

The Black-Scholes Option Pricing Model: A Comprehensive Introduction

Abstract

This document provides a complete mathematical treatment of the Black-Scholes option pricing model. We derive the Black-Scholes-Merton partial differential equation, solve it to obtain the famous Black-Scholes formula for European options, analyze the sensitivity measures (Greeks), and discuss the model's assumptions and limitations. All derivations are presented in full detail with step-by-step explanations.

1 Introduction and Historical Context

The Black-Scholes model, developed by Fischer Black, Myron Scholes, and Robert Merton in 1973, revolutionized financial economics by providing the first widely accepted framework for pricing options. Merton and Scholes received the 1997 Nobel Prize in Economics for this work (Black had passed away by then).

The model provides a theoretical estimate of the price of European-style options and has become fundamental in both theoretical and applied finance.

2 Core Assumptions

The Black-Scholes model makes several key assumptions:

1. The underlying asset price follows geometric Brownian motion with constant drift μ and volatility σ .
2. No arbitrage opportunities exist.
3. The risk-free interest rate r is constant and known.
4. The underlying pays no dividends (this assumption can be relaxed).
5. Markets are frictionless: no transaction costs or taxes.
6. Trading is continuous.
7. Short selling is permitted with full use of proceeds.
8. There are no restrictions on borrowing or lending at the risk-free rate.

3 Geometric Brownian Motion

The underlying asset price S_t is modeled by the stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (1)$$

where W_t is a Wiener process.

Lemma 1 (Itô's Lemma). *Let X_t be an Itô process $dX_t = \mu_t dt + \sigma_t dW_t$ and $f(t, X_t)$ be a $C^{1,2}$ function. Then:*

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t \quad (2)$$

4 The Black-Scholes-Merton Differential Equation

Consider a portfolio Π consisting of a short position in one option and a long position in Δ shares:

$$\Pi = -V + \Delta S \quad (3)$$

The change in portfolio value over dt is:

$$d\Pi = -dV + \Delta dS \quad (4)$$

Apply Itô's Lemma to $V(S, t)$:

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t \quad (5)$$

Substituting dS and dV into $d\Pi$:

$$d\Pi = \left(-\frac{\partial V}{\partial t} - \mu S \frac{\partial V}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \Delta \mu S \right) dt + \left(-\sigma S \frac{\partial V}{\partial S} + \Delta \sigma S \right) dW_t \quad (6)$$

To eliminate risk, set $\Delta = \frac{\partial V}{\partial S}$ (delta hedging). The portfolio becomes riskless and must earn the risk-free rate:

$$d\Pi = r\Pi dt \quad (7)$$

This leads to the Black-Scholes-Merton PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (8)$$

5 The Black-Scholes Formula

The solution to the BSM PDE with boundary condition $V(S, T) = \max(S - K, 0)$ for a call option is:

Theorem 2 (Black-Scholes Formula). *The price of a European call option is:*

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \quad (9)$$

where:

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t} = \frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

and N is the standard normal CDF.

Proof. We transform the PDE using:

$$\begin{aligned} \tau &= T - t \\ x &= \ln S \\ v(x, \tau) &= V(S, t) \end{aligned}$$

This converts the BSM PDE to the heat equation:

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 v}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{\partial v}{\partial x} - rv \quad (10)$$

Further transformation with $v(x, \tau) = e^{-\alpha x - \beta \tau} u(x, \tau)$ for appropriate α, β yields:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} \quad (11)$$

The solution is obtained via the convolution of the initial condition with the fundamental solution, then reversing all transformations. \square

For a put option, by put-call parity:

$$P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1) \quad (12)$$

6 The Greeks

The sensitivities of the option price to various parameters:

6.1 Delta

$$\Delta_C = \frac{\partial C}{\partial S} = N(d_1) \quad (13)$$

For puts:

$$\Delta_P = \frac{\partial P}{\partial S} = N(d_1) - 1 \quad (14)$$

6.2 Gamma

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{n(d_1)}{S\sigma\sqrt{T-t}} \quad (15)$$

where n is the standard normal PDF.

6.3 Vega

$$\nu = \frac{\partial C}{\partial \sigma} = Sn(d_1)\sqrt{T-t} \quad (16)$$

6.4 Theta

$$\Theta_C = \frac{\partial C}{\partial t} = -\frac{Sn(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2) \quad (17)$$

6.5 Rho

$$\rho_C = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2) \quad (18)$$

7 Limitations of the Model

- Constant volatility assumption (volatility smile/skew observed in markets)
- Constant interest rate assumption
- No transaction costs or taxes
- Continuous trading assumption
- Log-normal distribution may not match empirical asset returns
- American options and early exercise not handled

8 Conclusion

The Black-Scholes model remains foundational in financial mathematics despite its limitations. Its mathematical elegance and relative simplicity have made it a cornerstone of modern option pricing theory, with numerous extensions developed to address its shortcomings.