Introduction to Applied Bayesian Statistics
Part I: History, Philosophy, and Motivation
Part II: Introduction to Probability

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Outline

Who were Bayes, Price, and Laplace?

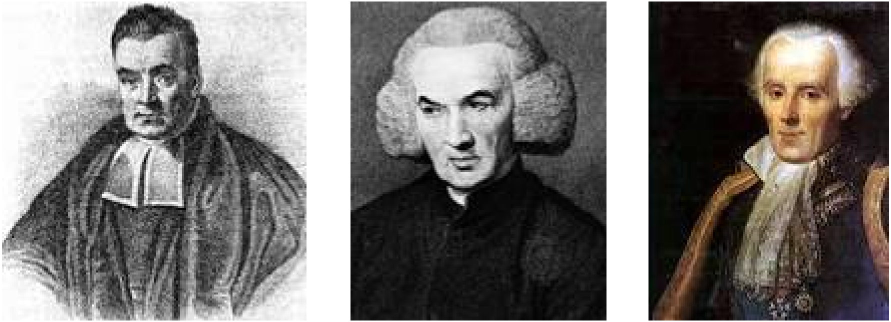


Figure: Major players in creation of Bayes' rule

Sharon Bertsch McGrayne [†]

- **McGrayne Bios (5:41-9:55)**

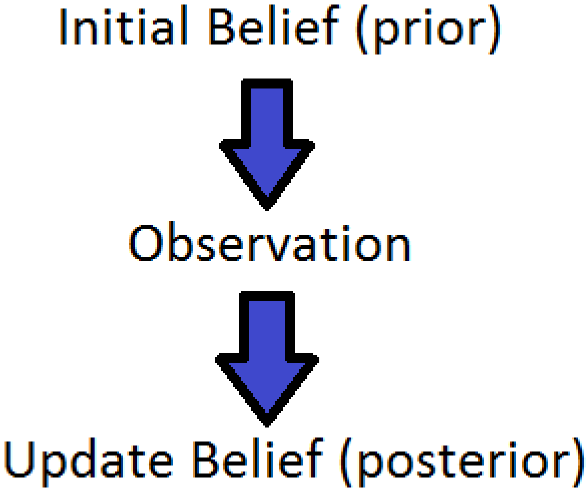
[†] "The Theory That Would Not Die" How Bayes' Rule Cracked the Enigma Code, Hunted Down Russian Submarines, and Emerged Triumphant from Two Centuries of Controversy"

Bayesian Paradigm

Initial Belief (prior)

Bayesian Paradigm

Bayesian Paradigm



Motivation and History

The allegory of our statistical lives

- Most of us are born Bayesians.

> *"It is remarkable that this science (probability), which originated in the consideration of games of chance, should have become the most important object of human knowledge."*
>
> $\sim$ Pierre-Simon de Laplace (1749-1827)

- **McGrayne WWII (16:02-30:23)**
- **McGrayne Air France Flight 447 (2:30-4:00)**

Motivation and History

- But by adolescence...

  - Fisher's work at Rothamstad advanced experimental design.
    McGrayne Fisher (12:44-16:00)
    McGrayne Obscurity (29:57-36:31)
  - Are these methods appropriate for observational studies?
- Today's message- "It's never too late to have a happy childhood."

Motivation and History

Why go Bayesian?

- Another powerful tool for your tool kit.
  - measurement error from misclassification
  - complex dependencies among observations
  - missing data
- Frequentist probabilities don't always make intuitive sense.
  - **McGrayne Frequentist (9:56-12.44)**
  - Example: Probability of an H-bomb accident.
- Bayesian "credible intervals" can be correctly interpreted by intro stat students.
- P-values and confidence intervals are somewhat ill defined.
  - Example: A study finds that out of 24 subjects with lung cancer 7 are female.

Motivation and History

Example: Out of 24 air-to-air refueling attempts for an F-16 pilot, 7 involve a premature disconnect.

Image courtesy of the Edward Air Force Base homepage (www.edwards.af.mil)

Motivation and History

Example: 24 air-to-air refueling attempts involve 7 premature disconnects.

Status	Status
0	0
1	0
0	0
0	0
1	0
0	0
1	0
1	0
1	0
0	1
0	0
1	0

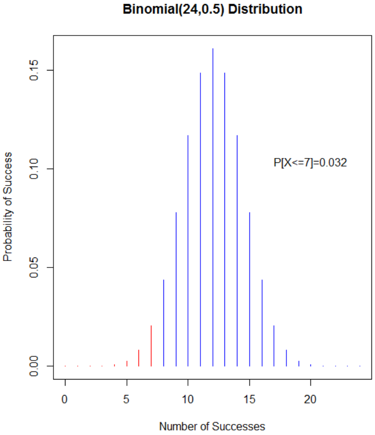


Figure: One-tailed Binomial p-value.

Motivation and History

Example: 24 air-to-air refueling attempts involve 7 premature disconnects.

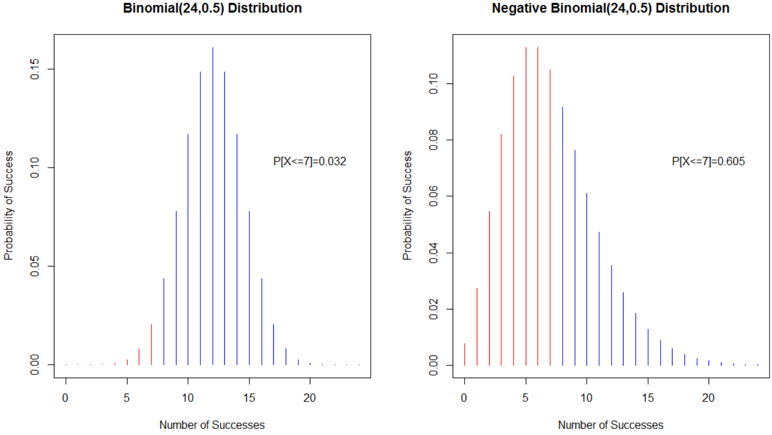


Figure: P-values depend on the researcher's intention.

Experiment, Outcome, Sample Space, and Event

Experiment

A process or procedure for which there is more than one possible outcome.

Outcome

The results of a single trial of an experiment.

Sample Space S

The collection of all possible outcomes of an experiment.

Event

A subset of the S. A collection of outcomes of interest.

Examples of Sample Space

- Experiment 1: Tossing one coin one time: $S = \{H, T\}$

- Experiment 2: Tossing one coin 4 times:
- How many ways to get
  - all Tails?
  - at least 2 Tails?
  - at least 1 Head?

$$S = \left\{ \begin{array}{cccc} H & H & H & H \\ H & H & H & T \\ H & H & T & H \\ H & T & H & H \\ H & H & T & T \\ H & T & T & H \\ H & T & H & T \\ H & T & T & T \\ T & H & H & H \\ T & H & H & T \\ T & H & T & H \\ T & T & H & H \\ T & H & T & T \\ T & T & H & T \\ T & H & T & T \\ T & T & T & T \end{array} \right\}$$

Examples of Sample Space

- Rolling one die: $S = \{1, 2, 3, 4, 5, 6\}$
- Rolling two dice:

S

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Figure: The sample space for rolling two dice.

Example of an Event

Let A be the event that the sum of the dice is 6.

S

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
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Figure: The sample space for rolling two dice.

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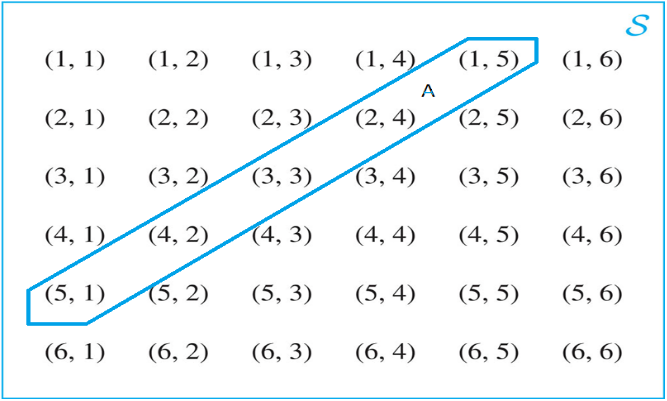


Figure: The sample space for rolling two dice.

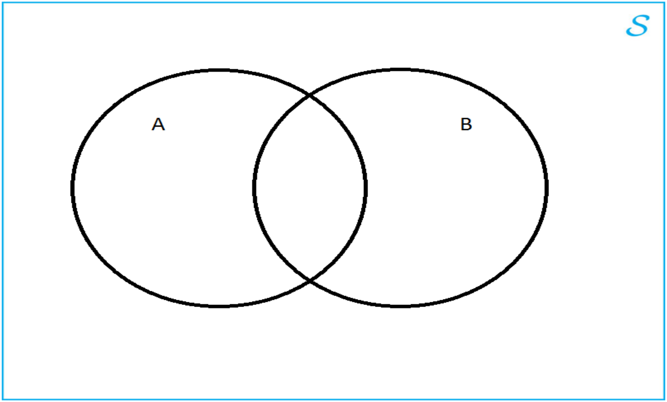
Axioms of Probability

For a sample space S

1. the probability of the i^{th} outcome is p_i
2. $0 \leq p_i \leq 1$
3. $\sum_{i=1}^{n} p_i = 1$

Venn Diagrams

It is often useful to express events in S as a Venn Diagram

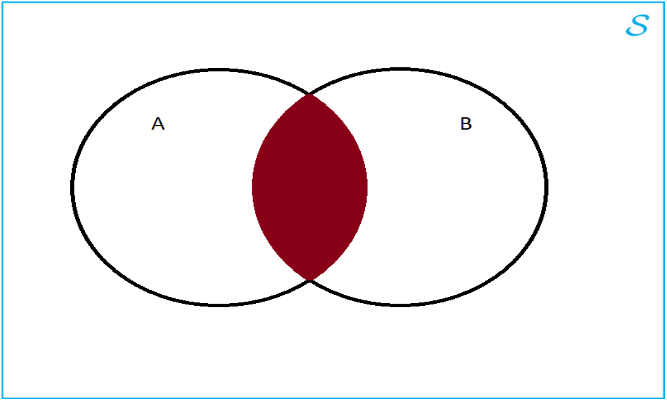


- gives you a feel for the outcomes common to the events.
- depicts event/S ratio.

Combinations of Events

Intersection of Events ($A \cap B$)

The set of outcomes that belong to both A "AND" B.



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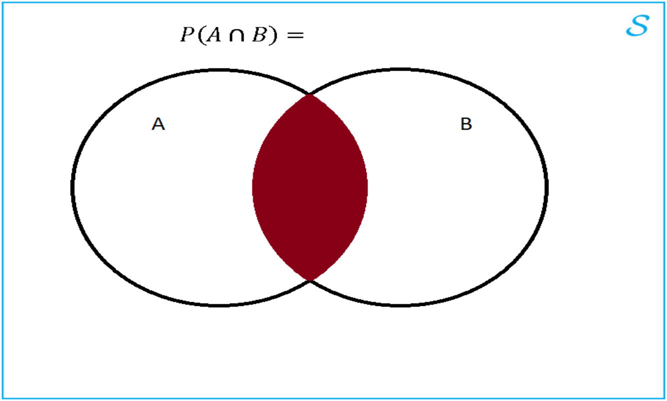
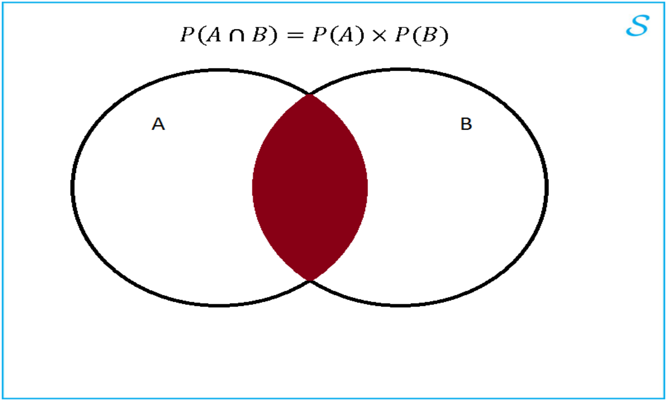


Combinations of Events

Intersection of Events $(A \cap B)$

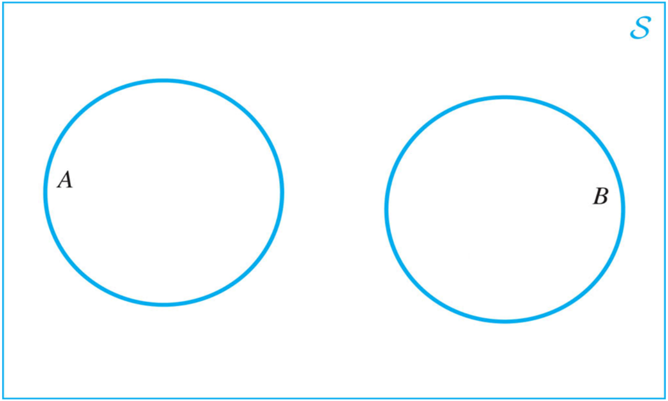
For **independent** A and B



Combinations of Events

Mutually Exclusive (disjoint) Events

Two events A and B are said to be mutually exclusive if $(A \cap B) = \emptyset$.



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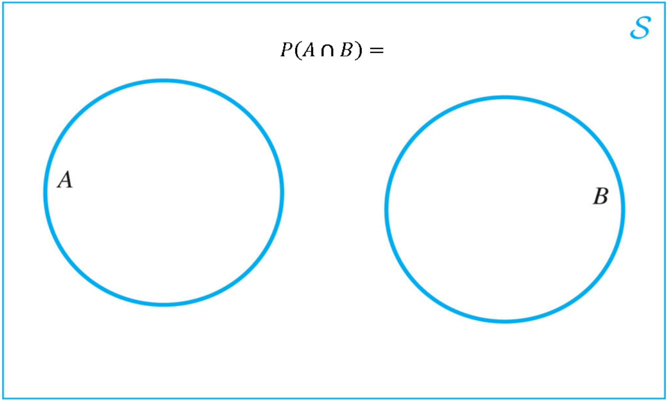

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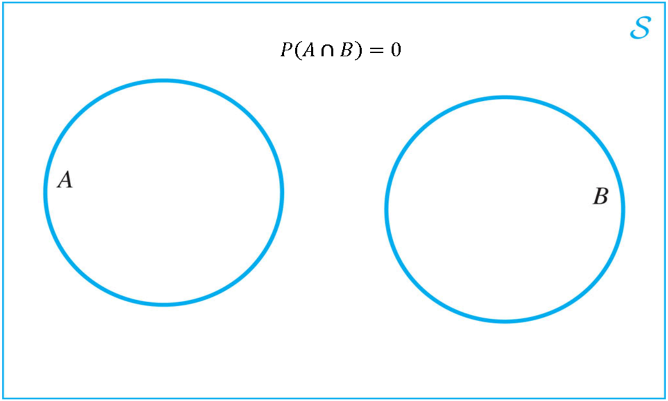

Combinations of Events

Union of Events ($A \cup B$)

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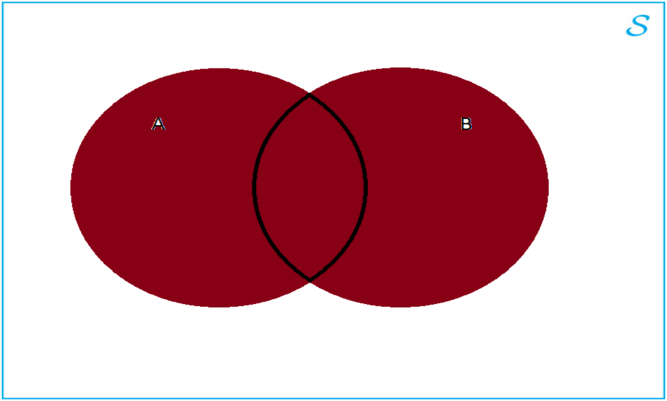

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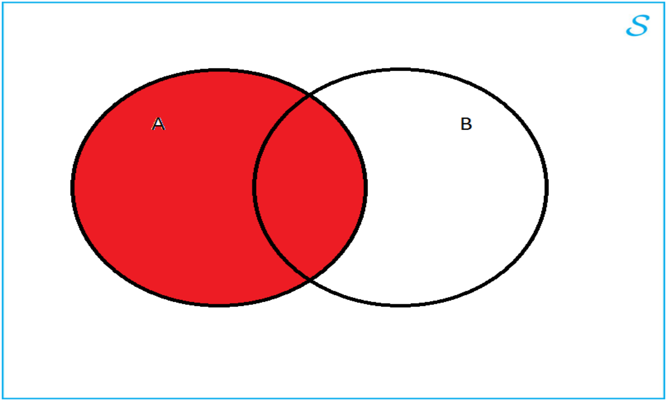

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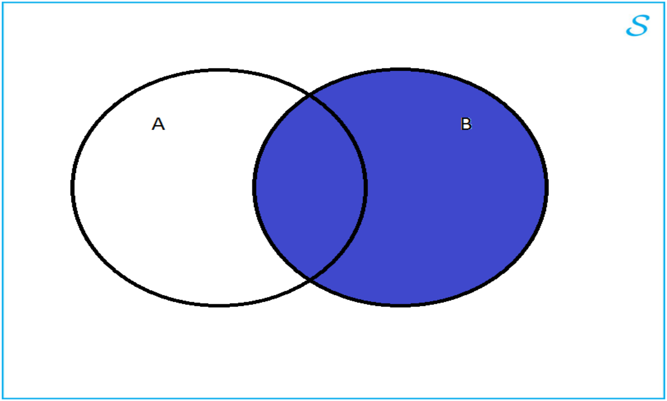

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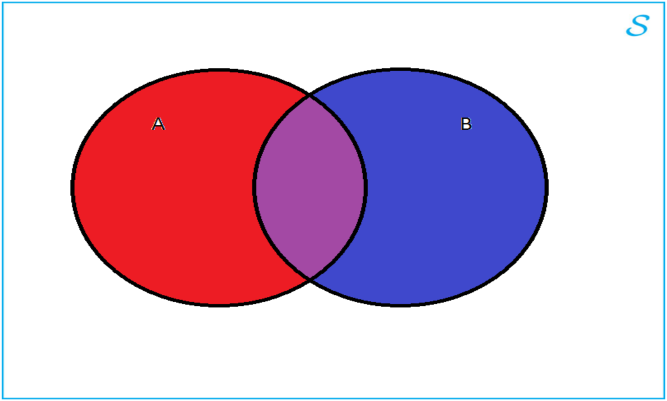

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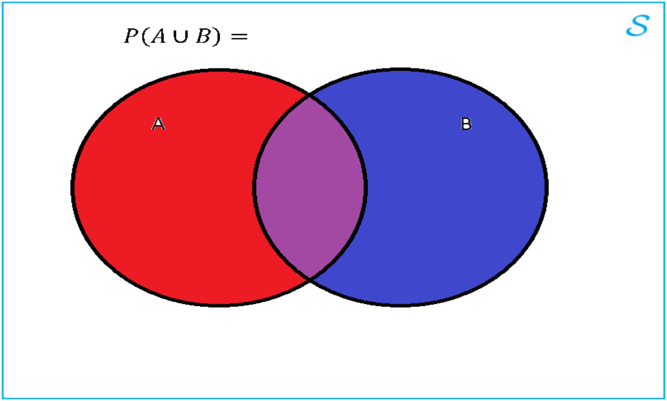

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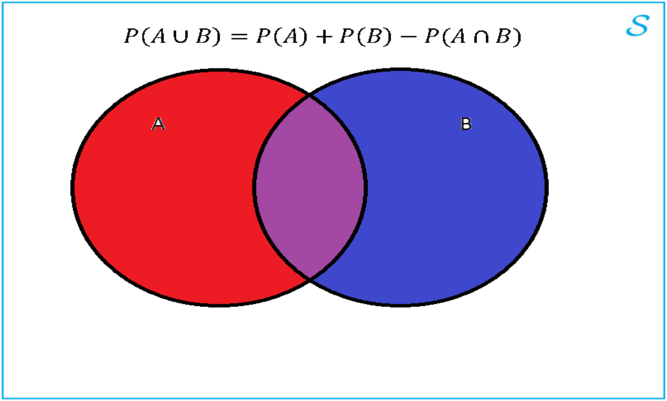
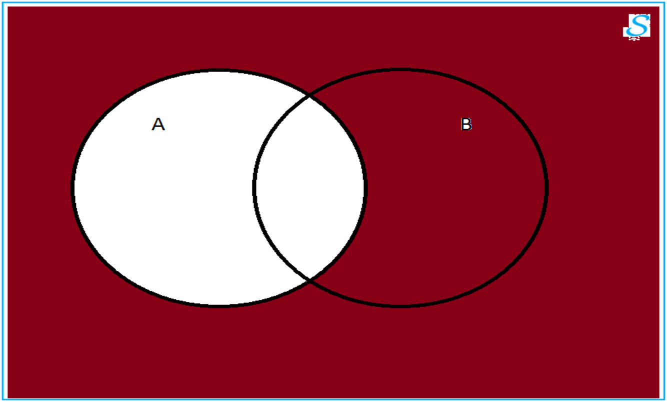


Combinations of Events

Complement of A, A^c

The set of all outcomes that are not in A.



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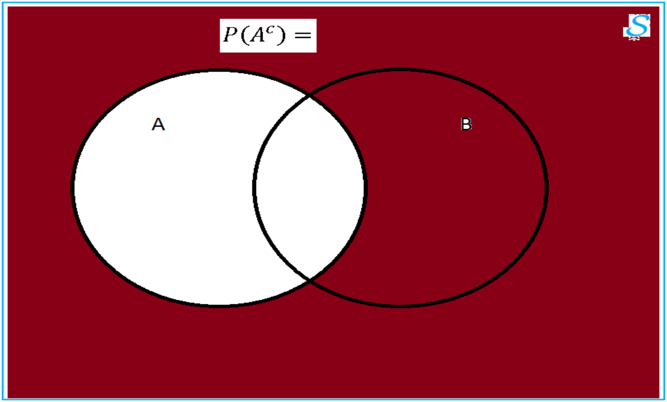

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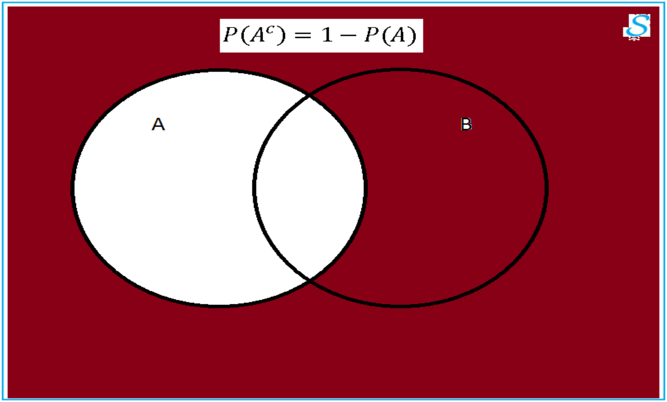

Example: Risk of Mortality during the West Africa Ebola Virus Outbreak of 2014†

	Total
Yes	11310
No	17306
Total	28616

P(Yes)=11310/28616≈0.40.
What if we have additional information such as location (country) of the case?
What is the probability of mortality given the location of the case (i.e. $P(Yes|location)$)?

- $P(Yes|Guinea) = 2544/3814 \approx 0.67$.
- $P(Yes|SierraLeone) = 3956/14124 \approx 0.28$.
- $P(Yes|Liberia) = 4810/10678 \approx 0.45$.

† http://www.cdc.gov/vhf/ebola/outbreaks/2014-west-africa/case-counts.html

What is a Conditional Probability?

Conditional Probability of Events A and B

If A and B are events in S, and $P(B) > 0$, then the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

and

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Note: This gives us the general formulas for computing $P(A \cap B)$.

- $P(A \cap B) = P(A)P(B|A)$.
- $P(A \cap B) = P(B)P(A|B)$.
- Note: $P(A)P(B|A) = P(B)P(A|B)$

Independence

Independence

Two events A and B are said to be independent if the occurence of one event in no way influences the probability of occurence of the other event. More formally this means

$$P(A|B) = P(A)$$

or equivalently

$$P(B|A) = P(B)$$

So, for independent events A and B, $P(A \cap B) = P(A)P(B)$.

Let's contrast independent events with mutually exclusive events.

	Independent	Mutually Exclusive
$P(A \cap B)$	$P(A) * P(B)$	$\emptyset$
$P(A\|B)$	$P(A)$	0
$P(B\|A)$	$P(B)$	0

Example

In 2012, the prevalence of HIV among the general U.S. population was estimated to be 0.38% (www.cdc.gov). Rapid diagnostic tests have been developed to test for the presence of HIV infection in as little as 10 minutes. Suppose one such diagnostic test was evaluated in a case-control study and it was observed that out of 1000 subjects with HIV, 967 obtained positive test results and out of 1000 subjects with out HIV, 985 obtained negative test results.

In summary

- $P(D+) = 0.0038 \Rightarrow P(D-) = 1 - 0.0038 = 0.9962$
- $P(T+|D+) = 0.967 \Rightarrow P(T-|D+) = 1 - 0.967 = 0.033$
- $P(T-|D-) = 0.985 \Rightarrow P(T+|D-) = 1 - 0.0985 = 0.015$

Suppose a person in the U.S. is selected at random, given the test, and the result is positive. Should this person begin treatment?

- $P(D+|T+)$?

Venn Diagram of the Example

Venn Diagram of the Example

Note

- $P(D+\cap T+) = P(D+)P(T+|D+)$
- $P(D-\cap T+) = P(D-)P(T+|D-)$
- $P(T+) = P(D+)P(T+|D+) + P(D-)P(T+|D-)$

$$P(D+|T+) = \frac{P(D+\cap T+)}{P(T+)} = \frac{P(D+)P(T+|D+)}{P(D+)P(T+|D+)+P(D-)P(T+|D-)}.$$

Example (continued)

Recall

- $P(D+) = 0.0038 \Rightarrow P(D-) = 1 - 0.0038 = 0.9962$
- $P(T+|D+) = 0.967 \Rightarrow P(T-|D+) = 1 - 0.967 = 0.033$
- $P(T-|D-) = 0.985 \Rightarrow P(T+|D-) = 1 - 0.985 = 0.015$

$P(D+|T+) = \frac{P(D+)P(T+|D+)}{P(D+)P(T+|D+)+P(D-)P(T+|D-)} =$
$\frac{(0.0038)(0.967)}{(0.0038)(0.967)+(0.9962)(0.015)} = 0.1974$

What does this mean?