the data of a graph $G = (V, E)$ and a map $l \colon E \to M$, called *edge-labelling*. The image of an edge by this map is called the *label* of that edge.

Definition 1.2. Let X be an algebraic space. We call *geometric point* of X a morphism $\operatorname{Spec} \bar{k} \to X$ where the image of $\operatorname{Spec} \bar{k}$ is a point with residue field k, and $\bar{k}$ is a separable closure of k (notice the "separable" instead of "algebraic"). Given two geometric points s, t of X, we say that t is an *étale generization* of s (or that s is an *étale specialization* of t) when the image of $t \to X$ is a generization of the image of $s \to X$. We will often omit the word "étale" and just call them specializations and generizations.

Definition 1.3. A *curve* over a separably closed field k is a proper morphism $X \to \operatorname{Spec} k$ with X of pure dimension 1. It is called *nodal* if it is connected, and for every point x of X, either X/k is smooth at x, or x is an ordinary double point (i.e. the completed local ring of X at x is isomorphic to $k[[u, v]]/(uv)$).

A *curve* (resp. *nodal curve*) over a scheme S is a proper, flat, finitely presented morphism $X \to S$ whose geometric fibers are curves (resp. nodal curves).

Remark 1.3.1. By [Liu02], Proposition 10.3.7, our definition of nodal curves

is unchanged if one defines geometric points with algebraic closures instead of separable closures.

Definition 1.4. Let S be a scheme, s a point of S, and $\bar{s}$ a geometric point mapping to s. We will call *étale neighbourhood of $\bar{s}$ in S* the data of an étale morphism of schemes $V \to S$, a point v of V, and a factorization $\bar{s} \to v \to s$ of $\bar{s} \to s$. Etale neighbourhoods naturally form a codirected system, and we call *étale stalk of S at s* the limit of this system. The étale stalk of S at s is an affine scheme, and we call *étale local ring at s*, and note $\mathcal{O}^{\text{et}}_{S,s}$, its ring of global sections. We will sometimes keep the choice of geometric point $\bar{s}$ implicit and abusively call (V, v), or even V, an étale neighbourhood of s in S.

Remark 1.4.1. The étale local ring of S at s is a strict henselization of the Zariski local ring $\mathcal{O}_{S,s}$. The étale local ring of S at $\bar{s}$ is the strict henselization determined by the separable closure $k(s) \to k(\bar{s})$.

1.2   The local structure

Proposition 1.5. *Let S be a locally noetherian scheme and X/S be a nodal curve. Let s be a geometric point of S and x be a non-smooth point of X_s. There exists a unique principal ideal (Δ) of the étale local ring $\mathcal{O}^{\text{et}}_{S,s}$, called the singular ideal of x, such that*

$$\widehat{\mathcal{O}^{\text{et}}_{X,x}} \simeq \widehat{\mathcal{O}^{\text{et}}_{S,s}}[[u, v]]/(uv - \Delta)$$

Proof. This is [Hol19b], Proposition 2.5. $\qquad\square$

Remark 1.5.1. The singular ideal of x is generated by a nonzerodivisor if and only if X/S is generically smooth in a neighbourhood of x.

1.3   The dual graph at a geometric point

Definition 1.6. Let X, S be as above and s be a geometric point of S. We define a graph, called *dual graph of X at s*, labelled by the semigroup of nontrivial principal ideals of $\mathcal{O}^{\text{et}}_{S,s}$, as follows: its vertices are the irreducible components of X_s; its edges are the singular points x of X_s with endpoints the two (not necessarily distinct) irreducible components containing x; and the label of x is its singular ideal.

When S is strictly local, we will sometimes refer to the dual graph of X at the closed point as simply "the dual graph of X".

Proposition 1.7. *Let $S' \to S$ be a morphism of locally noetherian schemes, X/S a nodal curve, s a geometric point of S, and s' a geometric point of S' mapping to a generization of s in S. Let X' be the base change of X to S', Γ and Γ' be the dual graphs, respectively of X at s and of X' at s'.*

Let $R := \mathcal{O}^{\text{et}}_{S,s}$; $R' := \mathcal{O}^{\text{et}}_{S',s'}$, and ϕ be the natural map $\operatorname{Spec} R' \to \operatorname{Spec} R$. Then Γ' is obtained from Γ by contracting all edges whose label becomes invertible in R', and pulling back the labels of the other edges by ϕ.

In particular, if s' has image s, Γ and Γ' are isomorphic as non-labelled graphs, and the labels of Γ' are obtained by pulling back those of Γ.

Proof. This is [Hol19b], Remark 2.12. We reprove it here.

We can reduce to $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ affine and strictly local[2], with respective closed points s and s'.

Let x be a singular point of X of image s, and Δ be a generator of its (principal) singular ideal. Then we can choose an isomorphism $\widehat{\mathcal{O}_{X,x}} = \widehat{R}[[u,v]]/(uv - \Delta)$.

This yields $\widehat{\mathcal{O}_{X,x}} \otimes_R R' = \widehat{R} \otimes_R R'[[u,v]]/(uv - \Delta)$. The ring $\widehat{R} \otimes_R R'$ is local, with completion $\widehat{R'}$ with respect to the maximal ideal: as desired, if Δ is invertible in R', then X' is smooth above a neighbourhood of x, and otherwise, X' has exactly one singular closed point of image x, with singular ideal $\Delta R'$. $\qquad\square$

Example 1.8. With notations as above, in the case $S = S'$, we have defined the *specialization maps* of dual graphs: take s, ξ geometric points of S with s specializing ξ, we have a canonical map from the dual graph at s to the dual graph at ξ, contracting an edge if and only if its label becomes the trivial ideal in $\mathcal{O}_{S,\xi}^{\mathrm{et}}$.

It can be somewhat inconvenient to always have to look at geometric points. We can often avoid it as in [Hol15], by reducing to a case in which the dual graphs already make sense without working étale-locally on the base.

1.4 Quasisplitness, dual graphs at non-geometric points

Definition 1.9 (see [Hol15], Definition 4.1). We say a nodal curve $X \to S$ is *quasisplit* if the two following conditions are met:

1. for any point $s \in S$ and any irreducible component E of X_s, there is a smooth section $S \to (X/S)^{sm}$ intersecting E;

2. the singular locus $\operatorname{Sing}(X/S) \to S$ is of the form

$$\coprod_{i \in I} F_i \to S,$$

where the $F_i \to S$ are closed immersions.

Example 1.10. Consider the real conic

$$X = \operatorname{Proj}(\mathbb{R}[x,y,z]/(x^2 + y^2)).$$

It is an irreducible nodal curve over $\operatorname{Spec} \mathbb{R}$, but the base change $X_{\mathbb{C}}$ has two irreducible components: X is not quasisplit over $\operatorname{Spec} \mathbb{R}$.

On the other hand, consider the real projective curve

$$Y = \operatorname{Proj}(\mathbb{R}[x,y,z]/(x^3 + xy^2 + xz^2)).$$

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[2]isomorphic to spectra of strictly henselian local rings

It has two irreducible components (respectively cut out by x and by $x^2+y^2+z^2$), both geometrically irreducible. The singular locus of $Y/\mathbb{R}$ consists of two $\mathbb{C}$-rational points, with projective coordinates $(0:i:1)$ and $(0:-i:1)$, at which $Y_{\mathbb{C}}$ is nodal. Since $\mathrm{Sing}(Y/\mathbb{R})$ is not a disjoint union of $\mathbb{R}$-rational points, Y is not quasisplit over $\mathrm{Spec}\,\mathbb{R}$. However, both X and Y become quasisplit after base change to $\mathrm{Spec}\,\mathbb{C}$.

Remark 1.10.1. Our definition of quasisplitness is slightly more restrictive than that of [Hol15].

Remark 1.10.2. Consider a quasisplit nodal curve $\pi\colon X \to S$ and a point s of S. For any geometric point s' of S of image s, the irreducible components of $X_{s'}$ are in canonical bijection with those of X_s by the first condition defining quasisplitness, and the singular ideals of X at s' come from principal ideals of the Zariski local ring $\mathcal{O}_{S,s}$ by the second condition. Thus, from now on, we will freely talk about the singular ideal (in $\mathcal{O}_{S,s}$) of X/S at any singular point $x \in X_s$, and about the dual graph of X at s.

This may clash with Definition 1.6 when the point of S we consider has separably closed residue field. Unless specified otherwise, when there is an ambiguity, we will always privilege Definition 1.6.

Lemma 1.11. *Quasisplit curves are stable under arbitrary base change.*

Proof. Both conditions forming the definition of quasisplitness are stable under base change. $\qquad\square$

Lemma 1.12. *Let S be a strictly local scheme and X/S a nodal curve. Then X/S is quasisplit.*

Proof. There is a section through every closed point in the smooth locus of X/S, so in particular there is a smooth section through every irreducible component of every fiber. Proposition 1.5 implies the map $\mathrm{Sing}(X/S) \to S$ is a disjoint union of closed immersions. $\qquad\square$

Corollary 1.13. *Let S be a locally noetherian scheme and X/S a nodal curve. Then there is an étale cover $V \to S$ such that X_V/V is quasisplit.*

Lemma 1.14. *Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Quasisplitness of X/S gives a factorization*

$$x \to F \to \mathrm{Sing}(X/S) \to X \to S,$$

where $F \to S$ is a closed immersion and $F \to \mathrm{Sing}(X/S)$ is the connected component containing x. Then, there is an étale neighbourhood (V,y) of x in X; two effective Cartier divisors C, D on V; and an isomorphism $V \times_X F = C \times_V D$ such that $V \times_S F$ is the union of C and D.

Proof. Let $\bar{s}$ be a geometric point of S mapping to s, and $\bar{x} = x \times_s \bar{s}$. By Proposition 1.5, we have an isomorphism $\widehat{\mathcal{O}^{\mathrm{et}}_{X,\bar{x}}} = \widehat{\mathcal{O}^{\mathrm{et}}_{S,s}}[[u,v]]/(uv - \Delta)$, where Δ is a generator of the singular ideal of x in $\mathcal{O}^{\mathrm{et}}_{S,\bar{s}}$. The base change of F/S to $\mathrm{Spec}\,\mathcal{O}^{\mathrm{et}}_{S,\bar{s}}$ is cut out by Δ, and the zero loci C_u of u and C_v of v are effective

Cartier divisors on $\widehat{\mathcal{O}_{X,\bar{x}}^{\text{et}}}$, intersecting in $\widehat{\mathcal{O}_{X,\bar{x}}^{\text{et}}}/(u,v) = F \times_X \operatorname{Spec} \widehat{\mathcal{O}_{X,\bar{x}}^{\text{et}}}$. The union of C_u and C_v is $\widehat{\mathcal{O}_{S,\bar{s}}^{\text{et}}}[[u,v]]/(\Delta, uv)$, so the proposition follows by a limit argument. $\qquad\qquad\square$

2 Primality and base change

In this section, we will discuss questions of permanence of primality (of an element of an integral regular ring) under étale maps, smooth maps and completions. One of the reasons these considerations are important to talk about Néron models of nodal curves is that a nodal curve over an excellent and regular base admits local sections through a singular point if and only if the label of this singular point is reducible (Lemma 4.3).

An element Δ of a regular local ring R is prime in R^{sh} when the quotient $R/(\Delta)$ is *unibranch* (i.e. has integral strict henselization), so we are interested in questions of permanence of unibranch rings under tensor product. For a more detailed discussion on unibranch rings or counting geometric branches in general, see [Ray70], chapitre IX. We also refer to [GD64], 23.2.

In [Swe75], Sweedler gives a necessary and sufficient condition for the tensor product of two local algebras over a field to be local. We are interested in a sufficient condition for algebras over a strictly local ring. The proof of [Swe75] carries over without much change: we reproduce it here.

Lemma 2.1. *Let R be a strictly henselian local ring, $R \to A$ an integral morphism of local rings with purely inseparable residue extension, and $R \to B$ any morphism of local rings. Then $A \otimes_R B$ is local, and its residue field is purely inseparable over that of B.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $A \otimes_R B$. The map $B \to A \otimes_R B$ is integral so it has the going-up property ([Sta17, Tag 00GU]), therefore the inverse image of $\mathfrak{m}$ in B is a maximal ideal: it must be $\mathfrak{m}_B$. Thus $\mathfrak{m}$ contains $A \otimes_R \mathfrak{m}_B$.

In particular, $\mathfrak{m}$ also contains the image of $\mathfrak{m}_R$ in $A \otimes_R B$: it corresponds to a maximal ideal of $A \otimes_R B/(\mathfrak{m}_R A \otimes_R B)$, that we will still call $\mathfrak{m}$. We have a commutative diagram

$$
\begin{array}{ccc}
k_R & \longrightarrow & B/\mathfrak{m}_R B \\
\downarrow & & \downarrow \\
A/\mathfrak{m}_R A & \longrightarrow & A \otimes_R B/(\mathfrak{m}_R A \otimes_R B).
\end{array}
$$

Since $A/\mathfrak{m}_R A$ is local and integral over the field k_R, its maximal ideal m_A is nilpotent and is its only prime. The inverse image of $\mathfrak{m}$ in $A/\mathfrak{m}_R A$ is a prime ideal, so it can only be $\mathfrak{m}_A$. This shows that, as an ideal of $A \otimes_R B$, $\mathfrak{m}$ also contains $\mathfrak{m}_A \otimes_R B$.

Every maximal ideal of $A \otimes_R B$ contains both $\mathfrak{m}_A \otimes_R B$ and $A \otimes_R \mathfrak{m}_B$, so the maximal ideals of $A \otimes_R B$ are in bijective correspondence with those of

$k_A \otimes_{k_R} k_B = A \otimes_R B/(\mathfrak{m}_A \otimes_R B + A \otimes_R \mathfrak{m}_B)$. We will now show the latter is local, with purely inseparable residue extension over k_B.

By hypothesis, the extension k_A/k_R is purely inseparable. If k_R has characteristic 0, then $k_A = k_R$ and we are done. Suppose k_R has characteristic $p > 0$. For any $x \in k_A \otimes_{k_R} k_B$, we can write x as a finite sum $\sum_{i \in I} \lambda_i \otimes \mu_i$ with the λ_i, μ_i in k_A, k_B respectively. There is an integer $N > 0$ such that for all i, $\lambda_i^{p^N}$ is in k_R. Therefore $x^{p^N} = \sum_{i \in I} \lambda_i^{p^N} \mu_i^{p^N}$ is in k_B, and x is either nilpotent or invertible. It follows that $k_A \otimes_{k_R} k_B$ is local, with maximal ideal its nilradical, and that its residue field is purely inseparable over k_B as claimed. $\square$

Lemma 2.2. *Let $(R, \mathfrak{m})$ be an integral and strictly local noetherian ring, and $R \to R'$ a smooth ring map. Let $\mathfrak{p}$ be a prime ideal of R' containing $\mathfrak{m}R'$. Call $\tilde{R}'$ a strict henselization of $R'_\mathfrak{p}$. Then $R'_\mathfrak{p}$ is geometrically unibranch, i.e. $\tilde{R}'$ is an integral domain.*

Proof. We know $\tilde{R}'$ is reduced since it is a filtered colimit of smooth R-algebras. Let B, B' be the integral closures of $R, R'_\mathfrak{p}$ in their respective fraction fields. The ring $R'_\mathfrak{p}$ is integral so by [Ray70], chapitre IX, corollaire 1, $\tilde{R}'$ is an integral domain if and only if B' is local and the extension of residue fields of $R'_\mathfrak{p} \to B'$ is purely inseparable. But any smooth base change of B/R remains normal (see [Liu02], Corollary 8.2.25), so $B \otimes_R R'_\mathfrak{p}$ is normal as a filtered colimit of normal B-algebras. Moreover, any normal algebra over $R'_\mathfrak{p}$ must factor through $B \otimes_R R'_\mathfrak{p}$, so we have $B' = B \otimes_R R'_\mathfrak{p}$. Applying Lemma 2.1, we find that B' is local and the extension of residue fields of $R'_\mathfrak{p} \to B'$ is purely inseparable, which concludes the proof. $\square$

Corollary 2.3. *Let S be a regular scheme, $Y \to S$ a smooth morphism and consider a commutative square*

$$
\begin{array}{ccc}
y & \longrightarrow & s \\
\downarrow & & \downarrow \\
Y & \longrightarrow & S
\end{array}
$$

where $s \to S$ and $y \to Y$ are geometric points. Then for any prime element Δ of $\mathcal{O}^{\mathrm{et}}_{S,s}$, the image of Δ in $\mathcal{O}^{\mathrm{et}}_{Y,y}$ is prime.

Proof. Base change to $\operatorname{Spec} \mathcal{O}^{\mathrm{et}}_{S,s}/(\Delta)$, replace Y by an affine neighbourhood of y in Y, and apply Lemma 2.2. $\square$

Lemma 2.4. *Let R be a strictly henselian excellent local ring and $\widehat{R}$ its completion with respect to the maximal ideal. Then an element Δ of R is prime in R if and only if it is in $\widehat{R}$.*

Proof. The nontrivial implication is the direct sense. Suppose Δ is prime in R. Since R is excellent, the morphism $R \to \widehat{R}$ is regular. Then by Popescu's thorem ([Sta17, Tag 07GB]), $\widehat{R}$ is a directed colimit of smooth R-algebras. We

conclude by writing $R/(\Delta) \to \widehat{R}/(\Delta)$ as a colimit of smooth $R/(\Delta)$-algebras and applying Lemma 2.2. $\qquad\square$

Lemma 2.5. *Let S be an excellent and regular scheme, X/S an S-scheme of finite presentation, $\bar{s}$ a geometric point of S, and x a closed point of $X_{\bar{s}}$ with an isomorphism $\widehat{\mathcal{O}^{\mathrm{et}}_{X,x}} = \widehat{\mathcal{O}^{\mathrm{et}}_{S,\bar{s}}}[[u,v]]/(uv - \Delta)$ for some $\Delta \in \mathfrak{m}_{\bar{s}} \subset \mathcal{O}^{\mathrm{et}}_{S,\bar{s}}$. For every $t_1, t_2 \in \mathfrak{m}_{\bar{s}}$ such that $t_1 t_2 = \Delta$, there exists an étale neighbourhood $S' \to S$ of $\bar{s}$ and a section $S' \to X$ through x such that the induced map $\widehat{\mathcal{O}^{\mathrm{et}}_{X,x}} \to \widehat{\mathcal{O}^{\mathrm{et}}_{S',\bar{s}}}$ sends u, v respectively to a generator of (t_1) and a generator of (t_2).*

Proof. Put $R = \widehat{\mathcal{O}^{\mathrm{et}}_{S,\bar{s}}}$ and consider the map $\widehat{\mathcal{O}^{\mathrm{et}}_{X,x}} \to \widehat{R}$ that sends u, v to t_1, t_2 respectively. Compose it with $\mathcal{O}^{\mathrm{et}}_{X,x} \to \widehat{\mathcal{O}^{\mathrm{et}}_{X,x}}$ to get a map $f_0 : \mathcal{O}^{\mathrm{et}}_{X,x} \to \widehat{R}$.

For noetherian local rings, quotients commute with completion with respect to the maximal ideal, so two distinct ideals are already distinct modulo some power of the maximal ideal. Let $\prod_{i=1}^{n} \Delta_i^{\nu_i}$ be the prime factor decomposition of Δ in R. Principal ideals of R of the form $(\Delta_i^{\mu_i})$ with $0 \le \mu_i \le \nu_i$ are pairwise distinct and in finite number, so there exists some $N \in \mathbb{N}$ such that their images in $R/\mathfrak{m}_R^N$ are pairwise distinct. Since R is henselian and excellent, it has the Artin approximation property ([Sta17, Tag 07QY]), so there exists a map $f : \mathcal{O}^{\mathrm{et}}_{X,x} \to R$ that coincides with f_0 modulo m_R^N. This f induces a map $\widehat{f} : \widehat{\mathcal{O}^{\mathrm{et}}_{X,x}} \to \widehat{R}$. Call a, b the respective images of u, v by $\widehat{f}$, we have $a = t_1$ and $b = t_2$ in $R/\mathfrak{m}_R^N$. But $ab = \Delta$ in $\widehat{R}$ and, by Lemma 2.4, Δ has the same prime factor decomposition in R and $\widehat{R}$, so the only principal ideals of $\widehat{R}$ containing Δ are of the form $(\Delta_i^{\mu_i})$ with $0 \le \mu_i \le \nu_i$. By definition of N we get $a\widehat{R} = t_1\widehat{R}$ and $b\widehat{R} = t_2\widehat{R}$. Since X/S is of finite presentation, f comes from an S-morphism $S' \to X$ where S' is an étale neighbourhood of $\bar{s}$ in S. $\qquad\square$

3   Sections of nodal curves

We present a few technicalities regarding sections of a quasisplit nodal curve X/S. There is a double interest in this. First, blow-ups of nodal curves in the ideal sheaves of sections will be an important tool to construct Néron models of smooth curves with nodal reduction. Second, we can view the Néron mapping property as a condition of extension of rational points into sections after smooth base change. Thus, we will be interested in conditions under which a section of a nodal curve factors through the blowing-up in the ideal sheaf of another section. The basis for this formalism was thought of together with Giulio Orecchia.

3.1   Type of a section

We will define a combinatorial invariant, the *type* of a section, summarizing information about the behavior of said section around the singular locus of a

nodal curve X/S. Later on, we will show that sections of all types exist étale-locally on the base (Proposition 3.9) and that the type of a section locally characterizes the blowing-up of X in the ideal sheaf of that section (Corollary 4.7).

Definition 3.1. Let S be a regular scheme and $s \to S$ a geometric point. Let X/S be a nodal curve, smooth over a dense open subscheme U of S. Let x be a singular point of X_s, we call *thickness* of x (in X/S) the image of the singular ideal of x in the monoid $\mathcal{O}_{S,s}^{\mathrm{et}}/(\mathcal{O}_{S,s}^{\mathrm{et}})^*$ of principal ideals of $\mathcal{O}_{S,s}^{\mathrm{et}}$.

Definition 3.2. Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Let F be the connected component of $\mathrm{Sing}(X/S)$ containing x. Then the set of connected components of $(X\backslash F) \times_X \mathrm{Spec}\,\mathcal{O}_{X,x}^{\mathrm{et}} \times_S F$ is a pair $\{C, D\}$ (see Proposition 1.5 or Lemma 1.14), on which the Galois group $\mathrm{Aut}_{O_S}(\mathcal{O}_{S,s}^{\mathrm{et}}) = \mathrm{Gal}(k(s)^{sep}/k(s))$ acts. If this action is trivial, we say X/S is *orientable at* x, and we call *orientations of X/S at* x the ordered pairs (C, D) and (D, C). The scheme-theoretical closures of C and D in $\mathrm{Spec}\,\mathcal{O}_{X,x}^{\mathrm{et}}$ are effective Cartier divisors, and we will often also call them C and D.

Remark 3.2.1. The curve X/S is orientable at x if and only if the preimage of x in the normalization of X_s consists of two $k(s)$-rational points, in which case an orientation is the choice of one of these points. Roughly speaking, this also corresponds to picking an orientation of the edge corresponding to x in the dual graph of X at s. The "roughly speaking" is due to the case of loops: there is an ambiguity on how to orient them. We could get rid of this ambiguity by using a heavier notion of dual graphs (such as the tropical curves often used in log geometry), but this work does not require it.

Remark 3.2.2. Since the base change of X/S to $\mathrm{Spec}\,\mathcal{O}_{S,s}^{\mathrm{et}}$ is orientable at x, there is an étale neighbourhood (V, v) of s in S such that X_V/V is orientable at $x \times_s v$.

Lemma 3.3. *Let S be a locally noetherian scheme, X/S a quasisplit nodal curve, s a point of S and x a singular point of X_s. Let x' be a singular point of X/S specializing to x and s' the image of x' in S. Suppose X/S is orientable at x, then it is orientable at x'. Moreover, in that case, there is a canonical bijection between orientations at x and orientations at x'.*

Proof. Suppose X/S is orientable at x and let (C_1, C_2) be an orientation at x. Let F be the connected component of $\mathrm{Sing}(X/S)$ containing x and x'. Pick a (non-canonical) isomorphism

$$\mathcal{O}_{X,x}^{\mathrm{et}} = \mathcal{O}_{S,s}^{\mathrm{et}}[[u, v]]/(uv - \Delta). \tag{1}$$

Permuting u and v if necessary, we can assume C_1 and C_2 come from the zero loci of u and v respectively. Pick a (canonical up to the residue extension $k(s)^{sep} \to k(s')^{sep}$) factorization

$$\mathrm{Spec}\,\mathcal{O}_{S,s'}^{\mathrm{et}} \to \mathrm{Spec}\,\mathcal{O}_{S,s}^{\mathrm{et}} \to S.$$

Then, tensoring by $\mathcal{O}_{S,s'}^{\mathrm{et}}$ in equation (1), we get an isomorphism

$$\mathcal{O}_{X,x'}^{\mathrm{et}} = \mathcal{O}_{S,s'}^{\mathrm{et}}[[u, v]]/(uv - \Delta),$$

and the two connected components C_1' and C_2' of $(X \backslash F) \times_X \operatorname{Spec} \mathcal{O}_{X,x'}^{\text{et}} \times_S F$ come from the zero loci of u and v. Therefore, in order to prove that X/S is orientable at x' and that the order (C_1', C_2') is canonically induced by the order (C_1, C_2), we only need to know that the action of $G = \operatorname{Aut}_{\mathcal{O}_{S,s}^{\text{et}}}(\mathcal{O}_{S,s'}^{\text{et}})$ on $\{C_1', C_2'\}$ is trivial, which is true since the action of G on $\mathcal{O}_{S,s'}^{\text{et}}[[u,v]]/(uv - \Delta)$ preserves the zero loci of u and v. $\qquad \square$

From now on, given an orientation (C_1, C_2) at a singular point x, we will also write (C_1, C_2) for the induced orientation at a singular generization of x.

Definition 3.4 (type of a section)**.** Let X/S be a quasisplit nodal curve with S regular. Suppose X is smooth over a dense open subscheme U of S. Let s be a point of S and x a singular point of X_s at which X/S is orientable. Pick an orientation (C_1, C_2) at x and an isomorphism

$$\widehat{\mathcal{O}_{X,x}^{\text{et}}} = \widehat{\mathcal{O}_{S,s}^{\text{et}}}[[u,v]]/(uv - \Delta_x),$$

where C_1 corresponds to $u = 0$ and Δ_x is a generator of the singular ideal of x in $\mathcal{O}_{S,s}^{\text{et}}$. We call *type at x* any element of the monoid $\mathcal{O}_{S,s}^{\text{et}}/(\mathcal{O}_{S,s}^{\text{et}})^*$ strictly comprised between 1 and the thickness of x (for the order induced by divisibility). There are only finitely many types at x, given by the association classes of the strict factors of Δ_x in $\mathcal{O}_{S,s}^{\text{et}}$ (which is a unique factorization domain).

Let σ be a section of X/S through x. It induces a morphism

$$\widehat{\sigma^{\#}} \colon \widehat{\mathcal{O}_{S,s}^{\text{et}}}[[u,v]]/(uv - \Delta_x) \to \widehat{\mathcal{O}_{S,s}^{\text{et}}}$$

By Lemma 2.4, Δ_x has the same prime factor decomposition in $\mathcal{O}_{S,s}^{\text{et}}$ and in $\widehat{\mathcal{O}_{S,s}^{\text{et}}}$, so there is a canonical embedding of the submonoid of $\widehat{\mathcal{O}_{S,s}^{\text{et}}}/(\widehat{\mathcal{O}_{S,s}^{\text{et}}})^*$ generated by the factors of Δ_x into $\mathcal{O}_{S,s}^{\text{et}}/(\mathcal{O}_{S,s}^{\text{et}})^*$. We call *type of σ at x relatively to (C_1, C_2)* the image of u in in $\mathcal{O}_{S,s}^{\text{et}}/(\mathcal{O}_{S,s}^{\text{et}})^*$. It is a type at x, and since X/S is orientable at x, it does not depend on our choice of isomorphism $\widehat{\mathcal{O}_{X,x}^{\text{et}}} = \widehat{\mathcal{O}_{S,s}^{\text{et}}}[[u,v]]/(uv - \Delta_x)$ as long as C_1 is given by $u = 0$. When they are clear from context, we will omit x and (C_1, C_2) from the notation and just call it the *type of σ*. In general, given a type T at x, there need not exist a section of type T.

If the type of a section σ at x relatively to (C_1, C_2) is equal to the type of a section σ' at x relatively to (C_2, C_1), we say σ and σ' are of *opposite type* at x.

Lemma 3.5. *Let X/S, s, x, (C_1, C_2) and U be as in Definition 3.4, and σ be a section $S \to X$ of type T at x. Let s' be a generization of s. Then there is a singular point of $X_{s'}$ specializing to x if and only if the thickness of x does not map to 1 in $\mathcal{O}_{S,s'}^{\text{et}}/(\mathcal{O}_{S,s'}^{\text{et}})^*$. Suppose it is the case and write x' this singular point, then*

- *if the image of T in $\mathcal{O}_{S,s'}^{\text{et}}/(\mathcal{O}_{S,s'}^{\text{et}})^*$ is either 1 or the thickness of x', then $\sigma(s')$ is a smooth point of $X_{s'}$;*

- *otherwise, the image of T is a type at s', that we still write T, and σ is of type T at x' relatively to (C_1, C_2).*

15

Proof. By Proposition 1.5, if the thickness of x maps to 1 in $\mathcal{O}^{\text{et}}_{S,s'}/(\mathcal{O}^{\text{et}}_{S,s'})^*$, then all points of X mapping to s' and specializing to x are S-smooth, and otherwise there is a unique singular point x' of $X_{s'}$ specializing to s. Suppose the latter holds, then by Lemma 3.3, (C_1, C_2) induces an orientation of X/S at x', and the lemma follows from the definition of types. $\qquad\square$

Remark 3.5.1. One can think of the thickness of x as the relative version of a length, and of the type of a section σ relatively to an orientation (C_1, C_2) as a measure of the intersection of σ with C_1, seen as an effective Cartier divisor locally around x as in Lemma 1.14. In other words, the type is a measure of "how close to C_1" the section is.

Proposition 3.6. *Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over some dense open subscheme U of S. Let σ and σ' be two S-sections of X. Then the union of $(X/S)^{sm}$ with the set of singular points x of X/S at which σ and σ' have the same type (resp. opposite types) is an open subscheme of X.*

Proof. Since the singular locus $\mathrm{Sing}(X/S)$ is finite over S, every singular point x of X/S has a Zariski-open neighbourhood $V \subset X$ containing only S-smooth points and singular generizations of x. Thus, since the smooth locus of X/S is open in X, the proposition reduces to the following claim: if σ and σ' have the same type (resp. opposite types) at a singular geometric point $x \to X$, then they have the same type (resp. opposite types) at every singular generization of x. This claim is true by Lemma 3.5. $\qquad\square$

Definition 3.7. The open subschemes of X described in Proposition 3.6 above are called respectively the *same type locus* and the *opposite type locus* of σ and σ'.

3.2 Admissible neighbourhoods

Here we will show that, when one works étale-locally on the base (in a sense that we will make precise), one can always assume sections of all types exist.

Definition 3.8. Let S be a regular scheme and X/S a nodal curve, smooth over a dense open U of S. Let s be a point of S and (V, v) an étale neighbourhood of s in S. We say (V, v) is an *admissible neighbourhood of s* (relatively to X/S) when the following conditions are met:

1. The curve X_V/V is quasisplit, and orientable at all the singular points of X_v.

2. For any singular point x of X_v with singular ideal $(\Delta_x) \subset \mathcal{O}^{\text{et}}_{S,s}$, all prime factors of Δ_x in $\mathcal{O}^{\text{et}}_{S,s}$ come from global sections of $\mathcal{O}_V$.

3. For every singular point x of X_v (however oriented), there are sections $V \to X_V$ of all types at x.

When $\bar{s} \to S$ is a geometric point with image s and (V, v) an admissible neighbourhood of s with a factorization $\bar{s} \to v$, we will also sometimes call V an *admissible neighbourhood of $\bar{s}$*.

Remark 3.8.1. In the situation of Definition 3.8, if S is strictly local, then it is an admissible neighbourhood of its closed point.

Proposition 3.9. *Let X/S be a nodal curve, where S is a regular and excellent scheme. Then any point $s \in S$ has an admissible neighbourhood.*

Proof. We can assume X/S is quasisplit by Corollary 1.13. By Remark 3.2.2 and since X_s has finitely many singular points $x_1, ..., x_n$, we can assume X/S is orientable at all the x_i. The singular ideal of each x_i has only finitely many prime factors in $\mathcal{O}_{S,s}^{\text{et}}$, so we can shrink S again into a neighbourhood satisfying condition 2. of the definition of admissibility. The fact this V can be shrinked again until it meets all three conditions follows from Lemma 2.5. $\qquad\square$

Remark 3.9.1. If (V, v) is an admissible neighbourhood of a point s of S, then V is not necessarily admissible even at generizations of v (see Example 3.10). Thus, it is not easy a priori to find a good global notion of admissible cover.

Example 3.10. Let $R = \operatorname{Spec} \mathbb{C}[[u, v, w]]$. Then R is regular, local and strictly henselian, and $\Delta := u^2(v - w) - v^2(u + w)$ is prime in R. Let X_K be an elliptic curve over $K := \operatorname{Frac} R$, cut out in $\mathbb{P}_K^2$ by an equation with coefficients in R and with discriminant Δ. The minimal Weierstrass model of X_K is a nodal curve over $\operatorname{Spec} R$, whose closed fiber has exactly one singular point with label ΔR. Let t be the point of $S := \operatorname{Spec} R$ corresponding to the prime ideal (u, v) of R. Then Δ has two prime factors in the étale local ring of S at t (because the invertible elements $v - w$ and $u + w$ of this étale local ring admit square roots). Hence, S is an admissible neighbourhood of its closed point, but not of t.

The next proposition states that admissible neighbourhoods are compatible with smooth (and not just étale) morphisms.

Proposition 3.11. *Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over some dense open subscheme U of S. Let $\pi \colon Y \to S$ be a smooth morphism, y a point of Y and $s = \pi(y)$. Let V be an admissible neighbourhood of s in S, then $V \times_S Y$ is an admissible neighbourhood of y in Y.*

Proof. This follows from Corollary 2.3 and the definition of admissible neighbourhoods. $\qquad\square$

4 Refinements and resolutions

This section is dedicated to techniques aiming at constructing inductively nodal models of a smooth curve with prime singular ideals, starting from any nodal model. When Néron models are concerned, the interest of nodal models with prime labels lies in two facts: they are locally factorial, which is a crucial hypothesis in the existence result of [Hol19b] for separated Néron models of Jacobians, and all their sections factor through their smooth locus, which will allow us to construct Néron models of curves as gluings of smooth loci of nodal models.

4.1 Arithmetic complexity and motivation for refinements

We start by defining what will be our recursion parameter, the *arithmetic complexity* of a nodal curve.

4.1.1 Arithmetic complexity

Definition 4.1. Let M be the free commutative semigroup over a set of generators G. We call *word length* of $m \in M$, and note $wl(m)$, the (unique) $n \in \mathbb{N}^*$ such that we can write $m = \prod_{i=1}^{n} g_i$ with all the g_i in G.

Given a labelled graph $\Gamma = (V, E, l)$ over M and an edge $e \in E$, we call *arithmetic complexity* of e and note n_e the natural integer $wl(l(e)) - 1$. We call *arithmetic complexity* of Γ and note n_Γ the sum of the arithmetic complexities of all its edges.

Given a nodal curve X/S where $S = \operatorname{Spec} R$ is a local unique factorization domain, the semigroup of nontrivial principal ideals of R is the free commutative semigroup over the set of primes of height 1. From now on, we will talk freely about arithmetic complexities of edges of dual graphs, always implicitly referring to this set of generators.

When X/S is quasisplit, we define the *arithmetic complexity* of a closed singular point x, noted n_x, as the arithmetic complexity of the corresponding edge of the dual graph: If S is local, we define the *arithmetic complexity* of X, noted n_X, to be that of its dual graph.

Note that X is of arithmetic complexity 0 if and only if every singular ideal is prime: it is an integer measuring "how far away from being prime" the singular ideals are. Arithmetic complexity is not stable even under étale base change.

4.1.2 Factoriality of completed étale local rings

Lemma 4.2. *Let R be a regular and complete local ring, and Δ be an element of $\mathfrak{m}_R$. Then $\widehat{A} := R[[u,v]]/(uv - \Delta)$ is a unique factorization domain if and only if Δ is prime in R.*

Proof. Suppose that $\widehat{A}$ is a unique factorization domain, and let d be a prime factor of Δ in R. Call S the complement of the prime ideal (u, d) in $\widehat{A}$. Let $\mathfrak{p}$ be a nonzero prime ideal of $S^{-1}\widehat{A}$. Then, p contains a nonzero element $x = ux_u + x_v$, with x_u and x_v in $R[[u]]$ and $R[[v]]$ respectively. Since $\mathfrak{p} \neq S^{-1}\widehat{A}$, we have $d|x_v$. Call n and m respectively the maximal elements of $\mathbb{N}^* \cup \{+\infty\}$ such that $u^n|ux_u$ and $d^m|x_v$. Since x is nonzero, we know that either n or m is finite. If $n \leq m$, then $v^n x = \Delta^n \frac{x_u}{u^{n-1}} + d^n \frac{v^n x_v}{d^n}$ is in $\mathfrak{p}$, and is associated to d^n in $S^{-1}\widehat{A}$, so we obtain $d \in \mathfrak{p}$, from which it follows that $\mathfrak{p} = (u, d)$. If $m < n$, a similar argument shows that $\mathfrak{p}$ contains u^m and thus equals (u, d). Therefore, $S^{-1}\widehat{A}$ has Krull dimension one, i.e. (u, d) has height 1 in $\widehat{A}$. Since $\widehat{A}$ is a unique factorization

domain, it follows that (u, d) is principal in it, from which we deduce that Δ and d are associated in $\widehat{A}$. In particular, Δ is prime in R.

The interesting part is the converse: let us assume that Δ is prime in R. We want to show that $\widehat{A}$ is a unique factorization domain. We first prove that $A := R[u, v]/(uv - \Delta)$ is a unique factorization domain: let p be a prime ideal of A of height 1, we have to show p is principal in A. We observe that u is a prime element of A, since the quotient $A/(u) = R/(\Delta)[v]$ is an integral domain. Therefore, if p contains u, then $p = (u)$ is principal. Otherwise, p gives rise to a prime ideal of height 1 in $A_u := A[u^{-1}]$, which is principal since $A[u^{-1}] = R[u, u^{-1}]$ is a unique factorization domain. In that case, write $pA_u = fA_u$ for some $f \in A_u$. Multiplying by a power of the invertible u of A_u, we can choose the generator f to be in $A \backslash uA$. Since p is a prime ideal of A not containing u, we know p contains f and thus fA. We will now prove the reverse inclusion. Let x be an element of p. The localization $pA_u = fA_u$ contains x, so x satisfies a relation of the form $u^n x = fy$ for some $n \in \mathbb{N}$ and some $y \in A$. But since u is prime in A, we know u^n divides y and x is in fA.

Now, we will deduce the factoriality of $\widehat{A}$ from that of A. The author would like to thank Ofer Gabber for providing the following proof. Let q be a prime ideal of $\widehat{A}$ of height 1, we will show that q is principal. We put $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$, $\widehat{X} = \operatorname{Spec} \widehat{A}$, and $Z = \operatorname{Spec}(\widehat{A}/q)$, so that Z is a prime Weil divisor on $\widehat{X}$. Let η, η' be the generic points of the respective zero loci of u, v in the closed fiber $\operatorname{Spec} k_R[[u, v]]/(uv)$ of $\widehat{X} \to S$. Since u and v are prime elements of $\widehat{A}$, we can once again assume that Z contains neither η nor η'. It follows that the closed fiber of $Z \to S$ is of dimension 0: the morphism $Z \to S$ is quasi-finite, hence finite by [GD60], chapter 0, 7.4. A fortiori, $\widehat{A}/q$ is finite over A, so by Nakayama's lemma, the morphism $A \to \widehat{A}/q$ is surjective. Call p its kernel. Then $A/p = \widehat{A}/q$ is $\mathfrak{m}_A$-adically complete and separated, so it maps isomorphically to its completion $\widehat{A}/p\widehat{A}$. The prime ideal p of A is of height 1 since $\widehat{X} \to X$ is a flat map of normal noetherian schemes. Therefore, p is principal in A, and $q = p\widehat{A}$ is principal in $\widehat{A}$. $\qquad\square$

4.1.3 Factoring sections through the smooth locus

Lemma 4.3. *Let X/S be a quasisplit nodal curve, where $S = \operatorname{Spec} R$ is a regular, strictly local and excellent scheme. Let σ be a section of X/S. Let $s \in S$ be the closed point and $x = \sigma(s)$, then x is either smooth over S, or singular of arithmetic complexity ≥ 1.*

Proof. Suppose by contradiction that x is singular of arithmetic complexity 0. The section σ factors through $\operatorname{Spec} \mathcal{O}_{X,x}$, and gives rise to a $\operatorname{Spec} \widehat{R}$-section of $\operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$. As $\widehat{\mathcal{O}_{X,x}}$ is of the form $\widehat{R}[[u, v]]/(uv - \Delta)$ for some prime $\Delta \in R$, this section is given by a morphism $\widehat{R}[[u, v]]/(uv - \Delta) \to \widehat{R}$, which is fully described by the images of u and v. But Δ is prime in $\widehat{R}$ by Lemma 2.4, so either u or v has invertible image in $\widehat{R}$. Thus the image of s factors through the complement of one of the two irreducible components of $\operatorname{Spec} \widehat{X}_{s,x}$, a contradiction. $\qquad\square$

4.2 Refinements of graphs

In order to reap the benefits from the properties of nodal models with prime labels, all we need is an algorithm that takes an arbitrary nodal model as an input, and returns another one with strictly lower arithmetic complexity.

Definition 4.4. As in [Hol19b], Definition 3.2, for a graph $\Gamma = (V, E, l)$ with edges labelled by elements of a semigroup M, we call *refinement* of Γ the data of another labelled graph $\Gamma' = (V', E', l')$ labelled by M and two maps

$$E' \to E$$
$$V' \to E \coprod V$$

such that:

- every vertex v in V has a unique preimage v' in V';

- for every edge $e \in E$ with endpoints $v_1, v_2 \in V$, there is a chain $C(e)$ from v_1', v_2' in Γ' such that the preimage of $\{e\}$ in $V' \coprod E'$ consists of all edges and intermediate vertices of $C(e)$;

- for all $e \in E$, the length of e is the sum of the lengths of all edges of $C(e)$.

We will often keep the maps implicit in the notation, in which case we call Γ' a *refinement* of Γ and write $\Gamma' \preceq \Gamma$. We say Γ' is a *strict refinement* of Γ and note $\Gamma' \prec \Gamma$, if in addition the map $E' \to E$ is not bijective.

Remark 4.4.1. Informally, a refinement of a graph is obtained by "replacing every edge by a chain of edges of the same total length". Suppose $\Gamma' \preceq \Gamma$, then $\Gamma' \prec \Gamma$ if and only if at least one of the chains $C(e)$ is of length ≥ 2, i.e. if and only if Γ' has strictly more edges than Γ.

Now we want to blow up X in a way that does not affect X_U, but refines the dual graph. We will define *refinements* of curves (Definition 4.5). We can obtain any refinement of a dual graph of X by iterating these refinements of curves, but they exist only étale-locally on the base.

4.3 Refinements of curves

Definition 4.5. Let S be a regular scheme and X/S a quasisplit nodal curve, smooth over a dense open subscheme U of S. Let s be a point of S and x a singular point of X_s at which X/S is orientable, and (C, D) an orientation of X/S at x. Let T be a type at x. We will call T-*refinement* of X (at x, relatively to (C, D)) the blowing-up of X in the sheaf of ideals of a section $S \to X$ through x, of type T. We will often omit x and (C, D) in the notation and just call these T-refinements of X.

A map $X' \to X$ is called a *refinement* if it is a T-refinement for some such $X, x, (C, D), T$.

Remark 4.5.1. • If S is excellent, then any geometric point $s \in S$ has an admissible neighbourhood V by Proposition 3.9, so X_V/V has a T-refinement for any type T at any singular point of X_s.

• Consider any morphism $S' \to S$ where S' is still regular (e.g. any smooth map $S' \to S$). Let x be a singular point of X and x' a singular point of X' of image x. Then any orientation of X/S at x pulls back to an orientation of $X_{S'}/S'$ at x'; any type T at x pulls back to a type T' at x'; and any T-refinement pulls back to a T'-refinement.

• Let $x \in X$ be a singular point at which X/S is orientable, and y a generization of x. By Lemma 3.5, for any type T at x and any T-refinement $X' \to X$, either T corresponds to a type (still noted T) at y, in which case $X' \to X$ is a T-refinement at y, or T becomes trivial at y, in which case $X' \to X$ restricts to an isomorphism above a Zariski neighbourhood of y.

Lemma 4.6. *Let $f : X \to S$ be a quasisplit nodal curve with S regular and excellent. Suppose X is smooth over some dense open $U \subset S$. Let $\sigma \colon S \to X$ be a section and $\phi \colon X' \to X$ be the blow-up in the ideal sheaf of σ.*

*Then ϕ is an isomorphism above the complement in X of $\mathrm{Sing}(X/S) \cap \sigma(S)$. In particular, it is an isomorphism above the smooth locus of X/S, which contains X_U, so X' is a model of X_U.*

Moreover, X' is a nodal curve, and its dual graphs are refinements of those of X. More precisely, let s be a point of S and suppose $\sigma(s)$ is a singular point x of X_s. Choose an orientation (C, D) of $X_{\mathcal{O}^{\mathrm{et}}_{S,s}}$ at x, and call T the type of σ at x relatively to (C, D). Then, the singular ideal of x in $\mathcal{O}^{\mathrm{et}}_{S,s}$ is generated by

$$\Delta_x = \Delta\Delta',$$

where Δ and Δ' are lifts to $\mathcal{O}^{\mathrm{et}}_{S,s}$ of T and of the opposite type of T respectively. Let Γ, Γ' be the respective dual graphs of X and X' at s, and let e be the edge of Γ corresponding to x. Then e has label generated by $\Delta_x = \Delta\Delta'$, and one obtains Γ' from Γ as follows:

• *if e is not a loop, then C and D come from two distinct irreducible components of X_s (that we still call C and D). In that case, Γ' is obtained from Γ by replacing e by a chain*
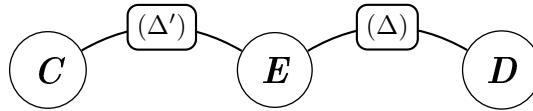
where the strict transforms of C and D are still called C and D, and E is the inverse image of x.

• *if e is a loop, i.e. x belongs to only one component L of X_s, then Γ' is obtained from Γ by replacing e by a cycle*
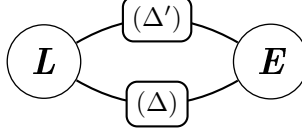
where the strict transform of L is still called L and E is the inverse image of x.

Proof. The ideal sheaf of σ is already Cartier above the smooth locus of X/S and outside the image of σ, so by the universal property of blow-ups ([Sta17, Tag 0806]), we only need to describe ϕ above the étale localizations $\operatorname{Spec} \mathcal{O}_{X,x}^{\mathrm{et}}$, where $x, s, (C, D)$ are as in the statement of the lemma. We can assume $S = \operatorname{Spec} R$ is strictly local, with closed point s. Pick an isomorphism

$$\widehat{\mathcal{O}_{X,x}^{\mathrm{et}}} = \widehat{R}[[u,v]]/(uv - \Delta\Delta')$$

such that C is locally given by $u = 0$. The map

$$\widehat{\sigma} \colon \widehat{\mathcal{O}_{X,x}^{\mathrm{et}}} \to \widehat{R}$$

yielded by σ sends u to a generator of $\Delta\widehat{R}$ and v to a generator of $\Delta'\widehat{R}$. Scaling u and v by a unit of $\widehat{R}$ if necessary, we can assume $\widehat{\sigma}(u) = \Delta$ and $\widehat{\sigma}(v) = \Delta'$.

The completed local rings of $\operatorname{Spec} \mathcal{O}_{X,x}^{\mathrm{et}} \times_X X'$ can be computed using the blowing-up of the algebra $B := R[u,v]/(uv - \Delta\Delta')$ in the ideal $(u - \Delta, v - \Delta')$ (since the completion of B at $(u, v, \mathfrak{m}_R)$ is $\widehat{\mathcal{O}_{X,x}}$).

The latter is covered by two affine patches:

- the patch where $u - \Delta$ is a generator, given by the spectrum of

  $$R[u,v,\alpha]/((v - \Delta') - \alpha(u - \Delta), u\alpha + \Delta') \simeq R[u,\alpha]/(u\alpha + \Delta')$$

  since, in the ring $R[u,v,\alpha]/((v - \Delta') - \alpha(u - \Delta))$, the element $uv - \Delta\Delta'$ is equal to $(u - \Delta)(u\alpha + \Delta')$

- and the patch where $v - \Delta'$ is a generator, where we obtain symmetrically the spectrum of
  $$R[v,\beta]/(v\beta + \Delta)$$

with the obvious gluing maps. Thus we see that X' remains nodal, and that the edge e of Γ (of label $(\Delta\Delta')$) is replaced in Γ' by a chain of two edges, one labelled (Δ) and one labelled (Δ'). It also follows from this description that the strict transform of C (resp. D) in $X' \times_X \operatorname{Spec} \widehat{\mathcal{O}_{X,x}^{\mathrm{et}}}$ contains the singularity of label (Δ') (resp. (Δ)). □

Corollary 4.7. *With the same hypotheses and notations as in Lemma 4.6, for any two sections σ, σ' of X/S, the blow-ups $Y \to X$ and $Y' \to X$ in the respective ideal sheaves of σ and σ' are canonically isomorphic above the same type locus of σ and σ' in X.*

Proof. It suffices to exhibit, for any point $s \to S$ and any singular point x of X_s such that $\sigma(s) = \sigma'(s) = x$ and σ, σ' have the same type T at x, a Zariski neighbourhood V of x in X and an isomorphism $Y \times_X V \to Y' \times_X V$ compatible with the canonical identifications $Y \times_X X^{sm} = X^{sm} = Y' \times_X X^{sm}$. Since X, Y, Y' are of finite presentation over S, this can be done assuming $S = \operatorname{Spec} R$ is strictly local, with closed point s. Using the universal property of blow-ups ([Sta17, Tag 0806]), we reduce to proving that the pull-back of the ideal sheaf of σ' (resp. σ) to Y (resp. Y') is Cartier. The proofs are symmetric, so we will only show that the pull-back to Y of the ideal sheaf of σ' is Cartier. This, in turn, reduces to proving that the ideal sheaf of σ' in $\operatorname{Spec} \widehat{\mathcal{O}^{\text{et}}_{X,x}}$ becomes Cartier in $Y \times_X \operatorname{Spec} \widehat{\mathcal{O}^{\text{et}}_{X,x}}$. Pick an isomorphism

$$\widehat{A} := \widehat{R}[[u,v]]/(uv - \Delta_x) = \widehat{\mathcal{O}^{\text{et}}_{X,x}},$$

where $\Delta_x \in R$ is a generator of the singular ideal of x. The map

$$\widehat{A} \to \widehat{R}$$

corresponding to σ sends u, v to elements Δ, Δ' of $\widehat{R}$ with $\Delta\Delta' = \Delta_x$. Since σ and σ' have the same type at x, there is a unit $\lambda \in \widehat{R}^\times$ such that the map

$$\widehat{A} \to \widehat{R}$$

corresponding to σ' sends u and v to $\lambda\Delta$ and $\lambda^{-1}\Delta'$ respectively. We have reduced to proving that the sheaf given by the ideal $(u - \lambda\Delta, v - \lambda^{-1}\Delta')$ of $\widehat{A}$ becomes Cartier in the blow-up of $\widehat{A}$ in $(u - \Delta, v - \Delta')$. Put

$$A = \widehat{R}[u,v]/(uv - \Delta\Delta'),$$

then it is enough to prove that the ideal $I = (u - \lambda\Delta, v - \lambda^{-1}\Delta')$ of A becomes invertible in the two affine patches (as described in the proof of Lemma 4.6) forming the blowing-up of A in $(u - \Delta, v - \Delta')$. By symmetry, we only check it in the patch generated by $u - \Delta$, which is the spectrum of

$$A_1 = \widehat{R}[u,\alpha]/(u\alpha + \Delta'),$$

where v maps to $\Delta' + \alpha(u - \Delta)$. We have $I = (u - \lambda\Delta, \lambda v - \Delta')$, and in A_1 we can write

$$\begin{aligned}
\lambda v - \Delta' &= \lambda(\Delta' + \alpha(u - \Delta)) + u\alpha \\
&= -\lambda\alpha\Delta + u\alpha \\
&= \alpha(u - \lambda\Delta).
\end{aligned}$$

Thus, the preimage of I in A_1 is the invertible ideal $(u - \lambda\Delta)$, and we are done. $\qquad\square$

4.4   Resolutions of nodal curves

Lemma 4.8. *Let Γ, Γ' be two labelled graphs over a free commutative semigroup. If $\Gamma' \preceq \Gamma$ (Definition 4.4), then $n_{\Gamma'} \leq n_\Gamma$. If $\Gamma' \prec \Gamma$, then $n_{\Gamma'} < n_\Gamma$.*

Proof. Suppose $\Gamma' \preceq \Gamma$. Then, by definition, the sum of lengths of edges of Γ is equal to the sum of lengths of edges of Γ' and Γ' has at least as many edges as Γ, so $n_{\Gamma'} \leq n_\Gamma$.

If equality holds in the latter, then Γ' and Γ have the same number of edges, which, combined with the fact $\Gamma' \preceq \Gamma$, implies they are isomorphic. $\square$

Now we want, starting from X, to find a model of X_U "refining Γ_s as much as possible", in the sense that it will be of arithmetic complexity 0 at s and its dual graph will be a refinement of Γ_s. We do it following the ideas of [dJ96], proposition 3.6, as follows:

Definition 4.9. Let X be a nodal curve over a regular scheme S, smooth over a dense open subscheme $U \subset S$. Let s be a point of S. We call *resolution of X at s* any nodal S-model X' of X_U, obtained by a finite sequence of refinements, and of arithmetic complexity 0 at s. If S is strictly local, we call *resolution of X* any resolution of X at the closed point.

Proposition 4.10. *With the same notations and hypotheses as above, if S is an admissible neighbourhood of s, then X admits a resolution at s.*

Proof. For a nodal model Y of X_U, write n_Y for the arithmetic complexity of Y at s. If $Y \to X$ is a refinement, then S is still an admissible neighbourhood of s relatively to Y/S. Let $X' \to X$ be a finite sequence of refinements minimizing $n_{X'}$, then $X' \to X$ is a resolution. Indeed, suppose it was not, then there would be a closed singular point x of X' of arithmetic complexity ≥ 1. There would exist a type T at x, and a T-refinement $X'' \to X$. We would have $n_{X''} < n_{X'}$, a contradiction. $\square$

Remark 4.10.1. Resolutions are not unique in general. For example, consider $S = \operatorname{Spec} \mathbb{C}[[u,v]]$, and suppose X is a nodal curve over S with dual graph
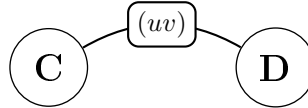
There are two types at the closed singular point x of X/S with respect to (C,D), namely the class T of u and the class T' of v. The T-refinement and the T'-refinement of X are both resolutions, but they are not isomorphic as models of X_U (they are not even isomorphic as schemes as soon as C,D are not isomorphic, e.g. of distinct genera).

Part II

Néron models of nodal curves and their Jacobians

5   Generalities about Néron models

5.1   Definitions

Definition 5.1. Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a U-algebraic space. An *S-model* of Z (or just *model* if there is no ambiguity) is an S-algebraic space X together with an isomorphism $X_U = Z$. A *morphism of S-models* between two models X and Y of Z is an S-morphism $X \to Y$ that commutes over U with the given isomorphisms $X_U = Z$ and $Y_U = Z$.

Definition 5.2. Let S be a scheme and U a scheme-theoretically dense open subscheme of S. Let Z/U be a smooth U-scheme. A *S-Néron model* of Z (or just *Néron model* if there is no ambiguity) is a smooth S-model N satisfying the following universal property, called the *Néron mapping property*:

For each smooth S-algebraic space Y, the restriction map

$$\mathrm{Hom}_S(Y, N) \to \mathrm{Hom}_U(Y_U, Z)$$

is bijective.

Remark 5.2.1. In the litterature, Néron models are often required to be separated and of finite type over the base. Néron models without a quasi-compactness condition are sometimes referred to as Néron-lft models, where "lft" stands for "locally of finite type". We justify the definition above by observing that our Néron models are still unique up to a unique isomorphism by virtue of the universal property: we can always discuss their separatedness or quasi-compactness a posteriori.

Remark 5.2.2. Let S, U, Z be as above, and N be a smooth, separated S-model of Z. Consider a smooth S-algebraic space Y/S and two morphisms $f_1, f_2 \colon Y \to N$ that coincide over U. The separatedness of N/S implies that the equalizer of f_1 and f_2 is a closed subspace of Y containing Y_U, and the latter is scheme-theoretically dense in Y by [GD66, Théorème 11.10.5]. Thus, we automatically have *uniqueness in the Néron mapping property for N/S*, i.e. injectivity of the restriction map

$$\mathrm{Hom}_S(Y, N) \to \mathrm{Hom}_U(Y_U, Z).$$

In particular, a separated S-space satisfying existence in the Néron mapping property (i.e. surjectivity of the restriction map) is automatically the Néron model.

Remark 5.2.3. By descent, Definition 5.2 is unchanged if we only require the Néron mapping property to hold when Y/S is a scheme. Therefore, when we ask if a Néron model exists for X_U/U, we are asking if the functor $Y \mapsto X_U(Y_U)$ on the small smooth site of S is representable by a smooth algebraic space. Similarly, we would get a looser (but still universal) definition of Néron models by asking for them to be smooth algebraic stacks.

5.2   Base change and descent properties

Proposition 5.3 (The formation of Néron models is compatible with smooth base change). *Consider a smooth morphism $S' \to S$, a scheme-theoretically dense open $U \subset S$ and a smooth S-model X of X_U with uniqueness in the Néron mapping property. Then, the base change $X_{S'}$ is a smooth S'-model of $X_{U'}$ with uniqueness in the Néron mapping property. If X is the Néron model of X_U, then X' is the Néron model of $X_{U'}$.*

Proof. First, note that $X_{S'}/S'$ is smooth since X/S is, and that U' is scheme-theoretically dense in S' by [GD66], théorème 11.10.5. Thus, we only need to check that X'/S' has uniqueness in the Néron mapping property, and existence if X/S does.

Let Y' be a smooth S'-scheme. A morphism $Y' \to X'$ is uniquely determined by the two projections $Y' \to S'$ and $Y' \to X$. Since $Y' \to S$ is smooth, it follows that X' has uniqueness in the Néron mapping property. Now, suppose that X is the Néron model of X_U and consider a U'-morphism $u' : Y'_{U'} \to X_{U'}$. Composing with the projection: $X_{U'} \to X_U$, we get a U-morphism $Y'_{U'} \to X_U$, which extends to a unique S-morphism $Y' \to X$ by the Néron mapping property since Y'/S is smooth. Then the induced morphism $Y' \to X'$ extends u'. $\square$

Corollary 5.4. *If S'/S is a cofiltered limit of smooth S-schemes (indexed by a cofiltered partially ordered set, e.g. a localization, a henselization when S is local...), and X is the S-Néron model of X_U, then $X_{S'}$ is the S'-Néron model of $X_{U'}$.*

Lemma 5.5 (Néron models are compatible with disjoint unions on the base). *Let I be a set, $(S_i)_{i \in I}$ a family of schemes, and $(N_i \to S_i)_{i \in I}$ a family of morphisms of algebraic spaces. Write $S = \coprod_{i \in I} S_i$ and $N = \coprod_{i \in I} N_i$. Let U be a scheme-theoretically dense open of S, and write $U_i = U \times_S S_i$ for every i. Then N is the S-Néron model of N_U if and only if for all i in I, N_i is the S_i-Néron model of $N_i \times_{S_i} U_i$.*

Proof. Suppose N is the Néron model of N_U. Then, by Proposition 5.3, for all i in I, N_i is the Néron model of $N_i \times_{S_i} U_i$. Conversely, suppose that for every i in I, N_i/S_i is the Néron model of its restriction to U, and consider a smooth S-algebraic space Y with a morphism $f_u : Y_U \to N_U$. For each i, we

write $Y_i = Y \times_S S_i$. We have

$$
\begin{aligned}
\operatorname{Hom}_S(Y, N) &= \prod_{i \in I} \operatorname{Hom}_{S_i}(Y_i, N_i) \\
&= \prod_{i \in I} \operatorname{Hom}_{U_i}(Y_i \times_{S_i} U_i, N_i \times_{S_i} U_i) \\
&= \operatorname{Hom}_U(Y_U, N_U),
\end{aligned}
$$

where the second equality holds because each Y_i/S_i is smooth. Since N/S is smooth if and only if all the N_i/S_i are smooth, we are done. $\qquad\square$

Proposition 5.6 (Néron models descend along smooth covers)**.** *Let S be a scheme and U a scheme-theoretically dense open of S. Let $S' \to S$ be a smooth surjective morphism and $U' = U \times_S S'$. Let X_U be a smooth U-algebraic space, and let X' be a smooth S'-model of $X_{U'}$. Suppose X' has uniqueness in the Néron mapping property, then it comes via base-change from a smooth S-model X of X_U with uniqueness in the Néron mapping property. If X' is the Néron model of $X_{U'}$, then X is the Néron model of X_U.*

Proof. Call p_1, p_2 the two projections $S'' := S' \times_S S' \to S'$. They are smooth morphisms, so by Proposition 5.3 and uniqueness in the Néron mapping property, we know $p_1^* X' = p_2^* X'$ as both coincide over $U'' = U \times_S S''$. By the effectiveness of fppf descent for algebraic spaces ([Sta17, Tag 0ADV]), X' comes via base change from an S-algebraic space X/S.

The morphism $X \to S$ is smooth since X'/S' is (smoothness is even fpqc local on the base, see [Sta17, Tag 02VL]). Therefore, we only need to prove that X/S has uniqueness in the Néron mapping property, and existence if X' does. Take Y a smooth S-algebraic space, and let

$$ f, g \colon Y \longrightarrow X $$

be two morphisms that coincide over U. Then the pull-backs of f and g by $S' \to S$ coincide over U', hence are equal. Since $S' \to S$ is a cover, we have $f = g$. Now, suppose X' is a Néron model and consider a generic morphism $f_U \colon Y_U \to X_U$. Write Y' (resp. f'_U) for the pullback of Y (resp. f_U) under $S' \to S$. Then Y'/S' is smooth so f'_U extends to a unique $f' \colon Y' \to X'$. We have a cartesian diagram

$$
\begin{array}{ccccc}
Y'' & \rightrightarrows & Y' & \longrightarrow & Y \\
\downarrow\downarrow & & \downarrow & & \big\downarrow \\
X'' & \rightrightarrows & X' & \longrightarrow & X \\
\downarrow & & \downarrow & & \big\downarrow\downarrow \\
S'' & \rightrightarrows & S' & \longrightarrow & S
\end{array}
$$

where $S'' := S' \times_S S'$ and the arrows $S'' \to S'$ are the two projections p_1, p_2, so that all horizontal rows are equalizers. We only need to show that $p_1^* f' = p_2^* f'$, which follows from uniqueness in the Néron mapping property of X''/S'' since they coincide over U''. $\qquad\square$

Proposition 5.7. *Let S be a scheme, $U \subset S$ a scheme-theoretically dense open subscheme, X_U/U a smooth U-scheme and N/S a model of X_U of finite type. Then N is the Néron model of X_U if and only if for every point $s \in S$, $N \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{\mathrm{et}}$ is the $\operatorname{Spec} \mathcal{O}_{S,s}^{\mathrm{et}}$-Néron model of its restriction to U.*

Proof. The "only if" part is a special case of Corollary 5.4. We prove the "if" part. Suppose that for all $s \in S$, N is the Néron model over $\operatorname{Spec} \mathcal{O}_{S,s}^{\mathrm{et}}$ of its restriction to U. Then N/S is smooth at any geometric point, hence smooth. Let Y/S be a smooth S-algebraic space and $f_u : Y_U \to X_U$ a U-morphism. Since Y/S is locally of finite presentation, by [GD66], théorème 8.8.2, every point $s \in S$ has an étale neighbourhood $V_s \to S$ such that f_u extends uniquely to a morphism $Y \times_S V_s \to N \times_S V_s$. By [GD66], théorème 11.10.5, U remains scheme-theoretically dense in every V_s, so these maps glue as in the proof of Proposition 5.6 and f_U extends to a morphism $Y \to N$. $\qquad\square$

5.3   Group models with injective restriction map

In this subsection, we show that any smooth group model of a group algebraic space has a biggest quotient with uniqueness in the Néron mapping property, namely the quotient by the étale part of its unit section (Corollary 5.10). Given a scheme S, we will write S_{sm} (resp. $(\operatorname{Sch}/S)_{\mathrm{sm}}$) for the small smooth site (resp. big smooth site) of S.

We write Ab for the category of abelian groups. We will use without further mention the fact that a smooth S-algebraic space is determined up to a unique isomorphism by the corresponding functor $(S_{\mathrm{sm}})^{\mathrm{op}} \to \operatorname{Ab}$ (by combining the Yoneda lemma with a descent argument).

Definition 5.8. Let S be a scheme and E/S an S-algebraic space. We call *étale part of E over S* the union $E^{\mathrm{et}/S}$ of all open subspaces of E that are étale over S. When there is no ambiguity, we simply call it the *étale part of E* and write it E^{et}.

Lemma 5.9 ([Hol19b], Lemma 5.17)**.** *Let S be a scheme, $U \subset S$ a scheme-theoretically dense open and $f : E \to S$ a S-algebraic space. Suppose that f restricts to an isomorphism over U and that $U \times_S E$ is scheme-theoretically dense in E. Then any section of f factors through E^{et}.*

Proof. The claim is étale-local on S and E, so we may assume that f is a morphism of affine schemes. In particular, f is separated. Then $S \to E$ is a closed immersion through which U factors, hence an isomorphism. A fortiori, f is étale. $\qquad\square$

Corollary 5.10. *Let S be a scheme, $U \subset S$ a scheme-theoretically dense open subscheme, $N_U \to U$ a smooth, separated U-group algebraic space and $f : N \to S$ a smooth S-group model of N_U. Call E the scheme-theoretical closure of the unit section in N. Then, for any smooth S-scheme Y, the sequence of abelian groups*

$$0 \to \operatorname{Hom}(Y, E^{\mathrm{et}}) \to \operatorname{Hom}(Y, N) \to \operatorname{Hom}(Y_U, N_U)$$

is exact. In particular, the quotient space N/E^{et} is a smooth S-group model of N_U with uniqueness in the Néron mapping property.

Remark 5.10.1. In the setting of Corollary 5.10, if N has existence in the Néron mapping property, it follows that N/E^{et} is the Néron model of N_U.

6 Néron models of Jacobians

If $X \to S$ is a morphism of schemes, its relative Picard functor is the fppf sheafification of the functor sending a S-scheme T to the group of isomorphism classes of line bundles on X_T. When X/S is a nodal curve, by [BLR90, Theorem 8.3.1; Theorem 9.4.1] the Picard functor is representable by a smooth, quasi-separated S-group algebraic space $\text{Pic}_{X/S}$, the *Picard space*. We write $\text{Pic}_{X/S}^{\text{tot}\,0}$ for the kernel of the degree map from $\text{Pic}_{X/S}$ to the constant sheaf $\mathbb{Z}$ on S; and $\text{Pic}_{X/S}^0$ for the fiberwise-connected component of identity of $\text{Pic}_{X/S}$, parametrizing line bundles of degree 0 on every irreducible component of every fiber.

A classical way of obtaining a Néron model for the Jacobian J of a proper smooth curve X_U/U with a nodal model X/S, when X is "nice enough", is to consider the quotient P/E, where $P = \text{Pic}_{X/S}^{\text{tot}\,0}$ and E is the closure of the unit section in P, so that P/E is the biggest separated quotient of P (see for example [BLR90, 9.5]). This works well when three conditions are met: P is representable by an S-algebraic space; E is flat over S (so that the quotient is also representable); and $\text{Pic}_{X/S}^{\text{tot}\,0}$ satisfies existence in the Néron mapping property (e.g. X is regular). However, this approach fails most of the time when S is of arbitrary dimension, since E is rarely S-flat ([Hol19b, Theorem 5.16]). The reason is that this method is designed to produce *separated* Néron models, and most Néron models over higher-dimensional bases turn out to be non-separated.

In this section, we will work assuming S is a regular scheme and $U \subset S$ a dense open subscheme. In view of Corollary 5.10, it is tempting to try to construct the Néron model as the quotient of P by the étale part of E. This works when P has existence in the Néron mapping property, i.e. when X is parafactorial along X_U after any smooth base change (e.g. regular). However, even if X_U has nodal models, it may be that none of them remains parafactorial after every smooth base change. We will show that such a model exists after passing to some étale cover of the base, from which it follows by descent that the Néron model exists. We will give a simple combinatorial criterion for the Néron model to be separated, related to the alignment condition of [Hol19b].

6.1 Construction of the Néron model

Proposition 6.1. *Let S be a regular scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U. There exist morphisms of schemes $V \to S$ and $X' \to X_V$ such that*

- $V \to S$ *is an étale cover.*

- X'/V is a nodal model of $X_U \times_S V$.

- For any geometric point $s \to S$, there exists a geometric point $v \to V$ above s such that the singular ideals of X' at v are prime in $\mathcal{O}_{V,v}^{\mathrm{et}}$.

Proof. For any point $s \in S$, pick an admissible neighbourhood V^s of s (Definition 3.8) and a resolution X^s of X (Proposition 4.10). Then $V = \coprod_{s \in S} V^s$ and $X' = \coprod_{s \in S} X^s$ satisfy all three conditions. $\qquad\square$

Theorem 6.2. *Let S be a regular scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U. Then, the Jacobian J of X_U admits a Néron model N over S. Moreover, there exists an étale cover $V \to S$ such that N_V is of the form $\mathrm{Pic}_{X'/V}^{\mathrm{tot}\,0} / E^{\mathrm{et}}$, where X'/V is a nodal model of $X_U \times_S V$ and E is the closure of the unit section in $\mathrm{Pic}_{X'/V}^{\mathrm{tot}\,0}$.*

Proof. We may assume that X/S has a section. Let $V \to S$ and $X' \to X_V$ be as in Proposition 6.1, and put $P = \mathrm{Pic}_{X'/V}^{\mathrm{tot}\,0}$. By Proposition 5.6 and Corollary 5.10, the quotient P/E^{et} comes via base change from a S-group model N of J with uniqueness in the Néron mapping property. Let Y be a smooth S-scheme, we will show that any morphism $f_U \colon Y_U \to J$, extends to a morphism $Y \to N$. By uniqueness in the Néron mapping property for N, it suffices to extend f_U étale-locally on Y around an arbitrary geometric point $y \to Y$. By Corollary 2.3, and by the definition of X', there is an étale neighbourhood Y' of y in $Y \times_S V$ such that the singular ideals of $X' \times_V Y'$ at y are prime. By Lemma 4.2 and Lemma 2.4, shrinking Y', we may assume that $X' \times_V Y'$ is locally factorial. Hence, the generic line bundle corresponding to f_U extends to a line bundle on all of $X' \times_V Y'$, which gives a morphism $Y' \to P$. Composing with $P \to P/E^{\mathrm{et}} \to N$, we obtain a morphism $Y' \to N$ extending f_U. $\qquad\square$

Remark 6.2.1. In the proof above, the quotients of the Picard spaces of the X^s/V^s by the étale parts of the closures of their respective unit sections naturally glue in the étale topology into a S-space, but the X^s themselves do not necessarily glue. However, we could glue them into a locally finite number of local models with similar properties as follows.

Consider the relation R on S given by sRt whenever t specializes s and for some (equivalently, any) geometric points $\bar{t}$ above t and $\bar{s}$ above s, the restriction map of étale stalks induces a canonical isomorphism between prime factors of singular ideals of X at $\bar{t}$ and at $\bar{s}$. Call $\sim$ the transitive closure of R, then it follows from [GD66, Corollaire 9.7.10] that the equivalence classes for $\sim$ are locally constructible subsets of S. In particular, they are locally in finite number. Suppose X/S is orientable everywhere and admits a family of orientations at every singular point compatible with the generization isomorphisms of Lemma 3.3 (which holds on some étale cover of S). For any equivalence class F of $\sim$, using Corollary 4.7, we can find an étale morphism $\pi \colon V \to S$ whose image contains F, and a nodal model of X_U over V whose singular ideals at *every* singular point above F are prime.

Remark 6.2.2. In [HMOP20], the authors describe the *strict logarithmic Jacobian* of a logarithmic curve, an algebraic space with a moduli interpretation.

They show that when X/S is a nodal curve over a toroidal variety, smooth over the complement U of the boundary divisor, then there are canonical log structures on X and S such that the strict logarithmic Jacobian is the Néron model of X_U. In particular, this gives a moduli interpretation for the Néron model constructed in Theorem 6.2 when U is the complement in S of a divisor with normal crossings. The same holds when $U \subset S$ is arbitrary. Indeed, let M_S be the étale subsheaf of monoids of $\mathcal{O}_S$ whose étale stalks are generated by the units and by the prime factors of the singular ideals of X. Let M_X be the submonoid of $\mathcal{O}_X$ whose étale stalk at a geometric point $x \to X$ above a geometric point $s \to S$ is:

- The submonoid of $\mathcal{O}_{X,x}^{\mathrm{et}}$ spanned by $(\mathcal{O}_{X,x}^{\mathrm{et}})^\times$ and $M_{S,s}$ if x is smooth over S.

- The submonoid of $\mathcal{O}_{X,x}^{\mathrm{et}}$ spanned by $(\mathcal{O}_{X,x}^{\mathrm{et}})^\times$, $M_{S,s}$ and local parameters for the two branches of X/S at x if x is singular.

Then, $M_S \to \mathcal{O}_S$ and $M_X \to \mathcal{O}_X$ are logarithmic structures in the sense of [Kat89], but they do not necessarily admit étale-local charts (cf example 3.10). Therefore, $(X, M_X) \to (S, M_S)$ is not a logarithmic curve in the sense of [HMOP20]. However, replacing U by the maximal open subscheme of S over which X is smooth, i.e. on which $M_S = \mathcal{O}_S^\times$, we find that the groupification M_X^{gp} coincides with the direct image of $\mathcal{O}_{X_U}^\times$ in X. Hence, two isomorphism classes of M_X^{gp}-torsors which coincide over X_U are equal. Write $H^1(X, M_X^{gp})^\dagger$ for the subgroup of $H^1(X, M_X^{gp})$ consisting of torsors which, locally on S, come from a line bundle on a refinement of X. Then it follows that $H^1(X, M_X^{gp})^\dagger = \mathrm{Hom}(U, \mathrm{Pic}^0_{X_U/U})$. Combining this with the fact that the formation of M_S and M_X commutes with smooth base change by Corollary 2.3, we find that

$$(\mathrm{Sch}\,/S)^{\mathrm{op}} \to \mathrm{Set}$$
$$T \mapsto H^1(X, M_X^{gp})^\dagger$$

is the Hom functor of the Néron model of $\mathrm{Pic}^0_{X_U/U}$.

6.2 A criterion for separatedness

In this subsection, we exhibit a necessary and sufficient combinatorial condition for the Néron model of Theorem 6.2 to be separated, closely related to the alignment condition of [Hol19b].

Definition 6.3. Suppose S is a regular scheme. Let s be a geometric point of S and $R = \mathcal{O}_{S,s}^{\mathrm{et}}$. Following [Hol19b], Definition 2.11, we say that a labelled graph Γ is *aligned* when for every cycle Γ^0 in Γ, all the labels figuring in Γ^0 are positive powers of the same principal ideal; and that a nodal curve X/S is *aligned at s* when its dual graph Γ_s at s is aligned. We say X/S is *aligned* if it is aligned at every geometric point of S.

We say that Γ_s is *strictly aligned*, or that X is *strictly aligned at s*, when it satisfies the following condition: for any cycle $\Gamma^0 \subset \Gamma_s$, there exists a *prime*

element $\Delta \in R$ such that all the labels of Γ^0 are powers of the principal ideal (Δ) of R. We say that X is *strictly aligned* if it is strictly aligned at every geometric point of S.

Example 6.4. Over $S = \operatorname{Spec}\mathbb{C}[[u,v]]$, at the closed point, among the three following dual graphs, the first is not aligned; the second is aligned but not strictly, and the third is strictly aligned.
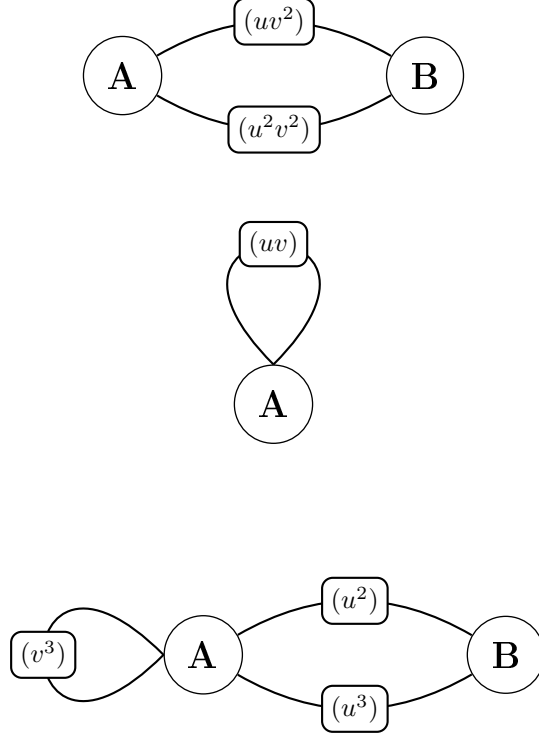
We have to be a little careful about the fact that alignment and strict alignment both deal with étale neighbourhoods. Let us consider two examples.

Example 6.5. The curve over $R = \mathbb{C}[u,v]_{(u,v)}$, given in the weighted projective space $\mathbb{P}_S(1,2,1)$ (in affine coordinates (x,y)) by

$$y^2 = \left((x-1)^2 - (1+v)v^2 + u^2\right)\left((x+1)^2 + (1+v)v^2 - u^2\right)$$

is quasisplit, and its dual graph at the closed point is the following 2-gon:
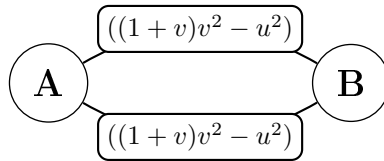
but it is not strictly aligned, even though $((1 + v)v^2 - u^2)$ is a prime element of R. Indeed, $((1 + v)v^2 - u^2)$ is prime in R but has two distinct prime factors in a strict henselization, since $(1 + v)$ becomes a square, and it is the prime factor decomposition in the étale local rings that counts in the definition of strict alignment.

Example 6.6. On the other hand, the equation

$$y^2 = \left((x - 1)^2 - v^2 + u^3\right)\left((x + 1)^2 + v^2 - u^3\right)$$

defines a nodal curve over $\operatorname{Spec} R$ with dual graph

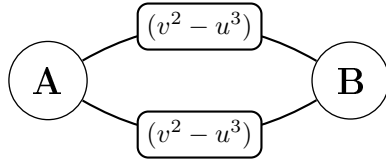which is strictly aligned at the closed point, because $v^2 - u^3$ remains prime in R^{sh}.

Proposition 6.7. *Let S be a regular scheme, $U \subset S$ a dense open, and X/S a nodal curve, smooth over U. Let $P = \operatorname{Pic}^{\operatorname{tot} 0}_{X/S}$, and E be the scheme-theoretical closure in P of its unit section. Then the following conditions are equivalent:*

1. *E/S is flat.*

2. *E/S is étale.*

3. *X/S is aligned.*

Proof. This is [Hol19b], Theorem 5.17. $\qquad\square$

Theorem 6.8. *Let S be a regular and excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U. Call J the Jacobian of X_U/U. Then, the S-Néron model N of J exhibited in Theorem 6.2 is separated if and only if X/S is strictly aligned.*

Proof. First, suppose that N is separated. Let $s \to S$ be a geometric point, we will show that X/S is aligned at s. Since alignment is an étale-local condition on the base, we may assume that S is an admissible neighbourhood of s. Let $Y \to X$ be a resolution of X at s, then the closure of the unit section in $\operatorname{Pic}^{\operatorname{tot} 0}_{Y/S}$ is étale over S by [Hol19b], Theorem 6.2. Hence, Y is aligned at s by Proposition 6.7. Since the singular ideals of Y at s are prime, it follows that Y is strictly aligned at s. By Lemma 4.6, it follows that X is strictly aligned at s.

Conversely, suppose that X is strictly aligned. We will show that N is separated. Since separatedness is a local property on the target for the étale topology, and since strict alignment is preserved by refinements and étale base change, by Theorem 6.2 we may assume that N is the quotient of $P = \operatorname{Pic}^{\operatorname{tot} 0}_{X/S}$ by the étale part of the scheme-theoretical closure E of its unit section. By Proposition 6.7, E is étale over S, so $N = P/E$ is separated. $\qquad\square$

7 Néron models of curves with nodal models

Let S be a regular base scheme, U a dense open subscheme of S, and X/S a nodal relative curve, smooth over U. In what follows, we are interested constructing and describing the S-Néron model of the curve X_U/U.

Under some restrictions on the rational curves appearing in the geometric fibers of X/S, we will show that a Néron model N for X_U exists (Theorem 7.38). When X/S is quasisplit, we will show that if N is separated, then almost all of the connected components of $\mathrm{Sing}(X/S)$ are irreducible. We will refine this into a necessary and sufficient condition for the separatedness of N (Theorem 7.45).

7.1 Factoring sections through refinements

A first question, easier to tackle than existence of a Néron model, is "given a U-point of X_U, can we extend it to a section of a smooth S-model of X_U".

We answer with a two-step strategy: first, when X has no rational component in its geometric fibers, all U-points of X extend to sections by the following result from [GLL14]:

Proposition 7.1 ([GLL14], Proposition 6.2)**.** *Let X/S be a proper morphism of schemes, where S is noetherian, regular and integral. Let K be the function field of S, and suppose that no geometric fiber of X/S contains a rational curve. Then every K-rational point of X_K extends to a section $S \to X$.*

The S-section of X we obtain might meet the singular locus. Our second step consists in finding a refinement of X such that the section comes (at least locally on S) from a smooth section of this refinement.

Lemma 7.2. *Let X be a quasisplit nodal curve over a regular scheme S. Suppose X is smooth over a scheme-theoretically dense open subscheme $U \subset S$. Let σ, τ be sections of X/S, and $\phi \colon X' \to X$ be the blowing-up in the ideal sheaf of τ. Let s be a point of S and suppose $\tau(s)$ is a singular point x of X_s at which X/S is orientable. Then the three following conditions are equivalent:*

1. *The restriction of σ to the étale local ring $\mathrm{Spec}\,\mathcal{O}_{S,s}^{\mathrm{et}}$ factors through a smooth section of $X' \times_S \mathrm{Spec}\,\mathcal{O}_{S,s}^{\mathrm{et}}/\mathrm{Spec}\,\mathcal{O}_{S,s}^{\mathrm{et}}$.*

2. *There exists an étale neighbourhood V of s such that the restriction of σ to V factors through a smooth section of $X' \times_S V/V$.*

3. *Either $\sigma(s)$ is a smooth point of X_s, or $\sigma(s) = x$ and σ and τ are of opposite types at x.*

Proof. Conditions (1) and (2) are equivalent since nodal curves are of finite presentation. We will now prove that (1) and (3) are equivalent. This can be done assuming $S = \mathrm{Spec}\,R$ is strictly local, with closed point s. We can also assume that $\sigma(s) = x$ since otherwise, the equivalence of (1) and (3) follows from observing that ϕ is an isomorphism away from x.

Under these additional hypotheses, let us assume (3) and prove (1). We know that σ factors uniquely through $\operatorname{Spec} \mathcal{O}_{X,x}^{\mathrm{et}}$.

Let us note $\widehat{S} = \operatorname{Spec} \widehat{R}$. We have the following commutative diagram:

$$
\begin{array}{ccc}
W & \longrightarrow & \operatorname{Spec} \widehat{\mathcal{O}_{X,x}} \\
\downarrow & & \downarrow \\
\widehat{S} \quad W_0 & \longrightarrow \operatorname{Spec} \mathcal{O}_{X,x}^{\mathrm{et}} \times_S \widehat{S} \longrightarrow & \widehat{S} \\
\downarrow \qquad \downarrow & & \downarrow \\
S \xleftarrow{\sigma'} \phi^* \operatorname{Spec} \mathcal{O}_{X,x}^{\mathrm{et}} \xrightarrow[\sigma]{\phi} & \operatorname{Spec} \mathcal{O}_{X,x}^{\mathrm{et}} \longrightarrow & S
\end{array}
$$

where $W_0 = \phi^* \operatorname{Spec} \mathcal{O}_{X,x}^{\mathrm{et}} \times_S \widehat{S}$ and $W = W_0 \times_{X \times_S \widehat{S}} \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$, so that all squares are pullbacks, and σ' is the strict transform of σ in X'. Then σ' is a rational map (defined at least over U) and our goal is to prove that it is defined everywhere.

There are sections $\widehat{\sigma}$ and $\widehat{\tau}$ of $\operatorname{Spec} \widehat{\mathcal{O}_{X,x}}/\widehat{S}$ induced by σ and τ respectively. Pick an isomorphism

$$
\operatorname{Spec} \widehat{\mathcal{O}_{X,x}} \simeq \widehat{R}[[u,v]]/(uv - \Delta\Delta'),
$$

where the comorphism of $\widehat{\tau}$ sends u, v to Δ, Δ' respectively. The section $\widehat{\sigma}$ is fully described by the images t_1 of u and t_2 of v in $\widehat{R}$ by its comorphism. Since σ and τ have opposite types at x, there is a unit λ such that $t_1 = \lambda\Delta'$ and $t_2 = \lambda^{-1}\Delta$.

We claim that $\widehat{\sigma}$ factors through $W \to \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$. Since $W \to \operatorname{Spec} \widehat{\mathcal{O}_{X,x}}$ is the blow-up in the ideal $I_{\widehat{\tau}} = (u - \Delta, v - \Delta')$ defining $\widehat{\tau}$, by the universal property of blow-ups ([Sta17, Tag 085U]), it suffices to show that the pull-back of $I_{\widehat{\tau}}$ to $\widehat{S}$ by $\widehat{\sigma}$ is Cartier. Blow-ups commute with completions, so our claim reduces to proving that the ideal $(u - \Delta, v - \Delta)$ of

$$
A := \widehat{R}[u,v]/(uv - \Delta\Delta')
$$

becomes invertible in $\widehat{R}$ when we map A to $\widehat{R}$ via

$$
\begin{aligned}
A &\to \widehat{R} \\
u &\mapsto \lambda\Delta' \\
v &\mapsto \lambda^{-1}\Delta.
\end{aligned}
$$

The image of $I_{\widehat{\tau}}$ under this map is the ideal $I = (\lambda\Delta' - \Delta)$. If $\lambda\Delta' \neq \Delta$, then I is invertible and the claim holds. Otherwise, we reduce to this case by observing that the blow-up of A in $(u - \Delta, v - \Delta')$ is canonically isomorphic to the blow-up of A in $(u - \mu\Delta, v - \mu^{-1}\Delta')$ for any unit μ of $\widehat{R}$, as can be seen in the proof of Corollary 4.7.

Now, let us check that σ factors through X' if and only if $\widehat{\sigma}$ factors through W. Looking at the diagram above, we see that a factorization of $\widehat{\sigma}$ through W yields a factorization of $\sigma \times_S \widehat{S}$ through W_0, which means the (faithfully flat) base change to $\widehat{S}$ of the rational map σ' is defined everywhere, so σ' itself is defined everywhere. Conversely, if σ' is an actual S-section, it yields a section from $\widehat{S}$ to a completed local ring of X', and all completed local rings of X' at points above x factor through W.

We have proven σ factors through a section $\sigma' \colon S \to X'$. We need to show this section is smooth. It suffices to show that $\sigma'(s)$ is a smooth point of X'/S. Call E the preimage of x in X'_s. The point $\sigma'(s)$ must be in E since $\sigma(s) = x$. Looking at the local description of X' in the proof of Lemma 4.6, we see E contains exactly two non-smooth points y and y' of X_s, and there is an isomorphism $\widehat{\mathcal{O}_{X',y}} = \widehat{R}[[\beta, v]]/(\beta v + \Delta)$ such that the natural map $\widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{X',y}}$ sends u, v to $\beta(v - \Delta') + \Delta$ and v respectively. It follows that $\sigma'(s) = y$ if and only if t_2 strictly divides Δ, i.e. if and only if Δ' strictly divides t_1. Symmetrically, $\sigma'(s) = y'$ if and only if Δ strictly divides t_2. Thus, $\sigma'(x)$ is in the smooth locus of X'/S as claimed, and we have proven that (3) implies (1).

Conversely, suppose σ comes from a section $\sigma' \colon (X'/S)^{sm} \to S$. By our additional hypothesis that $\sigma(s) = x$, we know $\sigma'(s)$ is a point of E that is neither y nor y', and it follows from the discussion in the preceding paragraph that σ and τ are of opposite types at x. $\qquad\square$

7.2 First construction of the Néron model

In the previous subsection, we have seen how to factor locally a section of X through the smooth locus of some refinement of X. If we want to approach the Néron mapping property, we would rather have a smooth model of X_U, mapping to X, through which *all* sections will simultaneously factor. Intuitively speaking, we need this model to contain the smooth loci of all possible refinements of X, at all singular points and of all types, after any smooth base change. We will now present the formal construction.

Construction 7.3. Let S be a regular and excellent scheme and X/S a quasi-split nodal curve, smooth over a dense open U of S. For each point s of S, pick an admissible neighbourhood $V^{(s)}$ of s in S (Definition 3.8). We will write $V^{(s,s')}$ the fiber product $V^{(s)} \times_S V^{(s')}$. For each s and each singular point x of X_s, pick an orientation of $X_{V^{(s)}}$ at x. For each type T at x, pick a $V^{(s)}$-section $\tau^{(x,T)}$ of $X_{V^{(s)}}$ of type T at x, and write $X^{(x,T)} \to X_{V^{(s)}}$ the blowing-up in that section. Write
$$X^{tot} = \coprod_{(s,x,T)} (X^{(x,T)}/V^{(s)})^{sm}.$$

Then X^{tot} is a X-scheme, smooth over S. Consider two index triples (s,x,T) and (s',x',T') and call R' the same type locus of $\tau^{(x,T)}|_{V^{(s,s')}}$ and $\tau^{(x',T')}|_{V^{(s,s')}}$. Then R' is an open subscheme of $X_{V^{(s,s')}}$ by Proposition 3.6, and the pull-back $R^{(x,T,x',T')}$ of R' to $(X^{(x,T)}_{V^{(s,s')}}/V^{(s,s')})^{sm}$ is canonically isomorphic to the pull-back of R' to $(X^{(x',T')}_{V^{(s,s')}}/V^{(s,s')})^{sm}$ by Corollary 4.7. Therefore, we have étale

maps

$$R^{(x,T,x',T')} \to (X^{(x,T)}/V^{(s)})^{sm}$$
$$R^{(x,T,x',T')} \to (X^{(x',T')}/V^{(s')})^{sm}.$$

These maps define an étale equivalence relation on X^{tot}. We write N the quotient algebraic space (see [Sta17, Tag 02WW]), and call it the *smooth aggregate* of X.

Proposition 7.4. *With the same hypotheses and notations as in Construction 7.3, N is well-defined, smooth over S, and depends only on X (i.e. if one makes different choices of admissible neighbourhoods $V^{(s)}$ and of sections $\tau^{(x,T)}$, the resulting smooth aggregate N' is canonically isomorphic to N). The map $N \to X$ is an isomorphism above the smooth locus of X/S.*

Proof. First, let us prove that N is well-defined, i.e. that we have indeed given an étale equivalence relation on X^{tot}. For any pair of index triples (s, x, T) and (s', x', T'), the maps

$$R^{(x,T,x',T')} \to (X^{(x,T)}/V^{(s)})^{sm},$$
$$R^{(x,T,x',T')} \to (X^{(x',T')}/V^{(s')})^{sm}$$

are étale since $V^{(s)} \to S$ and $V^{(s')} \to S$ are. These maps jointly form an étale equivalence relation since for any quasisplit nodal curve Y/R with R regular, and any singular point y at which Y/R is orientable, "having the same type at y" is an equivalence relation on the set of sections $R \to Y$. Since X^{tot} is S-smooth, N is also S-smooth. The fact that $N \to X$ is an isomorphism above the smooth locus of X/S follows from observing that all the $X^{(x,T)} \to X_{V^{(s)}}$ are isomorphisms above said smooth locus, and that the $V^{(s)}$ form an étale cover of S.

Now, we have to show that N only depends on X. For every (s, x, T), consider another admissible neighbourhood $W^{(s)}$ of s and a section $\sigma^{(x,T)}$ of $X_{W^{(s)}}$ of type T at x. This gives rise to another smooth aggregate N', and we will prove N and N' are canonically isomorphic. We can assume $V^{(s)}$ and $W^{(s)}$ are admissible neighbourhoods of the same geometric point $\bar{s} \to S$ mapping to s.

First, we will do so assuming that the $W^{(s)}$ are smaller than the $V^{(s)}$ and that the $\sigma^{(x,T)}$ are obtained from the $\tau^{(x,T)}$ via pullback. In that case, there is a canonical map from $X'^{tot} := \coprod_{(s,x,T)} (X^{(x,T)}/V^{(s)})^{sm}$ to X^{tot}, compatible with the étale equivalence relations defining N and N', so we get a canonical map $N' \to N$ of S-algebraic spaces. This map restricts to an isomorphism over the étale stalks of all geometric points $\bar{s} \to S$, so it is an isomorphism.

Now, let us drop the assumption that the $\sigma^{(x,T)}$ are obtained from the $\tau^{(x,T)}$ via pullback. For all (s, x, T), by the special case proven above, we can assume $V^{(s)} = W^{(s)}$. Using Proposition 3.6 and the special case proven above, we can assume (shrinking $V^{(s)}$ if necessary) that $\sigma^{(x,T)}$ and $\tau^{(x,T)}$ have the same type everywhere. It follows from Corollary 4.7 that the blowing-ups in the sheaves

of ideals of $\sigma^{(x,T)}$ and $\tau^{(x,T)}$ are canonically isomorphic, and this holds for all (s, x, T), so $N = N'$ by construction.

Finally, we also drop the assumption that there exist maps of étale neighbourhoods $W^{(s)} \to V^{(s)}$. Then, N and N' are still canonically isomorphic by the special cases above since $W^{(s)} \times_S V^{(s)}$ is an admissible neighbourhood of s that factors through both $V^{(s)}$ and $W^{(s)}$. □

Proposition 7.5. *The formation of smooth aggregates commutes with smooth base change, i.e. if S is a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, N the smooth aggregate of X/S, and Y/S a smooth morphism of schemes, then $N \times_S Y$ is the smooth aggregate of X_Y/Y.*

Proof. Immediate from Proposition 3.11 and Proposition 7.4. □

Corollary 7.6. *If S is a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, N the smooth aggregate of X/S, and Y/S a cofiltered limit of smooth morphisms, then $N \times_S Y$ is the smooth aggregate of X_Y/Y.*

Proposition 7.7. *Let S be a regular and excellent scheme, X/S a nodal curve, smooth over a dense open $U \subset S$, and N the smooth aggregate of X/S. Then every S-section of X/S factors uniquely through N.*

Proof. First, we prove uniqueness: suppose σ comes from two sections σ_0, σ_1 of N/S an let us show that $\sigma_0 = \sigma_1$. Since σ_0 and σ_1 coincide on $N_U = X_U$, it is enough to show that for any $t \in S$ we have $\sigma_0(t) = \sigma_1(t)$. This can be done assuming S is strictly local with closed point t. Describe N as in Construction 7.3 using admissible neighbourhoods $V^{(s)}$ of every point s of S and sections $\tau^{(x,T)}$ of $X_{V^{(s)}}$ of type T at x for every singular point x of X_s and every type T at x. By Proposition 7.4, we can assume none of the $V^{(s)}$ contains t except $V^{(t)}$ and $V^{(t)} = S$. Put $y = \sigma(t)$. If y is a smooth point of X/S, then σ factors through the smooth locus of X/S, above which $N \to X$ is an isomorphism, so we are done. Otherwise, by Lemma 7.2, we see that σ_0 and σ_1 must both factor through the Zariski-open subscheme $X^{(t,y,T)}$ of N, where T is the type at y opposite to that of σ. Since $X^{(t,y,T)}$ is a nodal curve over S, it is S-separated, and we conclude using the fact σ_0 and σ_1 coincide over U.

Next, we have to show existence. We recycle the notations of Construction 7.3. By descent, using the uniqueness part we have already proven, it is enough to show that for all $s \in S$, the section $\sigma^{(s)} := (\sigma, \mathrm{Id})$ of $X_{V^{(s)}}/V^{(s)}$ comes from a map $V^{(s)} \to N$. Put $y = \sigma(s)$. By Lemma 7.2 and Proposition 7.4, we can assume (shrinking $V^{(s)}$ if necessary) that $\sigma^{(s)}$ factors through $X^{(y,T)}$, where T is the type at y opposite to that of σ, so we are done. □

The properties of smooth aggregates proven above allow us to see them as the solutions of a universal problem:

Proposition 7.8. *Let S be a regular, excellent scheme and X/S a quasisplit nodal curve. Then for any smooth S-algebraic space Y together with a morphism $f: Y \to X$, f factors uniquely through the canonical map $N \to X$.*

Proof. The section (f, Id) of X_Y/Y factors uniquely through N_Y by Proposition 7.7 since the latter is the smooth aggregate of X_Y/Y by Proposition 7.5. Projecting onto N, we get the unique map $Y \to N$ through which f factors. $\quad\square$

Corollary 7.9. *Let S be a regular and excellent scheme and $X' \to X$ a morphism between two quasisplit nodal curves over S. Let N be the smooth aggregate of X, then $N \times_X X'$ is the smooth aggregate of X'.*

Now, we are equipped to prove the following result, which is a weak version of our main theorem of existence for Néron models of nodal curves:

Proposition 7.10. *Let S be a regular and excellent scheme and X/S a nodal curve, smooth over a dense open subscheme U of S, with no rational component in any geometric fiber. Then X_U has a Néron model N/S, and there is a canonical morphism $N \to X$ of models of X_U. When X/S is quasisplit, N is the smooth aggregate of X.*

Proof. By Corollary 1.13, Proposition 5.6 and Lemma 5.5, we can assume X/S is quasisplit and S is integral. Let N be the smooth aggregate of X/S. Then N is a smooth S-model of X_U with a canonical S-map $N \to X$. Consider a smooth S-scheme Y, then we have

$$\mathrm{Hom}_U(Y_U, X_U) = \mathrm{Hom}_S(Y, X)$$
$$= \mathrm{Hom}_S(Y, N),$$

where the first equality holds by Proposition 7.1 applied to the connected components of X_Y/Y, and the second by Proposition 7.5. Thus, N/S has the Néron mapping property. $\quad\square$

Remark 7.10.1. As in Remark 6.2.1, we could have constructed the nodal aggregate by gluing only a locally finite number of local nodal models of X_U/U.

The remainder of this section will be dedicated to improving Proposition 7.10 by weakening the hypothesis that the geometric fibers of X/S have no rational components, and determining conditions under which N/S is separated.

7.3 Exceptional components and minimal proper regular models

It is known that an elliptic curve over the fraction field of discrete valuation ring has a Néron model, given by the smooth locus of its minimal proper regular model. It is proven in [LT13] that the same holds for any smooth curve of positive genus. In particular, rational components of the special fiber that can be contracted to smooth points have a "special status": they must map to a mere point of the Néron model. In order to weaken the hypotheses of Proposition 7.10 to allow for rational components, we must take this phenomenon into account. Over a discrete valuation ring, these components of the special fiber that can be contracted to smooth points, the so-called *exceptional components*,

are characterized by Castelnuovo's criterion ([Liu02], Theorem 9.3.8). This criterion uses intersection theory on fibered surfaces, so it is not easy to generalize to higher-dimensional bases for arbitrary relative curves, but the nodal case is much simpler. In this subsection, we discuss an analogue of the notion of exceptional components for nodal curves over bigger regular base schemes.

7.3.1   Definition

Definition 7.11. Let k be a separably closed field and X_k/k a nodal curve. We define a sequence of subsets of the (finite) set I of irreducible components of X_k as follows. Put $J_0 = \emptyset$, and for all $n \in \mathbb{N}$, let J_{n+1} be the subset of I consisting of the components C meeting one of the following conditions:

- C is in J_n;

- C is rational and k-smooth, and intersects $\left(\bigcup_{D \in I - J_n - \{C\}} D \right)$ in exactly one point.

The sequence $(J_n)_{n \in N}$ is increasing, so it is stationary at some subset J of I, which we call the set of *exceptional components* of X.

We call *exceptional trees* the connected components of $\bigcup_{C \in J} C$.

A non-smooth point of X/k is called *exceptional* if it belongs to at least one exceptional component.

When X/S is a nodal relative curve, smooth over a schematic dense open $U \subset S$, we call *exceptional point* of X a singular point, exceptional in a fiber of X over a separably closed field-valued point of S.

If X/S is quasisplit, for any $s \in S$, we define the exceptional components (resp. exceptional points, resp. exceptional trees) of X_s as those giving rise to the exceptional components (resp. points, resp. trees) of $X_{\bar{s}}$ for some geometric point $\bar{s} \to s$.

Remark 7.11.1. Neither the components of X lying in a cycle of the dual graph, nor its components of genus ≥ 1 are exceptional. In particular, the exceptional trees correspond to actual trees of the dual graph, and they are not covering as soon as X is of genus ≥ 1.

7.3.2   The minimal proper regular model

Here, we discuss briefly the case of one-dimensional bases, where there is a canonical *minimal proper regular model* of which the Néron model is the smooth locus.

Proposition 7.12. *Let R be a discrete valuation ring, with field of fractions K and residue field k, and X_K a smooth K-curve of genus ≥ 1. Then X_K admits*

a unique minimal proper regular model X_{min} over S (i.e. X_{min} is a terminal object in the category of proper regular S-models of X_K).

Moreover, if X_K has a regular nodal model X, then X_{min} is nodal and the map $X \to X_{min}$ is just a contraction of every exceptional tree of the special fiber of X into a smooth point (i.e. the image of an exceptional tree of the special fiber of X is a smooth point of X_{min}/S, and $X \to X_{min}$ restricts to an isomorphism over the rest of X_{min}).

Proof. The existence of the minimal proper regular model is [Liu02], Theorem 9.3.21.

For the second part of the proposition, suppose X_K has a nodal regular model X/S. It follows from [Liu02], Definition 3.1 and Theorem 3.8, that there exists a regular proper model X'/S of X_K and a map $X \to X'$ that is just a contraction of every exceptional tree into a smooth point. In particular, X'/S is nodal. But then X' is relatively minimal in the sense of [Liu02], Definition 3.12, so it is X_{min} and we are done. $\qquad\square$

Theorem 7.13 ([LT13], Theorem 4.1)**.** *Let S be a connected Dedekind scheme (i.e. a regular scheme of dimension 1) with field of functions K. Let X_K/K be a proper regular connected curve of genus ≥ 1. Suppose that either S is excellent, or X_K/K is smooth, and let X_{min} be the minimal proper regular S-model of X_K. Then $(X_{min}/S)^{sm}$ is the Néron model of $(X_K/K)^{sm}$.*

7.3.3   Van der Waerden's purity theorem

We will define the exceptional locus of a birational morphism, and cite a result of purity of this exceptional locus when the target is factorial. This will allow us to describe explicitly some open subsets of the Néron model (when it exists) of a curve with a nodal model.

Definition 7.14. Let $f : X \to Y$ be a morphism locally of finite type between two locally noetherian algebraic spaces. We say f is a *local isomorphism* at some $x \in X$ when f induces an isomorphism $\mathcal{O}_{Y,f(x)} = \mathcal{O}_{X,x}$ (or, equivalently, if x has a Zariski open neighbourhood $V \subset X$ such that f induces an isomorphism from V onto its image in Y). The set of all points at which f is a local isomorphism is an open subscheme W of X, and we call its complement the *exceptional locus* of f. If $W = X$, we say f is a *local isomorphism*.

Example 7.15. Let Y be a noetherian integral scheme and $X \to Y$ be the blowing-up along a closed subscheme $Z \to Y$ of codimension ≥ 1. Then the exceptional locus of $X \to Y$ is the preimage of the set of all $z \in Z$ around which Z is not a Cartier divisor.

Example 7.16. Let k be a field, and glue two copies of the identity $\mathbb{A}_k^1 \to \mathbb{A}_k^1$ along the complement of the origin. The resulting map $A \to \mathbb{A}_k^1$, where A is the affine line with double origin, is a local isomorphism.

In Example 7.16, the birational map $f \colon A \to \mathbb{A}_k^1$ has empty exceptional locus, but it is not separated, so in particular not an open immersion. In the follow-

ing lemma, we will see that non-separatedness is essentially the only possible obstruction preventing local isomorphisms from being open immersions.

Lemma 7.17. *Let $f\colon X \to Y$ be a separated local isomorphism between two locally noetherian integral algebraic spaces. Then f is an open immersion.*

Proof. We need to show that f is injective. Call η the generic point of X. Since f is a local isomorphism at η, $f(\eta)$ is the generic point of Y. Consider two points x, x' of X with the same image y in Y. There are Zariski-open neighbourhoods U, U' of x and x' respectively, such that $U \to Y$ and $U' \to Y$ are open immersions. By separatedness of f, the canonical map $U \times_X U' \to U \times_Y U'$ is a closed immersion. But it follows from the fact f is a local isomorphism that $U \times_Y U'$ is integral, with generic point (η, η). Since this point is in the image of $U \times_X U'$, the map $U \times_X U' \to U \times_Y U'$ is an isomorphism, so the point $(x, x') \to X \times_Y X'$ factors through $U \times_X U'$, i.e. $x = x'$. $\qquad\square$

Theorem 7.18 (Van Der Waerden). *Let X, Y be locally noetherian integral schemes with Y locally factorial and $f : X \to Y$ a birational morphism of finite type. Then the exceptional locus of f is of pure codimension one in X.*

Proof. This is [GD67], Theorem 21.12.12. $\qquad\square$

Lemma 7.19. *Let S be a regular scheme, U a dense open subscheme of S, and X/S a quasisplit nodal curve, smooth over U. Let E be the union in X of the exceptional components of all fibers X_s (which are well-defined by quasisplitness). Suppose that X_U admits a Néron model N/S, then the map $(X \backslash E)^{sm} \to N$ extending the identity over U is an open immersion.*

Proof. The scheme $(X \backslash E)^{sm}$ is separated over S, hence separated over N. Therefore, using Lemma 7.17, we only need to prove that the exceptional locus of $(X \backslash E)^{sm} \to N$ is empty. By Proposition 1.7, E is Zariski-closed in X. The morphism of algebraic spaces $g : X^{sm} \to N$ extending the identity over U is birational and of finite type, and the domain and codomain are S-smooth, hence regular. It suffices to show that the exceptional locus E_0 of g is contained in E. Since N has an étale cover by a scheme, it is enough to prove that for any integral scheme V and any étale map $V \to N$, we have $E_0 \times_N V \subset E \times_N V$. The scheme V is smooth over S so it is regular, and $g_V : X^{sm} \times_N V \to V$ is birational and of finite type since g is. Furthermore, since the property "being an isomorphism" is local on the target for the fpqc topology, the exceptional locus of g_V is precisely $E_0 \times_N V$. Thus $E_0 \times_N V$ is either empty or pure of codimension one in $X^{sm} \times_N V$ by Theorem 7.18. If it is empty, we are done. Otherwise, since $E \times_X X^{sm} \times_N V$ is closed in $X^{sm} \times_N V$, it is enough to prove that every point of $E_0 \times_N V$ of codimension 1 in $X^{sm} \times_N V$ is contained in $E \times_N V$. Since $V \to N$ is an étale cover, this is true if and only if every point of E_0 of codimension 1 in X^{sm} is contained in E.

Let x be a point of E_0 of codimension 1. The image ξ of x in S has codimension ≤ 1 in S. Since X/S is smooth over U, ξ must have codimension 1: $\mathcal{O}_{S,\xi}$ is a discrete valuation ring. Since $N \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ is the $\operatorname{Spec} \mathcal{O}_{S,\xi}$-Néron model of its generic fiber by Proposition 5.4, it must coincide with the smooth locus of

the minimal proper regular model of $X \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ by Theorem 7.13. Now, by Proposition 7.12, the minimal proper regular model of $X \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ is the contraction of the exceptional trees of its special fiber into smooth points, and in particular contains $(X \backslash E) \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$ as an open subscheme. In particular, g is an isomorphism at every point of $(X \backslash E)^{sm} \times_S \operatorname{Spec} \mathcal{O}_{S,\xi}$, so x must be in E. □

Remark 7.19.1. With the hypotheses and notations of Lemma 7.19, if E is empty, it follows that the canonical morphism $N_0 \to N$, where N_0 is the smooth aggregate of X, is an open immersion. We will see in the next subsection that one can always reduce to this situation: if E is not empty, one can always contract X into a new nodal model of X_U with no exceptional components. However, we will also see that nodal models with no rational components at all do not always exist, so Néron models cannot always be easily described in terms of smooth aggregates.

7.4 Contractions and stable models

So far, we have met two features of a nodal curve X/S, smooth over a dense open $U \subset S$, that can cause problems for us: one is the complexity of its singularities (for example because there can be sections through a singular point of positive arithmetic complexity, meaning we lose relevant information if we take the smooth locus and forget this point), and the other one is the presence of rational components in its geometric fibers (in the absence of such components, we can construct explicitly a Néron model, see 7.2). Refinements allow us to "turn the first problem into the second": we get a new model of X_U with less complex singularities, but more rational components. Looking at Proposition 7.10, it is clear that we also have an interest in the inverse problem: if X/S has rational components, is it possible to blow them down and obtain a new nodal model with thicker singularities, but less rational components?

This question finds its answer in [F.K83], in which the author introduces and studies contraction morphisms for the moduli stacks of n-pointed stable curves. In this subsection, we will see how this translates into the algorithm we need.

7.4.1 The stack of n-pointed stable curves and the contraction morphism

Definition 7.20 ([F.K83], Definition 1.1.)**.** Let n, g be natural integers such that $2g - 2 + n > 0$. A *n-pointed stable curve of genus g over S* is a nodal relative curve X/S of genus g, together with n pairwise disjoint smooth sections $\sigma_1, ..., \sigma_n$ of X/S, such that for every geometric fiber X_s and every nonsingular rational component C of X_s, the sum of the number of intersection points between C and the union of all other irreducible components of X_s, and of the number of σ_i passing through C, is at least 3. When the sections are clear from context, we will sometimes omit them in the notation.

We define a *morphism* between two n-pointed stable curves $(X'/S', \sigma'_1, ..., \sigma'_n)$

and $(X/S, \sigma_1, ..., \sigma_n)$ as a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ f\ } & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\ g\ } & S \end{array}$$

such that $f\sigma_i' = \sigma_i g$ for all i.

Remark 7.20.1. The condition on the number of special points appearing on a rational component guarantees that the group of S-automorphisms of X as a n-pointed stable curve is finite.

Theorem 7.21. *Call* $\mathcal{M}_{g,n}$ *the category of stable n-pointed curves of genus g. As a category fibered in groupoids over schemes, it is a separated Deligne-Mumford stack, smooth and proper over* $\operatorname{Spec}\mathbb{Z}$.

Proof. This is [F.K83], Theorem 2.7. $\qquad\square$

Definition 7.22 ([F.K83], Definition 1.3.)**.** Let S be a scheme, g a natural integer, and $f : X \to X'$ a morphism of S-schemes between stable pointed S-curves of genus g. We say f is a *contraction*, or *contraction of X*, if:

- X is $n + 1$-pointed and X' is n-pointed, with $2g - 2 + n > 0$, and their respective sections $(\sigma_i)_{1 \leq i \leq n+1}, (\sigma_i')_{1 \leq i \leq n}$ satisfy $f \circ \sigma(i) = \sigma'(i)$ for all $0 \leq i \leq n$.

- For any geometric point $s \in S$, either $X_s \to X_s'$ is an isomorphism, or $\sigma_{n+1}(s)$ is in a rational component C of X_s such that $f(C)$ is a point $x \in X_s'$, and $X_s \backslash C \to X_s' \backslash \{x\}$ is an isomorphism.

Remark 7.22.1. We do not use the same notion of geometric point as [F.K83], but the two subsequent definitions of contractions are equivalent by [Liu02], Proposition 10.3.7.

Theorem 7.23. *Let S be a scheme and X/S a $n + 1$-pointed stable curve of genus g with $2g - 2 + n > 0$. Then X admits a contraction, unique up to a canonical isomorphism.*

Proof. This is [F.K83], Proposition 2.1. $\qquad\square$

7.4.2 The stable model

Definition 7.24. Let S be a scheme and $U \subset S$ a scheme-theoretically dense open. Let X_U/U be a smooth curve of genus $g \geq 2$. We call *stable S-model* of X_U a 0-pointed stable curve X of genus g with an isomorphism $X \times_S U = X_U$.

Lemma 7.25. *Let S be a normal, noetherian and strictly local scheme, $U \subset S$ a scheme-theoretically dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Then X_U has a unique stable model X^{stable}, and there is a unique morphism of models $X \to X^{stable}$.*

Proof. Let $s \in S$ be the closed point. The fiber X_s has finitely many rational components, and there are infinitely many disjoint smooth sections of X/S through each of those. If a rational component E of X_s contains only one singular point, consider two disjoint smooth sections through E, and if it contains two singular points, consider one smooth section through E. This gives a finite number $\sigma_1, ..., \sigma_n$ of sections through X^{sm}. Applying Proposition 1.7, and using the fact that the irreducible components of X are geometrically irreducible by quasisplitness, we see that for any geometric fiber X_t of X/S, any rational component of X_t intersecting the other components in two points contains $\sigma_i(t)$ for some i, and any rational component of X_t intersecting the other components in one point contains $\sigma_i(t)$ and $\sigma_j(t)$ for some $i \neq j$.

Thus X/S endowed with the $(\sigma_i)_{0 \leq i \leq n}$ becomes a n-pointed stable curve of genus g, restricting over U to the data of X_U and the U-sections $\sigma_i|_U : U \to X_U$, and we can apply repeatedly Theorem 7.23 to get a stable 0-pointed curve $X' \to S$ with a map $X \to X'$. Since X_U/U is smooth, the restriction of $X \to X'$ to U just forgets the sections (and induces an isomorphism on the curves), so $X^{stable} := X'$ is a stable model of X_U.

Consider two stable models of X_U, corresponding to two maps $a, b \colon S \rightrightarrows \mathcal{M}_{g,0}$ extending $U \to \mathcal{M}_{g,0}$. Call Z the equalizer of f and g, we have a cartesian diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & S \\
\downarrow & & \downarrow {\scriptstyle (a,b)} \\
\mathcal{M}_{g,0} & \longrightarrow & \mathcal{M}_{g,0} \times \mathcal{M}_{g,0}
\end{array}
$$

where the bottom arrow is the diagonal. Since $\mathcal{M}_{g,0}$ is Deligne-Mumford and separated, its diagonal is finite. Thus, $Z \to S$ is a finite and birational morphism of algebraic spaces, hence an isomorphism by Zariski's main theorem: the stable model X^{stable} is unique up to a unique isomorphism. As for uniqueness of the morphism $X \to X^{stable}$ of models of X_U, let f, g be two such morphisms, then their equalizer is a closed subscheme of X (by separatedness of X^{stable}/S), which contains X_U. But X_U is scheme-theoretically dense in X since U is scheme-theoretically dense in S and X/S is flat, so f and g must be equal. $\square$

Proposition 7.26. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$. Then*

1. *X_U has a stable S-model X^{stable}, unique up to a unique isomorphism, and there is a canonical map $X \to X^{stable}$.*

2. *The formation of X^{stable} commutes with any base change $S' \to S$ such that S' is normal and locally noetherian and $U \times_S S'$ is scheme-theoretically dense in S'.*

Proof. (2) is a consequence of Proposition 1.7. In (1), uniqueness of X^{stable} and of the map $X \to X^{stable}$ holds by the same argument as in the proof of Lemma 7.25 above. We will now prove their existence. Let s be a point of S. By Lemma 7.25, $X_U \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{\text{et}}$ admits a stable model $X^{0,s}$ over $\operatorname{Spec} \mathcal{O}_{S,s}^{\text{et}}$.

45

But then $X^{0,s}/\operatorname{Spec}\mathcal{O}^{\mathrm{et}}_{S,s}$ is of finite presentation, thus comes via base change from a morphism $X^s \to V^s$, where V^s is an étale neighbourhood of s in S. Moreover, by [GD66], Proposition 8.14.2, restricting V^s if necessary, the map $X \times_S \operatorname{Spec}\mathcal{O}^{\mathrm{et}}_{S,s} \to X^{0,s}$ extends to a V^s-map $X \times_S V^s \to X^s$. Restricting V^s once again if necessary, we take this map to be an isomorphism over U. Now, the locus on X^s where X^s/V^s is at-worst nodal is open in X^s and contains X_s so, restricting V_s again if necessary, we can assume X^s/V^s is a nodal curve. Finally, the union of all nonsingular rational components of fibers of X^s/V^s meeting the other components of their fiber in at most two points is closed in X^s, and does not meet X_s, so, restricting V_s one last time, we can assume X^s/V^s is stable. In particular, it corresponds to a morphism $V^s \to \mathcal{M}_{g,0}$.

For any $s, s' \in S$, the diagram of stacks

$$
\begin{array}{ccc}
V^s \times_S V^{s'} & \longrightarrow & V^s \\
\downarrow & & \downarrow \\
V^{s'} & \longrightarrow & \mathcal{M}_{g,0}
\end{array}
$$

commutes by uniqueness of the stable model of $X_{U \times_S V^s \times_S V^{s'}}$.

Applying a similar argument to the triple fibered products, we see the maps $V^s \to \mathcal{M}_{g,0}$ and their gluing isomorphisms satisfy the cocycle condition with respect to the (étale) covering of S by the V^s. Therefore, they come via base change from a map $S \to \mathcal{M}_{g,0}$ i.e. the X^s are obtained via base change from a stable curve X^{stable}/S (which is a model of X_U as desired). Likewise, the local maps $X \times_S V^s \to X^s$ glue to a morphism $X \to X^{stable}$. $\qquad\square$

7.4.3 Rational components of the stable model

We will now determine conditions on X guaranteeing that X^{stable} has no rational components in any geometric fiber. When said conditions are met, this allows us to use Proposition 7.10 to describe explicitly the Néron model of X_U.

Definition 7.27. Let k be a separably closed field and X/k a nodal curve. We say X *has rational cycles* if there is a union of rational components of X that is 2-connected, and *no rational cycles* otherwise. If S is a scheme and X/S a nodal curve, we say X/S *has rational cycles* if a fiber over some geometric point of S does.

Remark 7.27.1. The curve $X/\operatorname{Spec}k$ has rational cycles if and only if there is a cycle of its dual graph in which each vertex corresponds to a rational component. We call such cycles the *rational cycles* of the dual graph.

Remark 7.27.2. If $X/\operatorname{Spec}k$ is of genus $g \geq 2$ and has rational cycles, then every rational cycle of the dual graph either is a loop, or contains a rational component meeting the non-exceptional other components in at least three points.

Definition 7.28. Let k be a separably closed field and X/k a nodal curve. We call *rational loop* of X any singular rational irreducible component of X. If S is a scheme and X/S a nodal curve, we say X/S *has rational loops* if a fiber over some geometric point of S does.

Lemma 7.29. *Let k be a separably closed field, $(Y/k, y_1, ..., y_{n+1})$ a stable $n+1$-pointed curve of genus g over k with $2g - 2 + n > 0$, and $Y \to Z$ the contraction. Then Z has rational cycles if and only if Y does.*

Proof. If $Y \to Z$ is an isomorphism of schemes, it is obvious. Otherwise, there is a rational component C of Y whose image is a point $z \in Z$, and $Y \backslash C \to Z \backslash \{z\}$ is an isomorphism. If C does not belong to a 2-connected union of rational components of Y, we are done. Otherwise, let Γ be said union, the image of Γ in Z is still 2-connected and still contains only rational components. $\square$

Corollary 7.30. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$. Then X^{stable} has rational cycles if and only if X does.*

Rational cycles are an example of rational components we cannot get rid of by contracting. There is another family of such "problematic components": suppose for example that a rational component intersects three other non-rational components, then no contraction will get rid of it. The ones we *can* get rid of are described in the two following lemmas:

Lemma 7.31. *Let k be a separably closed field and $(Y/k, y_1, ..., y_{n+1})$ a stable $(n + 1)$-pointed curve of genus g with $2g - 2 + n > 0$. Suppose Y/k has no rational cycles, and $Y \to Z$ is the contraction. Let F_Y, F_Z denote the union of all rational components meeting the non-exceptional other components in at most two points, in Y and Z respectively. Then $Y \backslash F_Y$ is isomorphic to its image in Z, and the image of F_Y in Z is either F_Z or the union of F_Z and a point.*

Proof. If $Y \to Z$ is an isomorphism of schemes, this is obvious. Otherwise, there is a rational component C of Y whose image is a point $z \in Z$, and $Y \backslash C \to Z \backslash \{z\}$ is an isomorphism. We will prove that the image of F_Y in Z is $F_Z \cup \{x\}$.

The contracted component C meets the other irreducible components of Y in at most two points so C is in F_Y. There is a canonical bijection between irreducible components of Y that are not C and irreducible components of Z (namely, it sends a component of Y to its image in Z), so we just need to prove that a component $D \neq C$ of Y is in F_Y if and only if it is sent to a component of F_Z.

If D is not rational or does not meet C, this is true. Suppose D is rational and meets C. Then $D \cap C$ is exactly one point c_1 (otherwise $D \cup C$ would be 2-connected and Y would have rational cycles).

It is enough to show that D has as many intersection points with non-exceptional other irreducible components of Y than its image D' has in Z. Since the map $Y \backslash C \to Z \backslash \{z\}$ is an isomorphism, it comes down to saying that if C is not exceptional, then the other irreducible component of Y that C meets is not exceptional either, which follows from the definition of exceptional components. $\square$

Lemma 7.32. *Let S be a normal and locally noetherian scheme and $U \subset S$ a scheme-theoretically dense open. Let X/S be a quasisplit nodal curve, smooth over U, of genus $g \geq 2$, with no rational cycles. Let s be a field-valued point of*

S, F the union of all rational components of X_s intersecting the non-exceptional other components in at most two points, and F' its image in X^{stable}. Then

1. *$X \to X^{stable}$ induces an isomorphism between $X_s \backslash F$ and $X_s^{stable} \backslash F'$.*

2. *F' is a disjoint union of points, one for each connected component of F.*

Proof. By quasisplitness we can assume $k(s)$ is separably closed, and base-changing to $\operatorname{Spec} \mathcal{O}_{S,s}^{et}$ (which preserves X^{stable} by Proposition 7.26), we can assume S is strictly local, with closed point s. Then there are S-sections $\sigma_1, ..., \sigma_n$ of X making it a n-pointed stable curve. Consider the sequence $X = X_n \to X_{n-1} \to ... \to X_0 = X^{stable}$, where $X_i \to X_{i-1}$ is the contraction of σ_i. Call F_i the image of F in $(X_i)_s$ for all i, and call G_i the union of all rational components of $(X_i)_s$ meeting the non-exceptional other ones in at most two points. It follows from the preceding lemma that F_i is the union of G_i and a finite number of points. In particular, since $(X_i)_s \to (X_{i-1})_s$ induces an isomorphism from $(X_i)_s \backslash G_i$ to its image, we have $(X_i)_s \backslash F_i = (X_{i-1})_s \backslash F_{i-1}$, and (1) follows inductively. Then, (2) follows from observing that G_0 is empty: indeed, X_s^{stable} is a stable 0-pointed curve over $k(s)$, so it has no exceptional components, thus a component in G_0 would be rational, unmarked, and meet the other irreducible components in at most two points, which is forbidden by the definition of stable curves. $\square$

Proposition 7.33. *Let S be a normal and locally noetherian scheme and X/S a quasisplit nodal curve of genus $g \geq 2$, smooth over a scheme-theoretically dense open $U \subset S$. Let X^{stable} be the stable model of X_U. Then the two following conditions are equivalent:*

1. *There is a geometric point $s \to S$ such that X_s^{stable} has a rational component.*

2. *One of the fibers of X over S contains either a rational loop, or a rational component meeting the non-exceptional other irreducible components in at least three points.*

Proof. If X has rational loops, then X^{stable} has a geometric fiber with a rational component by Corollary 7.30. If a fiber X_s has a rational component E meeting the other non-exceptional other components in at least three points, then any contraction of X_s is an isomorphism over the open subscheme $E \cap (X/S)^{sm}$ of X_s, so X_s^{stable} has a rational component (thus some geometric fiber of X^{stable} also does).

Conversely, suppose X has no rational loops, and every rational component of a fiber X_s meets the other non-exceptional irreducible components of X_s in at most two points. By Remark 7.27.2, X has no rational cycles, so we conclude with Lemma 7.32. $\square$

7.4.4 Singular ideals of the stable model

Now we want to understand precisely what the stable model looks like as a nodal curve, i.e. we want to compute its singular ideals. This will be done in Lemma 7.37. We build up to it with a few technicalities.

Lemma 7.34. *Let $S = \operatorname{Spec} R$ be a trait with generic point η. Let X be a nodal S-model of a smooth η-curve X_η. Then the sum of the thicknesses of the non-exceptional singular points of X is the number of singular points of the special fiber of X_{min}, where X_{min} is the minimal proper regular model of X_η.*

Proof. The sum of the thicknesses of the non-exceptional singular points does not change when one blows up in a singular point of the closed fiber, and after a finite sequence of such blow-ups, we obtain a regular model $X_{reg} \to X$ of X_η. Therefore, we can assume that X is regular. Then, we conclude by Proposition 7.12. $\qquad\square$

Corollary 7.35. *If R is a discrete valuation ring with fraction field K and X_K is a smooth K-curve, then the sum of thicknesses of non-exceptional singular points is invariant among all nodal R-models of X_K.*

Proposition 7.36. *Let $S = \operatorname{Spec} R$ be a regular local scheme with closed point s, U a dense open subscheme of S and $X/S, Y/S$ two quasisplit nodal curves, with $X_U = Y_U$ smooth over U. Then the product of singular ideals of non-exceptional points of X_s is equal to that of Y_s.*

Remark 7.36.1. If we call *thickness* of a singular point its singular ideal, and note additively the monoid of principal prime ideals of R, we can rephrase this as "the sum of the thicknesses of the non-exceptional singular points in the closed fiber is a birational invariant for generically smooth quasisplit nodal curves over a regular local ring".

Proof. Since R is regular, it is a unique factorization domain. Let $\Delta_1, ..., \Delta_k$ be the prime elements of R such that the generic point of $\{\Delta_i = 0\}$ is not in U. Every singular point of X_s has singular ideal of the form $\left(\prod_{i=1}^k \Delta_i^{\nu_i} \right)$ for some integers ν_i, not all zero, and the same goes for Y. Therefore, if we call λ, μ the products of all singular ideals of non-exceptional points of X_s and Y_s respectively, we have integers $n_1, .., n_k, m_1, ..., m_k$ with $\lambda = \left(\prod_{i=1}^k \Delta_i^{n_i} \right)$ and $\mu = \left(\prod_{i=1}^k \Delta_i^{m_i} \right)$, and we only need to show $n_i = m_i$ for all i.

Pick some $1 \leq i \leq k$, let t be the generic point of the zero locus of Δ_i in S and set $T = \operatorname{Spec} \mathcal{O}_{S,t}$. Base-changing to T, we get nodal curves X_T, Y_T, with the same smooth generic fiber. Proposition 1.7 implies that the sum of the thicknesses of their non-exceptional singular points are respectively n_i and m_i, but they must be equal by Corollary 7.35, so $n_i = m_i$ for all i and we are done. $\qquad\square$

The next lemma describes how to compute the singular ideals of X^{stable} from the singular ideals of X.

Lemma 7.37. *Let X/S be a quasisplit nodal curve with no rational cycles, of genus ≥ 2, with S regular. Suppose X is smooth over a dense open $U \subset S$. Let s be a point of S and $F \subset X_s$ be the union of all rational components of X_s intersecting the non-exceptional other irreducible components in at most two points. Call Z the set of non-exceptional singular points of X_s. The image in X^{stable} of a connected component G of F is a smooth point if all singular points in G are exceptional, and a singular point of label $\prod\limits_{y \in Z \cap G} l(y)$ otherwise, where we note $l(y)$ the label of y.*

Remark 7.37.1. To put this in simpler words, in the formalism of Remark 7.36.1, with the additional convention that the points of thickness 0 are the S-smooth points, Lemma 7.37 says that the thickness of a point of X^{stable} is the sum of thicknesses of all non-exceptional singular points of X above it.

Proof. We can assume S is local, with closed point s. Let $\sigma_1, ..., \sigma_n$ be such that X/S endowed with the σ_i is in $\mathcal{M}_{g,n}$ (they exist by quasisplitness). Permuting the σ_i if necessary, we assume that there is an index $0 \leq m \leq n$ such that $\sigma_i(s)$ is in G if and only if $i > m$. Consider the sequence $X_n \to ... \to X_0$, where X_n is X endowed with the σ_i, and each $X_{i+1} \to X_i$ is the contraction of σ_{i+1}: we have $X_0 = X^{stable}$. Call F_i, Z_i the images of F, Z in $(X_i)_s$. For all $y \in Z_i$, we call $l_i(y)$ the singular ideal of y in X_i.

Call G_i the image of G in X_i for every i. Since none of the $\sigma_i(s)$ with $1 \leq i \leq m$ are in G_i, we know that $G_m \to G_0$ is an isomorphism. But G_0 is a point, since it is connected, X_0 is stable, and all components of G are rational and meet the others in at most two points. Thus, G_m is a $k(s)$-point x of X_m.

Now, observe that for all $m < i \leq n$, $\sigma_i(s)$ is in G_i. Therefore, $X \to X_m$ induces an isomorphism $X \backslash G \to X_m \backslash G_m$. In particular, the product of all singular ideals of non-exceptional points of X outside of G is equal to the product of all singular ideals of non-exceptional points of X_m distinct from x. But we also know by Proposition 7.36 that the product of singular ideals of non-exceptional singular points of X and X_m are the same: it follows that if G consists only of exceptional components, then x is smooth over S, and otherwise, x is singular of label $\prod\limits_{y \in Z \cap G} l(y)$. $\qquad\square$

7.4.5 The main theorem

We can now generalize Proposition 7.10 by applying it to the stable model: since the latter is less likely to have rational components than the original nodal model, it is more likely to fall under the hypotheses of the proposition.

Theorem 7.38. *Let S be a regular excellent scheme, $U \subset S$ a dense open subscheme and X/S a nodal curve, smooth over U, of genus $g \geq 2$. Suppose X has no rational loops, and suppose no geometric fiber of X contains a rational component meeting the non-exceptional other irreducible components in three points or more. Then X_U/U has a Néron model N/S. If in addition X/S is quasisplit, then N is the smooth aggregate of the stable model X^{stable} of X_U.*

Proof. We can assume X/S is quasisplit by Corollary 1.13, Proposition 5.6 and Lemma 5.5. Then X/S has a stable model X^{stable}, which has no rational components in any geometric fiber by Proposition 7.33. We conclude by applying Proposition 7.10 to X^{stable}/S. $\qquad\square$

Remark 7.38.1. Our hypotheses on the rational components of the geometric fibers of X/S are quite unnatural, and merely come from the fact the rational components we allow are the only ones that can be contracted *while staying in the realm of nodal curves.* One could maybe get rid of these hypotheses in the following way:

1. Locally on the base if necessary, obtain a (not necessarily nodal) model of X_U with no rational components in any geometric fiber.

2. Try to see if the models obtained this way always fit into a category in which we can solve the universal problem 7.8.

Over one-dimensional bases, the standard way to contract a set E of rational components is to consider the projectivisation of the symmetric graded algebra of a very ample divisor that does not meet E. In higher dimension, however, it is not obvious that the resulting scheme would even remain flat over the base. When $\pi\colon X \to S$ is a $n+1$-pointed curve of genus g with sections $\sigma_1, ..., \sigma_{n+1}$ satisfying $2g - 2 + n > 0$, the contraction of Theorem 7.23 is constructed in [F.K83] as the projectivization of the graded algebra

$$\bigoplus_{i \geq 0} \pi_*(\omega_{X/S}(\sigma_1 + ... + \sigma_n)^{\otimes i}),$$

where $\omega_{X/S}$ is the relative dualizing sheaf; and a key step in the proof consists in showing that $\pi_*(\omega_{X/S}(\sigma_1 + ... + \sigma_n)^{\otimes i})$ is S-flat for $i \geq 2$.

7.5   Separatedness of the Néron model

Here, we investigate when the Néron model of a smooth curve with nodal reduction is separated. Roughly speaking, the defect of separatedness of the Néron model comes from the existence of distinct locally factorial models. In fact, we will show that a separated Néron model exists precisely when there is a canonical "minimal étale-locally factorial model": parallels can be drawn to the case of one-dimensional bases studied in [LT13].

Definition 7.39. Let S be a regular scheme and $X \to S$ a nodal curve, smooth over a dense open $U \subset S$. Let s be a geometric point of S and x a singular point of X_s. We say $\mathrm{Sing}(X/S)$ is *irreducible around x* when the connected component of the singular locus of $X \times_S \mathrm{Spec}\,\mathcal{O}^{\mathrm{et}}_{S,s} / \mathrm{Spec}\,\mathcal{O}^{\mathrm{et}}_{S,s}$ containing x is irreducible. We say $\mathrm{Sing}(X/S)$ is *étale-locally irreducible* if it is irreducible around every singular geometric point of X. We will sometimes omit the word "étale" and just say $\mathrm{Sing}(X/S)$ is *locally irreducible.*

Lemma 7.40. *With the hypotheses and notations of Definition 7.39, $\mathrm{Sing}(X/S)$ is irreducible around x if and only if the singular ideal of x is of the form $(\Delta)^n$, where n is a positive integer and Δ a prime element of $\mathrm{Spec}\,\mathcal{O}^{\mathrm{et}}_{S,s}$.*

Proof. The base change of X/S to $\operatorname{Spec} \mathcal{O}_{S,s}^{\text{et}}$ is quasisplit, so if we write Y the connected component of the singular locus of $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{\text{et}} / \operatorname{Spec} \mathcal{O}_{S,s}^{\text{et}}$ containing x, then the structural morphism $Y \to \operatorname{Spec} \mathcal{O}_{S,s}^{\text{et}}$ is a closed immersion. Therefore, the irreducible components of Y are in bijection with the distinct irreducible factors of the singular ideal of x. $\qquad\square$

Remark 7.40.1. With notations as above, $\operatorname{Sing}(X/S)$ is irreducible around x if and only if the radical of the singular ideal of x is generated by a prime element of $\mathcal{O}_{S,s}^{\text{et}}$: if the singular ideal of x is of the form (Δ^n) with Δ prime in $\mathcal{O}_{S,s}^{\text{et}}$ and $n > 0$, its radical is precisely (Δ).

Proposition 7.41. *Let $S = \operatorname{Spec} R$ be a strictly local unique factorization domain, $U \subset S$ a dense open subscheme, and take two non-units Δ_1, Δ_2 of R with no common prime factor. There exists a trait $T = \operatorname{Spec} A \to S$ such that*

- *The generic point of T is sent to a point of U.*

- *The special point of T is sent to the closed point of S*

- *The images in A of Δ_1 and Δ_2 are equal, nonzero, and not units.*

Proof. Take $\pi : S' \to S$ the blowing-up in the (non-invertible) ideal (Δ_1, Δ_2) of R. Call D_1 and D_2 the strict transforms of the divisors cut out in S by Δ_1 and Δ_2 respectively, and E the exceptional divisor. Let s be a closed point of S' in the zero locus of $\frac{\Delta_1}{\Delta_2} - 1$ (which is contained in $E \backslash (D_1 \cup D_2)$). By [GD61], Proposition 7.1.9, there exists a trait $T \to S'$ such that the closed point is mapped to s and the generic point to a point of U. The map $T \to S'$ factors through $\mathcal{O}_{S',s}$ and $\Delta_1 = \Delta_2$ in $\mathcal{O}_{S',s}$, so $T \to S$ satisfies all the desired properties. $\qquad\square$

Remark 7.41.1. Though over one-dimensional bases, uniqueness in the Néron mapping property is already a weaker condition than separatedness, Proposition 7.41 illustrates the fact that the gap between these two conditions becomes much greater in higher base dimension. Indeed, as the base gets bigger, smooth morphisms of base change remain pretty rare and tame, while the quantity (and array of potentially wild behavior) of traits on the base gets much bigger. This is why non-separated Néron models are so prevalent in higher dimension, even though, to the author's knowledge, no examples are known over a Dedekind scheme.

Lemma 7.42. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. Suppose X_U has a separated Néron model over S. Then $\operatorname{Sing}(X/S)$ is irreducible around every non-exceptional (Definition 7.11) singular geometric point x of X/S.*

Proof. Call N the Néron model of X_U. We will work by contradiction, assuming there exist a geometric point $s \in S$ and a singular point $x \in X_s$ around which $\operatorname{Sing}(X/S)$ is not irreducible. Since $X_U \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{\text{et}}$ has a Néron model by 5.4, we can assume x is a closed point of X and $S = \operatorname{Spec} R$ is strictly local with closed point s. In particular, S is an admissible neighbourhood of s relatively

to X/S. Let $\prod\limits_{i=1}^{r} \Delta_i^{\nu_i}$ be the decomposition in prime factors of a generator Δ_x of the singular ideal of x in R. By hypothesis, we have $r \geq 2$.

Let (C_1, C_2) be an orientation of X/S at x. Define T_1 and T_2 to be the images in the (multiplicative) monoid $R/R^\times$ of $\prod\limits_{i=1}^{r-1} \Delta_i^{\nu_i}$ and $\Delta_r^{\nu_r}$ respectively. They are opposite types at x. For $j \in \{1, 2\}$, consider a section $\sigma_j : S \to X$ of type T_j at x relatively to (C_1, C_2). We define X_j as the blowing-up of X in the ideal sheaf of σ_{1-j} (note the index). Call F the union of all exceptional components of X, and F_1, F_2 the preimages of F in X_1 and X_2 respectively. Since S is strictly local, X/S is quasisplit so F, F_1, F_2 are well-defined closed subsets of X, X_1, X_2 respectively. By Lemma 4.6, we know that F_i is the union of all the exceptional components of X_i for $i = 1, 2$. By Lemma 7.19, the canonical maps

$$(X_i \backslash F_i)^{sm} \to N$$

are open immersions. Thus, they induce isomorphisms onto open subspaces V_1, V_2 of N. Put $V = V_1 \cup V_2$ of N. Then V is isomorphic to the gluing of $(X_1 \backslash F_1)^{sm}$ and $(X_2 \backslash F_2)^{sm}$ along the preimages of $V_1 \cap V_2$ in each of them. We will conclude by proving that V is not separated, which is absurd since it is an open subspace of N.

Using Lemma 7.2, we find that σ_1 factors through X_1^{sm}. Since x is not in F, σ_1 even factors through a section $\sigma_1' : S \to (X_1 \backslash F_1)^{sm}$. However, $X_2 \to X$ is precisely the blowing-up in the ideal sheaf of σ_1, so σ_1 does not factor through X_2. Symmetrically, σ_2 factors through a section $\sigma_2' : X_2^{sm} \backslash F_2$, but not through X_1. In particular, the image x_j of $\sigma_j'(s)$ in N is not in $V_1 \cap V_2$, so $x_1 \neq x_2$.

Proposition 7.41 gives a trait $T = \operatorname{Spec} A \to S$, with generic point η and closed point t, such that η is sent to a point of U and t to s, and such that the images in A of $\left(\prod\limits_{i=1}^{r-1} \Delta_i^{\nu_i} \right)$ and $\Delta_r^{\nu_r}$ are equal, nonzero, and not units. In particular, the T-sections $(\sigma_1)_T$ and $(\sigma_2)_T$ of X_T induced by σ_1 and σ_2 are equal (because $\left(\prod\limits_{i=1}^{r-1} \Delta_i^{\nu_i} \right)$ and $\Delta_r^{\nu_r}$ are equal in A). Since η is sent to a point of U, it follows that $(\sigma_1')_T(\eta) = (\sigma_2')_T(\eta)$ in V_T. However, $(\sigma_1')_T(t) = \sigma_1'(s) = x_1 \neq x_2 = \sigma_2'(s) = (\sigma_2')_T(t)$ since x_1 is in $V_1 \backslash V_2$ and x_2 in $V_2 \backslash V_1$: V does not satisfy the valuative criterion for separatedness, a contradiction. $\qquad\square$

Example 7.43.   • Local irreducibility of the singular locus is only necessary around non-exceptional points: for example, consider $S = \operatorname{Spec} \mathbb{C}[[a, b]]$ and $U = D(ab)$, and take the elliptic (so a fortiori nodal) curve X/S cut out in the weighted projective space $\mathbb{P}^2(2, 3, 2)$ by $y^2 = x(x^2 - z^2)$, so that X is the S-Néron model of X_U. Consider the blowing-up $X' \to X$ of X in the sheaf of ideals $\mathcal{I}$ given by (y, abz). Since $\mathcal{I}$ is Cartier outside of the zero locus of ab, we have $X_U' = X_U$: in particular, X_U' has a Néron model over S (namely X). However, computing the blowup explicitly, we find that X' is nodal and that its closed fiber consists of two irreducible components, intersecting in a point p of label (ab): the singular locus is not irreducible around p, but p is exceptional.

- Local irreducibility of the singular locus around non-exceptional points is not sufficient either: take X to be a nodal curve over $S = \operatorname{Spec} \mathbb{C}[[a, b]]$, whose closed fiber has two irreducible components C_1 and C_2 of genus 1, intersecting in a singular point of label (ab). In this case, X is smooth over the dense open $U = D(ab)$ of S and the singular point is not exceptional, so by (2), X_U has no Néron model over S. Take $X' \to X$ to be the (C_1, a)-refinement, this means X'_U has no Néron model over S, even though X' has étale-locally irreducible singular locus. However, we will see in Theorem 7.45 that if X has no rational components in any geometric fiber, the condition (which then just becomes "X has étale-locally irreducible singular locus") is necessary and sufficient.

As seen in Example 7.43, there are situations in which we cannot conclude to nonexistence of a Néron model by applying directly Lemma 7.42, but we can if we apply it to a different nodal model. This argument can be made systematic, and gives the following (more restrictive) necessary condition:

Lemma 7.44. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. Suppose X_U has a separated Néron model over S. Then the following two conditions are met:*

- *The singular locus $\operatorname{Sing}(X/S)$ is irreducible around every non-exceptional singular geometric point of X/S.*

- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}_{S,s}^{\mathrm{et}}$ have the same radical.*

Proof. By Corollary 5.4, we can assume S is strictly local. In particular, X/S is quasisplit. If X/S is of genus 1, this is a special case of Theorem 6.8. Otherwise, we can apply Lemma 7.42 to the stable model X^{stable} and conclude using Lemma 7.37. □

Remark 7.44.1. In the genus 1 case, there is no 0-pointed stable model, but we could use a 1-pointed stable model instead of referring to our work on Jacobians.

One could wonder if the necessary conditions of Lemma 7.44 are still too weak. We will now show that, when we know the Néron model exists, its separatedness is equivalent to the conjonction of these conditions.

Theorem 7.45. *Let X/S be a nodal curve of genus ≥ 1, smooth over some dense open $U \subset S$, with S regular and excellent. If X_U has a separated Néron model over S, then the two following conditions are met:*

- *The singular locus $\operatorname{Sing}(X/S)$ is irreducible around every non-exceptional singular geometric point of X/S.*

- *For any geometric point $s \to S$, if a rational component E of X_s meets the non-exceptional other components of X_s in exactly two points x and y, then the singular ideals of x and y in $\mathcal{O}_{S,s}^{\mathrm{et}}$ have the same radical.*

Conversely, suppose these conditions are met. Suppose in addition that no geometric fiber of X/S contains either a rational cycle or a rational component meeting the non-exceptional other components in at least three points. Then the Néron model of X_U/U exhibited in Theorem 7.38 is separated over S.

Proof. The first part of the theorem is Lemma 7.44. We will now prove the "conversely" part. Let N/S be the Néron model of X_U exhibited in Theorem 7.38. Separatedness of N/S can be checked over the étale stalks of S: we may assume that $S = \operatorname{Spec} R$ is strictly local, with closed point s. In particular, X/S is quasisplit. By Proposition 7.33, we know X^{stable} has no rational components in any geometric fiber, and by Lemma 7.37, we can assume that $X = X^{stable}$ while preserving all the hypotheses made on X. Then, N is the smooth aggregate of X/S. By Remark 7.40.1, and since $\operatorname{Sing}(X/S)$ is irreducible around every singular geometric point of X/S, we find that the radicals of the singular ideals of X at points of X_s remain prime in any étale localization of S. In particular, S is an admissible neighbourhood of *all* of its points. For every pair (x,T) where x is a singular point of X_s and T a type at x, call $X^{(x,T)}$ the blowing-up of X in a section of type T at x. It follows that N is the gluing of the $(X^{(x,T)}/S)^{sm}$ along the strict transforms of $(X/S)^{sm}$ in each of them.

Now, for any singular point y of X_s, the singular ideal of y in R is of the form $(\Delta_y^{\nu_y})$, where Δ_y is a prime element of R and ν_y a positive integer. Consider the morphism $X' \to X$ obtained as a composition of $\nu_y - 1$ blowing-ups in sections through points of positive arithmetic complexity above y, it follows that all the $X^{y,T}$ factor uniquely through X'. Repeating the process for every x, we find a nodal model X_{min} of X_U of arithmetic complexity 0, with a map $X_{min} \to X$, such that N is the smooth locus of X_{min}. In particular, N is separated. $\quad\square$

Remark 7.45.1. Here, we only constructed X^{min} locally, but when X/S is quasisplit and the hypotheses of the "conversely" part of Theorem 7.45 are met, these local models always glue into a canonical "minimal étale-locally factorial model" of X_U, of which the Néron model is the smooth locus.

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