

# A (not so) gentle introduction to networks in ecology

Timothée Poisot

Université du Québec à Rimouski

August 26, 2013

# Why should I care about networks?

- ▶ A good way to harness complexity
- ▶ A solid mathematical foundation
- ▶ Elegant algorithms
- ▶ Visualisations look *awesome*
- ▶ I'm going to talk about them for, like, two hours...

# What is a network?

## A mathematical approach

A *graph* is a **representation** of a **set of objects** where some pairs of objects are **connected** by **links**.

Or more formally,  $G = (V, E)$ , a *graph*  $G$  is an ordered pair of *vertices*  $V$  linked together by *edges*  $E$ .

Each element of  $E$  is a two-element subset of  $V$ .

The *order* of a graph is  $|V|$ , and its *size* is  $|E|$ .

# What is a network?

## An example

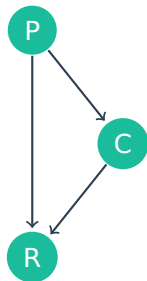
In an omnivory scenario, one top predator  $P$  consumes both an intermediate consumer  $C$  and a primary producer  $R$ . The intermediate consumer also consumes the producer.

This network is specified by

$$G = (\{P, C, R\}, \{\{P, C\}, \{P, R\}, \{C, R\}\})$$

Or for brevity

$$G = (\{P, C, R\}, \{PC, PR, CR\})$$



# What is a network?

## The adjacency matrix

Networks are often represented by their **adjacency matrix**.

The adjacency matrix **A** of a graph  $G = (V, E)$  has elements  $A_{ij}$  with value 1 if there is an edge between the node  $V_i$  and the node  $V_j$ , and 0 otherwise.

It follows that  $\sum_i \sum_j A_{ij} = |E|$ .

# What is a network?

## Edge direction

Edges can be *directed* (arcs, directed edges) or not. An edge between a vertex and itself (cannibalism) is a *self-loop*.

In an **undirected graph**, there are at most  $|V|(|V| - 1)/2$  edges if there are no *self-loops*.

In a **directed graph**, there are at most  $|V|(|V| - 1)$  edges if there are no *self-loops*.

**Exercise:** What is the maximal size of a graph of order  $n$  if there are self-loops?

# What is a network?

## Edge weight

Edges in a network can have a **weight** (for example, the number of contacts between individuals).

The elements of the adjacency matrix **A** can be given *continuous* values.

It's possible to work both on the *weighted* and *unweighted* properties of a graph. **However**, there are many methods that (as of now) can only be applied to **undirected, unweighted** networks.

## Number of partners

The number of vertices *receiving* a link from a focal vertex are called its **successors**

The number of vertices *establishing* a link towards a focal vertex are called its **predecessors**

The *total* number of edges connected to a focal vertex is this vertex **degree**



## Some remarks on degrees

In a graph with  $n$  nodes, the degree  $k$  of a node  $i$  is  $k_i = \sum_{j=1}^n A_{ij}$  (iff the graph is undirected).

If there are  $m$  edges, there are  $2m$  ends of edges, with  $2m = \sum_{i=1}^n k_i$ . In an undirected graph, this is equal to the sum of degrees.

$$\text{Thus, } m = \frac{1}{2} \sum_{i=1}^n k_i = \frac{1}{2} \sum_i \sum_j A_{ij}$$

As the mean degree of nodes is simply  $c = \frac{1}{n} \sum k_i$ , we can write  $c = \frac{2m}{n}$ .

# Some exercises on degrees

## Exercise(s):

Using the same approach, find **the expression of the mean number of predecessors and successors** in a directed graph.

Discuss the relationship between these two values.

Is the mean degree different in a directed/undirected network?

# Unipartite and bipartite graphs

In a **bipartite** graph, vertices are divided into two distinct sets, with edges established between two vertices from *different* sets.

This do not happen in **unipartite** graphs.

**Exercise:** Discuss the application of bipartite and unipartite networks to ecological systems.

In a bipartite graph, the adjacency matrix is (formally) called the **incidence matrix**.

# Trees and cycles

# Paths

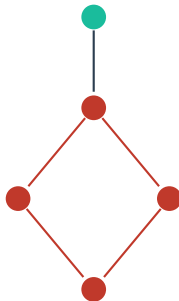
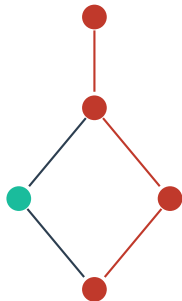
A **path** is a sequence of nodes, so that every consecutive pair of nodes are connected through an edge. You can “walk” along edges from the first to the last node along a path.

The **length** of a path is the number of *edges* in that path.

If nodes  $i$  and  $j$  are connected, then *there is a path of length 1 between them*. If they are connected through  $k$ , there is *a path of length 2*.

# Cycles

A **cycle** is a path from node  $i$  to itself, visiting any number of nodes in between.



## Reminder: multiplying matrices

- ▶ Only *square* matrices can be raised to an exponent (**Exercise:** What of bipartite rectangular networks?)
- ▶  $(\mathbf{AB})_{ij} = \sum_{p=1}^m A_{ip}B_{pj}$
- ▶  $\mathbf{A}^0 = \mathbf{I}$
- ▶  $\mathbf{A}^2 = \mathbf{AA}$
- ▶  $(r\mathbf{A})^k = r^k\mathbf{A}^k$
- ▶  $\det(\mathbf{A}^k) = \det(\mathbf{A})^k$  because  $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

# Number of paths

The number of paths of length 2 between  $i$  and  $j$  is given by

$$N_{ij}^{(2)} = \sum_k 1^n A_{ik} A_{kj}$$

.

This is generalized for paths of length  $n$  to

$$N_{ij}^{(n)} = (\mathbf{A}^n)_{ij}$$

.



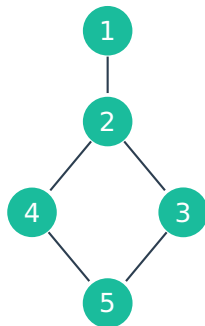
# Number of paths

A little bit of programming

## Exercise

Write this undirected network as an adjacency matrix, and calculate the number of paths of length 1, 2, ..., between species 1 and 3, and between species 1 and 5.

Reminder:  $\mathbf{A} \times \mathbf{B} = \mathbf{A} \%*\% \mathbf{B}$



# Number of paths

Find the number of cycles

Based on the previous result, we can identify the number of paths of size  $n$  from one node to itself,  $(\mathbf{A}^n)_{ii}$ .

Because for square matrices,  $\text{Tr}(\mathbf{A}) = \sum \text{diag}(\mathbf{A})$ , the *total number of cycles of length  $n$*  is given by  $L_n = \text{Tr}(\mathbf{A}^n)$ .

# Identify the shortest path

A **very crude** way to find the shortest path between  $i$  and  $j$  is to find the *smallest value of  $n$*  for which  $(\mathbf{A}^n)_{ij} > 0$ .

The shortest path is called the **geodesic** path.

This method is inefficient, and modern software implements much better search strategies.

## Other remarkable paths

An **Eulerian path** visits each *edge* once. The graph is Eulerian (unicursal) if this path is a cycle, and semi-Eulerian (traversable) if not.

A **Hamiltonian path** visits each *node* once. A graph with a Hamiltonian path is traceable.

**Exercise:** Prove that for a complete undirected graph with  $n$  nodes, there are  $(n - 1)!/2$  Hamiltonian paths.

# Where is the ecology in all that?

**graph** The whole community, *i.e.* the populations and their interactions

**vertices** The composition of the community (species present)

**edges** The interactions between the populations

# Where is the ecology in all that?

Example of “networkable” systems

- ▶ Trophic systems
- ▶ Plant–pollinators
- ▶ Hosts–parasites
- ▶ Mutualism
- ▶ Social interactions

Any system in which the **same ecological interaction** happens several time in a community can (should) be studied using network theory

# Connectance

Connectance is the *proportion of possible interactions realized* (also called the density  $\rho$  in graph theory)

The order of an ecological network is usually called  $S$ , and its size  $L$

Ecologists often define connectance as  $Co = L/S^2$ .

**Exercise:** What do you think of this definition based on results about the maximal order of a graph or size  $S$ ?

## Number of partners

The number of (e.g.) preys of a predator is its **generality** (number of successors)

The number of (e.g.) predators of a prey is its **vulnerability** (number of predecessors)



## Number of partners

```
1 web = read.table('web.dat')
2 generality = rowSums(web)
3 vulnerability = colSums(web)
4 degree = generality + vulnerability
```

# Null models

## The ecological use

Even just *by chance*, networks will have statistical properties. The truth is in the residuals.

**Null models** allow to disentangle the random and non-random properties of networks.

Example of a question: is a given network more or less nested than expected by chance?

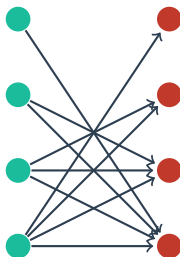
# Null models

WTF is nestedness?

In a *nested* network, a species with a degree  $d_i$  interacts with a subset of the species with a degree  $d_j$  so that  $d_j > d_i$ .

The more this is true, the more a network is nested.

```
1 library(vegan)
2 web = read.table('matrix.txt')
3 nestednodf(web)
```



# Null models

## The basic approach

We want to simulate *pseudo-random* networks that reproduce *some* features of the empirical (observed) network.

For any empirically measured property  $X$ , we will have a distribution of  $\mathcal{X}$ .

We might want to test that  $\sum(X - \mathcal{X})/n = 0$ , *i.e.* the observed property is not significantly different from the random expectation.

# Null models

## Controlling for mean degree

This is formally called a *Erdős–Renyi graph*. Each pair of nodes has a probability  $\rho$  of being connected.

**Exercise:** write a function in R filling a matrix with 0 and 1 at random.

# Null models

## Controlling for node degree

We want each node to receive and establish interactions (almost) as in the original network.

$$P(i \rightarrow j) = \frac{1}{2} \left( \frac{g_i}{d_i} + \frac{v_j}{d_j} \right)$$

**Exercise:** Using operations on matrices, write a function to generate a random network in R.

# The niche model of food webs