A Comprehensive Introduction to the Black–Scholes Option Pricing Model

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Abstract

This document provides a self-contained introduction and full derivations of the Black–Scholes option pricing model. We cover historical context, model assumptions, derivation of the Black–Scholes–Merton partial differential equation, the closed-form solution for European options, the Greeks, and model limitations.

Contents

1 Introduction and Historical Context

Options are financial derivatives granting the right, but not obligation, to buy (call) or sell (put) an underlying asset at a predetermined strike price K on or before maturity T. In 1973, Fischer Black and Myron Scholes published their seminal paper deriving a closed-form formula for European option prices under idealized market conditions; Robert Merton simultaneously extended the model using continuous-time stochastic calculus.

2 Core Assumptions of the Model

A. *Underlying Price Dynamics:* The asset price S_t follows a geometric Brownian motion under the real-world probability measure: $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$, where μ is the drift, $\sigma > 0$ the volatility, and W_t a standard Brownian motion.

B. *Frictionless Markets:* No transaction costs or taxes; trading is continuous.

C. *No Arbitrage:* Markets do not admit risk-free profit.

D. *Constant Parameters:* μ, σ, and the risk-free rate r are constant.

E. *Borrowing/Lending:* Unlimited borrowing and lending at the constant risk-free rate r.

F. *Dividends:* Underlying pays no dividends (extensions exist for continuous yield).

G. *European Exercise:* Options can only be exercised at maturity T.

3 Derivation of the Black–Scholes–Merton PDE

Let $V(S,t)$ denote the price of an option on S_t. Under assumptions above, we derive the governing PDE.

3.1 Itô's Lemma

Lemma 3.1 (Itô's Lemma)**.** *If $f = f(S_t, t)$ is twice continuously differentiable in S and once in t, then*

$$df = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial S}\,dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\,(dS_t)^2.$$

Since $(dW_t)^2 = dt$ and $(dt)^2 = dt\,dW_t = 0$, and $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$, apply to $V(S_t,t)$:

$$dV = V_t\,dt + V_S\,dS_t + \tfrac{1}{2}V_{SS}\,(dS_t)^2 = \Big(V_t + \mu S V_S + \tfrac{1}{2}\sigma^2 S^2 V_{SS}\Big)\,dt + \sigma S V_S\,dW_t.$$

3.2 Risk-Free Hedge

Construct a portfolio Π holding one option short and Δ shares of underlying:

$$\Pi = -V + \Delta S.$$

Its change in value:

$$d\Pi = -\,dV + \Delta\,dS = -\big(V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + \mu S V_S\big)\,dt - \sigma S V_S\,dW_t + \Delta(\mu S\,dt + \sigma S\,dW_t).$$

Choose $\Delta = V_S$ to eliminate the stochastic term $\sigma S(\,\Delta - V_S)\,dW_t = 0$. Then

$$d\Pi = -\Big(V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + \mu S V_S\Big)\,dt + V_S \mu S\,dt = -\Big(V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS}\Big)\,dt.$$

No arbitrage implies the risk-free return on Π must be $r\Pi\,dt$:

$$d\Pi = r\Pi\,dt = r(-V + V_S S)\,dt.$$

Hence

$$-\Big(V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS}\Big) = r(-V + S V_S) \quad\Longrightarrow\quad V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

This is the *Black–Scholes–Merton PDE*:

$$\boxed{V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.}$$

4 The Black–Scholes Formula for European Options

We now solve the PDE for European call $C(S,t)$ with terminal payoff $C(S,T) = \max(S-K,0)$. Two approaches exist: (i) risk-neutral expectation and (ii) transform to heat equation. We present the risk-neutral method.

4.1 Risk-Neutral Valuation

Under the risk-neutral measure $\mathbb{Q}$, the drift μ is replaced by r. Thus

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}},$$

and by the Feynman–Kac theorem,

$$C(S,t) = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}\big[\max(S_T - K, 0) \mid S_t = S\big].$$

Set $\tau = T - t$. Under $\mathbb{Q}$,

$$\ln S_T \sim \mathcal{N}\big(\ln S + (r - \tfrac{1}{2}\sigma^2)\tau,\ \sigma^2\tau\big).$$

Hence

$$C(S,t) = e^{-r\tau}\int_{-\infty}^{\infty} \max(e^x - K)\,\frac{1}{\sqrt{2\pi\sigma^2\tau}}\exp\Big(-\frac{(x-m)^2}{2\sigma^2\tau}\Big)\,dx,$$

with $m = \ln S + (r - \frac{1}{2}\sigma^2)\tau$. Split the integral at x^* where $e^{x^*} = K$, i.e. $x^* = \ln K$:

$$C = e^{-r\tau}\int_{x^*}^{\infty} (e^x - K)\,\phi(x)\,dx.$$

Compute separately:

$$I_1 = e^{-r\tau}\int_{x^*}^{\infty} e^x\,\phi(x)\,dx, \quad I_2 = Ke^{-r\tau}\int_{x^*}^{\infty} \phi(x)\,dx.$$

By completing the square,

$$I_1 = S\,\Phi(d_1), \quad I_2 = Ke^{-r\tau}\,\Phi(d_2),$$

where

$$d_{1,2} = \frac{\ln\frac{S}{K} + \big(r \pm \frac{1}{2}\sigma^2\big)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}.$$

Thus the celebrated *Black–Scholes formula*:

$$\boxed{C(S,t)\ =\ S\,\Phi(d_1)\ -\ Ke^{-r\tau}\,\Phi(d_2).}$$

A European put price follows by put–call parity:

$$P(S,t) = C(S,t) + Ke^{-r\tau} - S.$$

5 The Greeks: Sensitivities of Option Prices

Define $\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Then:

5.1 Delta

$$\Delta_C = \frac{\partial C}{\partial S} = \Phi(d_1), \qquad \Delta_P = \Phi(d_1) - 1.$$

5.2 Gamma

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\phi(d_1)}{S\sigma\sqrt{\tau}}.$$

5.3 Vega

$$\nu = \frac{\partial C}{\partial \sigma} = S\,\phi(d_1)\,\sqrt{\tau}.$$

5.4 Theta

$$\Theta_C = \frac{\partial C}{\partial t} = -\frac{S\phi(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2).$$

5.5 Rho

$$\rho_C = \frac{\partial C}{\partial r} = K\tau e^{-r\tau}\Phi(d_2).$$

Each Greek measures sensitivity to a parameter: Δ to spot, Γ to convexity, ν to volatility, Θ to time decay, ρ to interest rates.

6 Limitations of the Model

- *Constant Volatility:* Real markets exhibit volatility smiles/skews.
- *No Jumps:* Underlying with jumps or heavy tails violate lognormality.
- *Continuous Trading:* Discrete rebalancing introduces hedging error.
- *No Dividends:* Extensions exist for continuous or discrete yields.
- *European Only:* American options require free-boundary methods.
- *Frictionless Markets:* Ignores transaction costs and liquidity constraints.

7 Conclusion

The Black–Scholes model provides an elegant, tractable framework for European option pricing under idealized conditions. Its derivation via hedging arguments and closed-form solution via risk-neutral expectation are foundational in quantitative finance. Real-world deviations motivate extensions such as stochastic volatility, jump-diffusion models, and local volatility frameworks.

References

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